Projections of spherical Brownian motion[*]

Aleksandar Mijatović[†]   Veno Mramor[‡]   Gerónimo Uribe Bravo[§]

Abstract

We obtain a stochastic differential equation (SDE) satisfied by the first n coordinates of a Brownian motion on the unit sphere in $\mathbb{R}^{n+\ell}$. The SDE has non-Lipschitz coefficients but we are able to provide an analysis of existence and pathwise uniqueness and show that they always hold. The square of the radial component is a Wright-Fisher diffusion with mutation and it features in a skew-product decomposition of the projected spherical Brownian motion. A more general SDE on the unit ball in $\mathbb{R}^{n+\ell}$ allows us to geometrically realize the Wright-Fisher diffusion with general non-negative parameters as the radial component of its solution.

Keywords: non-Lipschitz stochastic differential equation; skew-product decomposition; pathwise uniqueness; Wright-Fisher diffusion.
AMS MSC 2010: 60H10; 58J65.
Submitted to ECP on June 1, 2018, final version accepted on August 8, 2018.
Supersedes arXiv:1806.00266.

1 Introduction and main results

The Theorem of Archimedes [2] states that the projection π_1 from the unit sphere $\mathbb{S}^2 \subset \mathbb{R}^3$ to any coordinate (in $\mathbb{R}^3$) preserves the uniform distribution; see [1] and the references therein for a very modern account or [11] for one with more probabilistic insight. In probabilistic language, if $U^{(2)}$ is a uniform random vector on $\mathbb{S}^2$, then $\pi_1(U^{(2)})$ is uniform on $[-1, 1]$. In fact, this holds in any dimension $d \geq 3$: for a uniform random vector $U^{(d-1)}$ on the Euclidean unit sphere $\mathbb{S}^{d-1} := \left\{ z \in \mathbb{R}^d \; ; \; |z| = 1 \right\}$, its projection $\pi_{d-2}(U^{(d-1)})$ onto any $d - 2$ coordinates is uniform on the unit ball $\mathbb{B}^{d-2} := \left\{ z \in \mathbb{R}^{d-2} \; ; \; |z| \leq 1 \right\}$. A more general version of this result for spheres in the p-norm can be found in [4].

Since the uniform distribution on $\mathbb{S}^{d-1}$ is the invariant measure for Brownian motion on the sphere, it is natural to investigate the process obtained by projecting it to the ball $\mathbb{B}^{d-2}$. Such a process ought to have a uniform distribution on $\mathbb{B}^{d-2}$ as its invariant measure. The aim of this paper is to give a complete characterization of such processes

in terms of SDEs they satisfy and to deduce certain structural consequences of this characterisation.

Let Z be Brownian motion on the sphere $\mathbb{S}^{n+\ell-1}$ ($n, \ell \in \mathbb{N}$); although this is an instance of a Brownian motion on a Riemannian manifold, we will just use the Stroock representation (in Itô form) and consider it as the solution of an equation

$$\mathrm{d}Z_t = (I - Z_t Z_t^\top)\mathrm{d}\widetilde{B}_t - \frac{n+\ell-1}{2}Z_t \mathrm{d}t, \qquad Z_0 \in \mathbb{S}^{n+\ell-1}, \tag{1.1}$$

where $\widetilde{B}$ a Brownian motion on $\mathbb{R}^{n+\ell}$ and I denotes the identity matrix of appropriate dimension (cf. [9, Ch.3§3, p. 83]).

Proposition 1.1. *Let X denote the first n coordinates of Z. Then, there exists a Brownian motion B on $\mathbb{R}^n$ such that the pair (X, B) satisfies the SDE*

$$\mathrm{d}X_t = \sigma(X_t)\mathrm{d}B_t - \frac{n+\ell-1}{2}X_t \mathrm{d}t, \qquad X_0 \in \mathbb{B}^n,$$

where the volatility matrix $\sigma(x)$ takes the form

$$\sigma(x) = I - \left(1 - \sqrt{1-|x|^2}\right)\frac{xx^\top}{|x|^2}\mathbb{1}(|x| > 0), \qquad x \in \mathbb{B}^n.$$

Pathwise uniqueness holds for this SDE and X is a strong Markov process with a unique invariant measure which admits the density

$$h(x) = \frac{\Gamma((n+\ell)/2)}{\pi^{n/2}\Gamma(\ell/2)}\left(1-|x|^2\right)^{(\ell-2)/2}\mathbb{1}(|x| \leq 1).$$

Furthermore, $U = |X|^2$ is a Wright-Fisher diffusion, i.e. there exists a scalar Brownian motion β such that the pair (U, β) satisfies the SDE

$$\mathrm{d}U_t = 2\sqrt{U_t(1-U_t)}\mathrm{d}\beta_t + [n(1-U_t) - \ell U_t]\,\mathrm{d}t.$$

Remark 1.2. In the case $\ell = 2$, the invariant measure of X in Proposition 1.1 is uniform on the unit ball $\mathbb{B}^n$, as expected from the theorem of Archimedes.

Remark 1.3. The process X enjoys a skew-product decomposition analogous to the one of Brownian motion in $\mathbb{R}^n$; it is a particular case of Theorem 1.5 below.

Remark 1.4. Since $\sigma(x)$ is the unique non-negative definite square root of the matrix $I - xx^\top$ for $x \in \mathbb{B}^n$, the SDE for the projected process X implies that its infinitesimal generator equals

$$\frac{1}{2}\sum_{i,j=1}^n (\delta_{ij} - x^i x^j)\frac{\partial^2}{\partial x^i \partial x^j} - \sum_{i=1}^n \frac{n-1+\ell}{2}\frac{\partial}{\partial x^i}.$$

This operator equals the generator $\frac{1}{2}\Delta_{n-1+\ell, n}$ of a process considered in [3]. Our results show that the martingale problem arising from this generator is well-posed.

More generally, and in order to state the skew-product decomposition, we will consider the SDE on the unit ball $\mathbb{B}^n$ given by

$$\mathrm{d}X_t = \gamma(|X_t|)\sigma(X_t)\mathrm{d}B_t - g(|X_t|)X_t\mathrm{d}t, \tag{1.2}$$

with starting point $X_0 = x_0 \in \mathbb{B}^n$. Assume that $\gamma\colon [0,1] \to (0,\infty)$ and $g\colon [0,1] \to \mathbb{R}$ are Lipschitz continuous[1] and satisfy $\frac{g(1)}{\gamma^2(1)} \geq \frac{n-1}{2}$. In particular, the above SDE extends the projected process X of Proposition 1.1 to non-integer dimensions. The coefficients of

---

[1]Note that by Lipschitz continuity of the euclidean norm $|\cdot|$, functions $\gamma(|\cdot|)$ and $g(|\cdot|)$ are also Lipschitz.

SDE (1.2) are bounded and continuous implying that weak existence holds (see e.g. [10, Ch. IV, Thm 2.2]). The function σ is locally Lipschitz only on the interior of the ball $\mathbb{B}^n$, making it impossible to apply the classical theory for the uniqueness of solutions of SDEs (note that we do not exclude the cases when either $X_0 \in \mathbb{S}^{n-1}$ or the boundary sphere is reached in finite time). The condition $\frac{g(1)}{\gamma^2(1)} \geq \frac{n-1}{2}$ turns out to be necessary for a solution to stay in the unit ball. In fact, if $\frac{g(1)}{\gamma^2(1)} = \frac{n-1}{2}$ and $X_0 \in \mathbb{S}^{n-1}$, the solution X is a Brownian motion on $\mathbb{S}^{n-1}$ time-changed by $t \mapsto \gamma^2(1)t$.

The radial component $R := |X|$ of a solution X of SDE (1.2) and its square $U := |X|^2$ are the unique strong solutions of the respective SDEs in (2.3) and (2.1) below; the latter reduces to the SDEs of Wright-Fisher diffusion with mutation in the setting of Proposition 1.1. In particular, both processes are strong Markov. After a time-change, a pathwise comparison of U with a Wright-Fisher diffusion implies that for $n \geq 2$ the process X never hits 0 (see Lemma 2.1 in Section 2 below). This enables us to define a time-change process $S_s(t) := \int_s^t \frac{\gamma^2(R_u)}{R_u^2} \mathrm{d}u$, satisfying $\lim_{t \to \infty} S_s(t) = \infty$, and its inverse $T_s \colon [0, \infty) \to [s, \infty)$ (see Lemma 2.3 below). Moreover, it turns out that X possesses a skew-product decomposition analogous to the one of Brownian motion on $\mathbb{R}^n$.

Theorem 1.5 (Skew-product decomposition)**.** *Let $n \geq 2$ and X be a solution of SDE* (1.2). *Pick $s \in \mathbb{R}_+ := [0, \infty)$ and assume that either $s > 0$ or $s = 0$ and $X_0 \neq 0$. Then the process $\widehat{V} = (\widehat{V}_t)_{t \in \mathbb{R}_+}$, given by $\widehat{V}_t := X_{T_s(t)} / R_{T_s(t)}$, is a Brownian motion on $\mathbb{S}^{n-1}$ (started at $\widehat{V}_0 = X_s / R_s$) independent of R. Hence we obtain the skew-product decomposition $X_t = R_t \widehat{V}_{S_s(t)}$ for $t \geq s$. Furthermore, if $X_0 = 0$, then $\widehat{V}_t$ is uniformly distributed on $\mathbb{S}^{n-1}$ for any $t > 0$ and subsequently evolves as a stationary Brownian motion on the sphere.*

Since the skew-product decomposition expresses X as a measurable functional of a pair of independent processes $(R, \widehat{V})$ with given distributions, the following result holds.

Corollary 1.6. *Uniqueness in law holds for SDE* (1.2).

The pathwise uniqueness of SDE (1.2) is more delicate because σ is not Lipschitz at the boundary of the ball $\mathbb{B}^n$. Since σ is locally Lipschitz, pathwise uniqueness holds up to the first hitting time of the boundary by well established arguments. Hence, if $\frac{g(u)}{\gamma^2(u)} \geq \frac{n-1}{2} + 1$ holds for u sufficiently close to one, $|X_0| < 1$, and X never visits the boundary of $\mathbb{B}^n$, then pathwise uniqueness holds. If $\frac{g(1)}{\gamma^2(1)} = \frac{n-1}{2}$, X behaves as time-changed Brownian motion on $\mathbb{S}^{n-1}$ after the first time it hits the boundary and hence pathwise uniqueness also holds in this case. If $n = 1$, the SDE (1.2) simplifies to $\mathrm{d}X_t^1 = \gamma(|X_t^1|)\sqrt{1 - (X_t^1)^2}\mathrm{d}B_t - g(|X_t^1|)X_t^1 \mathrm{d}t$ and pathwise uniqueness holds by a theorem of Yamada and Watanabe [12, IX.3.5]. Pathwise uniqueness of the similar looking equation $\mathrm{d}X_t = \gamma(|X_t|)(1 - |X_t|^2)^{1/2}\mathrm{d}B_t - g(|X_t|)X_t\mathrm{d}t$ on $\mathbb{B}^n$, where γ and g are positive Lipschitz function and $\frac{g(1)}{\gamma^2(1)}$ is sufficiently large, was established by DeBlassie [7] by a clever generalisation of the idea in [13]. Note that the diffusion coefficient in the SDE (1.2) depends on x and not just on its length $|x|$, making it impossible to apply the result of [7]. However, it is possible to adapt the method of [7] to our setting and obtain:

Theorem 1.7. *If $\frac{g(1)}{\gamma^2(1)} - \frac{n-1}{2} > \sqrt{2} - 1 \doteq 0.4142$, then pathwise uniqueness holds for the SDE* (1.2).

The remaining cases, when $n \geq 2$ and $\frac{g(1)}{\gamma^2(1)} - \frac{n-1}{2} \in (0, \sqrt{2} - 1]$, are left open.

2 Characterization of the projected process

We are interested in the process consisting of the first n coordinates of Brownian motion on the sphere $\mathbb{S}^{n-1+\ell}$. One way of constructing such a Brownian motion is via the Stroock representation, i.e. a solution to the SDE (1.1). Note that coefficients of SDE (1.1) are locally Lipschitz continuous, so pathwise uniqueness holds.

In the following proof as well as in the rest of the paper superscripts will denote components of vectors, e.g. X^i means i-th component of process X. When we wish to express powers we will enclose variables in additional sets of parentheses.

Proof of Proposition 1.1. Let X' denote last ℓ coordinates of Z and similarly split $\widetilde{B} = (\widetilde{B}^1, \widetilde{B}^2)$. We claim that the process B given by the equation

$$B_t = \int_0^t \sigma(X_s)\mathrm{d}\widetilde{B}_s^1 + \int_0^t \left(-X_s {X'_s}^\top (1 - |X_s|^2)^{-1/2}\mathbb{1}(|X_s| < 1) + X_s z^\top \mathbb{1}(|X_s| = 1) \right) \mathrm{d}\widetilde{B}_t^2,$$

where $z \in \mathbb{S}^{\ell-1}$ is an arbitrary (fixed) unit vector, is an n-dimensional Brownian motion.

Note that $|X'_t|^2 = |Z_t|^2 - |X_t|^2 = 1 - |X_t|^2$. Furthermore, let us consider the $n \times (n+\ell)$ matrix

$$A_t := \left[\sigma(X_t), -X_t {X'_t}^\top (1 - |X_t|^2)^{-1/2}\mathbb{1}(|X_t| < 1) + X_t z^\top \mathbb{1}(|X_t| = 1) \right] =: \left[\sigma(X_t), D_t \right].$$

The choice of the constant vector z in the definition of A_t will turn out not to be relevant as we will see that the time spent by X at the boundary of the ball has Lebesgue measure zero. We can compute $\sigma(X_t)^2 = \sigma(X_t)\sigma(X_t)^\top = I - X_t X_t^\top$ and $D_t D_t^\top = X_t X_t^\top$, so it follows that $A_t A_t^\top = \sigma(X_t)\sigma(X_t)^\top + D_t D_t^\top = I$. Since B is defined by $B_t = \int_0^t A_s \mathrm{d}\widetilde{B}_s$ and it is a continuous local martingale with quadratic variation $\langle B^i, B^j \rangle_t = \int_0^t (A_s A_s^\top)_{ij}\mathrm{d}s = \delta_{ij} t$, it is n-dimensional Brownian motion by Levy's characterization theorem. Further calculations show that $\sigma(X_t)D_t = -X_t {X'_t}^\top$ and finally, the facts and the definition of Z, imply the SDE satisfied by X:

$$\mathrm{d}X_t = (I - X_t X_t^\top)\mathrm{d}\widetilde{B}_t^1 - X_t {X'_t}^\top \mathrm{d}\widetilde{B}_t^2 - \frac{n-1+\ell}{2}X_t \mathrm{d}t$$

$$= \sigma(X_t)^2 \mathrm{d}\widetilde{B}_t^1 + \sigma(X_t)D_t \mathrm{d}\widetilde{B}_t^2 - \frac{n-1+\ell}{2}X_t \mathrm{d}t$$

$$= \sigma(X_t)A_t \mathrm{d}\widetilde{B}_t - \frac{n-1+\ell}{2}X_t \mathrm{d}t = \sigma(X_t)\mathrm{d}B_t - \frac{n-1+\ell}{2}X_t \mathrm{d}t.$$

The above SDE is just a special case of SDE (1.2) with $\gamma \equiv 1$ and $g \equiv \frac{n-1+\ell}{2}$, therefore pathwise uniqueness holds immediately by Theorem 1.7 since $\frac{g(1)}{\gamma^2(1)} - \frac{n-1}{2} = \frac{\ell}{2} > \sqrt{2} - 1$ for $\ell \in \mathbb{N}$. Consequently X is a strong Markov process. Furthermore, Lemma 2.1 shows that $U = |X|^2$ is Wright-Fisher diffusion with mutation rates n and ℓ. When $n \geq 2$ this also helps us find invariant measure for process X since we can use skew-product decomposition in Theorem 1.5. The invariant measure for Wright-Fisher diffusion U is given by $\mathrm{Beta}(n/2, \ell/2)$ distribution. Hence $g(r) = B_{n,\ell}r^{n-1}(1 - r^2)^{(\ell-2)/2}\mathbb{1}(r \in [0, 1])$ is the density of the invariant measure of R. The invariant measure for Brownian motion on a sphere is a normalised uniform measure. This continues to hold for the time-changed Brownian motion on a sphere as long as the time change is independent of Brownian motion. So let us suppose that the initial distribution of the process X has the density h from Proposition 1.1. Then, using polar coordinates and the skew-product decomposition, the density of R_0 is g. Since this density is invariant for R, R_t has density g for all $t \geq 0$. The time changed Brownian motion on a sphere also remains uniformly distributed and reversing polar coordinates we get that X_t has density h for any t.

In the case $n = 1$ we see that the process X^1 satisfies the SDE $\mathrm{d}X_t^1 = \sqrt{1 - (X_t^1)^2}\mathrm{d}B_t - \frac{\ell}{2}X_t^1 \mathrm{d}t$, and by the forward Kolmogorov equation the invariant density can easily be seen to be equal to

$$h(x) = \frac{\Gamma((1+\ell)/2)}{\pi^{1/2}\Gamma(\ell/2)}\left(1 - x^2\right)^{(\ell-2)/2}\mathbb{1}(|x| \leq 1). \qquad \square$$

2.1 Skew-product decomposition for SDE (1.2)

The solution of (1.2) naturally lives on the closed unit ball $\mathbb{B}^n$. To better understand such a process it is crucial to understand its radial component (or its square) and in particular if and when it hits the boundary (or zero). Since the square of the radial component will be shown to be closely related to Wright-Fisher diffusion let us first collect some known fact about the latter. Fix non-negative parameters α and β. The SDE $\mathrm{d}\mathcal{Z}_t = 2\sqrt{\mathcal{Z}_t(1-\mathcal{Z}_t)}\mathrm{d}B_t + (\alpha(1-\mathcal{Z}_t)-\beta\mathcal{Z}_t)\mathrm{d}t$ has a unique strong solution which is called the Wright-Fisher diffusion; denote it by $\mathrm{WF}(\alpha,\beta)$. It is possible to define $\mathrm{WF}(\alpha,\beta)$ for negative α (resp. β), but then the solution has a finite lifetime equal to the first hitting time of 0 (resp. 1). Pathwise uniqueness is a consequence of $1/2$-Hölder continuity of diffusion coefficient and then applying theorem of Yamada and Watanabe [12, IX.3.5]. Furthermore, it is well-known[2] that $\mathrm{WF}(\alpha,\beta)$ never hits 0 for $\alpha \geq 2$ and never hits 1 for $\beta \geq 2$.

Some of the facts about square of radial component of solution to SDE (1.2) are summarized in the following lemma.

Lemma 2.1. *Let X be a solution of* (1.2) *where $\frac{g(1)}{\gamma^2(1)} \geq \frac{n-1}{2}$. Then the process $U = |X|^2$ satisfies the SDE*

$$\mathrm{d}U_t = 2\widetilde{\gamma}(U_t)\sqrt{U_t(1-U_t)}\mathrm{d}\theta_t + \widetilde{\gamma}^2(U_t)\left(n(1-U_t) - \left(\frac{2\widetilde{g}(U_t)}{\widetilde{\gamma}^2(U_t)} - (n-1)\right)U_t\right)\mathrm{d}t, \quad (2.1)$$

where $\widetilde{g}(u) := g(\sqrt{u}), \widetilde{\gamma}(u) := \gamma(\sqrt{u})$, and the $\mathcal{F}_t$-Brownian motion θ is defined by

$$\theta_t = \sum_{i=1}^n \int_0^t \frac{X_s^i}{\sqrt{U_s}}\mathbb{1}(U_s > 0)\mathrm{d}B_s^i + \int_0^t \mathbb{1}(U_s = 0)\mathrm{d}\chi_s,$$

where the scalar Brownian motion χ is independent of B.

For $n \geq 2$, the process U never hits 0 and if $\frac{g(u)}{\gamma^2(u)} \geq \frac{n-1}{2} + 1$ holds near 1, then U never hits 1.

Proof. Since $U = |X|^2$ and $\langle X^i \rangle_t = \gamma^2(|X_t|)(1 - (X_t^i)^2)\mathrm{d}t$ we can apply Itô's formula and get

$$\mathrm{d}U_t = 2\gamma(|X_t|)\sum_{j=1}^n X_t^j\sqrt{1-U_t}\mathrm{d}B_t^j - 2g(|X_t|)\sum_{i=1}^n (X_t^i)^2\mathrm{d}t + \gamma^2(|X_t|)\sum_{i=1}^n (1-(X_t^i)^2)\mathrm{d}t$$

which in turn yields (2.1). Moreover, since $\langle\theta\rangle_t = t$, the continuous local martingale θ is a Brownian motion by Levy's characterization theorem.

We can slightly simplify the equation (2.1) by time-change without affecting the boundary hitting properties. Define q by

$$q_t := \int_0^t \gamma^2(|X_u|)\mathrm{d}u \text{ and its inverse } \widetilde{q}_t := \inf\{u \geq 0 ; q(u) = t\}.$$

Then $\widehat{U}_t := U_{\widetilde{q}_t}$ satisfies the SDE

$$\mathrm{d}\widehat{U}_t = 2\sqrt{\widehat{U}_t(1-\widehat{U}_t)}\mathrm{d}\widetilde{\theta}_t + \left(n(1-\widehat{U}_t) - \left(\frac{2\widetilde{g}(\widehat{U}_t)}{\widetilde{\gamma}^2(\widehat{U}_t)} - (n-1)\right)\widehat{U}_t\right)\mathrm{d}t \qquad (2.2)$$

where $\widetilde{\theta}_t = \int_0^{\widetilde{q}_t} \gamma(|X_u|)\mathrm{d}\theta_u$ is also a Brownian motion. This is almost the same equation as that of a Wright-Fisher diffusion. The volatility term is exactly the same so that

---

[2]It can for example be seen from Lemma 2.2 and well-known facts about hitting of 0 of Bessel processes.

the difference appears only in the drift term, which is, nevertheless, still Lipschitz continuous[3]. Again we can use the Yamada-Watanabe Theorem [12, Theorem 3.5 in Ch. IX] and conclude that pathwise uniqueness holds for the SDE (2.2). Since $\frac{g(1)}{\gamma^2(1)} \geq \frac{n-1}{2}$ we can consider SDE (2.2) with the increased drift

$$n(1-u) - \left(\frac{2\widetilde{g}(u)}{\widetilde{\gamma}^2(u)} - \frac{2\widetilde{g}(1)}{\widetilde{\gamma}^2(1)} \right) u$$

and for $\widehat{U}_0 = 1$ such an equation has the unique solution $\widehat{U}_t \equiv 1$. By using the comparison theorem for SDEs, as in [12, Ch. IX, (3.7)], we deduce that the inequality $\widehat{U}_t \leq 1$ holds for the solution of (2.2) as long as $\widehat{U}_0 \leq 1$ a.s. By denoting

$$M := \max_{u \in [0,1]} \left(\frac{2\widetilde{g}(u)}{\widetilde{\gamma}^2(u)} - (n-1) \right) \geq 0,$$

another application of the comparison theorem shows that the solution of (2.2) is always larger than $\mathrm{WF}(n, M)$ started at the same point, so it is always non-negative and therefore $U_t \in [0,1]$ for $U_0 \in [0,1]$ a.s. which also means it has infinite lifetime. This second use of comparison theorem also shows, that for $n \geq 2$, the processes $\widehat{U}$ and U never hit 0 unless they start there. Similarly, if $\frac{g(u)}{\gamma^2(u)} \geq \frac{n-1}{2} + 1$ holds near 1, we could at least locally (near 1) use comparison theorem to show that $\widehat{U}$ is smaller than $\mathrm{WF}(n, 2)$ started at the same point and therefore under such conditions $\widehat{U}$ (and U) never hits 1. □

Before we can prove Theorem 1.5 we need two additional lemmas. The first one expresses the Wright-Fisher diffusion as a certain skew-product of two independent squared Bessel processes. For notation and basic facts about (squared) Bessel processes see [12, Chapter XI].

Lemma 2.2. *Let $\alpha \geq 0, \beta \in \mathbb{R}$ and let $\mathcal{X}$ be BESQ^α process started at $x_0 \geq 0$ and $\mathcal{Y}$ be independent BESQ^β process started at $y_0 \geq 0$ such that $x_0 + y_0 > 0$. Let $T_0(\mathcal{Y}) := \inf \{ t \geq 0 \, ; \, \mathcal{Y}_t = 0 \}$. Define the continuous additive functional ρ as*

$$\rho_t = \int_0^t \frac{1}{\mathcal{X}_u + \mathcal{Y}_u} \mathrm{d}u \text{ and its inverse } \zeta_t = \inf \{ u \geq 0 \, ; \, \rho_u = t \} .$$

Let $U_t := \frac{\mathcal{X}_t}{\mathcal{X}_t + \mathcal{Y}_t}$ for $t < T_0(\mathcal{Y})$. Then $\widehat{U}_t := U_{\zeta_t}$ for $t < \rho_{T_0(\mathcal{Y})}$ is a $\mathrm{WF}(\alpha, \beta)$ started at $\frac{x_0}{x_0+y_0}$ and $\widehat{U}$ is independent of $\mathcal{X} + \mathcal{Y}$.

Proof. The lemma is proved in [14, Propositon 8] for non-negative coefficients α, β and the larger stopping time $T_0(\mathcal{X} + \mathcal{Y}) = \inf \{ t \geq 0 \, ; \, \mathcal{X}_t + \mathcal{Y}_t = 0 \}$, where they use the term Jacobi diffusion for particular Wright-Fisher diffusions. The same proof works for $\beta < 0$, since $T_0(\mathcal{Y}) < T_0(\mathcal{X} + \mathcal{Y})$ and hence on the stochastic interval $[0, T_0(\mathcal{Y}))$ all the processes in the proof of [14, Propositon 8] are well defined and all calculations stay exactly the same. □

Our next lemma summarizes facts about the time-change used in Theorem 1.5.

Lemma 2.3. *Let $n \geq 2$ and either $s > 0$ or $s = 0$ and $X_0 \neq 0$. Define*

$$S_s(t) := \int_s^t \frac{\gamma^2(R_u)}{R_u^2} \mathrm{d}u.$$

Then $S_s \colon [s, \infty) \to \mathbb{R}_+$ is continuous, strictly increasing, and $\lim_{t \to \infty} S_s(t) = \infty$. Its right continuous inverse $T_s \colon \mathbb{R}_+ \to [s, \infty)$ is also continuous and strictly increasing with $T_s(0) = s$. Furthermore, if $X_0 = 0$, then $\lim_{s \downarrow 0} S_s(t) = \infty$ holds for any $t > 0$.

---

[3]The function $u \mapsto uf(\sqrt{u})$ is Lipschitz if f is. Hence, $u \mapsto u\frac{\widetilde{g}(u)}{\widetilde{\gamma}^2(u)}$ is Lipschitz continuous.

Proof. Since a.s. $R_t^2 \in (0,1]$ for any $t \geq s$, R_t is continuous in t, and also $\gamma(R_t)$ is continuous in t and bounded away from 0, everything but the last claim follows from classical results on inverses of strictly increasing continuous functions. Since $\lim_{s \downarrow 0} \int_s^t \frac{\gamma^2(R_u)}{R_u^2} \mathrm{d}u = \int_0^t \frac{\gamma^2(R_u)}{R_u^2} \mathrm{d}u$, in order to prove the last claim it is enough to prove that $\int_0^t \frac{\gamma^2(R_u)}{R_u^2} \mathrm{d}u = \infty$ for any $t > 0$. Changing variables with the time change q_t from Lemma 2.1 we get

$$\int_0^t \frac{\gamma^2(R_u)}{R_u^2} \mathrm{d}u = \int_0^t \widehat{U}_{q_u}^{-1} \mathrm{d}q_u = \int_0^{q_t} \widehat{U}_u^{-1} \mathrm{d}u.$$

By the comparison theorem for SDEs and strict positivity of q_t it is therefore enough to prove that $\int_0^{t \wedge T_1(\mathcal{U})} \mathcal{U}_u^{-1} \mathrm{d}u = \infty$ for any $t > 0$, where $\mathcal{U}$ is Wright-Fisher(n, m) diffusion started at 0,

$$m := \min_{u \in [0,1]} \left(\frac{2\widetilde{g}(u)}{\widetilde{\gamma}^2(u)} - (n-1) \right)$$

is a possibly negative constant, and $T_1(\mathcal{U}) = \inf \{t \geq 0 \,;\, \mathcal{U}_t = 1\}$ is strictly positive. By Lemma 2.2 we can find a BESQ^n process $\mathcal{X}$ started at 0 and a BESQ^m process $\mathcal{Y}$ not started at 0 such that $\mathcal{U}_t = \frac{\mathcal{X}_{\zeta_t}}{\mathcal{X}_{\zeta_t} + \mathcal{Y}_{\zeta_t}}$ holds for $t < T_1(\mathcal{U}) = \rho_{T_0(\mathcal{Y})}$ with the time changes ζ and ρ defined as in Lemma 2.2. Using the time change formula for the Lebesgue-Stieltjes integral we get

$$\int_0^{t \wedge T_1(\mathcal{U})} \mathcal{U}_t^{-1} \mathrm{d}u = \int_0^{t \wedge T_1(\mathcal{U})} \frac{\mathcal{X}_{\zeta_u} + \mathcal{Y}_{\zeta_u}}{\mathcal{X}_{\zeta_u}} \mathrm{d}u = \int_0^{t \wedge T_1(\mathcal{U})} \mathcal{X}_{\zeta_u}^{-1} \mathrm{d}\zeta_u = \int_0^{\zeta_t \wedge \zeta_{T_1(\mathcal{U})}} \mathcal{X}_u^{-1} \mathrm{d}u.$$

This corresponds to the integral $\int_0^{\zeta_t \wedge \zeta_{T_1(\mathcal{U})}} h(\sqrt{\mathcal{X}_u}) \mathrm{d}u$ where $h(x) = x^{-2}$ and $\sqrt{\mathcal{X}}$ is Bessel process started at 0 with parameter $n \geq 2$. Since $\zeta_{T_1(\mathcal{U})} = T_0(\mathcal{Y}) > 0$, $\zeta_t > 0$ for any $t > 0$ and $-2 \leq -(2 \wedge n)$, Corollary 2.4 in [5] implies that the integral $S_0(t)$ diverges. $\qquad\square$

Proof of Theorem 1.5. Let us define function $\kappa(x^1, \ldots, x^n) := \left(\sum_{i=1}^n (x^i)^2 \right)^{1/2}$. Then $R_t = \kappa(X_t), V_t^i = \frac{\partial \kappa}{\partial x^i}(X_t)$ and a straightforward computation shows that

$$\frac{\partial^2 \kappa}{\partial x^i \partial x^j}(x) = \frac{\delta_{ij} r^2 - x^i x^j}{r^3} \text{ and } \frac{\partial^3 \kappa}{\partial x^i \partial x^j \partial x^k}(x) = 3r^{-5} x^i x^j x^k - r^{-3} \left(x^i \delta_{jk} + x^j \delta_{ik} + x^k \delta_{ij} \right),$$

where $r := \kappa(x)$. Note that also $\mathrm{d}\langle X^i, X^j \rangle_t = \gamma^2(|X_t|)(\delta_{ij} - X_t^i X_t^j)\mathrm{d}t$. Now we can use the Itô formula to get equations for R_t and V_t. First,

$$\mathrm{d}R_t = \gamma(|X_t|) \sum_{j=1}^n \frac{X_t^j}{R_t} \sqrt{1 - R_t^2} \mathrm{d}B_t^j - g(|X_t|) R_t \mathrm{d}t + \frac{1}{2} \sum_{i,j=1}^n \frac{\delta_{ij} R_t^2 - X_t^i X_t^j}{R_t^3} \mathrm{d}\langle X^i, X^j \rangle_t$$

$$= \gamma(R_t) \sum_{j=1}^n V_t^j \sqrt{1 - R_t^2} \mathrm{d}B_t^j + \frac{(n-1)\gamma^2(R_t) - 2g(R_t) R_t^2}{2R_t} \mathrm{d}t.$$

Since $\mathrm{d}\theta_t = \sum_{j=1}^n V_t^j \mathrm{d}B_t^j$ is also 1-dimensional Brownian motion (actually, it is the same process as in Lemma 2.1) we write the above equation in a more compact way as

$$\mathrm{d}R_t = \gamma(R_t)\sqrt{1 - R_t^2}\mathrm{d}\theta_t + ((n-1)\gamma^2(R_t) - 2g(R_t) R_t^2)/(2R_t)\mathrm{d}t. \qquad (2.3)$$

We can also write an equation for V_t^i.

$$\mathrm{d}V_t^i = \sum_{j=1}^n \frac{\delta_{ij} R_t^2 - X_t^i X_t^j}{R_t^3} \mathrm{d}X_t^j + \frac{1}{2} \sum_{j,k=1}^n \left(\frac{\partial^3 \kappa}{\partial x^i \partial x^j \partial x^k}(X_t) \right) \mathrm{d}\langle X^j, X^k \rangle_t$$

Similar calculations as above show that first sum is equal to $\gamma(R_t)\sum_{j=1}^{n}\frac{\delta_{ij}-V_t^i V_t^j}{R_t}\mathrm{d}B_t^j$ where the drift part vanishes and we compute the second sum as

$$\frac{\gamma^2(R_t)}{2}R_t^{-5}\sum_{j,k=1}^{n}\left(3X_t^i X_t^j X_t^k - (X_t^i \delta_{jk}+X_t^j \delta_{ik}+X_t^k \delta_{ij})R_t^2\right)(\delta_{jk}-X_t^j X_t^k)\,\mathrm{d}t =$$

$$=\frac{\gamma^2(R_t)}{2}R_t^{-5}\left(3X_t^i R_t^2 - (nX_t^i + X_t^i + X_t^i)R_t^2 - 3X_t^i R_t^4 + (X_t^i + X_t^i + X_t^i)R_t^4\right)\mathrm{d}t$$

$$=-\gamma(R_t)\frac{n-1}{2}\frac{X_t^i}{R_t^3}\mathrm{d}t = -\frac{\gamma(R_t)}{R_t^2}\frac{n-1}{2}V_t^i\mathrm{d}t.$$

Altogether, in vector form, we get $\mathrm{d}V_t = \frac{\gamma(R_t)}{R_t}(I-V_t V_t^\top)\mathrm{d}B_t - \frac{\gamma^2(R_t)}{R_t^2}\cdot\frac{n-1}{2}V_t\mathrm{d}t$. To show that the process $\widehat{V}_t = V_{T_t}$ is Brownian motion on a sphere it is sufficient to show that it satisfies SDE (1.1). Change of time formula for the Itô and the Lebesgue-Stieltjes integrals immediately shows that $\widehat{V}_t - \widehat{V}_0 = \int_0^t (I-\widehat{V}_u \widehat{V}_u^\top)\mathrm{d}\widetilde{W}_u - \int_0^t \frac{n-1}{2}\widehat{V}_u\mathrm{d}u$, where $\widetilde{W}_t = \int_s^{T_s(t)}\frac{\gamma(R_u)}{R_u}\mathrm{d}B_u$ is Brownian motion. This immediately implies that $\widehat{V}$ is Brownian motion on $\mathbb{S}^{n-1}$ but does not allow us to conclude that R and $\widehat{V}$ are independent. With this in mind we modify the Brownian motion driving the SDE for $\widehat{V}$. Let us enlarge the probability space to accommodate another scalar Brownian motion ξ which is independent of B and define a continuous local martingale $W_t = \int_s^{T_s(t)}\frac{\gamma(R_u)}{R_u}(I-V_u V_u^\top)\mathrm{d}B_u + \int_s^{T_s(t)}\frac{\gamma(R_u)}{R_u}V_u\mathrm{d}\xi_u$. Then we can compute $\langle W^i, W^j\rangle_t = \int_s^{T_s(t)}\delta_{ij}\frac{\gamma^2(R_u)}{R_u^2}\mathrm{d}u = \delta_{ij}t$, so W is $\mathcal{G}_t$-Brownian motion, where $\mathcal{G}_t := \mathcal{F}_{T_s(t)}$. Since $(I-V_u V_u^\top)\cdot\left[\frac{\gamma(R_u)}{R_u}(I-V_u V_u^\top),\ \frac{\gamma(R_u)}{R_u}V_u\right] = \left[\frac{\gamma(R_u)}{R_u}(I-V_u V_u^\top),\ 0\right]$ we can use change of time formula for stochastic and Lebesgue-Stieltjes integral [12, Chapter V, §1] and we get $\widehat{V}_t - \widehat{V}_0 = \int_0^t (I-\widehat{V}_u \widehat{V}_u^\top)\mathrm{d}W_u - \int_0^t \frac{n-1}{2}\widehat{V}_u\mathrm{d}u$ so that $\widehat{V}$ is a Brownian motion on $\mathbb{S}^{n-1}$.

To prove independence of $\widehat{V}$ and R it is enough to prove independence of Brownian motions W and θ which are driving the respective SDEs. Then $\widehat{V}$ and R are strong solutions to their corresponding SDEs which are driven by independent Brownian motions, so they are also independent. This holds since we note that for a strong solution X of some SDE there exists a measurable map Φ, such that $X = \Phi(\widetilde{B})$, where $\widetilde{B}$ is Brownian motion driving the SDE c.f. [6, 15]. Therefore we can find measurable maps Φ_1, Φ_2, such that $\widehat{V} = \Phi_1(W), R = \Phi_2(\theta)$ and independence does indeed follow from the independence of θ and W. The Markov property implies that W depends on $\mathcal{G}_0 = \mathcal{F}_s$ only through $W_0 = 0$, so W is independent of $\mathcal{F}_s$. Hence W is independent of $(\theta_t)_{t\in[0,s]}$. Therefore, it is enough to prove that W is independent of $(\theta_t - \theta_s)_{t\geq s}$. Define $\eta_t := \theta_{T_s(t)} - \theta_s = \int_s^{T_s(t)}V_u^\top\mathrm{d}B_u$ so that η is a $\mathcal{G}_t$-local martingale. Simple calculation shows that $\langle W_i, \eta\rangle_t = 0$ and $\langle\eta\rangle_t = T_s(t) - s$ with inverse $S_s(t+s)$. We then use Knight's Theorem (also known as the multidimensional Dambis-Dubins-Schwarz Theorem found in [12, Chapter V, Theorem 1.9]) to show that W and $(\eta_{S_s(t+s)})_{s\geq t} = (\theta_{s+t} - \theta_s)_{s\geq t}$ are independent Brownian motions.

To address the last statement we need to consider the situation when the solution is started from 0. The evolution of such a process is given by $(R_t\phi_{S_s(t)}, t\geq s)$ where R is a square root of a solution to SDE (2.1) and ϕ is an independent Brownian motion on the sphere started at $\phi_0 = X_s/R_s$. Due to *rapid spinning* (i.e. $\lim_{s\downarrow 0}S_s(t) = \infty$), the initial point $\phi_0 = X_s/R_s$ will be forced to be uniformly distributed on the sphere. This follows from the properties of the skew-product decomposition established in this proof, Lemma 2.3 above and [8, Lemma 3.12]. $\qquad\square$

Proof of Corollary 1.6. For $n = 1$, pathwise uniqueness holds so uniqueness in law follows trivially. Now let $n \geq 2$. When $x_0 \neq 0$, by Theorem 1.5, the solution X is a

measurable functional of two indpendent processes R and $\widehat{V}$ with given laws. Hence the law of X is unique.

Finally, we consider the case of $X_0 = 0$. What follows is almost direct application of the proof of Theorem 1.1 in [8]. For any $k \in \mathbb{N}$ and open set $U \subseteq \mathbb{R}^k$, define a measurable function $F_U \colon (0, \infty)^k \to [0, 1], F_U(t_1, \ldots, t_k) := \mathbb{P}_\Psi[(\psi_{t_1}, \ldots, \psi_{t_k}) \in U]$, where law $\mathbb{P}_\Psi[\cdot]$ is defined in [8, Lemma 3.7]. Letting $\mathcal{F}_\infty^R := \sigma(R_u, u \in \mathbb{R}_+)$, we apply Lemma 2.3 and Theorem 1.5 above and [8, Lemma 3.12] to get $\mathbb{P}\left[(X_{t_1}/R_{t_1}, \ldots, X_{t_k}/R_{t_k}) \in U \big| \mathcal{F}_\infty^R\right] = F_U(S_s(t_1), \ldots, S_s(t_k))$ a.s. for $0 < s < t_1 < \cdots < t_k$. So $\mathbb{P}\left[(X_{t_1}/R_{t_1}, \ldots, X_{t_k}/R_{t_k}) \in U\right] = \mathbb{E}[F_U(S_s(t_1), \ldots, S_s(t_k))]$ and therefore the finite-dimensional distributions of $(X_t/R_t, t > 0)$ are determined uniquely by $\mathbb{P}_\Psi[\cdot]$ and law of R. Moreover, law of X is determined uniquely by the law of $(R, X/R)$, therefore uniquely by $\mathbb{P}_\Psi[\cdot]$ and law of process R solving SDE (2.3) started at 0. $\qquad\square$

2.2 Pathwise uniqueness for SDE (1.2)

Let X and $\widetilde{X}$ be solutions to the SDE (1.2) driven by the same Brownian motion B and started at the same point $x_0 \in \mathbb{B}^n$. Pathwise uniqueness clearly holds up to the hitting time of boundary, so by restarting argument it is enough to prove pathwise uniqueness for starting points x_0 on the boundary. Furthermore, it is enough to prove that $X_t = \widetilde{X}_t$ for $t \leq \tau_\varepsilon$ where $\tau_\varepsilon = \inf\left\{t \geq 0 \, ; \, |X_t|^2 \wedge |\widetilde{X}_t|^2 \leq 1 - \varepsilon\right\}$ for some $\varepsilon > 0$ and without loss of generality we can clearly assume that $\varepsilon < \frac{1}{2}$. To prove equality of the processes we will apply method of DeBlassie [7]. Namely, we wish to use Gronwall's lemma, but due to non-Lipschitzness we cannot apply it directly to $\mathbb{E}[|X_t - \widetilde{X}_t|^2]$. The idea of DeBlassie (and Swart before with $p = \frac{1}{2}$) is to denote $Y := 1 - |X|^2, \widetilde{Y} := 1 - |\widetilde{X}|^2$ and look at the process $W := |X - \widetilde{X}|^2 + (Y^p - \widetilde{Y}^p)^2$ for some $p \in (\frac{1}{2}, 1)$. We then have

$$dY_t = -2\gamma(|X_t|)Y_t^{1/2}\sum_{i=1}^n X_t^i dB_t^i - n\gamma^2(|X_t|)Y_t dt + \left(2g(|X_t|) - (n-1)\gamma^2(|X_t|)\right)|X_t|^2 dt.$$

A slight modification of [7, Lemma 2.1] implies that for $p > 1 + \frac{n-1}{2} - \frac{g(1)}{\gamma^2(1)}$ a formal application of Itô's formula for the mapping $x \mapsto x^p$ is justified. Defining

$$G(u) := g(u) - \frac{n-1}{2}\gamma^2(u) + (p-1)\gamma^2(u),$$

we get

$$dY_t^p = -2p\gamma(|X_t|)Y_t^{p-1/2}\sum_{i=1}^n X_t^i dB_t^i + 2pY_t^{p-1}|X_t|^2 \mathbb{1}(Y > 0)G(|X_t|)dt - np\gamma^2(|X_t|)Y_t^p dt,$$

where $t \leq \tau_\varepsilon$, $|X_0|^2 > 1 - \varepsilon$, and $\varepsilon = \varepsilon(p)$ is chosen in such a way that $p > 1 + \frac{n-1}{2} - \frac{g(u)}{\gamma^2(u)}$ for $u \in (1 - \varepsilon(p), 1]$. The latter condition is necessary to keep second term on the right hand side negative to allow use of Fatou's lemma. Furthermore, $\int_0^t \mathbb{1}(Y_s = 0)ds = 0$ holds. Note that essentially all necessary calculations and results are the same as in [7] if we change their g for $g - \frac{n-1}{2}\gamma^2$. Subtracting the equations for Y^p and $\widetilde{Y}^p$ we get

$$\begin{aligned} d(Y^p - \widetilde{Y}^p)_t = {} & -2p\sum_{i=1}^n \left(\gamma(|X_t|)Y_t^{p-1/2}X_t^i - \gamma(|\widetilde{X}_t|)\widetilde{Y}_t^{p-1/2}\widetilde{X}_t^i\right)dB_t^i \\ & + 2p\left(Y_t^{p-1}|X_t|^2\mathbb{1}(Y > 0)G(|X_t|) - \widetilde{Y}_t^{p-1}|\widetilde{X}_t|^2\mathbb{1}(\widetilde{Y} > 0)G(|\widetilde{X}_t|)\right)dt \\ & - np\left(\gamma^2(|X_t|)Y_t^p - \gamma^2(|\widetilde{X}_t|)\widetilde{Y}_t^p\right)dt \\ = {} & : dM_t + \mathbf{I}_1 dt + \mathbf{I}_2 dt \end{aligned}$$

and Itô's formula yields

$$
\begin{aligned}
\mathrm{d}(Y^p - \widetilde{Y}^p)^2_t = {} & 2(Y_t^p - \widetilde{Y}_t^p)(dM_t + \mathbf{I}_1 dt + \mathbf{I}_2 dt) \\
& + 4p^2 \sum_{i=1}^{n} \left(\gamma(|X_t|) Y_t^{p-1/2} X_t^i - \gamma(|\widetilde{X}_t|) \widetilde{Y}_t^{p-1/2} \widetilde{X}_t^i \right) dt \\
=: {} & d\widetilde{M}_t + 2(Y_t^p - \widetilde{Y}_t^p)(\mathbf{I}_1 dt + \mathbf{I}_2 dt) + \mathbf{I}_3 dt.
\end{aligned}
$$

We can also compute

$$
\begin{aligned}
\mathrm{d}|X_t - \widetilde{X}_t|^2 = {} & 2\sum_{i=1}^{n}(X_t^i - \widetilde{X}_t^i) \sum_{j=1}^{n} \left(\gamma(|X_t|)\sigma_{ij}(X_t) - \gamma(|\widetilde{X}_t|)\sigma_{ij}(\widetilde{X}_t) \right) \mathrm{d}B_t^j \\
& - 2\sum_{i=1}^{n}(X_t^i - \widetilde{X}_t^i) \left(g(|X_t|)X_j^i - g(|\widetilde{X}_t|)\widetilde{X}_t^i \right) \mathrm{d}t \\
& + \sum_{i,j=1}^{n} (\gamma(|X_t|)\sigma_{ij}(X_t) - \gamma(|\widetilde{X}_t|)\sigma_{ij}(\widetilde{X}_t))^2 \mathrm{d}t \\
=: {} & \mathrm{d}N_t + \mathbf{I}_4 \mathrm{d}t + \mathbf{I}_5 \mathrm{d}t.
\end{aligned}
$$

The term $\mathbf{I}_5$ is the one which disallows direct use of Gronwall's lemma. It is singular in a sense that $\frac{\mathbf{I}_5}{W}$ can be arbitrarily large. Another singular term is $\mathbf{I}_3$, but fortunately we also have negative singular term $2(Y_t^p - \widetilde{Y}_t^p)\mathbf{I}_1$, which will ensure that altogether we stay non-singular. We will bound all the terms $\mathbf{I}_k$ and since $\mathbf{I}_1, \mathbf{I}_2, \mathbf{I}_3$ and $\mathbf{I}_4$ are exactly the same[4] as in [7] in Lemmas 3.1, 3.2, 3.4, and 3.5, respectively, we will not do the calculations but only summarize final results. Let us introduce non-negative process $Z := (Y^p - \widetilde{Y}^p)(\widetilde{Y}^{p-1} - Y^{p-1})$. To make sense of Z_t we implicitly multiply everything by $\mathbb{1}(Y_t > 0, \widetilde{Y}_t > 0)$. We will use this convention until the end of the proof. Then we have

$$
\begin{aligned}
(Y_t^p - \widetilde{Y}_t^p)\mathbf{I}_1 &\leq -2pZ_t |X_t|^2 G(|X_t|) + C_1 \varepsilon Z_t, \\
|\mathbf{I}_2| &\leq C_2 \left(|Y_t^p - \widetilde{Y}_t^p| + |X_t - \widetilde{X}_t| \right), \\
\mathbf{I}_3 &\leq \frac{p(2p-1)^2}{1-p} \gamma^2(|X_t|) |X_t|^2 Z_t + C_3 |X_t - \widetilde{X}_t|^2 + C_3 \varepsilon Z_t, \\
\mathbf{I}_4 &\leq C_4 |X_t - \widetilde{X}_t|^2,
\end{aligned}
$$

where constants C_1, C_2, C_3, and C_4 are independent of ε. Bound for $\mathbf{I}_5$ has to be done differently due to non-diagonal nature of our SDE. By straightforward computation e.g. by computing Frobenius norm of the matrix $\gamma(|X_t|)\sigma(X_t) - \gamma(|\widetilde{X}_t|)\sigma(\widetilde{X}_t)$, we see that

$$
\begin{aligned}
\mathbf{I}_5 = {} & \gamma^2(|X_t|) \left(1 - \sqrt{1 - |X_t|^2}\right)^2 + \gamma^2(|\widetilde{X}_t|) \left(1 - \sqrt{1 - |\widetilde{X}_t|^2}\right)^2 \\
& - 2\gamma(|X_t|)\gamma(|\widetilde{X}_t|) \left(1 - \sqrt{1 - |X_t|^2}\right) \left(1 - \sqrt{1 - |\widetilde{X}_t|^2}\right) \frac{\left(X_t \cdot \widetilde{X}_t\right)^2}{|X_t|^2 |\widetilde{X}_t|^2} \\
= {} & \left(\gamma(|X_t|)Y_t^{1/2} - \gamma(|\widetilde{X}_t|)\widetilde{Y}_t^{1/2} + \gamma(|\widetilde{X}_t|) - \gamma(|X_t|) \right)^2 \\
& + 2\gamma(|X_t|)\gamma(|\widetilde{X}_t|) \left(1 - \sqrt{1 - |X_t|^2}\right) \left(1 - \sqrt{1 - |\widetilde{X}_t|^2}\right) \frac{|X_t|^2 |\widetilde{X}_t|^2 - \left(X_t \cdot \widetilde{X}_t\right)^2}{|X_t|^2 |\widetilde{X}_t|^2}.
\end{aligned}
$$

---

[4] Our G is defined slightly differently but it is still Lipschitz, so everything works.

For the first term we can use Cauchy-Schwartz inequality to bound it from above by

$$2\left(\gamma(|X_t|)Y_t^{1/2} - \gamma(|\widetilde{X}_t|)\widetilde{Y}_t^{1/2}\right)^2 + 2\left(\gamma(|\widetilde{X}_t|) - \gamma(|X_t|)\right)^2 \le C\left(\varepsilon^{2-p}Z_t + |X_t - \widetilde{X}_t|^2\right),$$

where the inequality follows from the proof of [7, Lemma 3.6] and the Lipschitz continuity of γ. The second term can by our assumptions be bounded from above by $8(\sup\gamma)^2\left(|X_t|^2|\widetilde{X}_t|^2 - \left(X_t \cdot \widetilde{X}_t\right)^2\right)$. Using $2X_t \cdot \widetilde{X}_t = |X_t|^2 + |\widetilde{X}_t|^2 - |X_t - \widetilde{X}_t|^2$ we get $|X_t|^2|\widetilde{X}_t|^2 - \left(X_t \cdot \widetilde{X}_t\right)^2 \le |X_t - \widetilde{X}_t|^2$. Hence we find $\mathbf{I}_5 \le C_5(\varepsilon^{2-p}Z_t + |X_t - \widetilde{X}_t|^2)$ with C_5 independent of ε. Using above facts we get

$$\begin{aligned}
2(Y_t^p - \widetilde{Y}_t^p)\mathbf{I}_1 + \mathbf{I}_3 + \mathbf{I}_5 &\le -4pZ_t|X_t|^2 G(|X_t|) + C\varepsilon Z_t \\
&\quad + \frac{p(2p-1)^2}{1-p}|X_t|^2 Z_t + C|X_t - \widetilde{X}_t|^2 + C\varepsilon Z_t \\
&\quad + C(\varepsilon^{2-p}Z_t + |X_t - \widetilde{X}_t|^2) \\
&= 4pZ_t\gamma^2(|X_t|)|X_t|^2\left(1 - p + \frac{(2p-1)^2}{4(1-p)} + \frac{n-1}{2} - \frac{g(|X_t|)}{\gamma^2(|X_t|)}\right) \\
&\quad + C(2\varepsilon + \varepsilon^{2-p})Z_t + 2C|X_t - \widetilde{X}_t|^2.
\end{aligned}$$

Note that expression $1 - p + \frac{(2p-1)^2}{4(1-p)}$ is minimized at $p = 1 - \frac{\sqrt{2}}{4}$ and the value is then $\sqrt{2} - 1$. Therefore we use initial assumption that $\frac{g(1)}{\gamma^2(1)} - \frac{n-1}{2} > \sqrt{2} - 1$ to ensure the whole bracket is negative. Note also that such choice of p implies $p = 1 - \frac{\sqrt{2}}{4} > 1 - (\sqrt{2} - 1) > 1 + \frac{n-1}{2} - \frac{g(1)}{\gamma^2(1)}$ so all previous calculations are justifiable since we have necessary condition our use of Lemma 2.1 from [7]. Fixing $p = 1 - \frac{\sqrt{2}}{4}$ we then let ε possibly be even smaller to ensure that $\frac{g(u)}{\gamma^2(u)} - \frac{n-1}{2} > \sqrt{2} - 1 + \delta$ holds on $(1 - \varepsilon, 1]$ for some small fixed $\delta > 0$. The coefficient in front of Z_t equals

$$4p\gamma^2(|X_t|)|X_t|^2\left(\sqrt{2} - 1 + \frac{n-1}{2} - \frac{g(|X_t|)}{\gamma^2(|X_t|)}\right) + C(2\varepsilon + \varepsilon^{2-p}),$$

which is bounded from above by $-4p(\inf\gamma^2)(1-\varepsilon)\delta + C(2\varepsilon + \varepsilon^{2-p})$. Therefore, by letting ε be small enough we ensure that this coefficient in front of non-negative Z_t is negative and bound $2(Y_t^p - \widetilde{Y}_t^p)\mathbf{I}_1 + \mathbf{I}_3 + \mathbf{I}_5 \le C|X_t - \widetilde{X}_t|^2$ follows. Recall that

$$\mathrm{d}W_t = d\widetilde{M}_t + \mathrm{d}N_t + 2(Y_t^p - \widetilde{Y}_t^p)\mathbf{I}_1\mathrm{d}t + 2(Y_t^p - \widetilde{Y}_t^p)\mathbf{I}_2\mathrm{d}t + \mathbf{I}_3\mathrm{d}t + \mathbf{I}_4\mathrm{d}t + \mathbf{I}_5\mathrm{d}t$$

and let $\widetilde{\tau}_m$ be a localizing sequence of stopping times for local martingale $\widetilde{M} + N$. Then using above bounds and the fact that $\int_0^t \mathbb{1}(Y_s = 0 \text{ or } \widetilde{Y}_s = 0)\mathrm{d}s = 0$ yields

$$\begin{aligned}
\mathbb{E}[W_{t \wedge \tau_\varepsilon \wedge \widetilde{\tau}_m}] &= \mathbb{E}\left[\int_0^{t \wedge \tau_\varepsilon \wedge \widetilde{\tau}_m}\left(2(Y_s^p - \widetilde{Y}_s^p)(\mathbf{I}_1 + \mathbf{I}_2) + \mathbf{I}_3 + \mathbf{I}_4 + \mathbf{I}_5\right)\mathbb{1}(Y_s > 0, \widetilde{Y}_s > 0)\mathrm{d}s\right] \\
&\le C\mathbb{E}\left[\int_0^{t \wedge \tau_\varepsilon \wedge \widetilde{\tau}_m}\left(|X_s - \widetilde{X}_s|^2 + 2(Y_s^p - \widetilde{Y}_s^p)^2 + 2|Y_s^p - \widetilde{Y}_s^p||X_s - \widetilde{X}_s|\right)\mathrm{d}s\right] \\
&\le 3C\int_0^{t \wedge \tau_\varepsilon \wedge \widetilde{\tau}_m}\mathbb{E}[W_s]\mathrm{d}s
\end{aligned}$$

and Gronwall's lemma implies that $\mathbb{E}[W_t] = 0$ and by non-negativity also $W_t = 0$ for $t \le \tau_\varepsilon \wedge \widetilde{\tau}_m$. Letting $m \to \infty$ we get $W_t = 0$ for $t < \tau_\varepsilon$. Therefore $X_t = \widetilde{X}_t$ for $t < \tau_\varepsilon$ and pathwise uniqueness in Theorem 1.7 follows.

References

[1] T. M. Apostol and M. A. Mnatsakanian, *New balancing principles applied to circumsolids of revolution, and to n-dimensional spheres, cylindroids, and cylindrical wedges*, Amer. Math. Monthly **120** (2013), no. 4, 298–321. MR-3035124

[2] Archimedes, *On the sphere and cylinder*, ca. 225 BC.

[3] D. Bakry, *Remarques sur les semigroupes de Jacobi*, Astérisque **236** (1996), 23–40. MR-1417973

[4] F. Barthe, O. Guédon, S. Mendelson, and A. Naor, *A probabilistic approach to the geometry of the l_p^n-ball*, Ann. Probab. **33** (2005), no. 2, 480–513. MR-2123199

[5] A. S. Cherny, *Convergence of some integrals associated with Bessel processes*, Theory of Probability & Its Applications **45** (2001), no. 2, 195–209. MR-1967756

[6] A.S. Cherny, *On the strong and weak solutions of stochastic differential equations governing Bessel processes*, Stochastics and Stochastic Reports **70** (2000), no. 3-4, 213–219. MR-1800956

[7] D. DeBlassie, *Uniqueness for diffusions degenerating at the boundary of a smooth bounded set*, Ann. Probab. **32** (2004), no. 4, 3167–3190. MR-2094442

[8] N. Georgiou, A. Mijatović, and A. R. Wade, *Invariance principle for non-homogeneous random walks*, ArXiv e-prints, 1801.07882 (2018).

[9] E.P. Hsu, *Stochastic analysis on manifolds*, Graduate Studies in Mathematics 38, vol. 38, American Mathematical Society, 2002. MR-1882015

[10] N. Ikeda and S. Watanabe, *Stochastic differential equations and diffusion processes*, Kodansha scientific books, North-Holland, 1989. MR-1011252

[11] J. Pitman and N. Ross, *Archimedes, Gauss, and Stein*, Notices Amer. Math. Soc. **59** (2012), no. 10, 1416–1421. MR-3025901

[12] D. Revuz and M. Yor, *Continuous martingales and Brownian motion*, 3rd ed ed., Springer, 1999. MR-1725357

[13] J. M. Swart, *Pathwise uniqueness for a SDE with non-Lipschitz coefficients*, Stochastic Processes and their Applications **98** (2002), no. 1, 131–149. MR-1884927

[14] J. Warren and M. Yor, *The Brownian burglar: conditioning Brownian motion by its local time process*, Séminaire de Probabilités XXXII (Berlin, Heidelberg) (Jacques Azéma, Marc Yor, Michel Émery, and Michel Ledoux, eds.), Springer Berlin Heidelberg, 1998, pp. 328–342. MR-1655303

[15] T. Yamada and S. Watanabe, *On the uniqueness of solutions of stochastic differential equations*, J. Math. Kyoto Univ. **11** (1971), no. 1, 155–167. MR-0278420