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# Scaling of the Sasamoto-Spohn model in equilibrium*

Milton Jara† Gregorio R. Moreno Flores‡

**Abstract**

We prove the convergence of the Sasamoto-Spohn model in equilibrium to the energy solution of the stochastic Burgers equation on the whole line. The proof, which relies on the second order Boltzmann-Gibbs principle, follows the approach of [9] and does not use any spectral gap argument.

**Keywords:** KPZ equation; Burgers equation; Sasamoto-Spohn model.
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## 1 Model and results

The goal of this note is to show the convergence of a certain discretization of the stochastic Burgers equation:

$$\partial_t u = \tfrac{1}{2}\partial_x^2 u + \partial_x u^2 + \partial_x \mathscr{W}, \tag{1.1}$$

where $\mathscr{W}$ is a space-time white noise. This equation can be seen as the evolution of the slope of solutions to the KPZ equation [15] which is itself a model of an interface in a disordered environment. The KPZ/Burgers equation has been subject to an extensive body of work in the last years. It appears as the scaling limit of a wide range of particle systems [4, 8], directed polymer models [1, 20] and interacting diffusions [6], and constitutes a central element in a vast family of models known as the KPZ universality class [5, 21].

Due to the nonlinearity, a lot of care has to be taken to obtain a notion of solution for (1.1). There are today several alternatives, for instance, regularity structure [14], paracontrolled distributions [11] and energy solutions [8, 10, 12], which is the approach we will follow.

The discretization we consider corresponds to

$$du_j \quad = \quad \tfrac{1}{2}\Delta u_j + \gamma B_j(u) + d\xi_j - d\xi_{j-1}, \tag{1.2}$$

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†Instituto de Matemática Pura e Aplicada, Rio de Janeiro, Brasil. Partially supported by CNPq and FAPERJ. E-mail: mjara@impa.br

‡Pontificia Universidad Católica de Chile, Santiago, Chile. Partially supported by Fondecyt grant 1171257 and Núcleo Milenio 'Modelos Estocásticos de Sistemas Complejos y Desordenados'.
E-mail: grmoreno@mat.uc.cl

where $(\xi_j)_j$ is an i.i.d. family of standard one-dimensional Brownian motions,

$$\begin{aligned}\Delta u_j &= u_{j+1} + u_{j-1} - 2u_j,\\ B_j(u) &= w_j - w_{j-1} \quad \text{with} \quad w_j = \frac{1}{3}(u_j^2 + u_j u_{j+1} + u_{j+1}^2).\end{aligned}$$

This model, introduced in [16] (see also [17]) and further studied in [22], is nowadays often referred to as the Sasamoto-Spohn model.

While the discretization of the second derivative and noise are quite straightforward, there are a priori several ways to discretize the nonlinearity in Burgers equation. This particular choice is motivated by two reasons: first, it only involves nearest neighbor sites and, second, it yields the explicit invariant measure $\mu = \rho^{\otimes\mathbb{Z}}$, where $d\rho(x) = \frac{1}{\sqrt{2\pi}}e^{-x^2/2}dx$ (see Section 3).

Our result states the convergence of the discrete equations (1.2) to Burgers equation in the sense of energy solutions (see Section 2 for a precise definition).

**Theorem 1.1.** *For each $n \geq 1$, let $u^n$ be the solution to the system* (1.2) *for $\gamma = n^{-1/4}$ and initial law $\mu$, and let*

$$\mathcal{X}_t^n(\varphi) = \frac{1}{n^{1/4}} \sum_j u_j^n(tn)\varphi(\tfrac{j}{\sqrt{n}}).$$

*The sequence of processes $(\mathcal{X}^n_\cdot)_{n\geq 1}$ converges in distribution in $C([0,T], \mathcal{S}'(\mathbb{R}))$ to the unique energy solution of the Burgers equation.*

A similar result was shown in [11] for much more general initial conditions although restricted to the periodic setting.

At the technical level, our approach relies on the techniques of [9] and avoids the use of any spectral gap estimate. The core of the proof consists in deriving certain dynamical estimates among which the so-called second order Boltzmann-Gibbs principle plays a major role. A key ingredient is a certain integration-by-parts satisfied by the model.

The paper is organized as follows: in Section 2, we recall the notion of energy solution from [8]. We show the invariance of the measure $\mu$ in Section 3. In Section 4, we prove the dynamical estimates. Finally, in Sections 5 and 6, we show, respectively, tightness and convergence to the energy solution. The construction of the dynamics (1.2) is given in the appendix.

Notations: We denote by $\mathcal{S}(\mathbb{R})$ the space of Schwarz functions on $\mathbb{R}$. For $n \geq 1$ and a smooth function $\varphi$, we define $\varphi_j^n = \varphi(\frac{j}{\sqrt{n}})$, $\nabla^n\varphi_j^n = \sqrt{n}(\varphi_{j+1}^n - \varphi_j^n)$ and $\Delta^n\varphi_j^n = n(\varphi_{j+1}^n + \varphi_{j-1}^n - 2\varphi^n)$. We also define

$$\mathcal{E}(\varphi) = \int \varphi^2(x)\,dx, \quad \mathcal{E}_n(\psi) = \frac{1}{\sqrt{n}}\sum_{j\in\mathbb{Z}} \psi_j^2,$$

respectively, for $\varphi \in L^2(\mathbb{R})$ and $\psi \in l^2(\mathbb{Z})$.

## 2 Energy solutions of the Burgers equation

We will introduce the notion of an energy solution for Burgers equation [8]. We start with two definitions:

**Definition 2.1.** *We say that a process $\{u_t : t \in [0,T]\}$ satisfies condition (S) if, for all $t \in [0,T]$, the $\mathcal{S}'(\mathbb{R})$-valued random variable $u_t$ is a white noise of variance* 1.

For a stationary process $\{u_t : t \in [0,T]\}$, $0 \leq s < t \leq T$, $\varphi \in \mathcal{S}(\mathbb{R})$ and $\varepsilon > 0$, we define

$$\mathcal{A}_{s,t}^{\varepsilon}(\varphi) = \int_s^t \int_{\mathbb{R}} u_r(i_\varepsilon(x))^2 \partial_x\varphi(x)dxdr$$

where $i_\varepsilon(x) = \varepsilon^{-1}\mathbf{1}_{(x,x+\varepsilon]}$.

**Definition 2.2.** *Let $\{u_t : t \in [0,T]\}$ be a process satisfying condition (S). We say that $\{u_t : t \in [0,T]\}$ satisfies the energy estimate if there exists a constant $\kappa > 0$ such that:*

*(EC1) For any $\varphi \in \mathcal{S}(\mathbb{R})$ and any $0 \le s < t \le T$,*

$$\mathbb{E}\left[\left|\int_s^t u_r(\partial_x^2\varphi)\,dr\right|^2\right] \le \kappa(t-s)\mathcal{E}(\partial_x\varphi)$$

*(EC2) For any $\varphi \in \mathcal{S}(\mathbb{R})$, any $0 \le s < t \le T$ and any $0 < \delta < \varepsilon < 1$,*

$$\mathbb{E}\left[\left|\mathcal{A}^\varepsilon_{s,t}(\varphi) - \mathcal{A}^\delta_{s,t}(\varphi)\right|^2\right] \le \kappa(t-s)\varepsilon\mathcal{E}(\partial_x\varphi)$$

We state a theorem proved in [8]:

**Theorem 2.3.** *Assume $\{u_t : t \in [0,T]\}$ satisfies (S) and (EC2). There exists an $\mathcal{S}'(\mathbb{R})$-valued stochastic process $\{\mathcal{A}_t : t \in [0,T]\}$ with continuous paths such that*

$$\mathcal{A}_t(\varphi) = \lim_{\varepsilon\to 0}\mathcal{A}^\varepsilon_{0,t}(\varphi),$$

*in $L^2$, for any $t \in [0,T]$ and $\varphi \in \mathcal{S}(\mathbb{R})$.*

We are now ready to formulate the definition of an energy solution:

**Definition 2.4.** *We say that $\{u_t : t \in [0,T]\}$ is a stationary energy solution of the Burgers equation if*

- *$\{u_t : t \in [0,T]\}$ satisfies (S), (EC1) and (EC2).*
- *For all $\varphi \in \mathcal{S}(\mathbb{R})$, the process*

$$u_t(\varphi) - u_0(\varphi) - \tfrac{1}{2}\int_0^t u_s(\partial_x^2\varphi)\,ds - \mathcal{A}_t(\varphi)$$

  *is a martingale with quadratic variation $t\mathcal{E}(\partial_x\varphi)$, where $\mathcal{A}$ is the process from Theorem 2.3.*

Existence of energy solutions was proved in [8]. Uniqueness was proved in [12].

## 3 Generator and invariant measure

The construction of the dynamics given by (1.2) is detailed in Appendix A. We denote by $\mathscr{C}$ the set of cylindrical functions $F$ of the form $F(u) = f(u_{-n}, \cdots, u_n)$, for some $n \ge 0$, with $f \in C^2(\mathbb{R}^{2n+1})$ with polynomial growth of its partial derivatives up to order 2. The generator of the dynamics (1.2) acts on $\mathscr{C}$ as

$$L = \sum_j \left\{\frac{1}{2}(\partial_{j+1} - \partial_j)^2 - \frac{1}{2}(u_{j+1} - u_j)(\partial_{j+1} - \partial_j) + \gamma B_j(u)\partial_j\right\},$$

where $\partial_j = \frac{\partial}{\partial u_j}$. Let us introduce the operators

$$S \;=\; \sum_j \left\{\frac{1}{2}(\partial_{j+1} - \partial_j)^2 - \frac{1}{2}(u_{j+1} - u_j)(\partial_{j+1} - \partial_j)\right\}, \quad A = \sum_j \gamma B_j(u)\partial_j,$$

which formally correspond to the symmetric and anti-symmetric parts of $L$ with respect to $\mu = \rho^{\otimes\mathbb{Z}}$, where $d\rho(x) = \frac{1}{\sqrt{2\pi}}e^{-x^2/2}dx$. We note that our model satisfies the Gaussian integration-by-parts formula:

$$\int u_j f d\mu = \int \partial_j f d\mu,$$

which will be heavily used in the sequel.

We will also consider the periodic model $u^M$ on $\mathbb{Z}_M := \mathbb{Z}/M\mathbb{Z}$ and denote by $L_M$, $S_M$ and $A_M$ the corresponding generator and its symmetric and anti-symmetric parts respectively. Finally, denote $\mu_M = \rho^{\otimes \mathbb{Z}_M}$ and let $\rho_M$ be its density.

**Lemma 3.1.** *The measure $\mu_M$ is invariant for the periodic dynamics $u^M$.*

*Proof.* The lemma follows from Echeverría's criterion ([7], Thm 4.9.17) once we show

$$\int L_M f \, d\mu_M = 0,$$

for all $f \in C^2(\mathbb{R}^{\mathbb{Z}_M})$ with polynomial growth of its derivatives up to order $2$. By standard integration-by-parts,

$$\int S_M f \, d\mu_M \quad = \quad \int f(u) S_M^\dagger \rho_M(u) \, du_{-M} \cdots du_M,$$

where

$$S_M^\dagger = \frac{1}{2} \sum_{j \in \mathbb{Z}_M} \left\{ (\partial_{j+1} - \partial_j)^2 + (u_j - u_{j+1})(\partial_j - \partial_{j+1}) + 2 \right\}.$$

It is a simple computation to show that $S_M^\dagger \rho_M \equiv 0$. It then remains to verify that

$$\int A_M f \, d\mu_M = \int \sum_{j \in \mathbb{Z}_M} (w_j - w_{j-1}) \partial_j f(u) \rho_M(u) \, du_{-M} \cdots du_M = 0.$$

But, using standard integration-by-parts once again, we can verify that there exists a degree three polynomial in two variables $p(\cdot, \cdot)$ such that

$$\int A_M f \, d\mu_m = \int \sum_{j \in \mathbb{Z}_M} f(u) \left\{ p(u_j, u_{j+1}) - p(u_{j-1}, u_j) \right\} d\mu_M.$$

Finally, Gaussian integration-by-parts yields a degree two polynomial in two variables $\tilde{p}(\cdot, \cdot)$ such that

$$\int A_M f \, d\mu_M = \int \sum_{j \in \mathbb{Z}_M} \left\{ \tilde{p}(\partial_j, \partial_{j+1}) - \tilde{p}(\partial_{j-1}, \partial_j) \right\} f(u) \, d\mu,$$

which is telescopic. This ends the proof. □

By construction of the infinite volume dynamics and taking the limit $M \to \infty$, we obtain

**Corollary 1.** The measure $\mu$ is invariant for the dynamics (1.2).

## 4 The second-order Boltzmann-Gibbs principle

We recall the Kipnis-Varadhan inequality: there exists $C > 0$ such that

$$\mathbb{E} \left[ \sup_{0 \le t \le T} \left| \int_0^t F(u(sn)) \, ds \right|^2 \right] \le CT ||F(\cdot)||_{-1,n}^2 ds, \tag{4.1}$$

where the $|| \cdot ||_{-1,n}$-norm is defined through the variational formula

$$||F||_{-1,n}^2 = \sup_{f \in \mathscr{C}} \left\{ 2 \int F(u) f d\mu + n \int f L f d\mu \right\}$$

The proof of this inequality in our context follows from a straightforward modification of the arguments of [12], Corollary 3.5. In our particular model, we have

$$-\int fLf d\mu = \frac{1}{2}\sum_j \int \left((\partial_{j+1}-\partial_j)f\right)^2 d\mu$$

so that the variational formula becomes

$$||F||^2_{-1,n} = \sup_{f\in\mathscr{C}} \left\{ 2\int F(u) f d\mu - \frac{n}{2}\sum_j \int \left((\partial_{j+1}-\partial_j)f\right)^2 d\mu \right\}.$$

Denote by $\tau_j$ the canonical shift $\tau_j u_i = u_{j+i}$ and let $\overrightarrow{u}^l_j = \frac{1}{l}\sum_{k=1}^{l} u_{j+k}$.

**Lemma 4.1.** *Let $l \geq 1$ and let $g$ be a function with zero mean with respect to $\mu$ which support does not intersect $\{1,\cdots,l\}$. Let $g_j(s) = g(\tau_j u(s))$. There exists a constant $C>0$ such that*

$$\mathbb{E}\left[\left|\int_0^t ds \sum_j g_j(sn)[u_{j+1}(sn) - \overrightarrow{u}^l_j(sn)]\varphi_j\right|^2\right] \leq C\frac{tl}{\sqrt{n}}||g||^2_{L^2(\mu)}\mathcal{E}_n(\varphi) \tag{4.2}$$

*Proof.* Let $\psi_i = \frac{l-i}{l}$, $i = 0,\cdots,l-1$. Then,

$$u_{j+1} - \overrightarrow{u}^l_j = \sum_{i=1}^{l-1}(u_{j+i}-u_{j+i+1})\psi_i.$$

Hence,

$$\begin{aligned}
\sum_j \varphi_j g_j(u_{j+1}-\overrightarrow{u}^l_j) &= \sum_j \varphi_j g_j \sum_{i=0}^{l-1}(u_{j+i}-u_{j+i+1})\psi_i \\
&= \sum_k \left(\sum_{i=1}^{l-1}\varphi_{k-i}g_{k-i}\psi_i\right)(u_k - u_{k+1}) \\
&=: \sum_k F_k(u_k-u_{k+1})
\end{aligned}$$

Now, for $f\in\mathscr{C}$, using integration-by-parts,

$$\begin{aligned}
2\int\sum_j \varphi_j g_j(u_{j+1}-\overrightarrow{u}^l_j) f d\mu &= 2\int\sum_k F_k(u_k-u_{k+1}) f d\mu \\
&= 2\int\sum_k F_k(\partial_k-\partial_{k+1}) f d\mu \\
&\leq \int\sum_k \left\{\alpha F_k^2 + \frac{1}{\alpha}((\partial_k-\partial_{k+1})f)^2\right\} d\mu,
\end{aligned}$$

by Young's inequality. Taking $\alpha = 2/n$, we find that the above is bounded by

$$\frac{2}{n}\sum_k\int\sum_k F_k^2 d\mu + \frac{n}{2}\sum_k\int((\partial_k-\partial_{k+1})f)^2 d\mu,$$

which, thanks to the Kipnis-Varadhan inequality, shows that the left-hand-side of (4.2) is bounded by

$$C\frac{t}{n}\sum_k\int F_k^2 d\mu.$$

Finally, as $g$ is centered,

$$\sum_k \int F_k^2 d\mu \le \sum_k \sum_{i=1}^{l-1} \varphi_{k-i}^2 \int g^2 d\mu \le l\sqrt{n} \int g^2 d\mu \mathcal{E}_n(\varphi). \qquad \square$$

We now state the second-order Boltzmann-Gibbs principle: let $Q(l,u) = (\overrightarrow{u}^l_0)^2 - \frac{1}{l}$,

**Proposition 4.2.** *Let $l \ge 1$. There exists a constant $C > 0$ such that*

$$\mathbb{E}\left[\left|\int_0^t ds \sum_j \{u_j(sn)u_{j+1}(sn) - \tau_j Q(l,u(sn))\}\varphi_j\right|^2\right] \le C\frac{tl}{\sqrt{n}}\mathcal{E}_n(\varphi)$$

*Proof.* We use the factorization

$$u_j u_{j+1} - \tau_j Q(l,u) \quad = \quad u_j(u_{j+1} - \overrightarrow{u}^l_j) + \overrightarrow{u}^l_j(u_j - \overrightarrow{u}^l_j) + \frac{1}{l}.$$

We handle the first term with Lemma 4.1. The second term is treated in the following lemma. $\square$

**Lemma 4.3.** *Let $l \ge 1$. There exists a constant $C > 0$ such that*

$$\mathbb{E}\left[\left|\int_0^t ds \sum_j \left\{\overrightarrow{u}^l_j(sn)[u_j(sn) - \overrightarrow{u}^l_j(sn)] + \frac{1}{l}\right\}\varphi_j\right|^2\right] \le C\frac{tl}{\sqrt{n}}\mathcal{E}_n(\varphi)$$

*Proof.* Let $\psi_i = \frac{l-i}{l}$. Then,

$$\overrightarrow{u}^l_j[u_j - \overrightarrow{u}^l_j] = \sum_{i=0}^{l-1} \psi_i(u_{j+i} - u_{j+i+1})\overrightarrow{u}^l_j.$$

For $f \in \mathscr{C}$, using integration-by-parts,

$$\begin{aligned}\int \overrightarrow{u}^l_j[u_j - \overrightarrow{u}^l_j] f d\mu \quad &= \quad \int \sum_{i=0}^{l-1} \psi_i(u_{j+i} - u_{j+i+1})\overrightarrow{u}^l_j f d\mu \\ &= \quad \int \left\{\sum_{i=0}^{l-1} \psi_i \overrightarrow{u}^l_j(\partial_{j+i} - \partial_{j+i+1})f - \frac{1}{l}f\right\} d\mu\end{aligned}$$

The second summand comes from the term $i = 0$. Hence,

$$2\int \sum_j \varphi_j \left\{\overrightarrow{u}^l_j[u_j - \overrightarrow{u}^l_j] + \frac{1}{l}\right\} f d\mu = 2\int \sum_j \varphi_j \sum_{i=0}^{l-1} \psi_i \overrightarrow{u}^l_j(\partial_{j+i} - \partial_{j+i+1}) f d\mu$$

By Young's inequality, this last expression is bounded by

$$\begin{aligned}&\int \sum_j \sum_{i=0}^{l-1} \left\{\alpha\varphi_j^2(\overrightarrow{u}^l_j)^2 + \frac{1}{\alpha}\psi_i^2((\partial_{j+i} - \partial_{j+i+1})f)^2\right\} d\mu \\ \le \quad &\alpha l \int \sum_j \varphi_j^2(\overrightarrow{u}^l_j)^2 d\mu + \frac{l}{\alpha}\int \sum_j ((\partial_j - \partial_{j+1})f)^2 d\mu\end{aligned}$$

Taking $\alpha = 2l/n$, this is further bounded by

$$\begin{aligned}&\frac{2l^2}{n}\int (\overrightarrow{u}^l_j)^2 d\mu \sum_j \varphi_j^2 + \frac{n}{2}\int \sum_j ((\partial_j - \partial_{j+1})f)^2 d\mu \\ \le \quad &\frac{l}{\sqrt{n}}\mathcal{E}_n(\varphi) + \frac{n}{2}\int \sum_j ((\partial_j - \partial_{j+1})f)^2 d\mu.\end{aligned}$$

The result then follows from the Kipnis-Varadhan inequality. $\square$

## 5 Tightness

In the sequel, we let $\varphi \in \mathcal{S}$ be a test function. Remember the fluctuation field is given by

$$\mathcal{X}_t^n(\varphi) = \frac{1}{n^{1/4}} \sum_j u_j(nt)\varphi_j^n.$$

Recalling the definition of the operators $S$ and $A$ from Section 3, the symmetric and anti-symmetric parts of the dynamics are given by

$$\begin{aligned} d\mathcal{S}_t^n(\varphi) &= nS\mathcal{X}_t^n(\varphi)dt &&= \frac{1}{n^{1/4}} n \sum_j u_j(tn)\Delta\varphi_j^n dt = \frac{1}{n^{1/4}} \sum_j u_j(tn)\Delta^n\varphi_j^n dt \\ d\mathcal{B}_t^n(\varphi) &= nA\mathcal{X}_t^n(\varphi)dt &&= -\frac{1}{n^{1/2}} n \sum_j w_j(tn)(\varphi_{j+1}^n - \varphi_j^n)dt = \sum_j w_j(tn)\nabla^n\varphi_j^n dt \end{aligned}$$

where we used $\gamma = n^{-1/4}$. Then, the martingale part of the dynamics corresponds to

$$\mathcal{M}_t^n(\varphi) = \mathcal{X}_t^n(\varphi) - \mathcal{X}_0^n(\varphi) - \mathcal{S}_t^n(\varphi) - \mathcal{B}_t^n(\varphi) = n^{1/4} \int_0^t \sum_j (\varphi_j - \varphi_{j+1}) d\xi_j(s)$$

and has quadratic variation

$$\langle \mathcal{M}^n(\varphi) \rangle_t = n^{1/2} t \sum_j (\varphi_j^n - \varphi_{j+1}^n)^2 = t\mathcal{E}_n(\nabla^n\varphi^n)$$

We will use Mitoma's criterion [19]: a sequence $\mathcal{Y}^n$ is tight in $C([0,T], \mathcal{S}'(\mathbb{R}))$ if and only if $\mathcal{Y}^n(\varphi)$ is tight in $C([0,T], \mathbb{R})$ for all $\varphi \in \mathcal{S}(\mathbb{R})$.

### 5.1 Martingale term

We recall that $\langle \mathcal{M}^n(\varphi) \rangle = t\mathcal{E}_n(\nabla^n\varphi^n)$. From the Burkholder-Davis-Gundy inequality, it follows that

$$\mathbb{E}\left[|\mathcal{M}_t^n(\varphi) - \mathcal{M}_s^n(\varphi)|^p\right] \leq C|t-s|^{p/2}\mathcal{E}_n(\nabla^n\varphi^n)^{p/2},$$

for all $p \geq 1$. Tightness then follows from Kolmogorov criterion by taking $p$ large enough.

### 5.2 Symmetric term

Tightness is obtained via a second moment computation and Kolmogorov criterion:

$$\mathbb{E}\left[|\mathcal{S}_t^n(\varphi) - \mathcal{S}_s^n(\varphi)|^2\right] \leq |t-s|^2 \frac{1}{\sqrt{n}} \sum_j \mathbb{E}[u_j^2](\Delta^n\varphi_j^n)^2 = |t-s|^2\mathcal{E}_n(\Delta^n\varphi^n).$$

### 5.3 Anti-symmetric term

We study the tightness of the term

$$\begin{aligned} \mathcal{B}_t^n(\varphi) &= \int_0^t \sum_j w_j(sn)\nabla^n\varphi_j^n ds \\ &= \int_0^t \sum_j \frac{1}{3}[u_{j+1}^2(sn) + u_j(sn)u_{j+1}(sn) + u_j^2(sn)]\nabla^n\varphi_j^n ds. \end{aligned}$$

We begin with a lemma:

**Lemma 5.1.** *The process*

$$Y_t^n(\varphi) = \int_0^t ds \sum_j \varphi_j \left\{ (u_j(sn)u_{j+1}(sn) - u_j^2(sn)) + 1 \right\}$$

*goes to zero in the ucp topology.*

*Proof.* Using integration by parts,

$$\begin{aligned} \int \sum_j \varphi_j (u_j u_{j+1} - u_j^2) f d\mu &= \int \sum_j \varphi_j (u_{j+1} - u_j) u_j f d\mu \\ &= \int \sum_j \varphi_j (\partial_{j+1} - \partial_j)(u_j f) d\mu \\ &= \int \sum_j \varphi_j \left\{ u_j (\partial_{j+1} - \partial_j) f - f \right\} \end{aligned}$$

Hence,

$$\int \sum_j \varphi_j \left\{ (u_j u_{j+1} - u_j^2) + 1 \right\} f d\mu = \int \sum_j \varphi_j u_j (\partial_{j+1} - \partial_j) f d\mu$$

Using Young's inequality,

$$\begin{aligned} 2 \int \sum_j \varphi_j \left\{ (u_j u_{j+1} - u_j^2) + 1 \right\} f d\mu &\leq \int \sum_j \left\{ \alpha \varphi_j^2 u_j^2 + \frac{1}{\alpha} ((\partial_{j+1} - \partial_j) f)^2 \right\} d\mu \\ &\leq \frac{2}{\sqrt{n}} \mathcal{E}_n(\varphi) + \frac{n}{2} \sum_j \int ((\partial_{j+1} - \partial_j) f)^2 d\mu, \end{aligned}$$

by taking $\alpha = 2/n$. Into the Kipnis-Varadhan inequality, this yields

$$\mathbb{E}\left[ \sup_{0 \leq t \leq T} \left| \int_0^t ds \sum_j \varphi_j \left\{ (u_j(sn)u_{j+1}(sn) - u_j^2(sn)) + 1 \right\} \right|^2 \right] \leq \frac{CT}{\sqrt{n}} \mathcal{E}_n(\varphi)$$

which shows that this process goes to zero in the ucp topology. □

This means we can switch the term $w_j$ in the anti-symmetric part of the dynamics by $u_j u_{j+1}$ modulo a vanishing term. Note that, as we apply the previous lemma to a gradient, the constant term $1$ will disappear. We are then left to prove the tightness of

$$\widetilde{\mathcal{B}}_t^n(\varphi) = \int_0^t \sum_j u_j(sn) u_{j+1}(sn) \nabla^n \varphi_j^n ds.$$

From Proposition 4.2, we have

$$\mathbb{E}\left[ \left| \widetilde{\mathcal{B}}_t^n(\varphi) - \int_0^t \sum_j \tau_j Q(l, u(sn)) \nabla^n \varphi_j^n ds \right|^2 \right] \leq C \frac{tl}{\sqrt{n}} \mathcal{E}_n(\nabla^n \varphi^n)$$

where, here and below, $C$ denotes a constant which value can change from line to line. On the other hand, a careful $L^2$ computation, taking dependencies into account, shows that

$$\mathbb{E}\left[ \left| \int_0^t \sum_j \tau_j Q(l, u(sn)) \nabla^n \varphi_j^n ds \right|^2 \right] \leq C \frac{t^2 \sqrt{n}}{l} \mathcal{E}_n(\nabla^n \varphi^n).$$

Observe that $\lim_{n\to\infty} \mathcal{E}_n(\nabla^n \varphi^n) = \int \partial_x \varphi(x)^2\, dx < \infty$. Summarizing,

$$\mathbb{E}\left[\left|\widetilde{\mathcal{B}}_t^n(\varphi)\right|^2\right] \leq C\left\{\frac{tl}{\sqrt{n}} + \frac{t^2\sqrt{n}}{l}\right\}.$$

For $t \geq 1/n$, we take $l \sim \sqrt{tn}$ and get

$$\mathbb{E}\left[\left|\widetilde{\mathcal{B}}_t^n(\varphi)\right|^2\right] \leq Ct^{3/2}.$$

For $t \leq 1/n$, a crude $L^2$ bound gives

$$\mathbb{E}\left[\left|\widetilde{\mathcal{B}}_t^n(\varphi)\right|^2\right] \leq Ct^2\sqrt{n} \leq Ct^{3/2}.$$

This gives tightness.

## 6 Convergence

From the previous section, we get processes $\mathcal{X}$, $\mathcal{S}$, $\mathcal{B}$ and $\mathcal{M}$ such that

$$\begin{aligned} \lim_{n\to\infty} \mathcal{X}^n &= \mathcal{X}, \qquad \lim_{n\to\infty} \mathcal{S}^n = \mathcal{S}, \\ \lim_{n\to\infty} \mathcal{B}^n &= \mathcal{B}, \qquad \lim_{n\to\infty} \mathcal{M}^n = \mathcal{M}, \end{aligned}$$

along a subsequence that we still denote by $n$. We will now identify these limiting processes.

### 6.1 Convergence at fixed times

A straightforward adaptation of the arguments in [6], Section 4.1.1, shows that $\mathcal{X}_t^n$ converges to a white noise for each fixed time $t \in [0, T]$. This in turns proves that the limit satisfies property (S).

### 6.2 Martingale term

The quadratic variation of the martingale part satisfies

$$\lim_{n\to\infty} \langle \mathcal{M}^n(\varphi)\rangle_t = t||\partial_x \varphi||_{L^2}^2.$$

By a criterion of Aldous [2], this implies convergence to the white noise.

### 6.3 Symmetric term

A second moment bound shows that

$$\mathbb{E}\left[\left|\mathcal{S}_t^n(\varphi) - \int_0^t \mathcal{X}_s^n(\partial_x^2\varphi)\, ds\right|^2\right] \leq C\frac{t^2}{n},$$

which shows that

$$\mathcal{S}(\varphi) = \lim_{n\to\infty} \mathcal{S}^n(\varphi) = \int_0^{\cdot} \mathcal{X}_s(\partial_x^2 \varphi)\, ds.$$

### 6.4 Anti-symmetric term

We just have to identify the limit of the process $\widetilde{\mathcal{B}}^n(\varphi)$. Remembering the definition of the field $\mathcal{X}^n$, we observe that

$$\sqrt{n}Q(\varepsilon\sqrt{n}, u(nt)) = \mathcal{X}_t^n(i_\varepsilon(0))^2 - \frac{1}{\varepsilon},$$

from where we get the convergences

$$\lim_{n\to\infty} \sqrt{n} Q(\varepsilon\sqrt{n}, u(nt)) = \mathcal{X}_t(i_\varepsilon(0))^2 - \frac{1}{\varepsilon}$$

and

$$\mathcal{A}^\varepsilon_{s,t}(\varphi) := \lim_{n\to\infty} \int_s^t \sum_j \tau_j Q(\varepsilon\sqrt{n}, u(rn)) \nabla^n \varphi^n_j dr.$$

The second limit follows by a suitable approximation of $i_\varepsilon(x)$ by $\mathcal{S}(\mathbb{R})$ functions (see [8], Section 5.3 for details). Now, by the second-order Boltzmann-Gibbs principle and stationarity,

$$\mathbb{E}\left[\left|\widetilde{\mathcal{B}}^n_t(\varphi) - \widetilde{\mathcal{B}}^n_s(\varphi) - \int_s^t \sum_j \tau_j Q(l, u(rn)) \nabla^n \varphi^n_j dr\right|^2\right] \leq C\frac{(t-s)l}{\sqrt{n}}.$$

Taking $l \sim \varepsilon\sqrt{n}$ and the limit as $n \to \infty$ along the subsequence,

$$\mathbb{E}\left[\left|\mathcal{B}_t(\varphi) - \mathcal{B}_s(\varphi) - \mathcal{A}^\varepsilon_{s,t}(\varphi)\right|^2\right] \leq C(t-s)\varepsilon. \tag{6.1}$$

The energy estimate (EC2) then follows by the triangle inequality. Theorem 2.3 yields the existence of the process

$$\mathcal{A}_t(\varphi) = \lim_{\varepsilon\to 0} \mathcal{A}^\varepsilon_{0,t}(\varphi).$$

Furthermore, from (6.1), we deduce that $\mathcal{B} = \mathcal{A}$.

It remains to check (EC1). It is enough to check that

$$\mathbb{E}\left[\left|\int_0^t \mathcal{X}^n_s(\partial_x^2 \varphi)\right|^2\right] \leq \kappa t.$$

Using the smoothness of $\varphi$ and a summation by parts, it is further enough to verify that

$$\mathbb{E}\left[\left|\int_0^t n^{1/4} \sum_j [u_{j+1}(sn) - u_j(sn)] \nabla^n \varphi^n_j\right|^2\right] \leq \kappa t. \tag{6.2}$$

For that purpose, we will use Kipnis-Varadhan inequality one last time: let $f \in \mathscr{C}$,

$$\begin{aligned} 2\int n^{1/4} \sum_j (u_{j+1} - u_j) \nabla^n \varphi^n_j f d\mu &= 2\int n^{1/4} \sum_j \nabla^n \varphi^n_j (\partial_{j+1} - \partial_j) f d\mu \\ &\leq \sum_j \left\{\alpha\sqrt{n}(\nabla^n \varphi^n_j)^2 + \frac{1}{\alpha}\int ((\partial_{j+1} - \partial_j) f)^2 d\mu\right\} \\ &\leq 2\mathcal{E}_n(\nabla^n \varphi^n) + \frac{n}{2}\sum_j \int ((\partial_{j+1} - \partial_j) f)^2 d\mu, \end{aligned}$$

with $\alpha = 2/n$, from where (6.2) follows.

## A Construction of the dynamics

The system of equations (1.2) can be reformulated as

$$u_j(t) = \frac{1}{2}\int_0^t \Delta u_j(s)\,ds + \gamma \int_0^t B_j(u(s))\,ds + \xi_j(t) - \xi_{j-1}(t).$$

We consider the system $u^M$ on $\mathbb{Z}_M = \mathbb{Z}/M\mathbb{Z}$ evolving under its invariant distribution. We first check that, for all $j$ and $T > 0$

$$\mathbb{E}\left[\sup_{0\le t\le T} |u_j^M(t)|^2\right] < \infty,$$

so that the dynamics is well-defined. Everything boils down to estimates of type

$$\begin{aligned}
\mathbb{E}\left[\sup_{0\le t\le T}\left|\int_0^t u_j^M(s)\,ds\right|^2\right] &\le T\mathbb{E}\left[\sup_{0\le t\le T}\int_0^t |u_j^M(s)|^2 ds\right] \\
&\le T\mathbb{E}\left[\int_0^T |u_j^M(s)|^2 ds\right] \\
&\le T^2,
\end{aligned}$$

where we used invariance in the last step.

Next, we show tightness of the processes (in $M$) where we now identify $u^M$ with a periodic system on the line. This follows from Kolmogorov's criterion. It is enough to control expressions of type

$$\mathbb{E}\left[\left|\int_s^t u_j^M(r)\,dr\right|^4\right] \le |t-s|^3 \mathbb{E}\left[\int_s^t \left|u_j^M(r)\right|^4\, dr\right] \le C|t-s|^3.$$

Together with a standard estimate on the increments of the Brownian motion, this yields

$$\mathbb{E}\left[|u_j^M(t) - u_j^M(s)|^2\right] \le C|t-s|^2.$$

Hence, each coordinate is tight. By diagonalization, we can extract a subsequence of $M_k$ such that $(u_j^{M_k})$ converges in law in $C[0,T]$ for each $j$. This gives a meaning to the system (1.2).

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