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Uniform Hausdorff dimension result for the inverse images of stable Lévy processes*

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Abstract

We establish a uniform Hausdorff dimension result for the inverse image sets of real-valued strictly α-stable Lévy processes with $1 < \alpha \le 2$. This extends a theorem of Kaufman [11] for Brownian motion. Our method is different from that of [11] and depends on covering principles for Markov processes.

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1 Introduction

Let $X = \{X(t), t \ge 0, \mathbb{P}^x\}$ be a real-valued strictly α-stable Lévy process with $\alpha \in (0, 2]$. Its characteristic exponent is given by, for $\xi \in \mathbb{R}$,

$$-\log \mathbb{E}^0[e^{i\xi X(1)}] = \begin{cases} \sigma^\alpha |\xi|^\alpha \Big(1 - i\beta \tan \frac{\pi\alpha}{2} \mathrm{sgn}\xi\Big), & \text{if } \alpha \neq 1; \\ \sigma|\xi|, & \text{if } \alpha = 1 \end{cases}$$

with some constants $\sigma > 0$ and $\beta \in [-1, 1]$ which are respectively the scale parameter and the skewness parameter. Throughout $\log = \log_e$ denotes the natural logarithm. Notice that, in the case of $\alpha = 1$, X is a symmetric Cauchy process. When $\alpha = 2$, X is a (scaled) Brownian motion. For $0 < \alpha < 2$, X shares the properties of self-similarity, independence and stationarity of increments, with Brownian motion, but it has heavy-tailed distributions and its sample functions are discontinuous. As such, stable Lévy processes form an important class of Markov processes. Many authors have studied the asymptotic and sample path properties of Lévy processes. We refer to the monographs [2] and [21] for systematic accounts on Lévy processes, and to [24, 26] for information on their fractal properties.

This note is concerned with a uniform Hausdorff dimension result, Theorem 1.1, for the inverse images of real-valued strictly α-stable Lévy processes and is motivated by the following results of Hawkes [8] and Kaufman [11].

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Hawkes [8] considered the Hausdorff dimension of the inverse image $X^{-1}(F) = \{t \geq 0 : X(t) \in F\}$ and proved that if $1 \leq \alpha \leq 2$ and $F \subseteq \mathbb{R}$ is a fixed Borel set, then for every $x \in \mathbb{R}$,

$$\dim_{\mathrm{H}} X^{-1}(F) = 1 - \frac{1}{\alpha} + \frac{\dim_{\mathrm{H}} F}{\alpha}, \qquad \mathbb{P}^x\text{-a.s.} \tag{1.1}$$

Here $\dim_{\mathrm{H}}$ denotes Hausdorff dimension; see Falconer [6], or [24, 26] for the definitions and properties of Hausdorff measure and Hausdorff dimension.

Note that the null event on which (1.1) does not hold depends on F. It is natural to ask if the following uniform Hausdorff dimension result holds: For every $x \in \mathbb{R}$,

$$\mathbb{P}^x\left(\dim_{\mathrm{H}} X^{-1}(F) = 1 - \frac{1}{\alpha} + \frac{\dim_{\mathrm{H}} F}{\alpha} \text{ for all Borel sets } F \subseteq \mathbb{R}\right) = 1. \tag{1.2}$$

Such a result, when it is valid, is more useful than (1.1) because, outside of a single null event, the dimension formula holds not only for all deterministic Borel sets $F \subset \mathbb{R}$ but also for random sets F that depend on the sample path of X.

We claim that, in the case $0 < \alpha < 1$, there is no uniform result like (1.2). This is because $X^{-1}(F) = \emptyset$ $\mathbb{P}^x$-a.s. if $\dim_{\mathrm{H}} F < 1 - \alpha$. The referee has asked us the following question that complements the aforementioned claim:[1] For every $x \in \mathbb{R}$, does

$$\mathbb{P}^x\left(\dim_{\mathrm{H}} X^{-1}(F) = 1 - \frac{1}{\alpha} + \frac{\dim_{\mathrm{H}} F}{\alpha} \text{ for all } F \in \mathcal{C}\right) < 1? \tag{1.3}$$

Here $\mathcal{C}$ is the family of all deterministic Borel sets $F \subset \mathbb{R}$ with $\dim_{\mathrm{H}} F \geq 1 - \alpha$. To answer this question, we first recall Theorem 2 of Hawkes [8] : If $0 < \alpha < 1$ and $F \subset \mathbb{R}$ is deterministic and satisfies $\dim_{\mathrm{H}} F \geq 1 - \alpha$, then

(i) For every $x \in \mathbb{R}$,

$$\sup\left\{\theta : \mathbb{P}^x\big(\dim_{\mathrm{H}} X^{-1}(F) \geq \theta\big) > 0\right\} = 1 - \frac{1}{\alpha} + \frac{\dim_{\mathrm{H}} F}{\alpha}. \tag{1.4}$$

(ii) If $x \in F^*$ (see [8, p.93] for the notation), then

$$\mathbb{P}^x\left(\dim_{\mathrm{H}} X^{-1}(F) = 1 - \frac{1}{\alpha} + \frac{\dim_{\mathrm{H}} F}{\alpha}\right) = 1. \tag{1.5}$$

(iii) If $F \backslash F^*$ is polar, then for every $x \in \mathbb{R}$,

$$\mathbb{P}^x\left(\dim_{\mathrm{H}} X^{-1}(F) = 1 - \frac{1}{\alpha} + \frac{\dim_{\mathrm{H}} F}{\alpha} \Big| X^{-1}(F) \neq \emptyset\right) = 1. \tag{1.6}$$

The answer to the referee's question is "yes" because we can choose a Borel set $F \in \mathcal{C}$ such that $F \backslash F^*$ is polar for X (cf. [8, p.96]), then it follows from Hawkes' result (iii) that for any $x \in \mathbb{R}$ the probability in (1.3) is not more than

$$\mathbb{P}^x\left(\dim_{\mathrm{H}} X^{-1}(F) = 1 - \frac{1}{\alpha} + \frac{\dim_{\mathrm{H}} F}{\alpha}\right) = \mathbb{P}^x\Big(X^{-1}(F) \neq \emptyset\Big) < 1.$$

Motivated by the referee's question and Hawkes' result (1.5), one may further ask to characterize the family $\mathcal{G}$ of deterministic Borel sets F such that for some $x \in \mathbb{R}$ (depending on $\mathcal{G}$),

$$\mathbb{P}^x\left(\dim_{\mathrm{H}} X^{-1}(F) = 1 - \frac{1}{\alpha} + \frac{\dim_{\mathrm{H}} F}{\alpha} \text{ for all } F \in \mathcal{G}\right) = 1. \tag{1.7}$$

[1] We thank the anonymous referee for this interesting question. Since $X^{-1}(F) = \emptyset$ $\mathbb{P}^x$-a.s. if $\dim_{\mathrm{H}} F < 1-\alpha$, we have modified slightly the referee's question.

This question seems to be rather nontrivial. We can imagine that (1.7) may hold for certain family of self-similar sets on $\mathbb{R}$, but this goes beyond the scope of the present paper.

Our objective of this paper is to study the uniform dimension problem (1.2) for $1 \le \alpha \le 2$. The validity of (1.2) in the case $\alpha = 2$ (X is a Brownian motion) is due to Kaufman [11]. His proof relies on the uniform modulus of continuity of Brownian motion as well as the Hölder continuity of the Brownian local time in the time variable. For $1 \le \alpha < 2$, the sample paths of an α-stable Lévy process are discontinuous, hence Kaufman's method is not applicable.

In the special case of $F = \{z\}$, it follows from Barlow et al [1, (8.7)] that if $1 < \alpha \le 2$ then

$$\mathbb{P}^x\Big(\dim_{\mathrm{H}} X^{-1}(z) = 1 - \frac{1}{\alpha} \text{ for all } z \in \mathbb{R}\Big) = 1. \tag{1.8}$$

This gives a uniform Hausdorff dimension result for the level sets of X. However, for $1 \le \alpha < 2$, it had been an open problem to prove (1.2) for all Borel sets $F \subseteq \mathbb{R}$; see [26, Sec. 8.2] for a discussion.

In this note, we verify (1.2) by proving the following theorem.

Theorem 1.1. *Let X be a real-valued strictly α-stable Lévy process with $1 < \alpha \le 2$. For every $x \in \mathbb{R}$, (1.2) holds.*

As mentioned above, the case of $\alpha = 2$ has already been proved by Kaufman [11] whose proof relies on special properties of Brownian motion. Our proof of Theorem 1.1 provides an alternative proof of his theorem.

The proof is split naturally into the upper bound part and lower bound part. To show the upper bound, we design a new covering principle (see Lemma 2.2 below) for the inverse images of recurrent processes (thus it is applicable to $\alpha = 1$). This covering lemma constitutes the key technical contribution of the present paper, and we expect it to be useful for other discontinuous Markov processes. Note that Lemma 2.2 in this paper is different from the covering lemma of [22, Lemma 2.2], which is only applicable to transient Markov processes (see Remark 2.3 in Section 2 of this paper). To prove the lower bound in (1.2), we make use of the uniform modulus of continuity (in time) of the maximum local time of X due to Perkins [18], together with a covering principle for the range of X in [10, 26, 22]. Since X has no local time when $\alpha = 1$, the proof of the lower bound in Theorem 1.1 is valid only for $1 < \alpha \le 2$. We think that (1.2) holds for $\alpha = 1$ as well, but have not been able to give a complete proof.

2 Proof of the upper bound

In this section we assume that $1 \le \alpha \le 2$. We will show that

$$\mathbb{P}^x\bigg(\dim_{\mathrm{H}} X^{-1}(F) \le 1 - \frac{1}{\alpha} + \frac{\dim_{\mathrm{H}} F}{\alpha} \text{ for all Borel sets } F \subseteq \mathbb{R}\bigg) = 1. \tag{2.1}$$

For any Borel set B, we denote by T_B the first hitting time of B by the process X. We state an asymptotic result due to Port [19, Thm. 2 and Thm. 4] on the first hitting time of compact sets by recurrent strictly stable processes, see [20, Thm. 22.1] for similar results in a more general setting. Note that when $1 \le \alpha \le 2$, X is recurrent by the Chung-Fuchs criterion ([20, Thm. 16.2]), and any nonempty set has positive capacity, so the condition in [20, Thm. 22.1] is satisfied.

Lemma 2.1. *(1). If $1 < \alpha \le 2$, then for any bounded interval B and any $x \in \mathbb{R}$,*

$$\mathbb{P}^x(T_B > t) \sim L_B(x) t^{-1+\frac{1}{\alpha}}, \qquad \text{as } t \to \infty,$$

where $L_B(x)$ is bounded from above on compact sets and is positive for $x \notin \overline{B}$, the closure of the set B. Here, $f(t) \sim g(t)$ means $\lim_{t\to\infty} f(t)/g(t) = 1$.

(2). If $\alpha = 1$, then for any bounded interval B and any $x \in \mathbb{R}$,

$$\mathbb{P}^x(T_B > t) \sim \frac{L_B(x)}{\log t}, \qquad \text{as } t \to \infty,$$

where $L_B(x)$ is bounded from above on compact sets and is positive for $x \notin \overline{B}$.

The main tool to obtain our upper bound is the following covering lemma. Before stating this lemma, we introduce some notation. Let $\mathcal{U}_n$ be any partition of $\mathbb{R}$ with intervals of length 2^{-n} and $\mathcal{D}_n$ be any partition of $\mathbb{R}_+$ with intervals of length $2^{-n\alpha}$. The choices of partitions have no effect on the result.

Lemma 2.2. *Let $1 \le \alpha \le 2$. Let $\delta > \alpha - 1$ and $T > 0$. $\mathbb{P}^x$-a.s., for all n large enough and every $U \in \mathcal{U}_n$, $X^{-1}(U) \cap [0,T]$ can be covered by $2 \cdot 2^{n\delta}$ intervals from $\mathcal{D}_n$.*

Proof. (1) Suppose first $1 < \alpha \le 2$. For a fixed interval $U \in \mathcal{U}_n$, write $U = (z - \frac{2^{-n}}{2}, z + \frac{2^{-n}}{2})$ for some $z \in \mathbb{R}$. Let $\tau_0 = 0$ and, for all $k \ge 1$, define

$$\tau_k = \inf\left\{ s > \tau_{k-1} + 2^{-n\alpha} : |X(s) - z| < \frac{2^{-n}}{2} \right\},$$

with the convention that $\inf \emptyset = \infty$. It is clear that $X^{-1}(U) \subset \bigcup_{i=0}^{\infty} [\tau_i, \tau_i + 2^{-n\alpha}]$, which implies that

$$\{\tau_k \ge T\} \subset \left\{ X^{-1}(U) \cap [0,T] \text{ can be covered by } k \text{ intervals of length } 2^{-n\alpha} \right\}.$$

Therefore,

$$\left\{ X^{-1}(U) \cap [0,T] \text{ cannot be covered by } k \text{ intervals of length } 2^{-n\alpha} \right\} \subset \{\tau_k < T\}.$$

Note by spatial homogeneity and scaling, we have that

$$\mathbb{P}^x\left(\inf_{2^{-n\alpha} \le s \le T} |X(s) - x| \le 2^{-n} \right) = \mathbb{P}^0\left(\inf_{1 \le s \le T 2^{n\alpha}} |X(s)| \le 1 \right) := p_n.$$

Due to the right continuity of the sample paths, we have $X(\tau_{k-1}) \in \overline{U}$ as $\tau_{k-1} < T$. By the strong Markov property, we obtain

$$\begin{aligned}
\mathbb{P}^x(\tau_k < T) &= \mathbb{P}^x(\tau_k < T | \tau_{k-1} < T)\mathbb{P}^x(\tau_{k-1} \le T) \\
&\le \sup_{y \in \overline{U}} \mathbb{P}^y\left(\inf_{2^{-n\alpha} \le s \le T} |X(s) - z| \le 2^{-n}/2 \right) \mathbb{P}^x(\tau_{k-1} \le T) \\
&\le \sup_{y \in \overline{U}} \mathbb{P}^y\left(\inf_{2^{-n\alpha} \le s \le T} |X(s) - y| - |y - z| \le 2^{-n}/2 \right) \mathbb{P}^x(\tau_{k-1} \le T) \\
&\le \sup_{y \in \overline{U}} \mathbb{P}^y\left(\inf_{2^{-n\alpha} \le s \le T} |X(s) - y| \le 2^{-n} \right) \mathbb{P}^x(\tau_{k-1} \le T) \\
&= p_n \cdot \mathbb{P}^x(\tau_{k-1} \le T).
\end{aligned}$$

By induction, we obtain

$$\mathbb{P}^x(\tau_k < T) \le p_n^k.$$

Next we show that there exists a constant c_T such that $p_n \le 1 - c_T 2^{-n\alpha(1-\frac{1}{\alpha})}$. By the independence of increments and the fact that $X(1)$ is supported on $\mathbb{R}$ ([23, Thm. 1]),

$$\begin{aligned}
1 - p_n &\ge \mathbb{P}^0(2 \le X(1) \le 3, \inf\{t \ge 1 : X(t) - X(1) \in [-4,-1]\} \ge T 2^{n\alpha}) \\
&\ge c\, \mathbb{P}^0(T_{[-4,-1]} \ge T 2^{n\alpha}).
\end{aligned}$$

Lemma 2.1 implies that

$$1 - p_n \geq c_T 2^{-n\alpha(1-\frac{1}{\alpha})},$$

as desired. For $n, K \geq 1$, define the event A_n^δ by

$$\big\{\exists U \in \mathcal{U}_n \cap [-K, K], \text{ s.t. } X^{-1}(U) \cap [0,T] \text{ cannot be covered by } 2^{n\delta} \text{ intervals of length } 2^{-n\alpha}\big\}.$$

Here $U \in \mathcal{U}_n \cap [-K,K]$ means that $U \in \mathcal{U}_n$ and $U \subset [-K,K]$. We have for $\delta > \alpha - 1$,

$$\begin{aligned}\sum_{n=1}^{\infty} \mathbb{P}^x(A_n^\delta) &\leq \sum_{n=1}^{\infty} \sharp\{U \in \mathcal{U}_n : U \cap [-K,K] \neq \emptyset\}(p_n)^{2^{n\delta}} \\ &\leq 2K \sum_{n=1}^{\infty} 2^n (1 - c_T 2^{-n\alpha(1-\frac{1}{\alpha})})^{2^{n\delta}} \\ &\leq 2K \sum_{n=1}^{\infty} \exp\Big(n(\log 2) - c_T 2^{n(\delta-\alpha+1)}\Big) < \infty.\end{aligned}$$

Since any interval of length $2^{-n\alpha}$ is covered by two intervals from $\mathcal{D}_n$, the conclusion for all $U \subset [-K,K]$ follows from the Borel-Cantelli Lemma. Letting $K \to \infty$ completes the proof.

(2) Now consider $\alpha = 1$. The proof of this case is basically the same as that of Part (1), except that $1 - p_n \geq c_T/n$ by Lemma 2.1.(2), and

$$\sum_{n=1}^{\infty} \mathbb{P}^x(A_n^\delta) \leq 2K \sum_{n=1}^{\infty} \exp(n(\log 2) - c_T 2^{n\delta}/n) < \infty.$$

We omit the details. □

Remark 2.3. As is said in the Introduction, the covering principle in [22, Lemma 2.2] is not applicable here. Intuitively, a recurrent process visits a fixed interval infinitely often, hence we could not expect that the inverse images could be covered by finite number of intervals. Mathematically, the condition in [22] is

$$\mathbb{P}^x\left(\inf_{t_n \leq t < T} |X(s) - x| \leq r_n\right) \leq K r_n^\delta$$

for some $\delta, p > 0$ and $\sum_{n=1}^{\infty} r_n^p < \infty$, which is not satisfied for recurrent Markov processes.

Let us prove the upper bound (2.1).

Proof of Theorem 1.1: upper bound. We first consider the case $1 < \alpha \leq 2$. For any Borel set F, let $\theta > \dim_{\mathrm{H}} F$ and $\delta > \alpha - 1$. Then there exists a sequence of intervals $\{U_i\}$ of length 2^{-n_i} such that

$$F \subset \bigcup_{i=1}^{\infty} U_i \quad \text{and} \quad \sum_{i=1}^{\infty} 2^{-n_i\theta} < 1.$$

Fix a $T > 0$ for now. By Lemma 2.2, each $X^{-1}(U_i) \cap [0,T]$ can be covered by $2 \cdot 2^{n_i\delta}$ intervals $\{I_{i,k}\}$ (of length $2^{-n_i\alpha}$) in $\mathcal{D}_{n_i}$, thus we see that

$$X^{-1}(F) \cap [0,T] \subset \bigcup_{i=1}^{\infty} \bigcup_{k=1}^{2\cdot 2^{n_i\delta}} I_{i,k}.$$

Moreover, let $d = (\theta + \delta)/\alpha$,

$$\sum_{i=1}^{\infty} \sum_{k=1}^{2\cdot 2^{n_i\delta}} [\mathrm{diam}(I_{i,k})]^d = 2 \cdot \sum_{i=1}^{\infty} 2^{n_i\delta} 2^{-n_i\alpha d} = 2 \cdot \sum_{i=1}^{\infty} 2^{-n_i\theta} < 2.$$

This proves $\dim_{\mathrm{H}} X^{-1}(F) \cap [0,T] \le d$. Letting $\theta \downarrow \dim_{\mathrm{H}} F$, $\delta \downarrow (\alpha - 1)$ and $T \uparrow \infty$ yields the desired upper bound.

Now we consider the case of $\alpha = 1$. One could repeat the argument above and use Lemma 2.2 to get the desired conclusion. Here we present an alternative argument. It follows from Hawkes and Pruitt [10] (see also [22]) that the following uniform dimension result holds:

$$\mathbb{P}^x \left(\dim_{\mathrm{H}} X(E) = \dim_{\mathrm{H}} E \ \text{ for all Borel } E \subset \mathbb{R}_+ \right) = 1. \tag{2.2}$$

For any Borel set $F \subset \mathbb{R}$, let $E = X^{-1}(F)$. Then $X(E) \subseteq F$. On the event in (2.2), we have $\dim_{\mathrm{H}} E = \dim_{\mathrm{H}} X(E) \le \dim_{\mathrm{H}} F$. Hence, $\mathbb{P}^x(\dim_{\mathrm{H}} X^{-1}(F) \le \dim_{\mathrm{H}} F$ for all $F \subset \mathbb{R}) = 1$. □

3 Proof of the lower bound

We assume that $1 < \alpha \le 2$. It follows from Kesten [12] and Hawkes [9] that X hits points and has local times $\{L_t^x, t \ge 0, x \in \mathbb{R}\}$. The local times characterize the sojourn properties of X via the occupation density formula: For all $t \ge 0$ and all Borel measurable function $f : \mathbb{R} \to \mathbb{R}$,

$$\int_0^t f(X(s))\mathrm{d}s = \int_{\mathbb{R}} f(x) L_t^x \mathrm{d}x.$$

Moreover, there is a version of the local times, still denoted by $\{L_t^x, t \ge 0, x \in \mathbb{R}\}$, which is jointly continuous in (t, x); see e.g., [2, 16].

We use the Hölder continuity of the local times of X to prove the uniform lower bound for the inverse image sets. This approach has been previously used by Kaufman [11], which was extended by Monrad and Pitt [17] in their study of inverse images of recurrent Gaussian fields. In both articles, the uniform modulus of continuity of the sample paths were used. Since the sample paths of the α-stable Lévy process X are discontinuous, we will apply a covering principle in [26, 22] for the range of X. Denote $\mathcal{C}_n$ any partition of $\mathbb{R}_+$ of intervals of length 2^{-n}. We recall here the covering principle, tailored to our situation.

Lemma 3.1. *Let $0 < \gamma < \frac{1}{\alpha}$. There exists a finite positive integer K, such that $\mathbb{P}^x$-a.s., for all n large enough, $X(I)$ can be covered by K intervals of diameter $2 \cdot 2^{-n\gamma}$, for all $I \in \mathcal{C}_n$.*

Proof. It suffices to verify condition (2.1) in the statement of [22, Lem. 2.1], namely, there exist $\delta > 0$ and $K_0 < \infty$ such that

$$\mathbb{P}^x \Big(\sup_{0 \le s \le 2^{-n}} |X(s) - x| \ge 2^{-n\gamma} \Big) \le K_0 2^{-n\delta}.$$

By spatial homogeneity and scaling, the probability above is equal to

$$\mathbb{P}^0 \Big(\sup_{0 \le s \le 1} |X(s)| \ge 2^{n(\frac{1}{\alpha} - \gamma)} \Big),$$

which, by [4, Thm. 5.1], is bounded from above by $2^{-n\delta}$ with $\delta = 1 - \gamma\alpha$, as desired. □

Let $L^*([s,t]) = \sup_{x\in\mathbb{R}}(L_t^x - L_s^x)$ be the maximum local time of X on $[s,t]$. We recall now the following result due to Perkins [18] on the uniform modulus of continuity (in time) of the maximum local time of a strictly α-stable Lévy process X with index $\alpha \in (1,2]$.

Lemma 3.2. *There exists a finite positive constant c_1 such that*

$$\limsup_{r\to 0} \sup_{\substack{|s-t|<r \\ 0\le s<t\le 1}} \frac{L^*([s,t])}{r^{1-\frac{1}{\alpha}}(\log 1/r)^{\frac{1}{\alpha}}} = c_1, \quad \mathbb{P}^x\text{-a.s.} \tag{3.1}$$

We refer to Ehm [5, Thm. 2.1] or Khoshnevisan, Zhong and Xiao [13, Thm. 4.3] for related results; and to Marcus and Rosen [14, 15, 16] for more sample path properties (in the space variable) of the local times of symmetric Markov processes.

We are ready to give the proof of the lower bound in Theorem 1.1.

Proof of Theorem 1.1: lower bound. It suffices to consider compact set F. For any compact $F \subset \mathbb{R}$ and $\varepsilon > 0$, by Frostman's lemma (cf. [6]) there exists a probability measure μ supported on F such that $\mu(B) \le |\mathrm{diam}(B)|^{\dim_{\mathrm{H}} F - \varepsilon}$ for any interval $B \subset \mathbb{R}$ with $|B| \le 1$. Define the random measure λ by

$$\lambda([a,b]) = \int_{\mathbb{R}} (L_b^x - L_a^x)\mu(\mathrm{d}x) \quad \text{for } 0 \le a \le b. \tag{3.2}$$

It is clear that $\lambda(\mathrm{d}t)$ is supported on $X^{-1}(F) \subset \mathbb{R}^+$, $\lambda(\mathbb{R}^+) > 0$, and

$$\lambda([a,b]) \le L^*([a,b])\mu(\overline{X([a,b])}).$$

Let n be sufficiently large, we have by Lemma 3.2 that

$$L^*([a, a+2^{-n}]) \le 2^{-n(1-\frac{1}{\alpha}-\varepsilon)}$$

uniformly for $a \in [0, 1-2^{-n}]$. On the other hand, by Lemma 3.1, there exist a sequence of intervals $\{I_i\}_{1\le i\le K}$ of length $2^{-n\gamma}$ with $\gamma < 1/\alpha$ such that the closure of $X([a, a+2^{-n}])$ is covered by the union of I_i, therefore,

$$\mu(\overline{X([a, a+2^{-n}])}) \le \sum_{i=1}^{K} \mu(I_i) \le K 2^{-n\gamma(\dim_{\mathrm{H}} F - \varepsilon)}. \tag{3.3}$$

We thus obtain

$$\lambda([a, a+2^{-n}]) \le K 2^{-n(1-\frac{1}{\alpha}+\gamma \dim_{\mathrm{H}} F - 2\varepsilon)}.$$

It follows that $\lambda(B) \le \mathrm{diam}(B)^{1-\frac{1}{\alpha}+\gamma\dim_{\mathrm{H}} F - 2\varepsilon}$ for all Borel sets B with sufficiently small diameter. This and Frostman's lemma imply that

$$\mathbb{P}^x\left(\dim_{\mathrm{H}} X^{-1}(F) \ge 1 - \frac{1}{\alpha} + \gamma \dim_{\mathrm{H}} F - 2\varepsilon \text{ for all compact Borel } F\right) = 1.$$

Letting $\gamma \uparrow \frac{1}{\alpha}$, then $\varepsilon \downarrow 0$ yields the desired lower bound for $\dim_{\mathrm{H}} X^{-1}(F)$. This finishes the proof of Theorem 1.1. □

4 Concluding remarks

This note raises several interesting questions for further investigation. In the following, we list three of them and discuss briefly the main difficulties. Solutions of these questions will require developing new techniques for Lévy processes.

(i). As having mentioned in the Introduction, we think that Theorem 1.1 holds for $\alpha = 1$. However, without a local time, it is not clear to us how to construct a random Borel measure supported on $X^{-1}(F)$ such that Frostman's lemma is applicable.

(ii). In [20, Thm. 22.1], the asymptotic result for the hitting times was obtained for recurrent Lévy processes with regularly varying λ-potential densities, see also the recent development by Grzywny and Ryznar [7]. Our method for proving the upper bound of $\dim_{\mathrm{H}} X^{-1}(F)$ is still applicable if the characteristic exponent of X is regularly varying at zero with index $\alpha \in (1, 2]$. On the other hand, by modifying the methods in Ehm [5], Khoshnevisan, Zhong and Xiao [13, Thm. 4.3], we can prove an upper bound for the uniform modulus of continuity in the time variable for the maximum local time as the one in Lemma 3.2 for Lévy processes with regularly varying exponent $\alpha \in (1, 2]$. Hence, Theorem 1.1 is valid for Lévy processes with regularly varying exponent $\alpha \in (1, 2]$. We believe that a similar result also holds for a large class of more general Markov processes including stable jump diffusions, stable like processes and Lévy-type processes as considered in [22]. However, proving such a result would require establishing first the asymptotic results for the hitting times and local times of these Markov processes. This is pretty challenging and goes well beyond the scope of the present paper. We will try to tackle this in a subsequent paper.

(iii). It is natural to expect that the packing dimension analogue of Theorem 1.1 also holds. Namely, if X is a real-valued strictly α-stable Lévy process with $1 \le \alpha \le 2$, then for any $x \in \mathbb{R}$ one has

$$\mathbb{P}^x\left(\dim_{\mathrm{P}} X^{-1}(F) = 1 - \frac{1}{\alpha} + \frac{\dim_{\mathrm{P}} F}{\alpha} \text{ for all Borel sets } F \subseteq \mathbb{R} \right) = 1. \tag{4.1}$$

Here $\dim_{\mathrm{P}}$ denotes packing dimension; see Falconer [6, Chapter 3] for its definition and properties, and [24, 26] for examples of its applications in studying sample path properties of Markov processes.

By using the connection between packing dimension and the upper box-counting (Minkowski) dimension (cf. [6]), one can see that the proof of the upper bound of Theorem 1.1 also implies that $\mathbb{P}^x$-a.s.,

$$\dim_{\mathrm{P}} X^{-1}(F) \le 1 - \frac{1}{\alpha} + \frac{\dim_{\mathrm{P}} F}{\alpha} \text{ for all Borel sets } F \subseteq \mathbb{R}.$$

In order to prove the reverse inequality, one may apply the lower density theorem for packing measure in [25, Theorem 5.4] and prove that for any $\gamma < 1/\alpha$ and $\varepsilon > 0$,

$$\sup_{a \in X^{-1}(F)} \liminf_{r \to 0} \frac{\lambda([a, a + r])}{r^{1-\alpha^{-1}+\gamma \dim_{\mathrm{P}} F - 2\varepsilon}} \le c_2 < \infty,$$

where λ is the random measure defined in (3.2) and c_2 is a finite constant. We are not able to prove this because (unlike the Hausdorff dimension case) the terms $\mu(I_i)$ in (3.3) can not be controlled for all i by the same n.

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