Electron. Commun. Probab. **23** (2018), no. 72, 1–3.
https://doi.org/10.1214/18-ECP175
ISSN: 1083-589X

ELECTRONIC COMMUNICATIONS in PROBABILITY

A note on tail triviality for determinantal point processes

Russell Lyons*

Abstract

We give a very short proof that determinantal point processes have a trivial tail σ-field. This conjecture of the author has been proved by Osada and Osada as well as by Bufetov, Qiu, and Shamov. The former set of authors relied on the earlier result of the present author that the conjecture held in the discrete case, as does the present short proof.

Keywords: transference principle.
AMS MSC 2010: Primary 60K99, Secondary 60G55.
Submitted to ECP on July 17, 2018, final version accepted on October 2, 2018.

We give a very short proof that determinantal point processes have a trivial tail σ-field. This conjecture of Lyons [4] has been proved by Osada and Osada [5][1] as well as by Bufetov, Qiu, and Shamov [1]. Osada and Osada relied on the earlier result of Lyons [3] that the conjecture held in the discrete case, as does the present short proof. In the discrete case and under the restrictive assumption that the spectrum of K is contained in the open interval $(0, 1)$, Shirai and Takahashi [7] also proved that the tail σ-field is trivial. In the continuous setting, tail triviality is important in proving pathwise uniqueness of solutions of certain infinite-dimensional stochastic differential equations related to determinantal point processes [6].

Our proof here relies on an extension of Goldman's transference principle, as elucidated in [4].

1 Goldman's transference principle

We review some definitions. See [4] for more details.

Let E be a locally compact Polish space (equivalently, a locally compact second countable Hausdorff space). Let μ be a Radon measure on E, i.e., a Borel measure that is finite on compact sets. Let $\mathcal{N}(E)$ be the set of Radon measures on E with values in $\mathbb{N} \cup \{\infty\}$. We give $\mathcal{N}(E)$ the vague topology generated by the maps $\xi \mapsto \int f \, d\xi$ for continuous f with compact support; then $\mathcal{N}(E)$ is Polish. The corresponding Borel σ-field of $\mathcal{N}(E)$ is generated by the maps $\xi \mapsto \xi(A)$ for Borel $A \subseteq E$.

Let $\mathfrak{X}$ be a simple point process on E, i.e., a random variable with values in $\mathcal{N}(E)$ such that $\mathfrak{X}(\{x\}) \in \{0, 1\}$ for all $x \in E$. We call $\mathfrak{X}$ ***determinantal*** if for some measurable

*Department of Mathematics, 831 E. 3rd St., Indiana University, Bloomington, IN 47405-7106. E-mail: rdlyons@indiana.edu

[1] In fact, there is a gap in [5]: Lemmas 4 and 5 there do not follow from the reasoning given and can be false when the tail σ-field is nontrivial. This gap can be filled via reasoning similar to that used here. More precisely, one can use the partition $\mathscr{C}$ here in place of their sequence of partitions $\Delta(\ell)$.

$K\colon E^2 \to \mathbb{C}$ and all $k \geq 1$, the function $(x_1, \ldots, x_k) \mapsto \det[K(x_i, x_j)]_{i,j \leq k}$ is a k-point intensity function of $\mathfrak{X}$. In this case, we denote the law of $\mathfrak{X}$ by $\mathbf{P}^K$.

We consider only K that are locally square integrable (i.e., $|K|^2\mu^2$ is Radon), are Hermitian (i.e., $K(y,x) = \overline{K(x,y)}$ for all $x, y \in E$), and are positive semidefinite. In this case, K defines a positive semidefinite integral operator $(Kf)(x) := \int K(x,y)f(y)\,d\mu(y)$ on functions $f \in L^2(\mu)$ with compact support. We consider K as defined only up to changes on a μ^2-null set. For every Borel $A \subseteq E$, we denote by μ_A the measure μ restricted to Borel subsets of A and by K_A the compression of K to A, i.e., $K_A f := (Kf) \restriction A$ for $f \in L^2(A, \mu_A)$. If $\mathfrak{X}$ has law $\mathbf{P}^K$, then the restriction of $\mathfrak{X}$ to A has law $\mathbf{P}^{K_A}$. The operator K is locally trace-class, i.e., for every compact $A \subseteq E$, the compression K_A is trace class, having a spectral decomposition $K_A = \sum_k \lambda_k^A\, \phi_k^A \otimes \overline{\phi_k^A}$, where $\langle \phi_k^A\,;\ k \geq 1\rangle$ are orthonormal eigenfunctions of K_A with positive summable eigenvalues $\langle \lambda_k^A\,;\ k \geq 1\rangle$.

The following extends Goldman's transfer principle from trace-class operators, as given in [4, Section 3.6], to locally trace-class operators:

Theorem 1.1. *Let μ be a Radon measure on a locally compact Polish space, E. Let K be a locally trace-class positive contraction on $L^2(E, \mu)$. Let $\langle A_i\,;\ i \geq 1\rangle$ be a partition of E into precompact Borel subsets of E. Then there exists a denumerable set F with a partition $\langle B_i\,;\ i \geq 1\rangle$ and a positive contraction Q on $\ell^2(F)$ such that the joint distribution of the random variables $\langle \mathfrak{X}(A_i)\,;\ i \geq 1\rangle$ for $\mathfrak{X} \sim \mathbf{P}^K$ equals the joint $\mathbf{P}^Q$-distribution of the random variables $\langle \mathfrak{X}(B_i)\,;\ i \geq 1\rangle$ for $\mathfrak{X} \sim \mathbf{P}^Q$. Moreover, we can choose Q to be unitarily equivalent to K.*

Proof. For each i, fix an orthonormal basis $\langle w_{i,j}\,;\ j < n_i\rangle$ for the subspace of $L^2(E, \mu)$ of functions that vanish outside A_i. Here, $n_i \in \mathbb{N} \cup \{\infty\}$. Define $B_i := \{(i,j)\,;\ j < n_i\}$ and $F := \bigcup_i B_i$. Let T be the isometric isomorphism (i.e., unitary map) from $L^2(E, \mu)$ to $\ell^2(F)$ that sends $w_{i,j}$ to $\mathbb{1}_{\{(i,j)\}}$. Define $Q := TKT^{-1}$, so Q is unitarily equivalent to K. Note that for all $\phi \in L^2(E)$ and all $i \geq 1$, we have $T\mathbb{1}_{A_i}\phi = \mathbb{1}_{B_i} T\phi$.

For $m \geq 1$, write $E_n := \bigcup_{i=1}^m A_i$ and $F_m := \bigcup_{i=1}^m B_i$. Then K_{E_m} and Q_{F_m} are unitarily equivalent trace-class operators. If $\langle \phi_{k,m}\,;\ k \geq 1\rangle$ are orthonormal eigenvectors of K_{E_m}, so that $K_{E_m} = \sum_k \lambda_k^{E_m} \phi_{k,m} \otimes \overline{\phi_{k,m}}$, then $Q_{F_m} = \sum_k \lambda_k^{E_m} T\phi_{k,m} \otimes \overline{T\phi_{k,m}}$. Furthermore, for all $\phi, \psi \in L^2(E, \mu)$ and all $i \geq 1$, we have $(\mathbb{1}_{A_i}\phi, \psi)_{L^2(E,\mu)} = (T\mathbb{1}_{A_i}\phi, T\psi)_{\ell^2(F)} = (\mathbb{1}_{B_i}T\phi, T\psi)_{\ell^2(F)}$. Thus, [4, Theorem 3.4] shows that the $\mathbf{P}^{K_{E_m}}$-distribution of $\langle \mathfrak{X}(A_i)\,;\ i \leq m\rangle$ equals the $\mathbf{P}^{Q_{F_m}}$-distribution of $\langle \mathfrak{X}(B_i)\,;\ i \leq m\rangle$. But these are precisely the $\mathbf{P}^K$-distribution of $\langle \mathfrak{X}(A_i)\,;\ i \leq m\rangle$ and the $\mathbf{P}^Q$-distribution of $\langle \mathfrak{X}(B_i)\,;\ i \leq m\rangle$, respectively. Because these are equal for all $m \geq 1$, the desired result follows. $\square$

2 Tail triviality: deduction from the discrete case

For a Borel set $A \subseteq E$, let $\mathscr{F}(A)$ denote the σ-field on $\mathcal{N}(E)$ generated by the functions $\xi \mapsto \xi(B)$ for Borel $B \subseteq A$. The ***tail σ-field*** is the intersection of $\mathscr{F}(E \setminus A)$ over all compact $A \subseteq E$; it is said to be trivial when each of its events has probability 0 or 1. For a collection $\mathscr{A}$ of Borel subsets of E, write $\mathscr{G}(\mathscr{A})$ for the σ-field generated by the functions $\xi \mapsto \xi(B)$ for $B \in \mathscr{A}$.

Theorem 2.1 (conjectured by [4], proved by [5, 1])**.** *If K is a locally trace-class positive contraction, then $\mathbf{P}^K$ has a trivial tail σ-field.*

Proof. Consider a sequence of increasingly finer partitions $\mathscr{A}_m = \{A_{m,i}\,;\ i \geq 1\}$ of E by precompact Borel sets $A_{m,i}$ such that the sequence $\langle \mathscr{A}_m\,;\ m \geq 1\rangle$ separates points of E. (This can be obtained, for example, by writing E as a countable union of compact sets [2, Theorem 5.3] and partitioning each compact set by the fact that it is a continuous image of the Cantor set [2, Theorem 4.18].) Then the corresponding count σ-fields $\mathscr{G}(\mathscr{A}_m)$ increase to the Borel σ-field $\mathscr{F}(E)$, so Lévy's 0-1 law tells us that for every event $\mathcal{A} \in \mathscr{F}(E)$, we have $\mathbf{P}\big(\mathcal{A} \mid \mathscr{G}(\mathscr{A}_m)\big)$ converges in L^1 to $\mathbb{1}_{\mathcal{A}}$. Similarly, if $D^{(n)} := \bigcup_{i=1}^n A_{1,i}$ and $\mathscr{G}_m^{(n)} := \mathscr{G}\big(\{A_{m,i}\,;\ A_{m,i} \cap D^{(n)} = \varnothing,\ i \ge 1\}\big)$, then for each n and all $\mathcal{A} \in \mathscr{F}(E \setminus D^{(n)})$, we have $\mathbf{P}(\mathcal{A} \mid \mathscr{G}_m^{(n)})$ converges in L^1 to $\mathbb{1}_{\mathcal{A}}$ as $m \to \infty$. In particular, if $\mathcal{A}$ is a tail event, then there is a sequence $m_n \to \infty$ such that $\mathbf{P}(\mathcal{A} \mid \mathscr{G}_{m_n}^{(n)})$ converges in L^1 to $\mathbb{1}_{\mathcal{A}}$ as $n \to \infty$. It follows that $\mathcal{A}$ belongs to the completion of the σ-field $\bigvee_{n \ge k} \mathscr{G}_{m_n}^{(n)}$ for each $k \ge 1$.

Now let $\mathcal{A}$ be a tail event and $\langle m_n\,;\ n \ge 1\rangle$ be such a sequence. Let $\mathscr{C} := \langle C_k\,;\ k \ge 1\rangle$ be the parts of the partition of E generated by $\{A_{m_n,i}\,;\ A_{m_n,i} \cap D^{(n)} = \varnothing,\ n \ge 1,\ i \ge 1\}$. Write $\mathscr{H}_n := \mathscr{G}\big(\{C_k\,;\ k \ge n\}\big)$. Then $\mathcal{A}$ belongs to the completion of the σ-field $\mathscr{H}_n$ for each $n \ge 1$, whence $\mathbf{P}(\mathcal{A} \mid \bigcap_{n \ge 1} \mathscr{H}_n) = \lim_{n \to \infty} \mathbf{P}(\mathcal{A} \mid \mathscr{H}_n) = \mathbb{1}_{\mathcal{A}}$ a.s. by Lévy's downwards theorem. By Theorem 1.1, there is a partition $\langle B_k\,;\ k \ge 1\rangle$ of a denumerable set F and a positive contraction Q on $\ell^2(F)$ such that the $\mathbf{P}^Q$-distribution of $\langle \mathfrak{X}(B_k)\,;\ k \ge 1\rangle$ equals the $\mathbf{P}^K$-distribution of $\langle \mathfrak{X}(C_k)\,;\ k \ge 1\rangle$. Let $\mathscr{H}'_n := \mathscr{G}\big(\{B_k\,;\ k \ge n\}\big)$. Then $\bigcap_{n \ge 1} \mathscr{H}'_n$ is contained in the tail σ-field $\bigcap_{B \text{ finite}} \mathscr{F}(F \setminus B)$. Since the latter is trivial by [3, Theorem 7.15], so is the former. Therefore, so is $\bigcap_{n \ge 1} \mathscr{H}_n$, whence $\mathcal{A}$ has probability 0 or 1. $\square$

References

[1] Alexander I. Bufetov, Yanqi Qiu, and Alexander Shamov, *Kernels of conditional determinantal measures and the Lyons–Peres conjecture*, (2016), Preprint, arXiv:1612.06751.

[2] Alexander S. Kechris, *Classical descriptive set theory*, Graduate Texts in Mathematics, vol. 156, Springer-Verlag, New York, 1995. MR-1321597

[3] Russell Lyons, *Determinantal probability measures*, Publ. Math. Inst. Hautes Études Sci. **98** (2003), no. 1, 167–212. MR-2031202

[4] Russell Lyons, *Determinantal probability: Basic properties and conjectures*, Proceedings of the International Congress of Mathematicians. Volume IV (Sun Young Jang, Young Rock Kim, Dae-Woong Lee, and Ikkwon Yie, eds.), Kyung Moon Sa, Seoul, 2014, Invited lectures, Held in Seoul, August 13–21, 2014, pp. 137–161. MR-3727606

[5] Hirofumi Osada and Shota Osada, *Discrete approximations of determinantal point processes on continuous spaces: Tree representations and tail triviality*, J. Stat. Phys. **170** (2018), no. 2, 421–435. MR-3744393

[6] Hirofumi Osada and Hideki Tanemura, *Infinite-dimensional stochastic differential equations and tail σ-fields*, (2014), Preprint, arXiv:1412.8674.

[7] Tomoyuki Shirai and Yoichiro Takahashi, *Random point fields associated with certain Fredholm determinants. I. Fermion, Poisson and boson point processes*, J. Funct. Anal. **205** (2003), no. 2, 414–463. MR-2018415