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Exponential concentration of cover times

Alex Zhai*

Abstract

We prove an exponential concentration bound for cover times of general graphs in terms of the Gaussian free field, extending the work of Ding, Lee, and Peres [8] and Ding [7]. The estimate is asymptotically sharp as the ratio of hitting time to cover time goes to zero.

The bounds are obtained by showing a stochastic domination in the generalized second Ray-Knight theorem, which was shown to imply exponential concentration of cover times by Ding in [7]. This stochastic domination result appeared earlier in a preprint of Lupu [22], but the connection to cover times was not mentioned.

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1 Introduction

Let $G = (V, E)$ be a finite undirected graph, possibly with self-loops and multiple edges. For the continuous time simple random walk on G started at a given vertex $v_0 \in V$, define τ_{cov} to be the first time that all the vertices in V have been visited at least once. This quantity, known as the *cover time*, is of fundamental interest in the study of random walks.

Another fundamental object in the study of random walks on graphs is the *Gaussian free field* (GFF). For purposes of stating our main result, let us define the GFF $\{\eta_x\}_{x\in V}$ on G with $\eta_{v_0} = 0$ to be the Gaussian process given by covariances $\mathbb{E}(\eta_x - \eta_y)^2 = R_{\text{eff}}(x, y)$, where R_{eff} denotes effective resistance. More background is given in Section 2.

Our main result is the following concentration bound on the cover time in terms of the Gaussian free field.

Theorem 1.1. *Let $G = (V, E)$ be a finite undirected graph with a specified initial vertex $v_0 \in V$. Let $\{\eta_x\}_{x\in V}$ be the Gaussian free field on G with $\eta_{v_0} = 0$. Define the quantities*

$$M = \mathbb{E} \max_{x\in V} \eta_x, \quad R = \max_{x,y\in V} R_{\text{eff}}(x, y) = \max_{x,y\in V} \mathbb{E}\left(\eta_x - \eta_y\right)^2 .$$

*Stanford University. E-mail: azhai@stanford.edu

Then, there are universal constants c and C such that for the continuous time random walk started at v_0, we have

$$\mathbb{P}\left(\left|\tau_{\text{cov}} - |E|M^2\right| \geq |E|(\sqrt{\lambda R} \cdot M + \lambda R)\right) \leq Ce^{-c\lambda}$$

for any $\lambda \geq C$.

Remark 1.2. Our result is most easily stated for a continuous time random walk, i.e. a random walk having the same jump probabilities as a simple random walk, but whose times between jumps are i.i.d. unit exponentials. However, note that if a continuous time random walk has run for time t, then the number of jumps it has made has Poisson distribution with mean t, which exhibits Gaussian concentration with fluctuations of order $\sqrt{t}$. Thus, Theorem 1.1 can be easily translated into a similar bound for discrete random walks.

Remark 1.3. Note that the definition of M is given in terms of a starting vertex v_0, but it does not depend on v_0. Indeed, let v_0' be another starting vertex. Then, $\eta' = \{\eta_x - \eta_{v_0'}\}_{x\in V}$ has the law of a GFF with $\eta'_{v_0'} = 0$, and

$$\mathbb{E} \max_{x\in V} \eta'_x = \mathbb{E} \max_{x\in V} \eta_x.$$

Remark 1.4. We actually show Theorem 1.1 in the slightly more general setting of electrical networks, which are introduced in Section 2.

We prove Theorem 1.1 following the approach first appearing in a paper of Ding, Lee, and Peres [8] and later refined by Ding [7]. Indeed, Ding observed that Theorem 1.1 is implied by a certain stochastic domination; in [7], the domination was proved for trees, but the general case was left as conjecture ([7], Question 5.2). Thus, the key missing piece is to prove the stochastic domination for general graphs; this is done in Theorem 3.1, whose proof is the main content of this paper.

In relation to [8] and [7], Theorem 1.1 extends Theorem 1.2 of [7], which gave the same concentration bound for trees. It also sharpens Theorem 1.1 of [8], where the equivalence of cover times and $|E|M^2$ (in the notation of Theorem 1.1) was proven up to a universal multiplicative constant. By "sharpen", we mean that we are able to remove the constant factor under the assumption $\sqrt{R} \ll M$. We mention that this was done already for bounded-degree graphs in Theorem 1.1 of [7], albeit without exponential tail bounds.

The condition $\sqrt{R} \ll M$ is a relatively mild one. Indeed, define $\tau_{\text{hit}}(x, y)$ to be the time it takes for a random walk started at x to hit y, and define

$$t_{\text{hit}} = \max_{x,y\in V} \mathbb{E}\tau_{\text{hit}}(x, y), \quad t_{\text{cov}} = \max_{x\in V} \mathbb{E}_x\tau_{\text{cov}},$$

where in the definition of t_{cov}, $\mathbb{E}_x$ denotes the expectation for the random walk started at x. The well-known commute time identity ([21], Proposition 10.6) states that

$$\mathbb{E}\tau_{\text{hit}}(x, y) + \mathbb{E}\tau_{\text{hit}}(y, x) = 2|E| \cdot R_{\text{eff}}(x, y).$$

It follows that

$$t_{\text{hit}} \geq |E| \cdot R.$$

On the other hand, Theorem 1.1 of [8] states that for some universal constant $C > 0$, we have

$$C^{-1}|E| \cdot M^2 \leq t_{\text{cov}} \leq C|E| \cdot M^2. \tag{1.1}$$

It follows that

$$\frac{R}{M^2} \leq C \cdot \frac{t_{\text{hit}}}{t_{\text{cov}}},$$

so $\sqrt{R} \ll M$ holds whenever $t_{\text{hit}} \ll t_{\text{cov}}$. We obtain the following corollary.

Corollary 1.5. *Let $G = (V, E)$, v_0, η, M, and R be as in Theorem 1.1. Then,*

$$\left(1 - C\sqrt{\frac{t_{\text{hit}}}{t_{\text{cov}}}}\right) \cdot |E| \cdot M^2 \;\leq\; t_{\text{cov}} \;\leq\; \left(1 + C\sqrt{\frac{t_{\text{hit}}}{t_{\text{cov}}}}\right) \cdot |E| \cdot M^2,$$

for a universal constant C.

Remark 1.6. We can also deduce the statement (1.1) directly from Theorem 1.1. This yields a simpler proof of Theorem 1.1 in [8] and a self-contained derivation of Corollary 1.5.

To make the deduction, note that $R = O(M^2)$. Then, Theorem 1.1 implies $t_{\text{cov}} = |E| \cdot (M^2 + O(\sqrt{R}M))$. This immediately gives the upper bound $t_{\text{cov}} = O(|E| \cdot M^2)$ and also gives the lower bound $t_{\text{cov}} = \Omega(|E| \cdot M^2)$ as long as $R < cM^2$ for some sufficiently small constant c. In the remaining case that $R \geq cM^2$, we may instead use the inequality $t_{\text{cov}} \geq t_{\text{hit}} \geq |E| \cdot R \geq c|E| \cdot M^2$, establishing (1.1) in all cases.

Remark 1.7. There is a deterministic polynomial-time approximation scheme (PTAS) due to Meka [26] for computing the supremum of a Gaussian process. Applying this to the quantity M gives a PTAS for t_{cov} when $t_{\text{hit}} \ll t_{\text{cov}}$.

Conversely, it was shown by Aldous [1] that if t_{hit} is of the same order as t_{cov}, then the cover time cannot be concentrated about its expectation (see the introduction of [7] for a more detailed discussion).

The main tool for estimating cover times employed by [8] and [7] is the generalized second Ray-Knight theorem, which is an identity in law relating the Gaussian free field to the time spent at each vertex by a continuous time random walk. In fact, the upper bound on t_{cov} in Corollary 1.5 was previously established as Theorem 1.4 of [8] (the same argument also proves the corresponding upper tail estimate in Theorem 1.1).

As mentioned earlier, it was shown in [7] that the matching lower bound reduces to proving a stochastic domination in the generalized second Ray-Knight theorem, and we prove this in Theorem 3.1. The main idea for proving Theorem 3.1 is to view the random walk as Brownian motion on a metric graph. After writing up an early draft of the proof, it was pointed out to us that this idea appeared previously in a recent preprint of Lupu [22] to prove essentially the same result ([22], Theorem 3). In that context, the idea was mainly used to study the percolation of loop clusters ([22], Theorems 1 and 2; see also subsequent work by Sznitman [32]). However, the application to cover times was not mentioned.

Even though Theorem 3.1 uses the same ideas as Theorem 3 of [22], we include a proof in order to establish the result in the language of our specific application. Additionally, our exposition is intended to be more accessible to audiences interested in cover times of random walks.

1.1 Related work on cover times

Cover times have been studied in many papers over the last few decades. We highlight several of them below; see also §1.1 of [8] for further background.

We first mention some results relating cover times and hitting times. Clearly, $t_{\text{cov}} \geq t_{\text{hit}}$. A classical result of Matthews [25] is that on a graph of n vertices, $t_{\text{cov}} \leq t_{\text{hit}}(1+\log n)$. This was proved by a clever argument analogous to the analysis of the coupon collector's problem. Matthews also gave an expression for a lower bound, which was later shown by Kahn, Kim, Lovasz, and Vu [14] to approximate the cover time to within $(\log\log n)^2$.

In [1], Aldous analyzed a generalization of the coupon collector's problem. As a consequence, he showed that τ_{cov} is concentrated around its expectation with high probability as $\frac{t_{\text{hit}}}{t_{\text{cov}}} \to 0$. More precisely, for any $\epsilon > 0$, there is a small enough δ so that

$$\mathbb{P}\left(|\tau_{\text{cov}} - t_{\text{cov}}| \leq \epsilon t_{\text{cov}}\right) \geq 1 - \epsilon$$

whenever $\frac{t_{\text{hit}}}{t_{\text{cov}}} < \delta$. This shows qualitatively the concentration of cover times.

On the other hand, cover times have also been estimated for many specific classes of graphs, including regular graphs [15], lattices [33], and bounded degree planar graphs [13], to name a few. Precise asymptotics are known for the two-dimensional discrete torus [6] and regular trees [2].

More recently, a breakthrough was made by Ding, Lee, and Peres [8] whereby the cover time was given (up to a constant factor) in terms of the Gaussian free field. Their result gives in some sense a quantitative estimate of the cover time that works for any graph. As touched upon earlier, Ding [7] later removed the constant factor for trees and bounded degree graphs. We complete the picture by extending this to general graphs.

1.2 Outline

The remaining sections are organized as follows. In Section 2, we establish notation and provide a brief review of electrical networks, local times, Gaussian free fields, and the generalized second Ray-Knight theorem. The notation mostly follows [7]. Section 3 states the key stochastic domination result (Theorem 3.1) and gives a heuristic description of the proof. In Section 4, we apply Theorem 3.1 to analyze cover times and obtain Theorem 1.1. Finally, in Section 5, we provide the full details for proving Theorem 3.1.

2 Definitions and preliminaries

An *electrical network* G is a finite, undirected graph (V, E), allowing self-loops, together with positive weights on the edges called *conductances*. We use either c_{xy} or c_{yx} to denote the conductance of an edge (x, y), and for vertices $x, y \in V$ that do not share an edge, we define $c_{xy} = 0$. It is convenient to define the quantity $c_x = \sum_{y \in V} c_{xy}$, which we refer to as the *total conductance* at x.

The name "electrical network" comes from the fact that G can be used to model an electric circuit, where each edge (x, y) corresponds to placing a resistor with resistance $\frac{1}{c_{xy}}$ between vertices x and y. For any $x, y \in G$, we can define the *effective resistance* $R_{\text{eff}}(x, y)$ between x and y to be the physical resistance when a voltage is applied between x and y. Mathematically, this quantity can be defined as a certain minimum energy (see Chapter 9 of [21] for more background on effective resistance and electrical networks).

There is a canonical *discrete time random walk* on an electrical network defined by taking the transition probability from x to y to be $\frac{c_{xy}}{c_x}$. In the case where the non-zero conductances are all equal, this reduces to the simple random walk on the underlying graph.

We will also want to consider the *continuous time random walk* on an electrical network. This is a continuous time process $\{X_t\}_{t \in \mathbb{R}^+}$ which can be sampled by having the same transition probabilities as the discrete time walk but introducing unit exponential waiting times between transitions. (Contrast this with the discrete time random walk, which we can think of as having waiting times that are deterministically equal to 1.)

In what follows, unless otherwise specified, all the electrical networks we consider will have a distinguished vertex $v_0 \in V$, and all random walks will be assumed to start at v_0.

2.1 Local times

Let $X = \{X_t\}_{t \in \mathbb{Z}^+}$ be a discrete time random walk on an electrical network G. For each time t and vertex v, we define the quantity

$$L_t^X(v) = \sum_{i=0}^{t} \mathbf{1}_{\{X_i = v\}},$$

which counts the number of visits of X to v up to time t.

We also define a continuous analogue of $L_t^X(v)$. Suppose now that $X = \{X_t\}_{t\in\mathbb{R}^+}$ is a continuous time random walk on G. For any time $t \geq 0$ and vertex $v \in V$, we define the *local time* $\mathcal{L}_t^X(v)$ of X at v to be

$$\mathcal{L}_t^X(v) = \frac{1}{c_v}\int_0^t \mathbf{1}_{\{X_s=v\}}ds.$$

Note the factor of $\frac{1}{c_v}$; this is a convenient normalization for various formulas. When there is no risk of confusion about the random walk X, we will sometimes shorten the notation to $L_t(v)$ or $\mathcal{L}_t(v)$.

Clearly, the cover time is related to the local time; it is the first time that all local times are positive. For a continuous time random walk X, we have

$$\tau_{\text{cov}} = \inf\left\{t \geq 0 : \min_{x\in V}\mathcal{L}_t^X(x) > 0\right\}.$$

We will also frequently consider the first time that v_0 accumulates a certain amount of local time. We give a formal definition for this stopping time. For a continuous time random walk X and any $t > 0$, define the *inverse local time* $\tau^+(t)$ as

$$\tau^+(t) = \inf\{s \geq 0 : \mathcal{L}_s^X(v_0) \geq t\},$$

It will always be clear what X is, so it is not included in the notation for sake of brevity.

2.2 Gaussian free fields

For an electrical network $G = (V, E)$, the Gaussian free field η_S with boundary $S \subset V$ is defined to be a random variable taking values in the set $\mathbb{R}^{V\setminus S}$ of real-valued functions on $V \setminus S$. Its probability density at an element $f \in \mathbb{R}^{V\setminus S}$ is proportional to

$$\exp\left(-\frac{1}{4}\sum_{x,y\in V} c_{xy}(f(x) - f(y))^2\right), \tag{2.1}$$

where we define $f(x) = 0$ for each $x \in S$. For our purposes, Gaussian free fields will always have boundary $S = \{v_0\}$. Thus, if we refer to *the* Gaussian free field on some network, we will mean the one with this boundary, and we will drop the subscript S.

From (2.1) it is clear that η is a multidimensional Gaussian random variable. It is not too hard to calculate (e.g., Theorem 9.20 of [12]) that for all $x, y \in V$,

$$\mathbb{E}\left(\eta_x - \eta_y\right)^2 = R_{\text{eff}}(x, y),$$

which confirms that our definition of the GFF is consistent with the one given in the introduction. Noting that $\eta_{v_0} = 0$, the above formula completely determines the correlations of η in terms of the effective resistances.

The Gaussian free field comes into the picture via a class of identities known as Isomorphism Theorems. The first such theorems were proved independently by Ray [28] and Knight [16] relating the local times of Brownian motion to a 2-dimensional Bessel process. More generally, it turns out that for any strongly symmetric Borel right process, there is an identity relating its local times to an associated Gaussian process.

Inspired by formulas of Symanzik [29] and Brydges, Fröhlich, and Spencer [4], Dynkin [9] gave the first isomorphism of this type to be expressed in terms of Gaussian free fields. Various related identities were subsequently discovered by Marcus and Rosen [23], Eisenbaum [10], Le Jan [18], Sznitman [31], and others. There is a nice version of the isomorphism in the case of continuous time random walks on finite electrical networks, first appearing in [11] (see also Theorem 8.2.2 of the book by Marcus and Rosen [24]).

Theorem 2.1 (Generalized Second Ray-Knight Theorem)**.** *Let $G = (V, E)$ be an electrical network, with a given vertex $v_0 \in V$. Let $X = \{X_t\}_{t \geq 0}$ be a continuous time random walk on G started at v_0, and for any $t > 0$, define $\tau^+(t) = \inf\{s \geq 0 : \mathcal{L}^X_s(v_0) \geq t\}$ to be the first time that v_0 accumulates local time t. Then, we have*

$$\left\{ \mathcal{L}^X_{\tau^+(t)}(x) + \frac{1}{2}\eta_x^2 \right\}_{x \in V} \overset{law}{=} \left\{ \frac{1}{2}\left(\eta_x + \sqrt{2t}\right)^2 \right\}_{x \in V}.$$

For more background on isomorphism theorems, we refer the interested reader to [24] and [30]. See also [19] for information relating Gaussian free fields to loop measures.

3 Stochastic domination in the generalized second Ray-Knight theorem

We are now ready to state the key stochastic domination theorem, which is a variant of Theorem 3 in [22].

Theorem 3.1 (variant of [22], Theorem 3)**.** *Let $\tau^+(t)$ and η be as in Theorem 2.1. Then, we have*

$$\left\{ \sqrt{\mathcal{L}_{\tau^+(t)}(x)} : x \in V \right\} \preceq \frac{1}{\sqrt{2}} \left\{ \max\left(\eta_x + \sqrt{2t}, 0\right) : x \in V \right\},$$

where $\preceq$ denotes stochastic domination.

Theorem 3.1 extends Theorem 2.3 from [7], which proves the result for trees. The approach in [7] uses a Markovian property of local times for trees which does not seem to extend to general electrical networks. We take a different approach of embedding the finite-dimensional Gaussian free field inside a larger infinite-dimensional Gaussian free field, which has desirable continuity properties that were not apparent in the finite-dimensional setting. As mentioned in the introduction, we discovered while writing up our results that this idea appeared earlier in [22].

Let us first give a heuristic description of the approach. Recall that the continuous time random walk on an electrical network makes jumps at exponentially distributed random intervals. An equivalent way of sampling the continuous time random walk is to perform a Brownian motion along the edges of the network. By this we mean that our discrete state space V is replaced by a larger state space $\widehat{V}$ which includes not only the vertices in V but also each point along each edge of E (regarding the edges as line segments, so that $\widehat{V}$ is topologically a simplicial 1-complex). The object $\widehat{V}$ is known as a *metric graph* and arises in physics and chemistry (see e.g. §5 of [5]).

A Brownian motion on $\widehat{V}$ is, informally, a continuous Markov process $\widehat{X} = \{\widehat{X}(t)\}_{t \geq 0}$ taking values in $\widehat{V}$ that behaves like a one-dimensional Brownian motion on edges. The earliest rigorous development of this idea we could find was carried out by Baxter and Chacon [3]. See also [17] for a more recent treatment.

It turns out that the Gaussian free field $\widehat{\eta}$ on $\widehat{V}$ (without defining this precisely) is almost surely continuous in the topology of $\widehat{V}$.[1] We can also define a notion of local time $\mathcal{L}^{\widehat{X}}_t(v)$, and we can define the stopping time $\tau^+(t)$ analogously to the discrete case. For convenience, let us write $\widehat{\mathcal{L}}_t$ for $\mathcal{L}^{\widehat{X}}_{\tau^+(t)}$. With an appropriate normalization, the

[1] The Gaussian free field on $\widehat{V}$ can be constructed by sampling the GFF on V and then sampling Brownian bridges on each edge.

restrictions of $\widehat{\eta}$ and $\widehat{\mathcal{L}}_t$ to $V \subset \widehat{V}$ have the same laws as the corresponding objects on the original network $G = (V, E)$. The generalized second Ray-Knight theorem translates to

$$\left\{ \widehat{\mathcal{L}}_{\tau^+(t)}(v) + \frac{1}{2}\widehat{\eta}_v^2 : v \in \widehat{V} \right\} \stackrel{\text{law}}{=} \left\{ \frac{1}{2}\left(\widehat{\eta}'_v + \sqrt{2t}\right)^2 : v \in \widehat{V} \right\}, \tag{3.1}$$

where $\widehat{\eta}'$ is another copy of $\widehat{\eta}$, and c_v is a continuous analogue of the total conductance at a vertex.

Now, suppose that $\widehat{\eta}$ and $\widehat{\eta}'$ are coupled in a way so that the two sides in equation 3.1 are actually equal. Consider the function $f : \widehat{V} \to \mathbb{R}$ given by $f(x) = (\widehat{\eta}'_x + \sqrt{2t}) - \widehat{\eta}_x$. We have that $f(v_0) = \sqrt{2t} > 0$, f is continuous, and if $f(x) = 0$, then $\widehat{\mathcal{L}}_t(x) = 0$. It turns out that the set $U = \{v \in \widehat{V} : \widehat{\mathcal{L}}_t(v) > 0\}$ is connected, and clearly it includes v_0. It follows that $f(x) > 0$ for all $x \in U$, which is exactly the desired stochastic domination once we restrict to $V \subset \widehat{V}$.

The assertion that U is connected deserves some elaboration. It is intuitively clear that the closure of U should be connected, since any point $v \in \widehat{V}$ which accumulates positive local time must have been visited along some connected path from v_0 to v. Thus, every non-trivial segment along this path should have also accumulated positive local time.

On the other hand, it is not immediately obvious why U itself is connected, since there might be local times of 0 at isolated points. However, we can see heuristically that this pathology doesn't occur by the first Ray-Knight theorem (stated in Section 5.2 below). The first Ray-Knight theorem equates the local times of a certain stopped Brownian motion to the distance of a planar Brownian motion from the origin. Because planar Brownian motion is not point-recurrent, the local times are *all* positive almost surely, and in particular, the set of points with 0 local time does not have isolated points.

To avoid technicalities, we will not actually use Brownian motion in our proof. Instead, we will use a discrete approximation of Brownian motion and pass to the limit. Arguments involving the continuity of Gaussian free fields and positivity of local times will be translated into corresponding quantitative estimates.

4 Application to cover times

Before diving into the detailed proof of Theorem 3.1, let us explain how Theorem 3.1 implies Theorem 1.1. Essentially, by showing that various quantities are concentrated around their expectation, one can deduce results about cover times from statements about local times (such as Theorem 3.1). In fact, the exact same arguments used in proving Theorem 1.2 of [7] carry through, replacing Theorem 2.3 there with Theorem 3.1 of the previous section. For the sake of completeness, we repeat the main parts of the argument from [7]. It should be mentioned that the argument for the upper tail bound is originally from [8] (see §2.2).

First, we record two auxiliary results used in [7]. Recall the notation that $M = \mathbb{E}\max_{x\in V}\eta_x$ for the Gaussian free field η and $R = \max_{x,y\in V}\mathbb{E}\left(\eta_x - \eta_y\right)^2$.

Lemma 4.1 (Lemma 2.1 of [7])**.** *Let X be a continuous time random walk on an electrical network $G = (V, E)$. Let $c_{\text{tot}} = \sum_{x,y\in V} c_{xy}$ be the total conductance of G. For any $t \geq 0$ and $\lambda \geq 1$,*

$$\mathbb{P}\left(\left|\tau^+(t) - c_{\text{tot}}\cdot t\right| \geq \frac{1}{2}\left(\sqrt{\lambda R t} + \lambda R\right)c_{\text{tot}}\right) \leq 6\exp\left(-\frac{\lambda}{16}\right).$$

Proof. See Lemma 2.1 of [7] and the associated remark. We have replaced $2|E|$ by c_{tot}. □

The next result is a well-known Gaussian concentration bound. See for example Theorem 7.1, Equation (7.4) of [20].

Proposition 4.2. *Let $\{\eta_x : x \in S\}$ be a centered Gaussian process on a finite set S, and suppose $\mathbb{E}\eta_x^2 \le \sigma^2$ for all $x \in S$. Then, for $\alpha > 0$,*

$$\mathbb{P}\left(\left|\max_{x\in S}\eta_x - \mathbb{E}\max_{x\in S}\eta_x\right| \ge \alpha\right) \le 2\exp\left(-\frac{\alpha^2}{2\sigma^2}\right).$$

Note that by symmetry, $\max$ can be replaced by $\min$ in Proposition 4.2, which is the version that we will use. We now give a proof of Theorem 1.1, closely following the proof of Theorem 1.2 in [7].

Proof of Theorem 1.1. We will prove Theorem 1.1 in the slightly more general setting where $G = (V, E)$ is an electrical network. As before, define $c_{\text{tot}} = \sum_{x,y\in V} c_{xy}$.

We first estimate τ_{cov} in terms of τ^+. Let $\beta \ge 3$ be a parameter to be specified later. In what follows, we will often use the fact that

$$R = \max_{x,y\in V}\mathbb{E}\left(\eta_x - \eta_y\right)^2 \ge \max_{x\in V}\mathbb{E}\eta_x^2.$$

To prove an upper bound, let $t^+ = \frac{(M+\beta\sqrt{R})^2}{2}$, and define the event

$$E = \left\{\min_{x\in V}\left(\mathcal{L}_{\tau^+(t^+)}(x) + \frac{1}{2}\eta_x^2\right) \ge \frac{\beta^2 R}{8}\right\},$$

where η is an independent copy of the Gaussian free field as in Theorem 2.1. We also have by Proposition 4.2 that

$$\mathbb{P}\left(\min_{x\in V}\frac{1}{2}\left(\eta_x + \sqrt{2t^+}\right)^2 \le \frac{\beta^2 R}{8}\right) \le \mathbb{P}\left(\min_{x\in V}\left(\eta_x + \sqrt{2t^+}\right) \le \frac{\beta\sqrt{R}}{2}\right)$$

$$= \mathbb{P}\left(\min_{x\in V}\eta_x \le -M - \frac{\beta\sqrt{R}}{2}\right) \le 2e^{-\frac{\beta^2}{8}},$$

so that in light of the isomorphism theorem (Theorem 2.1),

$$\mathbb{P}\left(E^c\right) \le 2e^{-\frac{\beta^2}{8}}. \tag{4.1}$$

Suppose now that $\tau_{\text{cov}} > \tau^+(t^+)$. Then, $\mathcal{L}_{\tau^+(t^+)}(x) = 0$ for some $x \in V$. Since

$$\mathbb{P}\left(\eta_x^2 \ge \frac{\beta^2 R}{4}\right) \le 2e^{-\frac{\beta^2}{8}}$$

and η is independent of the random walk, it follows that

$$\mathbb{P}\left(E \,\middle|\, \tau_{\text{cov}} > \tau^+(t^+)\right) \le 2e^{-\frac{\beta^2}{8}}. \tag{4.2}$$

Combining equations (4.1) and (4.2), we conclude that

$$\mathbb{P}\left(\tau_{\text{cov}} > \tau^+(t^+)\right) \le \frac{2e^{-\frac{\beta^2}{8}}}{1 - 2e^{-\frac{\beta^2}{8}}} \le 6e^{-\frac{\beta^2}{8}}.$$

For the lower bound, let $t^- = \frac{(M-\beta\sqrt{R})^2}{2}$. By Theorem 3.1, we have

$$\mathbb{P}\left(\tau_{\text{cov}} < \tau^+(t^-)\right) = \mathbb{P}\left(\min_{x\in V}\mathcal{L}_{\tau^+(t^-)}(x) > 0\right) \le \mathbb{P}\left(\min_{x\in V}\left(\eta_x + \sqrt{2t^-}\right) > 0\right)$$

$$= \mathbb{P}\left(\min_{x\in V}\eta_x > -M + \frac{\beta\sqrt{R}}{2}\right) \le 2e^{-\frac{\beta^2}{2}},$$

where the last inequality follows again from Proposition 4.2.

Combining the upper and lower bounds, it follows that

$$\mathbb{P}\left(\tau^+(t^-) \le \tau_{\text{cov}} \le \tau^+(t^+)\right) \ge 1 - 8e^{-\frac{\beta^2}{8}}.$$

For $\lambda \ge 9$, we now take $\beta = \sqrt{\lambda}$. Note that

$$c_{\text{tot}} \cdot t^+ + \frac{1}{2}\left(\sqrt{\lambda R t^+} + \lambda R\right) c_{\text{tot}} = \frac{c_{\text{tot}}}{2}\left(M^2 + 3\sqrt{\lambda R}M + 3\lambda R\right)$$

$$c_{\text{tot}} \cdot t^- - \frac{1}{2}\left(\sqrt{\lambda R t^-} + \lambda R\right) c_{\text{tot}} = \frac{c_{\text{tot}}}{2}\left(M^2 - 3\sqrt{\lambda R}M - \lambda R\right),$$

so by Lemma 4.1,

$$\mathbb{P}\left(\tau^+(t^+) \ge \frac{c_{\text{tot}}M^2}{2} + \frac{3c_{\text{tot}}(\sqrt{\lambda R}M + \lambda R)}{2}\right) \le 6\exp\left(-\frac{\lambda}{16}\right)$$

$$\mathbb{P}\left(\tau^+(t^-) \le \frac{c_{\text{tot}}M^2}{2} - \frac{3c_{\text{tot}}(\sqrt{\lambda R}M + \lambda R)}{2}\right) \le 6\exp\left(-\frac{\lambda}{16}\right).$$

We thus conclude that for $\lambda \ge 9$,

$$\mathbb{P}\left(\left|\tau_{\text{cov}} - \frac{c_{\text{tot}}M^2}{2}\right| \ge \frac{3}{2}c_{\text{tot}}(\sqrt{\lambda R}M + \lambda R)\right) \le 20\exp\left(-\frac{\lambda}{16}\right).$$

We obtain Theorem 1.1 upon an appropriate rescaling of λ, noting that $c_{\text{tot}} = 2|E|$ in the case where all conductances are 1. □

5 Proof of Theorem 3.1

The goal of this section is to provide the detailed proof of Theorem 3.1. Our starting point is to form a discrete approximation of the metric graph described in the heuristic proof.

5.1 A discrete refinement of G

Recall our setting of an electrical network $G = (V, E)$ with conductances $\{c_{xy} : x, y \in V\}$. For each positive integer $N > 1$, we define a refinement $G_N = (V_N, E_N)$ by subdividing each edge $(x, y) \in E$ into a length N path whose vertices we denote by

$$\{x = v_{xy,0}, v_{xy,1}, \ldots, v_{xy,N} = y\}.$$

We thus have edges between $v_{xy,i}$ and $v_{xy,i+1}$ for each $0 \le i < N$. We will use $v_{yx,i}$ to denote the same vertex as $v_{xy,N-i}$, and we will regard V as a subset of V_N, so that a vertex $x \in V$ will sometimes be considered as a vertex in V_N.

We choose the conductances of G_N so that the effective resistance between $x, y \in V$ as vertices in G will be the same when they are considered as vertices in G_N. In particular, we set the conductance between $v_{xy,i}$ and $v_{xy,i+1}$ to be Nc_{xy}. Since the effective resistances are equivalent, G is in some sense a projection of G_N. The following proposition makes this explicit.

Proposition 5.1. *Let η be the GFF on G, and let X be a continuous time random walk on G. Let η_N and X_N denote the corresponding objects for G_N. Then, for any $t > 0$ we have the following two identities in law.*

$$\{\eta_{N,v} : v \in V\} \overset{law}{=} \{\eta_v : v \in V\}$$

$$\left\{\mathcal{L}^{X_N}_{\tau^+(t)}(x) : x \in V\right\} \overset{law}{=} \left\{\mathcal{L}^{X}_{\tau^+(t)}(x) : x \in V\right\}.$$

The identity between η_N and η is immediate from the equivalence of effective resistances. The identity between local times then follows from Theorem 2.1. However, there is also a very direct way to see the equivalence of local times which we now describe.

If $X_N(t)$ is a continuous time random walk on G_N started at v_0, then $X_N(t)$ induces a random walk $X_N^G(t)$ on G by only recording the time spent in V. More formally, define $t_0 = 0$, and for each $i \geq 0$, define

$$t_{i+1} = \inf\{t > t_i : X_N(t) \in V \text{ and } X_N(t) \neq X_N(t_i)\}.^2$$

Define also

$$s_i = \int_{t_i}^{t_{i+1}} \mathbf{1}_{\{X_N(s)=X_N(t_i)\}} ds$$

to be the amount of time spent in $X_N(t_i)$ during the time interval $[t_i, t_{i+1}]$.

Then, consider the V-valued process $X_N^G(t)$ which starts at v_0 and, for each i, jumps to $X_N(t_{i+1})$ at time $\sum_{j=1}^{i} s_j$. Note that if $X_N(t_i) = x \in V$, at the next jump X_N transitions to $v_{xy,1}$ with probability $\frac{c_{xy}}{c_x}$ for each y neighboring x in G. After that, X_N behaves like a simple random walk on $\mathbb{Z}$ started at 1 and stopped upon hitting either 0 (corresponding to $v_{xy,0} = x$) or N (corresponding to $v_{xy,N} = y$). Thus, with probability $\frac{N-1}{N}$ it returns to x, and with probability $\frac{1}{N}$ it hits y.

Consequently, between times t_i and t_{i+1}, the number of times X_N visits x is geometrically distributed with mean N, and so the accumulated local time s_i is exponentially distributed with mean N. Moreover, we see that

$$\mathbb{P}\left(X_N(t_{i+1}) = y \,|\, X_N(t_i) = x\right) = \frac{c_{xy}}{c_x},$$

so $X_N^G(t)$ has the same law as a continuous time random walk on G except that the waiting times between jumps are scaled by N. In particular, we have

$$\left\{\mathcal{L}^{X_N}_{\tau^+(t)}(x) : x \in V\right\} = \left\{\frac{1}{N} \cdot \mathcal{L}^{X_N^G}_{\tau^+(Nt)}(x) : x \in V\right\} \overset{law}{=} \left\{\mathcal{L}^{X}_{\tau^+(t)}(x) : x \in V\right\},$$

where X is a continuous time random walk on G. Note that the factor of N appearing in the middle expression comes from the normalization by total conductance at x, which differs for G and G_N.

5.2 Random walks on paths and the first Ray-Knight theorem

In preparation for our analysis of G_N, we need some technical results about random walks on paths. In this setting, it is a classical theorem proved independently by Ray and Knight that the local times of a continuous time random walk can be related to Brownian motion.

[2] We are taking our process X_N to be right continuous, so the infimum is achieved, and in particular $X_N(t_{i+1}) \in V$.

Theorem 5.2 (First Ray-Knight Theorem). *For any $a > 0$, let B_t be a standard one-dimensional Brownian motion started at $B_0 = a$, and let $T = \inf\{t : B_t = 0\}$. Let $\{W_t\}_{t\geq 0}$ be a standard two-dimensional Brownian motion. Then,*

$$\left\{\mathcal{L}_T^{B_t}(x) : x \in [0,a]\right\} \overset{law}{=} \left\{|W_x|^2 : x \in [0,a]\right\},$$

where $\mathcal{L}_T^{B_t}$ denotes the local time of Brownian motion.

In Section 2.1, we did not define the local time of Brownian motion, which requires some minor technicalities due to the fact that it can only be defined as a density. For background on Brownian local times and Theorem 5.2, we refer the reader to [27], Chapter 6. However, we will only use a discretized version of Theorem 5.2, where we restrict our attention to a finite set of values for x. This is equivalent to replacing the Brownian motion B_t with a continuous time random walk on a path.

Corollary 5.3. *Let $G = (V, E)$ be an electrical network whose underlying graph is a path, with vertices labeled $0, 1, 2, \ldots, N$ and conductances $c_{k,k+1}$ between k and $k+1$ for $0 \leq k < N$. Let X_t be a continuous time random walk on G started at $X_0 = N$, and let $T = \inf\{t : X_t = 0\}$. Define*

$$a_k = \sum_{i=0}^{k-1} \frac{1}{c_{i,i+1}},$$

and let $\{W_t\}_{t\geq 0}$ be a standard two-dimensional Brownian motion. Then,

$$\left\{\mathcal{L}_T^{X}(k) : 1 \leq k < N\right\} \overset{law}{=} \left\{|W_{a_k}|^2 : 1 \leq k < N\right\}.$$

Proof. The equivalence to Theorem 5.2 can be seen as follows. For the discrete time random walk on a path started at a vertex x, the time spent at x before hitting one of the endpoints is geometrically distributed. Analogously, for any $x \in \mathbb{R}$, let B_t be a Brownian motion started at x and stopped upon hitting $x - r$ or $x + s$. Then, a variant of [27], Lemma 6.30 (whose proof can be adapted) tells us that the local time accumulated at x is distributed as an exponential random variable with mean $\frac{rs}{r+s}$.

When $x = a_k$, $r = \frac{1}{c_{k,k-1}}$ and $s = \frac{1}{c_{k,k+1}}$, this corresponds to an exponential jump time from the vertex k in G, scaled by a factor of $\frac{1}{c_{k,k-1}+c_{k,k+1}}$ which appears in the definition of $\mathcal{L}_T^X(k)$. □

In light of Corollary 5.3, it is useful to know something about two-dimensional Brownian motion. For our purposes, we need the following estimate, which is a quantitative verson of the standard fact that two-dimensional Brownian motion is not point-recurrent.

Lemma 5.4. *Let W_t be a standard two-dimensional Brownian motion. For any $\epsilon \in (0,1)$ and $\lambda > 0$, we have*

$$\mathbb{P}\left(\inf_{\epsilon \leq t \leq 1} |W_t|^2 < \lambda\right) \leq \frac{2}{\log \epsilon^{-1}} + \frac{5}{\epsilon} \exp\left(-\frac{\log \lambda^{-1}}{\log \epsilon^{-1}}\right).$$

Proof. See Appendix. □

Finally, the next lemma shows that certain conditioned random walks on paths are equivalent to random walks on a path of different conductances. Thus, the first Ray-Knight theorem may be applied in a conditional setting as well. This will be important when we study random walk transitions on general electrical networks.

Lemma 5.5. *Let N be a positive integer and $r > 0$ a real number.*

Consider an electrical network $G = (V, E)$ whose underlying graph is a path, with vertices labeled $0, 1, 2, \ldots, N+1$. Suppose that the conductances are $c_{k,k+1} = 1$ for $0 \le k < N$ and $c_{N,N+1} = r$. Let $X = \{X_t\}_{t\ge 0}$ be a discrete time random walk on G started at N, and let τ be the first time that X hits 0 or $N+1$.

On the other hand, let G' be a path on vertices $0, 1, 2, \ldots, N$ with conductances

$$c'_{k,k+1} = \frac{\left(N - k - 1 + \frac{1}{r}\right)\left(N - k + \frac{1}{r}\right)}{\frac{1}{r}\left(1 + \frac{1}{r}\right)}$$

for $0 \le k < N$. Let $Y = \{Y_t\}_{t\ge 0}$ be a discrete time random walk on G' started at k. Then, the paths of Y stopped upon hitting 0 have the same distribution as the paths of X conditioned on $X_\tau = 0$.

Proof. This can be easily checked by calculating hitting probabilities, which can then be used to calculate transition probabilities for X_t conditioned on $X_\tau = 0$. See Appendix. ☐

Corollary 5.6. *Let N, r, and G be as in Lemma 5.5, and suppose further that $r < 1$. Let X be a continuous time random walk on G started at N, and let $\tau = \inf\{t \ge 0 : X_t = 0 \text{ or } N+1\}$. Then, for any $\epsilon \in (0,1)$ and $\beta > 0$,*

$$\mathbb{P}\left(\min_{\epsilon N \le k < N} \mathcal{L}^X_\tau(k) \le \beta N \,\middle|\, X_\tau = 0\right) \le \frac{2}{\log \epsilon^{-1} - C_\alpha} + \frac{C_\alpha}{\epsilon}\exp\left(-\frac{\log \beta^{-1} - C_\alpha}{\log \epsilon^{-1} + C_\alpha}\right)$$

where $\alpha = rN$, and $C_\alpha > 0$ is a number depending only on α.

Remark 5.7. The statement of Corollary 5.6 takes this somewhat awkward form because it will be used for r on the order of $\frac{1}{N}$.

Proof. By Lemma 5.5 (using the same notation), the paths of X are distributed as a random walk on a path of N edges with conductances

$$c'_{k,k+1} = \frac{\left(N - k - 1 + \frac{1}{r}\right)\left(N - k + \frac{1}{r}\right)}{\frac{1}{r}\left(1 + \frac{1}{r}\right)}$$

for $0 \le k < N$. Thus, by Corollary 5.3,

$$\mathbb{P}\left(\min_{\epsilon N \le k < N} \mathcal{L}^X_\tau(k) \le \beta N \,\middle|\, X_\tau = 0\right) = \mathbb{P}\left(\min_{\epsilon N \le k < N} |W_{a_k}|^2 \le \beta N\right),$$

where W_t is a two-dimensional Brownian motion, and

$$a_k = \sum_{i=0}^{k-1} \frac{1}{c'_{i,i+1}} = \sum_{i=0}^{k-1} \frac{1}{r}\left(1 + \frac{1}{r}\right)\left(\frac{1}{N - i - 1 + \frac{1}{r}} - \frac{1}{N - i + \frac{1}{r}}\right)$$

$$= \frac{1}{r}\left(1 + \frac{1}{r}\right)\left(\frac{1}{N - k + \frac{1}{r}} - \frac{1}{N + \frac{1}{r}}\right).$$

From the above equations, the following bounds are easy to verify for $\epsilon N \le k < N$.

$$c'_{k-1,k+1} + c'_{k,k+1} \ge 2$$

$$a_k \ge \frac{1}{r}\left(1 + \frac{1}{r}\right)\left(\frac{1}{N - \epsilon N + \frac{1}{r}} - \frac{1}{N + \frac{1}{r}}\right) > \frac{\epsilon N}{(1 + rN)^2}.$$

$$a_k \le a_N \le \frac{2}{r}.$$

It follows that

$$\mathbb{P}\left(\min_{\epsilon N \le k < N} \mathcal{L}_\tau^X(k) \le \beta N \,\middle|\, X_\tau = 0\right) \le \mathbb{P}\left(\inf_{\frac{\epsilon N}{(1+rN)^2} \le t \le \frac{2}{r}} |W_t|^2 \le \beta N\right)$$

$$= \mathbb{P}\left(\inf_{\frac{\epsilon r N}{2(1+rN)^2} \le t \le 1} |W_t|^2 \le \frac{\beta r N}{2}\right) = \mathbb{P}\left(\inf_{\frac{\epsilon \alpha}{2(1+\alpha)^2} \le t \le 1} |W_t|^2 \le \frac{\beta \alpha}{2}\right)$$

$$\le \frac{2}{\log \epsilon^{-1} - C_\alpha} + \frac{C_\alpha}{\epsilon} \exp\left(-\frac{\log \beta^{-1} - C_\alpha}{\log \epsilon^{-1} + C_\alpha}\right),$$

for C_α sufficiently large. In the second line, we have used the scale-invariance of Brownian motion, and the third line is an application of Lemma 5.4. □

5.3 Local times of G_N

Using the results of the previous subsection, we will establish two estimates concerning local times on G_N, stated as Lemmas 5.9 and 5.11 below. These correspond to our assertion that the set U is connected in the heuristic proof outline provided at the beginning of the section.

In the lemmas that follow, we consider a continuous time random walk $X_N(t)$ on G_N started at a vertex $x \in V$. Let τ_x denote the first time the walk hits another vertex $y \in V$ distinct from x. The first estimate states, roughly, that it is very likely for vertices near x to accumulate significant local time.

We will need a standard concentration estimate for sums of i.i.d. exponential random variables. Unfortunately, we were unable to find a reference that contained both tail bounds, so a short proof is included in the appendix.

Lemma 5.8. *Let $X_1, X_2, \ldots, X_N$ be i.i.d. exponential random variables with mean μ. Then, for any $\alpha \in [0,1]$, we have*

$$\mathbb{P}\left(\left|\sum_{i=1}^N X_i - \mu N\right| \ge \alpha \mu N\right) \le 2e^{-\frac{1}{4}\alpha^2 N}.$$

Proof. See Appendix. □

Lemma 5.9. *Let $y \in V$ be any neighbor of x in G, let $\epsilon \in \left(0, \frac{1}{2}\right), \lambda > 0$ be given, and define $k = \lfloor \epsilon N \rfloor$. Then,*

$$\mathbb{P}\left(\min_{0 \le i \le k} \mathcal{L}_{\tau_x}(v_{xy,i}) < \lambda\right) \le C_G \cdot \epsilon N \left(\lambda + \exp\left(-\frac{\lambda N}{8 C_G}\right)\right)$$

for some constant C_G depending on G but not N.

Proof. Recall the notation $L_{\tau_x}(x)$ for the number of visits to x up until time τ_x, and recall also from Section 5.1 that $L_{\tau_x}(x)$ is distributed as a geometric random variable with mean N. Conditioning on $L_{\tau_x}(x)$, we may decompose the walk up until time τ_x into $L_{\tau_x}(x)$ excursions from x and a path to a neighbor of x in G. Each excursion may be sampled independently.

Let us now consider one excursion. The first step of the excursion goes to some vertex $v_{xz,1}$, where z is a neighbor of x in G. As noted earlier, from there the walk behaves like a simple random walk on $\mathbb{Z}$ started at 1, stopped upon hitting 0 (corresponding to the return to x), and conditioned on hitting 0 before N (corresponding to avoiding z).

Let E_m denote the event that a simple random walk on $\mathbb{Z}$ started at 1 hits m before 0. By a standard martingale argument, we have $\mathbb{P}(E_m) = \frac{1}{m}$. Thus,

$$\mathbb{P}\left(E_k \,|\, E_N^c\right) = \frac{\mathbb{P}(E_k \cap E_N^c)}{\mathbb{P}(E_N^c)} \geq \frac{1}{k} - \frac{1}{N} > \frac{1}{2k}.$$

In particular, this implies that for each excursion, there is a $\frac{c_{xy}}{c_x}$ probability that the first step is $v_{xy,1}$, and with probability at least $\frac{1}{2k}$ the excursion will then hit $v_{xy,k}$. In other words, letting p be the probability that a single excursion includes $v_{xy,k}$, we have $p \geq \frac{c_{xy}}{2kc_x}$.

Let L denote the number of excursions which hit $v_{xy,k}$. By the preceding discussion, it is the sum of $L_{\tau_x}(x)$ i.i.d. Bernoulli random variables with expectation p. Since $L_{\tau_x}(x)$ is geometrically distributed with mean N, it follows that L is geometrically distributed with mean pN. We thus have

$$\mathbb{P}\left(L < 2\lambda c_x N\right) \leq \frac{2\lambda c_x N}{pN} \leq \frac{4\lambda c_x^2 \epsilon N}{c_{xy}}. \tag{5.1}$$

Note that for each $i \in \{0, 1, 2, \ldots, k\}$, the vertex $v_{xy,i}$ is visited at least L times, and the total conductance of $v_{xy,i}$ is at most Nc_x. Thus, $\mathcal{L}_{\tau_x}(v_{xy}, i)$ stochastically dominates $\frac{1}{Nc_x}$ times the sum of L i.i.d. unit exponentials. By Lemma 5.8 with $\alpha = \frac{1}{2}$, we have

$$\mathbb{P}\left(c_x N \cdot \mathcal{L}_{\tau_x}(v_{xy,i}) < \lambda c_x N \,\middle|\, L \geq 2\lambda c_x N\right) \leq 2\exp\left(-\frac{\lambda c_x N}{8}\right),$$

and so

$$\mathbb{P}\left(\min_{0 \leq i \leq k} \mathcal{L}_{\tau_x}(v_{xy,i}) < \lambda \,\middle|\, L \geq 2\lambda c_x N\right) \leq 2\epsilon N \exp\left(-\frac{\lambda c_x N}{8}\right).$$

Combining this with equation (5.1) gives

$$\mathbb{P}\left(\min_{0 \leq i \leq k} \mathcal{L}_{\tau_x}(v_{xy,i}) < \lambda\right) \leq \frac{4\lambda c_x^2 \epsilon N}{c_{xy}} + 2\epsilon N \exp\left(-\frac{\lambda c_x N}{8}\right),$$

which takes the desired form for C_G sufficiently large. □

Corollary 5.10. *Let $S = \{y \in V : (x, y) \in E\}$ be the set of neighbors of x in G. Then,*

$$\mathbb{P}\left(\min_{y \in S} \min_{0 \leq k \leq \frac{N}{\log^3 N}} \mathcal{L}_{\tau_x}(v_{xy,k}) < \frac{\log^2 N}{N}\right) \longrightarrow 0$$

as $N \to \infty$.

Proof. This follows immediately from Lemma 5.9 by taking $\lambda = \frac{\log^2 N}{N}$ and $\epsilon = \frac{1}{\log^3 N}$. □

The second estimate states that, conditioned upon $X_N(\tau_x) = y$, it is very likely that vertices along the path from x to y are visited a large number of times, as long as they are not very close to y. This essentially follows from Corollary 5.6.

Lemma 5.11. *Let y be a neighbor of x in G. Then, for any $\epsilon, \lambda \in (0, 1)$, we have*

$$\mathbb{P}\left(\min_{\epsilon N \leq k < N} \mathcal{L}_{\tau_x}(v_{yx,k}) < \lambda \,\middle|\, X_N(\tau_x) = y\right) \leq \frac{2}{\log \epsilon^{-1} - C_G} + \frac{C_G}{\epsilon} \exp\left(-\frac{\log \lambda^{-1} - C_G}{\log \epsilon^{-1} + C_G}\right)$$

for some constant C_G depending on G but not N.

Proof. Let $S = \{z \in V : (x, z) \in E\}$. Note that the process X_N up to time τ_x induces a continuous time random walk $Y = \{Y_t\}_{t \geq 0}$ on the vertices

$$\{v_{xy,0}, v_{xy,1}, \ldots, v_{xy,N}\} \cup S$$

by ignoring visits to vertices outside of that set (namely, those of the form $v_{xz,k}$ for $z \neq y$ and $1 \leq k < N$). We can define a stopping time T_x analogous to τ_x as the first time Y hits S.

For convenience, define $p_{xz} = \frac{c_{xz}}{c_x}$ for each $z \in S$. Note that

$$\mathbb{P}\left(X_N \text{ hits } v_{xy,1} \text{ before hitting } S \text{ or returning to } x\right) = p_{xy}$$

$$\mathbb{P}\left(X_N \text{ hits } S \text{ before hitting } v_{xy,1} \text{ or returning to } x\right) = \frac{1-p_{xy}}{N}.$$

Thus, we can interpret Y up to time T_x as a continuous time random walk on a path with vertices $(w_0, w_1, w_2, \ldots, w_{N+1})$, where all the conductances are 1 except that the conductance between w_N and w_{N+1} is $\frac{1-p_{xy}}{Np_{xy}}$. Here, w_k corresponds to $v_{yx,k}$ (so Y is started at w_N), and w_{N+1} corresponds to any vertex in $S \setminus \{y\}$ (we may combine all of these states because Y is stopped upon hitting this set anyway).

We are now in the setting of Corollary 5.6, as conditioning on $Y_{T_x} = y$ corresponds to conditioning on hitting w_0 before w_{N+1}. Following the notation of Corollary 5.6, we have $r = \frac{1-p_{xy}}{Np_{xy}}$, so that $\alpha = \frac{1-p_{xy}}{p_{xy}}$.

We apply the corollary with $\beta = \lambda c_{xy}$. Note that the total conductances at $v_{yx,k}$ are $2Nc_{xy}$ as opposed to 2 in the statement of Corollary 5.6, so the local times will be scaled accordingly. It follows that

$$\begin{aligned}
\mathbb{P}\left(\min_{\epsilon N \leq k < N} \mathcal{L}^{X_N}_{\tau_x}(v_{yx,k}) < \lambda \,\middle|\, X_N(\tau_x) = y\right) &= \mathbb{P}\left(\min_{\epsilon N \leq k < N} \mathcal{L}^{Y}_{T_x}(w_k) < \lambda c_{xy} N \,\middle|\, Y_{T_x} = y\right) \\
&\leq \frac{2}{\log \epsilon^{-1} - C_\alpha} + \frac{C_\alpha}{\epsilon} \exp\left(-\frac{\log \lambda^{-1} - \log c_{xy} - C_\alpha}{\log \epsilon^{-1} + C_\alpha}\right) \\
&\leq \frac{2}{\log \epsilon^{-1} - C_G} + \frac{C_G}{\epsilon} \exp\left(-\frac{\log \lambda^{-1} - C_G}{\log \epsilon^{-1} + C_G}\right),
\end{aligned}$$

whenever $C_G > \max(C_\alpha, C_\alpha + \log c_{xy})$. In particular, since there are only finitely many possible values of p_{xy} and hence of α, we can choose C_G sufficiently large so that this holds independently of N. This proves the lemma. □

Corollary 5.12. *Let y be a neighbor of x in G. Then, we have*

$$\mathbb{P}\left(\min_{\frac{N}{\log^3 N} \leq k \leq N} \mathcal{L}_{\tau_x}(v_{yx,k}) < \frac{\log^2 N}{N} \,\middle|\, X_{\tau_x} = y\right) \longrightarrow 0$$

as $N \to \infty$.

Proof. We apply Lemma 5.11 with $\epsilon = \frac{1}{\log^3 N}$ and $\lambda = \frac{\log^2 N}{N}$. It suffices to show that both terms on the right hand side tend to zero. Clearly,

$$\frac{2}{\log \epsilon^{-1} - C_G} \to 0$$

as $N \to \infty$. To bound the other term, note that for sufficiently large N, we have

$$\frac{\log \lambda^{-1} - C_G}{\log \epsilon^{-1} + C_G} = \frac{\log N - 2\log\log N - C_G}{3\log\log N + C_G} \geq \frac{\log N}{6 \log\log N},$$

in which case

$$\begin{aligned}
\frac{C_G}{\epsilon} \exp\left(-\frac{\log \lambda^{-1} - C_G}{\log \epsilon^{-1} + C_G}\right) &\leq C_G \log^3 N \exp\left(-\frac{\log N}{6\log\log N}\right) \\
&= C_G \exp\left(-\frac{\log N}{6\log\log N} + 3\log\log N\right) \longrightarrow 0.
\end{aligned}$$

□

5.4 Proof of the stochastic domination

We now prove Theorem 3.1, following the plan described in Section 3. Let us first prove an approximation of Theorem 3.1.

Lemma 5.13. *Let $t > 0$ be given. Let Ω_N be a probability space with random variables η_N, η'_N, and $X_N = \{X_N(t)\}_{t\geq 0}$ such that η_N and η'_N are distributed as Gaussian free fields on G_N, and X_N is distributed as a continuous time random walk on G_N. Furthermore, suppose that η_N and X_N are independent, and almost surely for each $v \in V_N$,*

$$\frac{1}{2}\eta_{N,v}^2 + \mathcal{L}^{X_N}_{\tau^+(t)}(v) = \frac{1}{2}\left(\eta'_{N,v} + \sqrt{2t}\right)^2.$$

(Theorem 2.1 ensures that such a construction is always possible.) Then, for any $\epsilon > 0$, we have

$$\mathbb{P}\left(\text{for some } x \in V\text{, both } \mathcal{L}^{X_N}_{\tau^+(t)}(x) > 0 \text{ and } \eta'_{N,x} + \sqrt{2t} < 0\right) \leq \epsilon$$

for N sufficiently large.

Remark 5.14. Note that the hypothesis of Lemma 5.13 implies for each $x \in V$ that

$$\sqrt{\mathcal{L}^{X_N}_{\tau^+(t)}(x)} \leq \frac{1}{\sqrt{2}}\left|\eta'_{N,x} + \sqrt{2t}\right|.$$

Consequently, the conclusion of the lemma may be expressed equivalently as

$$\mathbb{P}\left(\sqrt{\mathcal{L}^{X_N}_{\tau^+(t)}(x)} > \frac{1}{\sqrt{2}}\max\left(0, \eta'_{N,x} + \sqrt{\frac{2t}{c_{v_0}}}\right) \text{ for some } x \in V\right) \leq \epsilon.$$

Proof. To shorten notation, we use τ^+ to denote $\tau^+(t)$.

Call a vertex $x \in V$ *well-connected* at time s if there exists a sequence of vertices $v_0 = w_0, w_1, \ldots, w_n = x$ in V_N such that $(w_i, w_{i+1}) \in E_N$ and $\mathcal{L}^{X_N}_s(w_i) \geq \frac{\log^2 N}{N}$ for each i. We will show that with high probability, every vertex in V with positive local time at time τ^+ is well-connected. (This corresponds to connectedness of the set U from Section 5.13.)

Recall from the discussion in Section 5.1 that X_N induces a random walk on G which, when regarded as a sequence of visited vertices (disregarding holding times), has the same law as a discrete time random walk on G. Thus, one way of sampling from X_N is to first sample a path

$$P = (v_0 = x_0, x_1, x_2, \ldots)$$

of the discrete time random walk on G. Then, we construct X_N as follows. For each $i \geq 0$, let $Y_i(t)$ be a continuous time random walk on G_N started at x_i, and let τ_i be the first time that Y_i hits a neighbor of x_i in G.

Let Z_i have the law of a copy of Y_i conditioned on the event $Y_i(\tau_i) = x_{i+1}$. Then, we may form X_N by concatenating the walks Z_i up to time τ_i. More formally, we may define

$$n(s) = \max\left\{n \geq 1 : \sum_{i=1}^{n-1} \tau_i \leq s\right\}$$

and set $X_N(s) = Z_{n(s)}\left(s - \sum_{i=1}^{n(s)-1} \tau_i\right)$.

To lighten notation, let us write $\mathcal{L}_i = \mathcal{L}^{Y_i}_{\tau_i}$ and $\mathbb{P}_i(\cdot) = \mathbb{P}\left(\cdot \,|\, Y_i(\tau_i) = x_{i+1}\right)$, noting that the randomness of the Y_i are independent. Let $P(s) = (x_1, x_2, \ldots, x_{n(s)})$ denote the truncation of P up until time s. We will say that $P(s)$ is well-connected if each x_i

appearing in $P(s)$ is well-connected at time s. Then,

$$\begin{aligned}\mathbb{P}\Big(P(\tau^+) \text{ is not well-connected} \,\Big|\, P(\tau^+)\Big) \\ &\le \sum_{i=1}^{|P(\tau^+)|-1} \mathbb{P}_i\left(\min_{0\le k\le N} \mathcal{L}_i(v_{x_i x_{i+1},k}) < \frac{\log^2 N}{N}\right) \\ &= \sum_{i=1}^{|P(\tau^+)|-2} \mathbb{P}_i\left(\min_{0\le k\le N-\frac{N}{\log^3 N}} \mathcal{L}_i(v_{x_i x_{i+1},k}) < \frac{\log^2 N}{N}\right) + \\ &\quad \sum_{i=2}^{|P(\tau^+)|-1} \mathbb{P}_i\left(\min_{0\le k<\frac{N}{\log^3 N}} \mathcal{L}_i(v_{x_i x_{i-1},k}) < \frac{\log^2 N}{N}\right)\end{aligned} \tag{5.2}$$

Fix a number T sufficiently large so that $\mathbb{P}\Big(|P(\tau^+)| > T\Big) \le \frac{\epsilon}{4}$. Again, by the discussion of Section 5.1, the law of $P(\tau^+)$ does not depend on N, so the number T can be chosen independently of N. Note that by Corollaries 5.10 and 5.12, each summand in either sum of the last expression of (5.2) is bounded by $\frac{\epsilon}{8T}$ for sufficiently large N. Consequently, for sufficiently large N, the whole expression is bounded by $2|P(\tau^+)| \cdot \frac{\epsilon}{8T}$, and we have

$$\begin{aligned}\mathbb{P}\Big(P(\tau^+) \text{ is not well-connected}\Big) &\le \mathbb{P}\Big(|P(\tau^+)| > T\Big) + \\ &\quad \mathbb{P}\Big(P(\tau^+) \text{ is not well-connected} \,\Big|\, |P(\tau^+)| \le T\Big) \\ &\le \frac{\epsilon}{4} + 2T\cdot\frac{\epsilon}{8T} = \frac{\epsilon}{2}.\end{aligned}$$

Note that almost surely, the vertices $x \in V$ for which $\mathcal{L}_{\tau^+}(x) > 0$ are exactly those appearing in $P(\tau^+)$. Thus, we have

$$\mathbb{P}\Big(\text{for some } x\in V,\ \mathcal{L}_{\tau^+}(x) > 0 \text{ but } x \text{ is not well-connected}\Big) \le \frac{\epsilon}{2}. \tag{5.3}$$

We next show that with high probability, the values of η'_N at adjacent vertices do not differ by very much. (This corresponds to the continuity of the Gaussian free field on the metric graph and means that it is unlikely for $\eta'_N + \sqrt{2t}$ to change sign at two adjacent vertices with large local times.)

Consider any $(x, y) \in E$ and $0 \le k < N$. For notational convenience, let $u = v_{xy,k}$ and $w = v_{xy,k+1}$. We have

$$\mathbb{E}(\eta'_{N,u} - \eta'_{N,w})^2 = R_{\text{eff}}(u,w) \le \frac{1}{Nc_{xy}}.$$

Since $\eta'_{N,u} - \eta'_{N,w}$ has a Gaussian distribution, it follows that

$$\mathbb{P}\left(|\eta'_{N,u} - \eta'_{N,w}| \ge \frac{\log N}{\sqrt{N}}\right) \le \exp\left(-c_{xy}\log^2 N\right).$$

Taking a union bound over all adjacent pairs $(u, w) \in E_N$, we obtain

$$\mathbb{P}\left(\max_{(u,w)\in E_N} |\eta'_{N,u} - \eta'_{N,w}| \ge \frac{\log N}{\sqrt{N}}\right) \le N\exp\left(-\left(\min_{(x,y)\in E} c_{xy}\right)\log^2 N\right) \le \frac{\epsilon}{2} \tag{5.4}$$

for N sufficiently large.

Finally, we may combine equations (5.3) and (5.4) to deduce the lemma. Indeed, suppose that for some $x \in V$, we have $\mathcal{L}^{X_N}_{\tau^+}(x) > 0$ but $\sqrt{2t} + \eta'_{N,x} < 0$. If x is well-connected at time τ^+, which occurs with high probability by (5.3), then there exists a path $v_0 = w_0, w_1, \ldots, w_n = x$ in G_N such that each $\mathcal{L}^{X_N}_{\tau^+}(w_i)$ is at least $\frac{\log^2 N}{N}$. Observe that $\sqrt{2t} + \eta'_{N,v_0} = \sqrt{2t} > 0$, so for some i we must have

$$\sqrt{2t} + \eta'_{N,w_i} > 0 \text{ and } \sqrt{2t} + \eta'_{N,w_{i+1}} < 0.$$

However, we also have

$$\frac{1}{\sqrt{2}}\left|\sqrt{2t} + \eta'_{N,w_i}\right| = \sqrt{\mathcal{L}^{X_N}_{\tau^+}(w_i) + \frac{1}{2}\eta^2_{N,x_i}} \geq \frac{\log N}{\sqrt{N}}.$$

Therefore, this can only happen if

$$\left|\eta'_{N,w_i} - \eta'_{N,w_{i+1}}\right| \geq \frac{2\log N}{\sqrt{N}}.$$

But by equation (5.4), this is unlikely. Thus, we have

$$\begin{aligned}
&\mathbb{P}\left(\text{for some } v \in V, \text{ both } \mathcal{L}^{X_N}_{\tau^+}(v) > 0 \text{ and } \sqrt{2t} + \eta'_{N,v} < 0\right) \\
&\qquad \leq \mathbb{P}\left(\max_{(u,w)\in E_N} |\eta'_{N,u} - \eta'_{N,w}| \geq \frac{\log N}{\sqrt{N}}\right) + \\
&\qquad\quad \mathbb{P}\Big(\text{for some } x \in V,\ \mathcal{L}_{\tau^+}(x) > 0 \text{ but } x \text{ is not well-connected}\Big) \\
&\qquad \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,
\end{aligned}$$

proving the lemma. □

Theorem 3.1 is now an easy consequence of Lemma 5.13.

Proof of Theorem 3.1. Let $A \subset \mathbb{R}^V$ be any monotone set. We wish to show that

$$\mathbb{P}\left(\left\{\frac{1}{\sqrt{2}}\max\left(0, \eta_x + \sqrt{2t}\right)\right\}_{x\in V} \in A\right) \geq \mathbb{P}\left(\left\{\sqrt{\frac{1}{c_x}\mathcal{L}^{X}_{\tau^+(t)}(x)}\right\}_{x\in V} \in A\right). \tag{5.5}$$

Let $\epsilon > 0$ be given, and take N sufficiently large so that the conclusion of Lemma 5.13 holds.

Let η_N be the Gaussian free field on G_N, and let X_N be a continuous time random walk independent of η_N. We will now try to define another Gaussian free field $\eta'_{N,v}$ on the same probability space so as to satisfy the hypotheses of Lemma 5.13. In fact, by the isomorphism theorem, η'_N can be given in terms of η_N and the local times up to a choice of sign in taking the square root.

To determine the signs, we can artificially introduce some additional randomness. Fix an arbitrary ordering on $\{-1,1\}^{V_N}$. For each $\sigma = \{\sigma_v\}_{v\in V_N} \in \{-1,1\}^{V_N}$, define the function $f_\sigma : \mathbb{R}^{V_N} \to \mathbb{R}$ by

$$f_\sigma(Z) = \mathbb{P}\left(\eta_{N,v} = \sigma_v\sqrt{Z_v} - \sqrt{2t} \text{ for all } v \in V_N \,\middle|\, \left(\eta_{N,v} + \sqrt{2t}\right)^2 = Z_v \text{ for all } v \in V_N\right).$$

Let U be uniformly distributed on $[0,1]$ and independent of η_N and X_N. For any $u \in [0,1]$ and $Z \in \mathbb{R}^{V_N}$, we may define

$$\sigma^*(u, Z) = \max\left\{\sigma \in \{-1,1\}^{V_N} : u \geq \sum_{\rho<\sigma} f_\rho(Z)\right\}.$$

We can then define

$$\zeta_{N,v} = \frac{1}{2}\eta^2_{N,v} + \frac{1}{c_v}\mathcal{L}^{X_N}_{\tau^+(t)}(v)$$

$$\eta'_{N,v} = \sigma^*(U, 2\zeta_{N,v})\sqrt{2\zeta_{N,v}} - \sqrt{2t}.$$

We are now in the setting of Lemma 5.13, which gives

$$\mathbb{P}\left(\text{for some } v \in V, \text{ both } \mathcal{L}^{X_N}_{\tau^+(t)}(v) > 0 \text{ and } \eta'_{N,v} + \sqrt{2t} < 0\right) \leq \epsilon,$$

or equivalently (by Remark 5.14),

$$\mathbb{P}\left(\sqrt{\mathcal{L}^{X_N}_{\tau^+(t)}(x)} > \frac{1}{\sqrt{2}}\max\left(0, \eta'_{N,x} + \sqrt{2t}\right) \text{ for some } x \in V\right) \leq \epsilon.$$

Now, let η and X be the GFF and a continuous time random walk on G, respectively. By the relationship between G_N and G described in Proposition 5.1, we have

$$\begin{aligned}\mathbb{P}\left(\left\{\frac{1}{\sqrt{2}}\max\left(0, \eta_x + \sqrt{2t}\right)\right\}_{x\in V} \in A\right) &= \mathbb{P}\left(\left\{\frac{1}{\sqrt{2}}\max\left(0, \eta'_{N,x} + \sqrt{2t}\right)\right\}_{x\in V} \in A\right)\\ \geq \mathbb{P}\left(\left\{\sqrt{\mathcal{L}^{X_N}_{\tau^+(t)}(x)}\right\}_{x\in V} \in A\right) - \epsilon &= \mathbb{P}\left(\left\{\sqrt{\frac{1}{c_x}\mathcal{L}^{X}_{\tau^+(t)}(x)}\right\}_{x\in V} \in A\right) - \epsilon.\end{aligned}$$

This holds for each $\epsilon > 0$, so taking $\epsilon \to 0$, we obtain (5.5), as desired. ☐

6 Appendix

6.1 Proof of Lemma 5.4

To break up the proof, we first establish a lemma.

Lemma 6.1. *Let $r > 0$ be given, and consider any point $y \in \mathbb{R}^2$ such that $|y| > r$. Let $\{W^y_t\}_{t\geq 0}$ be a standard planar Brownian motion started at y. Then,*

$$\mathbb{P}\left(\inf_{t\in[0,1]} |W^y_t| \leq r\right) \leq \inf_{0\leq\alpha\leq|y|}\left(\frac{2\log\alpha^{-1}}{\log r^{-1}} + 4\alpha^2\right).$$

Proof. Note that $\mathbb{P}\left(\inf_{t\in[0,1]} |W^y_t| \leq r\right)$ is decreasing in $|y|$, so it suffices to show the inequality only for $\alpha = |y|$. Let $s = \frac{1}{|y|}$. Define two stopping times

$$T = \inf\{t \geq 0 : |W^y_t| \notin [r, s]\}$$
$$T' = \inf\{t \geq 0 : |W^y_t| \notin [r, \infty)\}$$

Now, consider the stopped martingale $X_t = \log|W^y_{T\wedge t}|$, noting that $X_0 = \log|y|$ and $X_t \in [\log r, \log s]$. By the martingale property, we have

$$\mathbb{P}(X_1 = \log r) \leq \frac{\log s - \log|y|}{\log s - \log r} = \frac{2\log|y|^{-1}}{\log|y|^{-1} + \log r^{-1}} \leq \frac{2\log|y|^{-1}}{\log r^{-1}}.$$

Moreover, by Doob's maximal inequality[3] on the submartingale $|W^y_t|^2$,

$$\mathbb{P}\Big(\min(T,1) \neq \min(T',1)\Big) \leq \mathbb{P}\left(\sup_{t\in[0,1]} |W^y_t| \geq s\right) \leq \min\left(1, \frac{|y|^2+2}{s^2}\right) \leq 4|y|^2.$$

It follows that

$$\begin{aligned}\mathbb{P}\left(\inf_{t\in[0,1]} |W^y_t| \leq r\right) = \mathbb{P}\Big(T' \leq 1\Big) &\leq \mathbb{P}\Big(X_1 = \log r \text{ or } \min(T,1) \neq \min(T',1)\Big)\\ &\leq \frac{2\log|y|^{-1}}{\log r^{-1}} + 4|y|^2,\end{aligned}$$

as desired. ☐

[3] We use Doob's maximal inequality for brevity only. Other methods such as the reflection principle would serve just as well; the bound on $\sup|W^y_t|$ does not need to be sharp for our purposes.

Proof of Lemma 5.4. Define $\lambda' = \lambda^{\frac{1}{\log \epsilon^{-1}}}$. Recall that the probability density of the standard two-dimensional Gaussian is bounded above by $\frac{1}{2\pi}$, and so the probability density of W_ϵ is bounded above by $\frac{1}{2\pi\epsilon}$. Thus,

$$\mathbb{P}\left(|W_\epsilon|^2 \leq \lambda'\right) \leq \frac{1}{2\pi\epsilon} \cdot \pi\lambda' = \frac{1}{2\epsilon} \exp\left(-\frac{\log \lambda^{-1}}{\log \epsilon^{-1}}\right).$$

We now apply Lemma 6.1 with $y = W_\epsilon$, $r = \sqrt{\lambda}$, and taking $\alpha = \sqrt{\lambda'}$ in the infimum. This gives

$$\mathbb{P}\left(\inf_{\epsilon \leq t \leq 1} |W_t|^2 < \lambda \,\middle|\, |W_\epsilon|^2 \geq \lambda'\right) \leq \frac{2\log \lambda'^{-1}}{\log \lambda^{-1}} + 4\lambda'$$
$$= \frac{2}{\log \epsilon^{-1}} + 4\exp\left(-\frac{\log \lambda^{-1}}{\log \epsilon^{-1}}\right) \leq \frac{2}{\log \epsilon^{-1}} + \frac{9}{2\epsilon}\exp\left(-\frac{\log \lambda^{-1}}{\log \epsilon^{-1}}\right).$$

This along with the previous inequality proves the lemma ☐

6.2 Proof of Lemma 5.5

Proof. Define

$$f(x) = \begin{cases} x & : 0 \leq x \leq N \\ N + \frac{1}{r} & : x = N+1 \end{cases}$$

Note that $f(X)$ is a martingale. Thus, for a walk started at k, the probability of hitting 0 before $N+1$ is

$$\frac{f(N+1) - f(k)}{f(N+1) - f(0)} = \frac{N - k + \frac{1}{r}}{N + \frac{1}{r}}.$$

It follows that for $1 \leq k < N$,

$$\frac{\mathbb{P}\Big(X_{t+1} = k+1 \,\Big|\, X_t = k, X_\tau = 0\Big)}{\mathbb{P}\Big(X_{t+1} = k-1 \,\Big|\, X_t = k, X_\tau = 0\Big)} = \frac{N - k - 1 + \frac{1}{r}}{N - k + 1 + \frac{1}{r}} = \frac{c'_{k,k+1}}{c'_{k-1,k}},$$

where

$$c'_{k,k+1} = \frac{\left(N - k - 1 + \frac{1}{r}\right)\left(N - k + \frac{1}{r}\right)}{\frac{1}{r}\left(1 + \frac{1}{r}\right)}.$$

Thus, the transition probabilities of X conditioned on $X_\tau = 0$ are exactly the unconditioned transition probabilities of Y. Consequently, their paths have the same distribution. ☐

6.3 Proof of Lemma 5.8

Proof. For any $t < \frac{1}{\mu}$, we have by direct calculation

$$\log\left(\mathbb{E}\exp\left(t\sum_{i=1}^{N} X_i\right)\right) = N\log\left(\frac{1}{1-\mu t}\right).$$

If in fact $|t| \leq \frac{\alpha}{2\mu}$, we have

$$\log\left(\frac{1}{1-\mu t}\right) = \sum_{k=1}^{\infty}\frac{\mu^k t^k}{k} \leq \mu t + \mu^2 t^2 = \mu t(1 + 2\mu t) - \mu^2 t^2.$$

and so

$$\frac{\mathbb{E}\exp\left(t\sum_{i=1}^{N} X_i\right)}{\exp\left(t(1+2\mu t)\mu N\right)} \leq e^{-\mu^2 t^2 N}.$$

By Markov's inequality with $t = \frac{\alpha}{2\mu}$ and $t = -\frac{\alpha}{2\mu}$, we obtain

$$\mathbb{P}\left((1-\alpha)\mu N \leq \sum_{i=1}^{N} X_i \leq (1+\alpha)\mu N\right) \leq 2e^{-\frac{1}{4}\alpha^2 N}. \qquad \square$$

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