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A radial invariance principle for non-homogeneous random walks

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Abstract

Consider non-homogeneous zero-drift random walks in $\mathbb{R}^d$, $d \geq 2$, with the asymptotic increment covariance matrix $\sigma^2(\mathbf{u})$ satisfying $\mathbf{u}^\top \sigma^2(\mathbf{u})\mathbf{u} = U$ and $\operatorname{tr} \sigma^2(\mathbf{u}) = V$ in all in directions $\mathbf{u} \in \mathbb{S}^{d-1}$ for some positive constants $U < V$. In this paper we establish weak convergence of the radial component of the walk to a Bessel process with dimension V/U. This can be viewed as an extension of an invariance principle of Lamperti.

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1 Introduction and main result

A genuinely d-dimensional, spatially homogeneous random walk on $\mathbb{R}^d$ whose increments have zero mean and finite second moments is recurrent if and only if $d \leq 2$. In [5] a class of spatially non-homogeneous random walks (Markov chains) exhibiting anomalous recurrence behaviour was described; the increments for such walks again have zero mean, but have a covariance that depends on the current position in a certain way. In any dimension $d \geq 2$, such walks can be recurrent or transient, depending on the model parameters. Note that this anomalous recurrence is a genuinely many-dimensional phenomenon, as it is driven by the increment covariance.

Random walks are fundamental stochastic processes and have seen numerous applications over many decades in, for example, acoustics, ecology, finance, and physical chemistry. Non-homogeneous random walks arise naturally when assumptions of homogeneity are relaxed, and present novel mathematical problems that demand new intuitions. More broadly, non-homogeneous random walks serve as prototypical 'near-critical' stochastic systems. See [10] for further discussion and references. An important element of the classical theory of spatially homogeneous random walks is the Donsker

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invariance principle, which describes the scaling limit for the class of homogeneous random walks whose increments have zero mean and positive-definite covariance in terms of Brownian motion on $\mathbb{R}^d$. It is natural to ask: is there an invariance principle for a suitable class of non-homogeneous random walks, in which the scaling limit is some universal class of diffusions?

The goal of this note is to establish an invariance principle for the *radial component* of the walks studied in [5]. The result can be seen as an extension of the classical picture, where the radial component of a walk in the Donsker class converges to a Bessel process, and also as an extension of work of Lamperti [8] on one-dimensional Markov chains with asymptotically zero drifts. The result is also an important ingredient in the much more involved proof of a full invariance principle that is the subject of [6], where the question posed at the end of the previous paragraph is answered positively. We explain these points in more detail once we have given a precise description of the model and stated the main result.

We work in $\mathbb{R}^d$, $d \geq 2$. Write $\mathbf{0}$ for the origin in $\mathbb{R}^d$, and let $\| \cdot \|$ denote the Euclidean norm and $\langle \cdot , \cdot \rangle$ the Euclidean inner product on $\mathbb{R}^d$. Write $\mathbb{S}^{d-1} := \{ \mathbf{u} \in \mathbb{R}^d : \| \mathbf{u} \| = 1 \}$ for the unit sphere in $\mathbb{R}^d$. For $\mathbf{x} \in \mathbb{R}^d \setminus \{ \mathbf{0} \}$, set $\hat{\mathbf{x}} := \mathbf{x} / \| \mathbf{x} \|$. For convenience, set $\hat{\mathbf{0}} := \mathbf{e}_1$, the first element of the standard orthonormal basis of $\mathbb{R}^d$. For definiteness, vectors $\mathbf{x} \in \mathbb{R}^d$ are viewed as column vectors throughout. Set $\mathbb{N} := \{1, 2, 3, \ldots\}$ and $\mathbb{Z}_+ := \mathbb{N} \cup \{0\}$.

We now define $X = (X_n, n \in \mathbb{Z}_+)$, a discrete-time, time-homogeneous Markov process on a (non-empty, unbounded) subset $\mathbb{X}$ of $\mathbb{R}^d$. Formally, $(\mathbb{X}, \mathcal{B}_{\mathbb{X}})$ is a measurable space, $\mathbb{X}$ is a Borel subset of $\mathbb{R}^d$, and $\mathcal{B}_{\mathbb{X}}$ is the σ-algebra of all $B \cap \mathbb{X}$ for B a Borel set in $\mathbb{R}^d$. Suppose that X_0 is some fixed (i.e., non-random) point in $\mathbb{X}$. Write

$$\Delta_n := X_{n+1} - X_n$$

for the increments of X. By assumption, given $X_0, \ldots, X_n$, the law of Δ_n depends only on X_n (and not on n); so often we ease notation by taking $n = 0$ and writing just Δ for Δ_0. We also use the shorthand $\mathbb{P}_{\mathbf{x}}[\,\cdot\,] = \mathbb{P}[\,\cdot \mid X_0 = \mathbf{x}]$ for probabilities when the walk is started from $\mathbf{x} \in \mathbb{X}$; similarly we use $\mathbb{E}_{\mathbf{x}}$ for the corresponding expectations.

We make the following moments assumption:

(A0) Suppose that $\sup_{\mathbf{x} \in \mathbb{X}} \mathbb{E}_{\mathbf{x}}[\| \Delta \|^4] < \infty$.

The assumption (A0) ensures that Δ has a well-defined mean vector $\mu(\mathbf{x}) := \mathbb{E}_{\mathbf{x}}[\Delta]$, and we suppose that the random walk has *zero drift*:

(A1) Suppose that $\mu(\mathbf{x}) = \mathbf{0}$ for all $\mathbf{x} \in \mathbb{X}$.

The assumption (A0) also ensures that Δ has a well-defined covariance matrix, which we denote by $M(\mathbf{x}) := \mathbb{E}_{\mathbf{x}}[\Delta \Delta^\top]$, where Δ is viewed as a column vector. To rule out pathological cases, we assume that Δ is *uniformly non-degenerate*, in the following sense.

(A2) There exists $v > 0$ such that $\operatorname{tr} M(\mathbf{x}) = \mathbb{E}_{\mathbf{x}}[\| \Delta \|^2] \geq v$ for all $\mathbf{x} \in \mathbb{X}$.

Write $\| \cdot \|_{\mathrm{op}}$ for the matrix (operator) norm given by $\| M \|_{\mathrm{op}} = \sup_{\mathbf{u} \in \mathbb{S}^{d-1}} \| M \mathbf{u} \|$. The following assumption on the asymptotic stability of the covariance structure of the process along rays is central.

(A3) Suppose that there exists a positive-definite matrix function σ^2 with domain $\mathbb{S}^{d-1}$ such that, as $r \to \infty$,

$$\varepsilon(r) := \sup_{\mathbf{x} \in \mathbb{X} : \| \mathbf{x} \| \geq r} \| M(\mathbf{x}) - \sigma^2(\hat{\mathbf{x}}) \|_{\mathrm{op}} \to 0.$$

Finally, we assume the following.

(A4) Suppose that there exist constants U, V with $0 < U < V < \infty$ such that, for all $\mathbf{u} \in \mathbb{S}^{d-1}$, $\mathbf{u}^\top \sigma^2(\mathbf{u})\mathbf{u} = U$ and $\operatorname{tr} \sigma^2(\mathbf{u}) = V$. In the case $2U = V$, suppose in addition that ε as defined in (A3) satisfies $\varepsilon(r) = O(r^{-\delta})$ for some $\delta > 0$.

Informally, V quantifies the total variance of the increments, while U quantifies the variance in the radial direction; necessarily $U \leq V$. The final condition in (A4) is necessary to deal with the critical-parameter case.

The main result of [5] (see also [10, §4.2]) stated that under the assumptions (A0)–(A4), we have that (i) if $2U < V$, then $\lim_{n\to\infty} \|X_n\| = +\infty$, a.s. (transience); and (ii) if $2U \geq V$, then $\liminf_{n\to\infty} \|X_n\| \leq r_0$, a.s., for some constant $r_0 \in \mathbb{R}_+$ (recurrence).

For $n \in \mathbb{N}$ and $t \in \mathbb{R}_+$, define

$$\widetilde{X}_n(t) := n^{-1/2} X_{\lfloor nt \rfloor}. \tag{1.1}$$

For each n, we view $\widetilde{X}_n$ as an element of the space $\mathcal{D}(\mathbb{R}_+; \mathbb{R}^d)$ of functions $f : \mathbb{R}_+ \to \mathbb{R}^d$ that are right-continuous and have left limits, endowed with the Skorokhod metric: see e.g. [4, §3.5].

Theorem 1.1. *Suppose that* (A0)–(A4) *hold. Without loss of generality assume that* $U = 1$. *Then* $\|\widetilde{X}_n\|$ *converges weakly to the* V*-dimensional Bessel process started at* 0.

Remark. As $\|\widetilde{X}_n\|$ is typically non-Markov, Theorem 1.1 may be viewed as an extension of the invariance principle in [8, Thm 5.1], describing the weak convergence of a sequence of non-negative Markov processes to a Bessel diffusion.

We describe the plan of the proof of Theorem 1.1, which is carried out in detail in Section 2 below, and also how Theorem 1.1 provides a crucial step in the proof in [6] of a full invariance principle for X_n, under additional conditions.

The proof of Theorem 1.1 is based on a general result on the convergence of Markov chains to diffusions, [4, Thm 7.4.1., p. 354] (reproduced as Theorem 2.5 below). In order to establish the invariance principle, the limit must be uniqely determined (formally, the corresponding martingale problem must be well posed). It is well known that the stochastic differential equation (SDE)

$$\mathrm{d}\rho_t = \frac{V-1}{2\rho_t} \mathbf{1}\{\rho_t \neq 0\} \mathrm{d}t + \mathrm{d}B_t, \quad \rho_0 = x_0, \tag{1.2}$$

satisfied by a V-dimensional Bessel process, does not possess uniqueness in law for any $V > 1$ if $x_0 = 0$. Furthermore, if $V \in (1, 2)$, uniqueness in law fails also in the case $x_0 > 0$ (see [3, Thm 3.2(iii)] for both assertions). Hence in the proof of Theorem 1.1, we work with the sequence $\|\widetilde{X}_n\|^2$ and show that it converges to the law $\mathrm{BESQ}^V(0)$ of the squared Bessel process, which is uniquely determined by its SDE (see e.g. [11, Ch. XI, Sec. 1]). Then to prove Theorem 1.1, we must verify the conditions of [4, Thm 7.4.1, p. 354]. This relies on several computations based on the assumptions (A0)–(A4), and the fact, established in [5], that X_n is *null* in the sense that the limiting normalized occupation measure of any bounded set tends to 0. The details are in Section 2.

Establishing the full invariance principle requires significantly more work; this is carried out in [6]. The limiting diffusion on $\mathbb{R}^d$ satisfies the SDE $\mathcal{X}_t = \sigma(\hat{\mathcal{X}}_t)\mathrm{d}W_t$, where $\sigma(\mathbf{u})\sigma^\top(\mathbf{u}) = \sigma^2(\mathbf{u})$ for all $\mathbf{u} \in \mathbb{S}^{d-1}$, and W is Brownian motion on $\mathbb{R}^d$. There are two key steps in the proof of the full invariance principle: (i) showing that the above SDE uniquely characterizes in law the limiting diffusion, and (ii) showing the convergence of $\widetilde{X}_n$ to the limit. Both of these steps are complicated essentially by the discontinuity of $\mathbf{x} \mapsto \sigma(\hat{\mathbf{x}})$ at $\mathbf{x} = \mathbf{0}$, which means that for step (ii) we cannot apply directly the results of [4], which are for diffusions with continuous coefficients, and for step (i) we make use of a skew-product structure attached to an excursion description of the diffusion. The fact that Theorem 1.1 in the present paper identifies the radial part of the limit as a Bessel process provides a crucial backbone upon which to construct this excursion description. The skew-product structure requires an additional assumption. See [6] for details.

To end this section, we remark that our situation bears comparison with random walk in random environment in the case of stationary and ergodic *balanced environments*, in which the increment distribution at each site is uniformly elliptic and symmetric, where a quenched central limit theorem is known to hold: see [9, 1] and [12, §3.3]. Such environments satisfy analogues of our (A0)–(A2), but our conditions (A3)–(A4) are incompatible with stationarity apart from the special case where $\sigma^2(\mathbf{u}) \equiv \sigma^2$ does not depend on $\mathbf{u}$.

2 Proofs

Recall that $\Delta_n := X_{n+1} - X_n$ for $n \in \mathbb{Z}_+$.

Lemma 2.1. *Under assumptions* (A0)–(A4)*, for any $k \in \mathbb{N}$ the following limits hold:*

$$\lim_{n\to\infty} \frac{1}{n^\ell} \sup_{\mathbf{x}\in\mathbb{X}} \mathbb{E}_\mathbf{x}\Big[\max_{0\le m\le kn} \|\Delta_m\|^{2\ell}\Big] = 0, \text{ for } \ell \in \{1,2\}, \tag{2.1}$$

$$\lim_{n\to\infty} \frac{1}{n^2} \sup_{\mathbf{x}\in\mathbb{X}\cap B} \mathbb{E}_\mathbf{x}\Big[\max_{0\le m\le kn} \|\Delta_m\|^2\|X_m\|^2\Big] = 0, \tag{2.2}$$

where B is any compact set in $\mathbb{R}^d$.

The following estimates will be useful in the proof of Lemma 2.1 and subsequently.

Lemma 2.2. *Under assumptions* (A0)–(A4)*, there exists a constant $D_0 \in \mathbb{R}_+$ such that*

$$\mathbb{E}_\mathbf{x}\left[\|X_m\|^\ell\right] \le D_0(m^{\ell/2} + \|\mathbf{x}\|^\ell)$$

for any $\ell \in \{1,\dots,4\}$, all $m \in \mathbb{N}$, and all $\mathbf{x} \in \mathbb{X}$.

Proof. First note that $\|\mathbf{x}+\Delta_m\|^2 - \|\mathbf{x}\|^2 = 2\langle \mathbf{x}, \Delta_m\rangle + \|\Delta_m\|^2$. Hence by (A0) and (A1), there exists a constant $C_0 > 0$ such that, a.s.,

$$\mathbb{E}[\|X_{m+1}\|^2 - \|X_m\|^2 \mid X_m] = \mathbb{E}[\|\Delta_m\|^2 \mid X_m] \le C_0, \text{ for all } m \in \mathbb{N}.$$

The inequality $\mathbb{E}_\mathbf{x}[\|X_{m+1}\|^2] \le \mathbb{E}_\mathbf{x}[\|X_m\|^2] + C_0$ follows, implying

$$\mathbb{E}_\mathbf{x}[\|X_m\|^2] \le \|\mathbf{x}\|^2 + C_0 m, \text{ for all } \mathbf{x} \in \mathbb{X} \text{ and all } m \in \mathbb{N}. \tag{2.3}$$

Similarly,

$$\begin{aligned}\|\mathbf{x}+\Delta_m\|^4 - \|\mathbf{x}\|^4 &= (\|\mathbf{x}\|^2 + 2\langle \mathbf{x},\Delta_m\rangle + \|\Delta_m\|^2)^2 - \|\mathbf{x}\|^4 \\ &\le 6\|\mathbf{x}\|^2\|\Delta_m\|^2 + \|\Delta_m\|^4 + 4\|\mathbf{x}\|^2\langle \mathbf{x},\Delta_m\rangle + 4\|\mathbf{x}\|\|\Delta_m\|^3.\end{aligned}$$

Then by (A0) and (A1) again, we get, for some $C_1 \in \mathbb{R}_+$, a.s.,

$$\mathbb{E}[\|X_{m+1}\|^4 - \|X_m\|^4 \mid X_m] \le C_1(1 + \|X_m\|^2)$$

for all $m \in \mathbb{N}$. Taking expectations and applying (2.3), we find

$$\mathbb{E}_\mathbf{x}[\|X_{m+1}\|^4] \le \mathbb{E}_\mathbf{x}[\|X_m\|^4] + C_2(1 + m + \|\mathbf{x}\|^2),$$

for some $C_2 \in \mathbb{R}_+$, which implies that, for some $C_3 \in \mathbb{R}_+$,

$$\begin{aligned}\mathbb{E}_{\mathbf{x}}[\|X_m\|^4] &= \mathbb{E}_{\mathbf{x}}[\|X_0\|^4] + \sum_{k=1}^{m-1} \left(\mathbb{E}_{\mathbf{x}}[\|X_{k+1}\|^4] - \mathbb{E}_{\mathbf{x}}[\|X_k\|^4]\right) \\ &\leq \|\mathbf{x}\|^4 + C_3(m^2 + m\|\mathbf{x}\|^2), \text{ for all } m \in \mathbb{N} \text{ and } \mathbf{x} \in \mathbb{X}.\end{aligned}$$

Since $m\|\mathbf{x}\|^2 \leq m^2 + \|\mathbf{x}\|^4$, the inequality in the lemma for $\ell = 4$ follows. The case $\ell = 2$ follows from (2.3). The remaining cases are a consequence of these bounds, the Lyapunov inequality $\mathbb{E}_{\mathbf{x}} \|X_m\| \leq \mathbb{E}_{\mathbf{x}}[\|X_m\|^2]^{1/2}$, the Cauchy–Schwarz inequality

$$\mathbb{E}_{\mathbf{x}}[\|X_m\|^3] \leq \mathbb{E}_{\mathbf{x}}[\|X_m\|^4]^{1/2}\, \mathbb{E}_{\mathbf{x}}[\|X_m\|^2]^{1/2},$$

and the fact that $(m^{\ell/2} + \|\mathbf{x}\|^{\ell})^{1/2} \leq 2^{1/2} \max(m, \|\mathbf{x}\|^2)^{\ell/4}$. □

Proof of Lemma 2.1. Recall that $\Delta = \Delta_0$. First we prove the statement for $\ell = 2$. Then

$$\mathbb{E}_{\mathbf{x}} \max_{0 \leq m \leq kn} \|\Delta_m\|^4 \leq \mathbb{E}_{\mathbf{x}} \sum_{m=0}^{kn} \|\Delta_m\|^4,$$

where, by the Markov property and (A0),

$$\mathbb{E}_{\mathbf{x}}[\|\Delta_m\|^4] = \mathbb{E}_{\mathbf{x}}\, \mathbb{E}[\|\Delta_m\|^4 \mid X_m] \leq \sup_{\mathbf{y} \in \mathbb{X}} \mathbb{E}_{\mathbf{y}}[\|\Delta\|^4] \leq C_1,$$

for some $C_1 < \infty$. It follows that

$$0 \leq \frac{1}{n^2}\, \mathbb{E}_{\mathbf{x}} \max_{0 \leq m \leq kn} \|\Delta_m\|^4 \leq \frac{C_1(kn+1)}{n^2} \to 0,$$

giving the $\ell = 2$ case of (2.1). Then Lyapunov's inequality shows that

$$\mathbb{E}_{\mathbf{x}} \max_{0 \leq m \leq kn} \|\Delta_m\|^2 \leq \left(\mathbb{E}_{\mathbf{x}} \max_{0 \leq m \leq kn} \|\Delta_m\|^4\right)^{1/2},$$

and the $\ell = 1$ case of (2.1) follows.

To prove (2.2), take $\gamma \in (0, 1/2)$ and observe that

$$\|\Delta_m\|^2 \leq n^{2\gamma} + \|\Delta_m\|^2 \mathbf{1}\{\|\Delta_m\| > n^\gamma\}, \text{ for all } m \in \{0, \ldots, kn\}. \tag{2.4}$$

Hence we have from (2.4) that

$$\max_{0 \leq m \leq kn} \|\Delta_m\|^2 \|X_m\|^2 \leq n^{2\gamma} \max_{0 \leq m \leq kn} \|X_m\|^2 + \sum_{m=0}^{kn} \|\Delta_m\|^2 \mathbf{1}\{\|\Delta_m\| > n^\gamma\} \|X_m\|^2. \tag{2.5}$$

To bound the first term on the right-hand side of (2.5), note that the Markov chain X is a martingale. Hence for any $X_0 = \mathbf{x} \in \mathbb{X}$, the non-negative process $\|X\|$ is a submartingale and Doob's L^2 inequality (see e.g. [7, p. 543]) yields

$$\mathbb{E}_{\mathbf{x}} \max_{0 \leq m \leq kn} \|X_m\|^2 \leq 4\, \mathbb{E}_{\mathbf{x}} \|X_{kn}\|^2. \tag{2.6}$$

For the second term on the right-hand side of (2.5), conditioning on X_m and using the Markov property gives

$$\begin{aligned}\mathbb{E}_{\mathbf{x}} \sum_{m=0}^{kn} \|\Delta_m\|^2 \mathbf{1}\{\|\Delta_m\| > n^\gamma\} \|X_m\|^2 &= \mathbb{E}_{\mathbf{x}} \sum_{m=0}^{kn} \|X_m\|^2\, \mathbb{E}\left[\|\Delta_m\|^2 \mathbf{1}\{\|\Delta_m\| > n^\gamma\} \;\middle|\; X_m\right] \\ &\leq \mathbb{E}_{\mathbf{x}} \sum_{m=0}^{kn} \|X_m\|^2 \sup_{\mathbf{y} \in \mathbb{X}} \mathbb{E}_{\mathbf{y}} \left[\|\Delta\|^2 \mathbf{1}\{\|\Delta\| > n^\gamma\}\right].\end{aligned}$$

Then by (A0) we have that

$$\begin{aligned}\mathbb{E}_{\mathbf{y}}\left[\|\Delta\|^2\mathbf{1}\{\|\Delta\|>n^\gamma\}\right] &= \mathbb{E}_{\mathbf{y}}\left[\|\Delta\|^4\|\Delta\|^{-2}\mathbf{1}\{\|\Delta\|>n^\gamma\}\right]\\ &\le n^{-2\gamma}\,\mathbb{E}_{\mathbf{y}}[\|\Delta\|^4]\\ &\le C_1 n^{-2\gamma},\end{aligned}$$

for $C_1 < \infty$ and all $\mathbf{y} \in \mathbb{X}$. It follows that

$$\mathbb{E}_{\mathbf{x}} \sum_{m=0}^{kn} \|\Delta_m\|^2 \mathbf{1}\{\|\Delta_m\|^2 > n^\gamma\} \|X_m\|^2 \le C_1 n^{-2\gamma}\, \mathbb{E}_{\mathbf{x}} \sum_{m=0}^{kn} \|X_m\|^2. \tag{2.7}$$

The bounds in (2.5), (2.6) and (2.7), together with the $\ell = 2$ case of Lemma 2.2, show that

$$\mathbb{E}_{\mathbf{x}} \max_{0\le m\le kn} \|\Delta_m\|^2\|X_m\|^2 \le 4D_0 n^{2\gamma}(kn+\|\mathbf{x}\|^2) + C_1 D_0 n^{-2\gamma}(kn+1)(kn+\|\mathbf{x}\|^2),$$

which in turn implies (2.2) since $\gamma \in (0, 1/2)$. □

An important ingredient is the following result from [5, Theorem 2.3].

Lemma 2.3. *Suppose that* (A0)–(A4) *hold. Then the random walk X is* null*, i.e., for any bounded $A \subset \mathbb{R}^d$,*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \mathbf{1}\{X_k \in A\} = 0, \text{ a.s. and in } L^q \text{ for any } q \ge 1. \tag{2.8}$$

The idea of the proof of Lemma 2.3 is as follows; see [5] for the details. The result is obvious if X is transient. If X is recurrent, then define $\tau_r := \min\{n \ge 0 : \|X_n\| \le r\}$, and take r sufficiently large so that $\mathbb{P}_{\mathbf{x}}[\tau_r < \infty] = 1$ for all $\mathbf{x} \in \mathbb{X}$. The key to Lemma 2.3 is to show that

$$\mathbb{P}_{\mathbf{x}}[\tau_r \ge n \mid X_0] \ge cn^{-1/2}, \text{ on } \{\|X_0\| > R\}, \tag{2.9}$$

for some constants $R > r$ and $c > 0$. Hence the excursions of X away from a bounded set have non-integrable durations. On the other hand, the assumptions (A0)–(A2) and a many-dimensional martingale version of Kolmogorov's 'other' inequality show that the expected number of times that a bounded set is visited by the walk between successive excursions is uniformly bounded. These two facts combine to give the null property. The property (2.9) formalizes the intuition that X is 'less recurrent' than one-dimensional simple symmetric random walk.

Lemma 2.4. *Suppose that* (A0)–(A4) *hold and let $k \in \mathbb{N}$. Then, for any linear functional ϕ on $d \times d$ matrices, i.e. $\phi : \mathbb{R}^{d\times d} \to \mathbb{R}$, and any $\mathbf{x} \in \mathbb{X}$, the following limits in probability under $\mathbb{P}_{\mathbf{x}}$ hold:*

$$\frac{1}{n}\sum_{m=0}^{kn} \left|\phi M(X_m) - \phi\sigma^2(\hat{X}_m)\right| \xrightarrow{\mathrm{P}} 0, \tag{2.10}$$

$$\frac{1}{n^2}\sum_{m=0}^{kn} \left|\langle[M(X_m) - \sigma^2(\hat{X}_m)]X_m, X_m\rangle\right| \xrightarrow{\mathrm{P}} 0. \tag{2.11}$$

Proof. Since ϕ is necessarily continuous, the corresponding operator norm $\|\phi\|_{\mathrm{op}}$ is finite, and so

$$\left|\phi M(\mathbf{x}) - \phi\sigma^2(\hat{\mathbf{x}})\right| \le \|\phi\|_{\mathrm{op}}\|M(\mathbf{x}) - \sigma^2(\hat{\mathbf{x}})\|_{\mathrm{op}}, \text{ for any } \mathbf{x} \in \mathbb{R}^d.$$

Hence, for any $\varepsilon > 0$, condition (A3) entails that there exists $C \in \mathbb{R}_+$ such that

$$\left|\phi M(X_m) - \phi\sigma^2(\hat{X}_m)\right| \le \varepsilon, \text{ a.s., on } \{\|X_m\| \ge C\}.$$

By (A0) and (A3) we have $B := \sup_{\mathbf{x}\in\mathbb{X}} \|M(\mathbf{x}) - \sigma^2(\hat{\mathbf{x}})\|_{\rm op} < \infty$, and hence

$$\begin{aligned}\frac{1}{n}\sum_{m=0}^{kn}\left|\phi M(X_m) - \phi\sigma^2(\hat{X}_m)\right| &\leq \frac{1}{n}\sum_{m=0}^{kn}\varepsilon + \frac{1}{n}\sum_{m=0}^{kn} B\|\phi\|_{\rm op}\mathbf{1}\{\|X_m\| \leq C\} \\ &\leq 2k\varepsilon + \frac{B\|\phi\|_{\rm op}}{n}\sum_{m=0}^{kn}\mathbf{1}\{\|X_m\| \leq C\}, \text{ for all } n \in \mathbb{N}. \quad (2.12)\end{aligned}$$

Now, by (2.8), for any $C < \infty$, as $n \to \infty$, $n^{-1}\sum_{m=0}^{kn}\mathbf{1}\{\|X_m\| \leq C\} \xrightarrow{\rm p} 0.$ Since $\varepsilon > 0$ was arbitrary, together with (2.12), this implies (2.10).

We now establish (2.11). First note that

$$\left|\langle[M(\mathbf{x}) - \sigma^2(\hat{\mathbf{x}})]\mathbf{x}, \mathbf{x}\rangle\right| \leq \|M(\mathbf{x}) - \sigma^2(\hat{\mathbf{x}})\|_{\rm op}\|\mathbf{x}\|^2, \text{ for any } \mathbf{x} \in \mathbb{X}.$$

Denote by Z_n the random variable in (2.11). By (A3), for any $\varepsilon > 0$ there exists a constant $C < \infty$ such that for all $n \in \mathbb{N}$ we have

$$\begin{aligned}Z_n &\leq \frac{B}{n^2}\sum_{m=0}^{kn}\|X_m\|^2\mathbf{1}\{\|X_m\| \leq C\} + \frac{\varepsilon}{n^2}\sum_{m=0}^{kn}\|X_m\|^2\mathbf{1}\{\|X_m\| > C\} \\ &\leq 2C^2Bkn^{-1} + Z_n', \text{ where } Z_n' := \frac{\varepsilon}{n^2}\sum_{m=0}^{kn}\|X_m\|^2, \quad (2.13)\end{aligned}$$

and B is defined above the display in (2.12). Fix $X_0 = \mathbf{x} \in \mathbb{X}$. Then by the $\ell = 2$ case of Lemma 2.2, there is a constant $D_1 < \infty$ (depending on k and $\|\mathbf{x}\|$) such that

$$\mathbb{E}_{\mathbf{x}}\frac{1}{n^2}\sum_{m=0}^{kn}\|X_m\|^2 \leq D_1, \text{ for all } n \in \mathbb{N}.$$

In order to prove $Z_n \xrightarrow{\rm p} 0$, pick arbitrary $\varepsilon' > 0$ and $\varepsilon'' > 0$, and set $\varepsilon := \varepsilon'\varepsilon''/(4D_1)$. Markov's inequality implies that

$$\mathbb{P}_{\mathbf{x}}[Z_n' > \varepsilon'/2] < \frac{2D_1}{\varepsilon'}\varepsilon < \varepsilon'', \text{ for all } n \in \mathbb{N}.$$

Given ε, we can find $C < \infty$ such that the inequality in (2.13) holds for all $n \in \mathbb{N}$. Then, for any $n \geq 4C^2Bk/\varepsilon'$, the following inequalities hold:

$$\mathbb{P}_{\mathbf{x}}[Z_n > \varepsilon'] \leq \mathbb{P}_{\mathbf{x}}[2C^2Bkn^{-1} + Z_n' > \varepsilon'] \leq \mathbb{P}_{\mathbf{x}}[Z_n' > \varepsilon'/2] < \varepsilon''.$$

Since ε'' is arbitrary, we have that $\lim_{n\to\infty}\mathbb{P}_{\mathbf{x}}[Z_n > \varepsilon'] = 0$ and (2.11) follows. □

Recall that $\widetilde{X}_n$ in (1.1) is a continuous-time process given in terms of the scaled Markov chain X, which is started at $X_0 = \mathbf{x} \in \mathbb{R}^d$. Let $Y_n := \|\widetilde{X}_n\|^2$ be the square of the radial component of $\widetilde{X}_n$. Since the square root is continuous, the mapping theorem [2, Sec. 2, Thm 2.7] implies that Theorem 1.1 follows if we prove that Y_n converges weakly to $\mathrm{BESQ}^V(0)$ on $\mathcal{D} := \mathcal{D}(\mathbb{R}_+;\mathbb{R})$, the space of right-continuous functions from $\mathbb{R}_+$ to $\mathbb{R}$ with left limits, equipped with the Skorokhod metric. This fact will be established using [4, Thm 7.4.1, p. 354], the relevant ($d = 1$) version of which we quote here for ease of reference. Let $\mathcal{C} := \mathcal{C}(\mathbb{R}_+;\mathbb{R})$ denote the space of continuous functions from $\mathbb{R}_+$ to $\mathbb{R}$.

Theorem 2.5. *Let $a : \mathbb{R} \to \mathbb{R}_+$ and $b : \mathbb{R} \to \mathbb{R}$ be continuous functions. Consider the generator G defined for smooth $f : \mathbb{R} \to \mathbb{R}$ by $Gf = bf' + \frac{1}{2}af''$, and suppose that the corresponding $\mathcal{C}$ martingale problem is well posed. For $n \in \mathbb{N}$ let Y_n, A_n, and B_n be processes with sample paths in $\mathcal{D}$ such that for each $n \in \mathbb{N}$, $t \mapsto A_n(t)$ is non-decreasing,* *and both $Y_n - B_n$ and $Y_n^2 - A_n$ are local martingales. Suppose that for each $T > 0$, the following conditions hold:*

$$\lim_{n\to\infty} \mathbb{E} \sup_{t\in[0,T]} |Y_n(t) - Y_n(t-)|^2 = 0, \tag{2.14}$$

$$\lim_{n\to\infty} \mathbb{E} \sup_{t\in[0,T]} |B_n(t) - B_n(t-)|^2 = 0, \tag{2.15}$$

$$\lim_{n\to\infty} \mathbb{E} \sup_{t\in[0,T]} |A_n(t) - A_n(t-)| = 0, \tag{2.16}$$

$$\sup_{t\in[0,T]} \left| B_n(t) - \int_0^t b(Y_n(s))\mathrm{d}s \right| \xrightarrow{\mathrm{p}} 0, \tag{2.17}$$

$$\text{and} \sup_{t\in[0,T]} \left| A_n(t) - \int_0^t a(Y_n(s))\mathrm{d}s \right| \xrightarrow{\mathrm{p}} 0. \tag{2.18}$$

Then if $Y_n(0)$ converges weakly to a law ν on $\mathbb{R}$, we have that Y_n converges weakly to the solution of the martingale problem for (G, ν).

To apply this result, we take $Y_n := \|\widetilde{X}_n\|^2$ as described above, and then choose B_n to be the predictable compensator of Y_n. Then $M_n := Y_n - B_n$ is the corresponding local martingale. Define A_n as the predictable compensator of the submartingale M_n^2. In particular, both A_n and B_n start at zero. The following proposition establishes the conditions necessary to apply Theorem 2.5 with $a(x) = 4|x|$ and $b(x) = V$.

Proposition 2.6. *Suppose that* (A0)–(A4) *hold, that $U = 1$, and that $X_0 = \mathbf{x} \in \mathbb{X}$. Let $T > 0$. Then with the choice of Y_n, A_n, and B_n as described above, the limits* (2.14)*,* (2.15)*, and* (2.16) *hold. Furthermore, under $\mathbb{P}_\mathbf{x}$, we have that*

$$\sup_{t\in[0,T]} |B_n(t) - Vt| \xrightarrow{\mathrm{p}} 0, \tag{2.19}$$

$$\sup_{t\in[0,T]} \left| A_n(t) - \int_0^t 4Y_n(s)\mathrm{d}s \right| \xrightarrow{\mathrm{p}} 0. \tag{2.20}$$

Proof. Without loss of generality we may assume that $T = 1$. By definition, B_n is a piece-wise constant right-continuous process started at zero with jumps at $t = k/n$, $k \in \{1, \dots, n\}$, given by

$$\begin{aligned} B_n(t) - B_n(t-) &= \frac{1}{n}\,\mathbb{E}[\|X_k\|^2 - \|X_{k-1}\|^2 \mid X_{k-1}] \\ &= \frac{2}{n}\,\mathbb{E}[\langle X_{k-1}, \Delta_{k-1}\rangle \mid X_{k-1}] + \frac{1}{n}\,\mathbb{E}[\|\Delta_{k-1}\|^2 \mid X_{k-1}] \\ &= \frac{1}{n}\,\mathbb{E}[\|\Delta_{k-1}\|^2 \mid X_{k-1}], \end{aligned} \tag{2.21}$$

using (A1), and writing $B_n(t-) = \lim_{s\uparrow t} B_n(s)$. By (A0), $\mathbb{E}[\|\Delta_{k-1}\|^2 \mid X_{k-1}]$ is uniformly bounded. Hence

$$\sup_{t\in[0,1]} |B_n(t) - B_n(t-)|^2 = \frac{1}{n^2} \max_{1\le k\le n} \left|\mathbb{E}[\|\Delta_{k-1}\|^2 | X_{k-1}]\right|^2$$

is a sequence of bounded random variables converging to zero point-wise. Therefore the limit in (2.15) follows.

Similarly, the jumps of Y_n occur at times $t = k/n$ (where $k \in \{1, \dots, n\}$) and, writing

$Y_n(t-) = \lim_{s\uparrow t} Y_n(s)$ as usual, can be bounded as follows:

$$\begin{aligned} |Y_n(t) - Y_n(t-)|^2 &= \frac{1}{n^2}(\|X_k\|^2 - \|X_{k-1}\|^2)^2 \\ &\le \frac{1}{n^2}(\|\Delta_{k-1}\|^2 + 2\|X_{k-1}\|\|\Delta_{k-1}\|)^2 \\ &\le \frac{2}{n^2}(\|\Delta_{k-1}\|^4 + 4\|X_{k-1}\|^2\|\Delta_{k-1}\|^2), \end{aligned} \tag{2.22}$$

using the inequality $(x+y)^2 \le 2(x^2+y^2)$. We therefore find that

$$\mathbb{E}_{\mathbf{x}} \sup_{t\in[0,1]} |Y_n(t) - Y_n(t-)|^2 \le \frac{2}{n^2}(\mathbb{E}_{\mathbf{x}} \max_{1\le k\le n} \|\Delta_{k-1}\|^4 + 4\,\mathbb{E}_{\mathbf{x}} \max_{1\le k\le n} \|X_{k-1}\|^2\|\Delta_{k-1}\|^2).$$

Hence (2.1)–(2.2) in Lemma 2.1 imply (2.14).

The process A_n is piece-wise constant and right-continuous with jumps $A_n(t) - A_n(t-)$ at $t = k/n$, $k \in \{1, \dots, n\}$, with $A_n(t-) = \lim_{s\uparrow t} A_n(s)$, satisfying

$$\begin{aligned} A_n(t) - A_n(t-) &= \mathbb{E}[M_n(t)^2 - M_n(t-)^2 \mid \mathcal{F}_{k-1}] \\ &= \mathbb{E}[(M_n(t) - M_n(t-))^2 \mid \mathcal{F}_{k-1}] \\ &= \mathbb{E}[(Y_n(t) - Y_n(t-))^2 \mid \mathcal{F}_{k-1}] - (B_n(t) - B_n(t-))^2, \end{aligned} \tag{2.23}$$

using the fact that $B_n(t) - B_n(t-) = \mathbb{E}[Y_n(t) - Y_n(t-) \mid \mathcal{F}_{k-1}]$, where $\mathcal{F}_{k-1}$ is the σ-algebra generated by $X_0, X_1, \dots, X_{k-1}$. Hence by (2.23) with (2.21) and (2.22), we find that

$$\begin{aligned} |A_n(t) - A_n(t-)| &\le \mathbb{E}[(Y_n(t) - Y_n(t-))^2 \mid X_{k-1}] + (B_n(t) - B_n(t-))^2 \\ &\le \frac{2}{n^2}\left(\mathbb{E}[\|\Delta_{k-1}\|^4 \mid X_{k-1}] + 4\|X_{k-1}\|^2\,\mathbb{E}[\|\Delta_{k-1}\|^2 \mid X_{k-1}]\right) \\ &\quad + \frac{1}{n^2}\,\mathbb{E}[\|\Delta_{k-1}\|^2 \mid X_{k-1}]^2, \end{aligned}$$

for $t = k/n$, $k \in \{1, \dots, n\}$. By (A0) we have that there exists a constant $C_1 < \infty$ such that both $\mathbb{E}[\|\Delta_{k-1}\|^2 \mid X_{k-1}]$ and $\mathbb{E}[\|\Delta_{k-1}\|^4 \mid X_{k-1}]$ are bounded by C_1, a.s., so that

$$\sup_{t\in[0,1]} |A_n(t) - A_n(t-)| \le \frac{2C_1 + C_1^2}{n^2} + \frac{8C_1}{n^2} \max_{1\le k\le n} \|X_{k-1}\|^2.$$

By Doob's L^2 submartingale inequality we have $\mathbb{E}_{\mathbf{x}}[\max_{1\le k\le n} \|X_{k-1}\|^2] \le 4\,\mathbb{E}_{\mathbf{x}}\|X_n\|^2$, and then (2.16) follows from the $\ell = 2$ case of Lemma 2.2.

We now prove the limit in (2.19). Note that (2.21) and the fact that $\operatorname{tr} M(\mathbf{x}) = \mathbb{E}_{\mathbf{x}}[\|\Delta\|^2]$ implies that, with the usual convention that an empty sum is zero,

$$B_n(t) = \frac{1}{n} \sum_{m=0}^{\lfloor nt\rfloor - 1} \operatorname{tr} M(X_m). \tag{2.24}$$

By (A4) it holds that $\operatorname{tr} \sigma^2(\mathbf{u}) = V$ for all $\mathbf{u} \in \mathbb{S}^{d-1}$. Hence by (2.24) we find

$$|B_n(t) - Vt| \le \frac{V}{n} + \frac{1}{n} \sum_{m=0}^{\lfloor nt\rfloor - 1} |\operatorname{tr} M(X_m) - \operatorname{tr} \sigma^2(\hat{X}_m)|, \text{ for all } t \in [0,1],$$

and, as trace is a linear functional on square matrices, (2.10) in Lemma 2.4 yields

$$\sup_{t\in[0,1]} |B_n(t) - Vt| \le \frac{V}{n} + \frac{1}{n} \sum_{m=0}^{n} |\operatorname{tr} M(X_m) - \operatorname{tr} \sigma^2(\hat{X}_m)| \xrightarrow{\text{p}} 0.$$

Finally, we establish (2.20). From (2.23) with (2.21) and the equality in (2.22), we find that

$$\begin{aligned} A_n(t) - A_n(t-) &= \frac{1}{n^2}\,\mathbb{E}[(2\langle X_{k-1}, \Delta_{k-1}\rangle + \|\Delta_{k-1}\|^2)^2 \mid X_{k-1}] - \frac{1}{n^2}\,\mathbb{E}[\|\Delta_{k-1}\|^2 \mid X_{k-1}]^2 \\ &= \frac{4}{n^2}\,\mathbb{E}[\langle X_{k-1}, \Delta_{k-1}\rangle^2 \mid X_{k-1}] + \frac{4}{n^2}\,\mathbb{E}[\langle X_{k-1}, \Delta_{k-1}\rangle \|\Delta_{k-1}\|^2 \mid X_{k-1}] \\ &\quad + \frac{1}{n^2}\,\mathbb{E}[\|\Delta_{k-1}\|^4 \mid X_{k-1}] - \frac{1}{n^2}\,\mathbb{E}[\|\Delta_{k-1}\|^2 \mid X_{k-1}]^2. \end{aligned}$$

For any $t \in [0,1]$, denote

$$D_n(t) := \frac{1}{n^2} \sum_{k=0}^{\lfloor nt \rfloor - 1} 4\,\mathbb{E}[\langle X_k, \Delta_k\rangle^2 \mid X_k].$$

It follows that

$$A_n(t) - D_n(t) = \frac{1}{n^2} \sum_{k=0}^{\lfloor nt \rfloor - 1} \left(4\,\mathbb{E}[\langle X_k, \Delta_k\rangle \|\Delta_k\|^2 \mid X_k] + \mathbb{E}[\|\Delta_k\|^4 \mid X_k] - \mathbb{E}[\|\Delta_k\|^2 \mid X_k]^2\right).$$

By (A0), there exists a constant $C_2 < \infty$ bounding uniformly all $\mathbb{E}[\|\Delta_k\|^\ell \mid X_k]$ for $2 \le \ell \le 4$ and all $k \in \{0, \ldots, n\}$. Furthermore, it holds that

$$\frac{1}{n^2} \left| \sum_{k=0}^{n} \mathbb{E}[\langle X_k, \Delta_k\rangle \|\Delta_k\|^2 \mid X_k] \right| \le \frac{1}{n^2} \sum_{k=0}^{n} \|X_k\|\,\mathbb{E}[\|\Delta_k\|^3 \mid X_k].$$

Hence, by the $\ell = 1$ case of Lemma 2.2, we find that

$$\sup_{t \in [0,1]} |A_n(t) - D_n(t)| \le \frac{C_2 + C_2^2}{n} + \frac{4C_2}{n^2} \sum_{k=0}^{n} \|X_k\| \xrightarrow{\text{p}} 0.$$

It remains to show $\sup_{t\in[0,1]} \left| D_n(t) - \int_0^t 4Y_n(s)\mathrm{d}s \right| \xrightarrow{\text{p}} 0$. With this in mind, note that the following identities hold for all $k \in \{0, \ldots, n\}$:

$$\mathbb{E}[\langle X_k, \Delta_k\rangle^2 \mid X_k] = \langle M(X_k)X_k, X_k\rangle, \text{ and } \|X_k\|^2 = \langle \sigma^2(\hat{X}_k)X_k, X_k\rangle;$$

the latter is a consequence of (A4), which states that $\langle \sigma^2(\hat{X}_k)\hat{X}_k, \hat{X}_k\rangle = U$, and the assumption that $U = 1$. Since $Y_n(t) = n^{-1}\|X_{\lfloor nt \rfloor}\|^2$, we have that

$$\begin{aligned} \int_0^t Y_n(s)\mathrm{d}s &= \sum_{k=0}^{\lfloor nt \rfloor - 1} \int_{\frac{k}{n}}^{\frac{k+1}{n}} Y_n(s)\mathrm{d}s + \int_{n^{-1}\lfloor nt \rfloor}^{t} Y_n(s)\mathrm{d}s \\ &= \frac{1}{n^2} \sum_{k=0}^{\lfloor nt \rfloor - 1} \|X_k\|^2 + \frac{nt - \lfloor nt \rfloor}{n^2} \|X_{\lfloor nt \rfloor}\|^2. \end{aligned}$$

Hence for any $t \in [0,1]$ it holds that

$$\begin{aligned} \left| D_n(t) - \int_0^t 4Y_n(s)\mathrm{d}s \right| &\le \frac{4}{n^2}\|X_{\lfloor nt \rfloor}\|^2 + \frac{4}{n^2} \sum_{k=0}^{\lfloor nt \rfloor - 1} \left| \langle [M(X_k) - \sigma^2(\hat{X}_k)]X_k, X_k\rangle \right| \\ &\le \frac{4}{n^2} \max_{0 \le k \le n} \|X_k\|^2 + \frac{4}{n^2} \sum_{k=0}^{n} \left| \langle [M(X_k) - \sigma^2(\hat{X}_k)]X_k, X_k\rangle \right|. \quad (2.25) \end{aligned}$$

Doob's L^2 submartingale inequality and the $\ell = 2$ case of Lemma 2.2 imply that the first term on the right-hand side of (2.25) converges to zero in L^1 and hence in probability. The second term converges to zero in probability by (2.11) in Lemma 2.4. □

Proof of Theorem 1.1. As noted in the discussion following the theorem, it is sufficient to prove that $Y_n \Rightarrow Y$, where Y is $\mathrm{BESQ}^V(0)$. Let $g : \mathbb{R} \to \mathbb{R}_+$ be given by $g(x) := \sqrt{|x|}$ and note that Y satisfies the SDE $\mathrm{d}Y_t = V\mathrm{d}t + 2g(Y_t)\mathrm{d}B_t$, where $Y_0 = 0$. It is easy to see that $|g(x) - g(y)|^2 \le |x - y|$ for all $x, y \in \mathbb{R}$. Hence pathwise uniqueness for this SDE holds for any starting point $Y_0 = x_0 \in \mathbb{R}$ by [11, Ch. IX, Thm (3.5)(ii)] (use $\rho : (0,\infty) \to (0,\infty)$, given by $\rho(z) = 4z$). Hence, by the Yamada–Watanabe theorem [11, Ch. IX, Thm (1.7)], the uniqueness in law holds. Thus the $\mathcal{C}$ martingale problem for (H, δ_0) is well-posed, where $Hf := Vf' + 2g^2 f''$ for any smooth $f : \mathbb{R} \to \mathbb{R}$ and δ_0 is the Dirac delta measure on $\mathbb{R}$ concentrated at zero. Furthermore, any solution of this $\mathcal{C}$ martingale problem has non-negative trajectories because of the support of the law of $\mathrm{BESQ}^V(0)$ (alternatively the positivity of the paths follows from the comparison theorem [11, Ch. IX, Thm (3.7)] and the fact that $\mathrm{BESQ}^0(0)$ is equal to zero at all times). Proposition 2.6, and the fact that the drift in H is constant and g^2 is continuous and non-negative on $\mathbb{R}$, shows that the conditions in Theorem 2.5 are satisfied with $G = H$, $a(x) = 4|x|$, and $b(x) = V$, and so Theorem 2.5 implies that Y_n converges weakly to the unique solution Y of the $\mathcal{C}$ martingale problem for (H, δ_0). This proves Theorem 1.1. □

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