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The maximum deviation of the Sine_β counting process

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Abstract

In this paper, we consider the maximum of the Sine_β counting process from its expectation. We show the leading order behavior is consistent with the predictions of log–correlated Gaussian fields, also consistent with work on the imaginary part of the log–characteristic polynomial of random matrices. We do this by a direct analysis of the stochastic sine equation, which gives a description of the continuum limit of the Prüfer phases of a Gaussian β–ensemble matrix.

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1 Introduction

The Sine_β point process ([16]), which arises as the local point process limit of the eigenvalues of β–ensembles, can be defined in terms of the SDE

$$d\alpha_{x,t} = x\frac{\beta}{4}e^{-\frac{\beta}{4}t}dt + \text{Re}\left[\left(e^{-i\alpha_{x,t}} - 1\right)dZ_t\right], \qquad \alpha_{x,0} = 0, \tag{1.1}$$

where Z is a complex Brownian motion normalized so that $[Z_t, \bar{Z}_t] = 2t$ for all $t \geq 0$. Specifically, sending $t \to \infty$, $\alpha_{x,t}/(2\pi)$ converges for all x to an integer valued limit, which is the counting function of the Sine_β point process.

We are interested in the question of whether this function is an example of a process that should satisfy log–correlated field predictions. For an overview on work related to log–correlated Gaussian and approximately Gaussian processes see [1, 19]. This question follows naturally from the fact that the counting function of Sine_β is a scaling limit of the imaginary part of the logarithm of the characteristic polynomial of random matrices. Such Gaussian log–correlated field predictions have been proven for a variety of matrix models [2, 13, 5, 10]. Similar work has been done for randomized models of the Riemann ζ function [4], and also for the ζ function itself [3, 11]. For further discussion of the connections between the ζ function and random matrix theory see [8].

We consider the process $N(x) = \lim_{t\to\infty} \frac{\alpha_{x,t} - \alpha_{-x,t}}{2\pi}$, which counts the number of points in the Sine_β point process between $[-x, x]$ for any $x > 0$. This process exhibits a purer analogy with log–correlated fields (see Remark 1.5 for details). We show that:

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Theorem 1.1.

$$\frac{\max_{0\leq\lambda\leq x}[N(\lambda)-\frac{\lambda}{\pi}]}{\log x}\xrightarrow[x\to\infty]{\Pr}\frac{2}{\sqrt{\beta}\pi}.$$

Moreover, we do this by a direct argument for the Sine$_\beta$ process. Another possible approach might be to use the recent [17], which gives a coupling between the Sine$_\beta$ and CβE point processes, to transfer estimates from the random matrix process to the continuum limit.

Observe that as the process $N(\lambda)$ is almost surely non–decreasing, we may immediately replace this maximum over all $0\leq\lambda\leq x$ by the maximum over any discrete net of $[0,x]$ with maximum spacing $o(\log x)$. Likewise, we may assume that x is an integer. Going forward, we will take λ and x to be integers. The monotonicity of $N(\lambda)$ may be seen from the SDE description by observing that the noise term vanishes at multiples of 2π and the drift is positive for $\lambda>0$ and negative for $\lambda<0$ ([16, Proposition 9(ii)]).

It should be noted there is another SDE description due to [9] (only recently proven to give rise to the same process by [12], while another proof follows from [18]), which can be related to (1.1) by a time–reversal. This arises due to an order reversal of the Prüfer phases, for which reason the correlation structure is reversed from the previously studied CβE model. The processes $\alpha_{x,t}$ and $\alpha_{y,t}$ are strongly correlated for large times and weakly correlated for small times. We elaborate upon the correlation structure in (1.6).

Heuristic

We will name the martingale part of $\alpha_{\lambda,t}-\alpha_{-\lambda,t}$ diffusion:

$$M_{\lambda,t}=\text{Re}\int_0^t(e^{-i\alpha_{\lambda,s}}-e^{-i\alpha_{-\lambda,s}})dZ_s. \tag{1.2}$$

As the process $\alpha_{x,t}$ converges for all $x\in\mathbb{R}$ when $t\to\infty$, so does $M_{\lambda,t}$ converge for all $\lambda\in\mathbb{R}$ when $t\to\infty$. Moreover,

$$2\pi N(\lambda)-2\lambda=\text{Re}\int_0^\infty(e^{-i\alpha_{\lambda,s}}-e^{-i\alpha_{-\lambda,s}})dZ_s=M_{\lambda,\infty}.$$

Therefore we can reformulate Theorem 1.1 as

$$\frac{\max_{0\leq\lambda\leq x}M_{\lambda,\infty}}{\log x}\xrightarrow[x\to\infty]{\Pr}\frac{4}{\sqrt{\beta}}. \tag{1.3}$$

Let $T_\lambda=\frac{4}{\beta}\log\lambda$. This is heuristically the length of time that $M_{\lambda,t}$ needs to evolve so that it is within bounded distance of its limit. Specifically, the variables $M_{\lambda,\infty}-M_{\lambda,T_\lambda}$ have a uniform–in–λ exponential tail bound:

Proposition 1.2. *There is a constant $C=C_\beta$ so that for all $\lambda,r\geq 0$,*

$$\mathbb{P}\left[M_{\lambda,\infty}-M_{\lambda,T_\lambda}\geq C+r\right]\leq e^{-r/C}.$$

Using the monotonoicity of $N(\lambda)$, we can also show that:

Proposition 1.3.

$$\frac{\max_{0\leq\lambda\leq x}|M_{\lambda,\infty}-M_{\lambda,T_\lambda}|}{\log x}\xrightarrow[x\to\infty]{\Pr}0.$$

Hence we need only consider the process $M_{\lambda,t}$ up to time $t = T_\lambda$. We delay the proofs of these propositions to Section 2.

Another representation for $M_{\lambda,t}$ is given by, for all $t \geq 0$

$$\begin{aligned} M_{\lambda,t} &= \text{Re} \int_0^t (e^{-\frac{i}{2}(\alpha_{\lambda,s}-\alpha_{-\lambda,s})} - e^{-\frac{i}{2}(\alpha_{-\lambda,s}-\alpha_{\lambda,s})})e^{-\frac{i}{2}(\alpha_{\lambda,s}+\alpha_{-\lambda,s})}dZ_s \\ &= \text{Re} \int_0^t (e^{-\frac{i}{2}(\alpha_{\lambda,s}-\alpha_{-\lambda,s})} - e^{-\frac{i}{2}(\alpha_{-\lambda,s}-\alpha_{\lambda,s})})(dV_s^{(\lambda)} + idW_s^{(\lambda)}) \\ &= \int_0^t 2\sin\left(\tfrac{\alpha_{\lambda,s}-\alpha_{-\lambda,s}}{2}\right) dW_s^{(\lambda)}. \end{aligned} \tag{1.4}$$

where $dV_s^{(\lambda)} + idW_s^{(\lambda)} = e^{-\frac{i}{2}(\alpha_{\lambda,s}+\alpha_{-\lambda,s})}dZ_s$ is a standard complex Brownian motion.

Hence, the bracket process is given by

$$[M_\lambda]_t = \int_0^t 4\sin\left(\tfrac{\alpha_{\lambda,s}-\alpha_{-\lambda,s}}{2}\right)^2 ds.$$

Applying the trig identity $2\sin(x)^2 = 1-\cos(2x)$, and treating the oscillating the term as negligible, we can consider $[M_\lambda]_t \approx 2t$, for $t \leq T_\lambda$. This allows us to roughly consider M_{λ,T_λ}, for the purpose of moderate deviations, as a centered Gaussian of variance $2T_\lambda$.

As for the correlation structure,

$$[M_\lambda, M_\mu]_t = \text{Re} \int_0^t (e^{-i\alpha_{\lambda,s}} - e^{-i\alpha_{-\lambda,s}})(e^{i\alpha_{\mu,s}} - e^{i\alpha_{-\mu,s}})\, ds \tag{1.5}$$

Approximating $\alpha_{\lambda,t}$ by its drift in the equation above, we are led to the heuristic that M_λ and M_μ behave approximately independently for $t \leq \frac{4}{\beta}\log_+|\lambda-\mu|$ and are maximally correlated for larger t. This leads to the cross variation heuristic:

$$[M_\lambda, M_\mu]_{T_\lambda \wedge T_\mu} \approx 2(T_\lambda \wedge T_\mu - \tfrac{4}{\beta}\log_+|\lambda-\mu|). \tag{1.6}$$

We can define a Gaussian process that has the exact correlation structure suggested by the heuristics in (1.6):

$$G_{\lambda,t} = \text{Re} \int_0^t (e^{-i\mathbb{E}\alpha_{\lambda,s}} - e^{-i\mathbb{E}\alpha_{-\lambda,s}})dZ_s. \tag{1.7}$$

For this process, we have correlation given by

$$[G_\lambda, G_\mu]_t = 4\int_0^t \sin\left(\lambda(1-e^{-\frac{\beta}{4}s})\right)\sin\left(\mu(1-e^{-\frac{\beta}{4}s})\right) ds.$$

On the supposition that the maximum of $(M_{\lambda,\infty}, 0 \leq \lambda \leq x)$ and the maximum of $(G_{\lambda,T_\lambda}, 0 \leq \lambda \leq x)$ agree up to order 1 corrections, we are led to the following conjecture.

Conjecture 1.4. *There is a random variable ξ so that*

$$\max_{0\leq\lambda\leq x}(M_{\lambda,\infty}) - \tfrac{4}{\sqrt{\beta}}\left(\log x - \tfrac{3}{4}\log\log x\right) \xrightarrow[x\to\infty]{(d)} \xi.$$

Indeed by a theorem of [6], full convergence could be proven for the G_{λ,T_λ} field. One should expect that the distribution of ξ is sensitive to the model and so should be different than in the Gaussian case.

Remark 1.5. If we instead considered the one–sided problem, we would instead see

$$\frac{\max_{0\leq\lambda\leq x}[\alpha_{\lambda,\infty}-\lambda]}{\log x} \xrightarrow[x\to\infty]{\Pr} \frac{4}{\sqrt{2\beta}}.$$

We would be led to considering the martingale

$$V_{\lambda,t} = \mathrm{Re}\int_0^t (e^{-i\alpha_{\lambda,s}} - 1)dZ_s.$$

which has quadratic variation $[V_\lambda]_t \approx 2t$ for $t < T_\lambda$ and cross variation:

$$[V_\lambda, V_\mu]_{T_\lambda \wedge T_\mu} = \mathrm{Re}\int_0^t (e^{-i\alpha_{\lambda,s}} - 1)(e^{i\alpha_{\mu,s}} - 1)\, ds \approx T_\lambda \wedge T_\mu + \frac{1}{2}[M_\lambda, M_\mu]_{T_\lambda \wedge T_\mu}. \quad (1.8)$$

Thus, the process has an additional positive correlation, which is heuristically equivalent to adding a common standard normal of variance $\frac{4}{\beta}\log x$ to every $V_{\lambda,\infty}$ for $\delta x \le \lambda \le x$. In particular this is too small to change the behavior of the maximum. As working with $V_{\lambda,t}$ does not materially change the argument, we have not pursued it here.

2 Background tools

We begin with the proofs of Propositions 1.2 and 1.3. These rely heavily on basic properties of the diffusion established in [16, Proposition 9].

Delayed proofs from introduction

Proof of Proposition 1.2. Observe first by integrating the drift

$$M_{\lambda,\infty} - M_{\lambda,T_\lambda} = \alpha_{\lambda,\infty} - \alpha_{\lambda,T_\lambda} - 1. \quad (2.1)$$

Consider the process v that satisies

$$dv_t = \lambda\frac{\beta}{4}e^{-\frac{\beta}{4}t}\mathbf{1}\left\{t \le T_\lambda\right\}dt + \mathrm{Re}\left[\left(e^{-iv_t} - 1\right)dZ_t\right], \qquad v_0 = 0.$$

Then $\alpha_{\lambda,t}$ and v_t are equal until T_λ. After this time, v never crosses another multiple of 2π. Moreover, it eventually converges to a multiple of 2π ([16, Proposition 9(iv)]). Hence we have

$$|v_\infty - \alpha_{\lambda,T_\lambda}| \le 2\pi. \quad (2.2)$$

On the other hand $\alpha_{\lambda,\infty} - v_\infty$ has the same law as $\alpha_{1,\infty}$. By [16, Proposition 9(viii)], this has an exponential tail bound. □

Proof of Proposition 1.3. By (2.1), it suffices to show the same for $\alpha_{\lambda,\infty} - \alpha_{\lambda,T_\lambda}$. The diffusion $\alpha_{\lambda,t}$ can not cross below an integer multiple of 2π. Hence if $s \le t$, for all $\lambda \ge 0$ $\alpha_{\lambda,s} \le \alpha_{\lambda,t} + 2\pi$. This implies

$$\min_{0<\lambda\le x}(\alpha_{\lambda,\infty} - \alpha_{\lambda,T_\lambda}) \ge -2\pi,$$

and it suffices to consider an upper bound. For $x/2 \le \lambda \le x$, we can estimate

$$\alpha_{\lambda,\infty} - \alpha_{\lambda,T_\lambda} \le \alpha_{\lambda,\infty} - \alpha_{\lambda,T_{x/2}} + 2\pi$$

Let v_λ satisfy

$$dv_{\lambda,t} = \lambda\frac{\beta}{4}e^{-\frac{\beta}{4}t}\mathbf{1}\left\{t \le T_{x/2}\right\}dt + \mathrm{Re}\left[\left(e^{-iv_{\lambda,t}} - 1\right)dZ_t\right], \qquad v_{\lambda,0} = 0.$$

As v_λ can not cross multiples of 2π, for any $\lambda \in \mathbb{R}$, after $T_{x/2}$, we have

$$\alpha_{\lambda,\infty} - \alpha_{\lambda,T_{x/2}} + 2\pi \le \alpha_{\lambda,\infty} - v_{\lambda,\infty} + 4\pi.$$

On the other hand $\alpha_{\lambda,t} - v_{\lambda,t}$ is monotone increasing in λ almost surely (as the difference for parameters $\lambda_1 > \lambda_2$ satisfies an SDE that can not cross below 0, c.f. [16, Proposition 9(ii)]). Combining the work so far, we have the bound

$$\max_{x/2\le\lambda\le x} (\alpha_{\lambda,\infty} - \alpha_{\lambda,T_\lambda}) \le \alpha_{x,\infty} - v_{x,\infty} + 4\pi.$$

Using the equality in law given by

$$\left(\alpha_{x,t+T_{x/2}} - v_{x,t+T_{x/2}}, t \ge 0\right) \stackrel{\mathscr{L}}{=} (\alpha_{2,t}, t \ge 0)\,,$$

and by [16, Proposition 9(viii)], $\alpha_{2,\infty}$ has an exponential tail bound depending only on β. Applying the same argument for $j \in \mathbb{N}$ and $x2^{-j-1} \le \lambda \le x2^{-j}$, we may use a union bound up to j on the order of $\log x$ to conclude that there is a constant C_β so that

$$\max_{0<\lambda\le x} (\alpha_{\lambda,\infty} - \alpha_{\lambda,T_\lambda}) \le C_\beta \log\log x \tag{2.3}$$

with probability going to 1 as $x \to \infty$. □

Oscillatory integrals

For each $\lambda \in \mathbb{R}$, suppose that $A_{\lambda,t}$ is an adapted finite variation process so that $|A_{\lambda,t}| \le \xi \in (0,\infty)$ for all time almost surely and suppose that $X_{\lambda,t}$ is a martingale satisfying $d[X_\lambda]_t \le 2$. Suppose that

$$du_{\lambda,t} = \lambda \tfrac{\beta}{4} e^{-\frac{\beta}{4}t} dt + A_{\lambda,t}dt + dX_{\lambda,t}, \qquad u_{\lambda,0} = 0. \tag{2.4}$$

Proposition 2.1. *Let $u_{\lambda,t}$ satisfy* (2.4) *and let $\mathfrak{f}(t) = \frac{\beta}{4}e^{-\frac{\beta}{4}t}$, then for each fixed $\beta > 0$ there exist constants R and γ uniform in T and $\lambda, a \in \mathbb{R}$ such that*

$$\mathbb{E}\left[\sup_{0\le t\le T}\left|\int_0^t e^{iau_{\lambda,s}}ds\right|\right] \le \frac{R(1+|\xi|)}{|a\lambda|\mathfrak{f}(T)}, \tag{2.5}$$

and for all $C > 0$

$$\mathbb{P}\left(\sup_{0\le t\le T}\left|\int_0^t e^{iau_{\lambda,s}}ds\right| - \frac{R(1+|\xi|)}{|a\lambda|\mathfrak{f}(T)} \ge C\right) \le \exp\left[-\gamma C^2 a^2 \lambda^2 \mathfrak{f}(T)^2\right]. \tag{2.6}$$

Proof. The theorem is vacuous if $a\lambda = 0$, so we may assume this is not the case. Writing u_t in its integrated form, we have

$$u_t = \lambda\left(1 - \frac{4}{\beta}\mathfrak{f}(t)\right) + \mathcal{R}_t, \qquad \text{where} \qquad \mathcal{R}_t = \int_0^t \{A_{\lambda,s}ds + dX_{\lambda,s}\}\,.$$

Let $H(t) = 1 - \frac{4}{\beta}\mathfrak{f}(t)$ and $\Lambda(t) = \int_0^t e^{ia\lambda H(s)}ds$, then we may use Itô integration by parts to get

$$\int_0^t e^{ia\lambda u_s}ds = \int_0^t e^{ia\lambda H(s)}e^{ia\mathcal{R}_s}ds = e^{ia\mathcal{R}_t}\Lambda(t) + \int_0^t \Lambda(s)e^{ia\mathcal{R}_s}\cdot\left\{-ia\,d\mathcal{R}_s + \frac{a^2}{2}\,d[\mathcal{R}]_s\right\}. \tag{2.7}$$

Now observe that $\Lambda(t)$ may be bounded in the following way:

$$\begin{aligned}\int_0^t e^{ia\lambda H(s)}ds &= \int_0^t \frac{1}{ia\lambda\mathfrak{f}(s)}\frac{d}{ds}e^{ia\lambda H(s)}ds \\ &= \frac{4e^{\frac{\beta}{4}t}}{\beta ia\lambda}\left\{e^{ia\lambda H(t)} - 1\right\} - \frac{1}{ia\lambda}\int_0^t e^{\frac{\beta}{4}s}\left\{e^{ia\lambda H(s)} - 1\right\}ds.\end{aligned}$$

This gives us $|\Lambda(s)| \le \frac{16}{\beta|a\lambda|}e^{\frac{\beta}{4}t}$. Applying this to our integrated equation we get for the finite variation terms

$$\left|\int_0^t \Lambda(s)e^{i\mathcal{R}_s}aA_{\lambda,s}ds + \frac{a^2}{2}\int_0^t \Lambda(s)e^{ia\mathcal{R}_s}d[\mathcal{R}]_s\right| \le \frac{16}{\beta a\lambda}e^{\frac{\beta}{4}t}\left(|a|\xi + a^2\right).$$

By (2.7) and the triangle inequality, it remains to show the desired tail bound and supremum bound for the martingale V_t given by

$$V_t = \int_0^t \Lambda(s)iae^{ia\mathcal{R}_s}\cdot dX_{\lambda,s}$$

Note we have an easy bracket bound, for $\sigma \in \{1,i\}$ given by

$$[\Re(\sigma V)]_t \le \int_0^t 2\Lambda(s)a^2\,ds \le \frac{C_\beta}{\lambda^2}|a|e^{\frac{\beta}{2}t}$$

for some constant C_β. Hence the desired bounds follow immediately from the Dambis–Dubins–Schwarz theorem ([15, Theorem V.1.6] or [14, Theorem II.42]) and Doob's inequality. □

Tilting

We now want to look at the measure tilted so that $W^{(\lambda)}$ (see (1.4)) has a drift. In particular for deterministic $\eta \in \mathbb{R}$, we consider the measure $Q_{\eta,\lambda}$ so that

$$dX_s = dW_s^{(\lambda)} - \eta\sin\left(\tfrac{\alpha_{\lambda,s}-\alpha_{-\lambda,s}}{2}\right)ds$$

is a standard Brownian motion up to time T under $Q_{\eta,\lambda}$. By Girsanov (see e.g. [14, Theorem III.8.46]) we get that

$$\frac{dQ_{\eta,\lambda}}{d\mathbb{P}} = \mathcal{E}(\eta M_\lambda) = \exp(\eta M_{\lambda,T} - \tfrac{\eta^2}{2}[M_\lambda]_T) \tag{2.8}$$

Since $\sin^2(x) \le 1$ we have that the bracket process of $[M_\lambda]_t \le T$ almost surely for all $t \ge 0$. In particular, the exponential martingale is uniformly integrable by Novikov's condition for all $\eta \in \mathbb{R}$.

Under $Q_{\eta,\lambda}$ the law of $\alpha_{\lambda,t} - \alpha_{-\lambda,t}$ changes; it can be succinctly described as the solution to

$$du_{\lambda,\eta,t} = 2\lambda\tfrac{\beta}{4}e^{-\frac{\beta}{4}t}dt + 2\eta\sin\left(\tfrac{u_{\lambda,\eta,t}}{2}\right)^2dt + 2\sin\left(\tfrac{u_{\lambda,\eta,t}}{2}\right)dX_t, \qquad u_0 = 0 \tag{2.9}$$

for a Brownian motion dX, which we call the *accelerated stochastic sine equation* with acceleration η. Let $M_{\lambda,\eta,t}$ be the martingale part of $u_{\lambda,\eta,t}$.

Martingale bounds

Using the Girsanov transformation, we now give a nearly sharp tail bound for M_λ.

Proposition 2.2. *For any $\eta \in \mathbb{R}$, there is an $R > 0$ so that for all $\lambda > 0$, all $T \le T_\lambda$*

$$\mathbb{P}\left(\sup_{0\le t\le T} M_{\lambda,\eta,t} \ge C\right) \le \exp\left[\frac{-C^2}{4(T+R)}\left(1 - \frac{C^2R}{2(T+R)^3}\right) \wedge \frac{-C^{4/3}}{4T^{1/3}}\right].$$

and

$$\mathbb{P}\left(\inf_{0\le t\le T} M_{\lambda,\eta,t} \le -C\right) \le \exp\left[-\frac{C^2}{4(T+R)}\left(1 - \frac{C^2R}{2(T+R)^3}\right) \wedge \frac{-C^{4/3}}{4T^{1/3}}\right]$$

Remark 2.3. For C up to the order of magnitude of $T^{3/2}$ the Gaussian tail majorizes the martingale tail. For larger C, the second term majorizes the martingale tail. For much much larger C (on the order T^2) a small change in the proof gives decay of order $e^{-cC^{4/3}}$. A large deviations principle for N_λ is proven in [7] which suggests a stronger tail bound ought to be true.

Proof. Let X_t be a standard Brownian motion, and let w solve (2.9) the accelerated stochastic sine equation with acceleration η. Let M be the martingale part of w. Let $\xi \in \mathbb{R}$, and apply Doob's inequality to the submartingale $e^{\xi M_t}$ to get

$$\mathbb{P}\left(\sup_{0\le t\le T} M_t \ge C\right) \le e^{-\xi C}\mathbb{E}(e^{\xi M_T}).$$

Applying (2.8), we have that

$$\mathbb{E}(e^{\xi M_T}) = \mathbb{E}\left(\mathcal{E}(\xi M_T)e^{\frac{\xi^2}{2}[M]_T}\right) = \hat{Q}_E\left(e^{\frac{\xi^2}{2}[M]_T}\right),$$

with $\hat{Q}_E(\cdot)$ the expectation under the probability measure $\hat{Q}$ defined by

$$\frac{d\hat{Q}}{d\mathbb{P}} = \mathcal{E}(\xi M_T).$$

By the Girsanov theorem,

$$dY_s = dX_s - \xi \sin\left(\tfrac{w_t}{2}\right)\, ds$$

is a $\hat{Q}$–Brownian motion. Hence,

$$M_t = \int_0^t 2\sin\left(\tfrac{w_s}{2}\right) dY_s + \int_0^t 2\xi \sin\left(\tfrac{w_s}{2}\right)^2 ds.$$

Further, the law of w_s changes under $\hat{Q}$, as we have that

$$dw_t = 2\lambda\tfrac{\beta}{4}e^{-\frac{\beta}{4}t}dt + 2(\xi+\eta)\sin\left(\tfrac{w_t}{2}\right)^2 dt + 2\sin\left(\tfrac{w_t}{2}\right) dY_t, \qquad w_0 = 0.$$

Hence, under $\hat{Q}$, w is a solution of the accelerated stochastic sine equation with acceleration $\xi + \eta$.

As for the bracket, we have that for $t \le T$

$$[M_\lambda]_t = \int_0^t 4\sin\left(\tfrac{w_s}{2}\right)^2\, ds = 2t - \int_0^t 2\cos\left(w_s\right)\, ds.$$

Using Proposition 2.1, we have that for $T \le T_\lambda$, there is an R independent of ξ and η so that for all $C > 0$

$$\hat{Q}\left(\int_0^T -2\cos\left(w_s\right)\, ds \ge R(1+|\xi+\eta|) + C\right) \le e^{-C^2/R}.$$

Therefore, we have that for $T \le T_\lambda$

$$\hat{Q}_E\left(e^{\frac{\xi^2}{2}[M_\lambda]_T}\right) = e^{\xi^2 T}\hat{Q}_E\left(\exp\left(\int_0^T -\xi^2\cos\left(w_s\right)\, ds\right)\right) \le e^{\xi^2(T+S)+S|\xi|^3}$$

for some constant $S > 0$ independent of ξ, λ or T but depending on η.

There remains to optimize in ξ. From the work so far, we have

$$\mathbb{P}\left(\sup_{0\le t\le T} M_t \ge C\right) \le e^{-\xi C}\mathbb{E}(e^{\xi M_T}) \le e^{-\xi C+\xi^2(T+S)+S|\xi|^3}.$$

Taking $\xi = \frac{C}{2(T+S)}$,

$$\mathbb{P}\left(\sup_{0\le t\le T} M_t \ge C\right) \le \exp\left[-\frac{C^2}{4(T+S)} + \frac{SC^3}{8(T+S)^3}\right],$$

and taking $\xi = (C/(4T+4S))^{1/3}$ gives

$$\mathbb{P}\left(\sup_{0\le t\le T} M_t \ge C\right) \le \exp\left[-\frac{3C^{4/3}}{4(4(T+S))^{1/3}} + \frac{C^{2/3}(T+S)^{1/3}}{4^{2/3}}\right].$$

Hence the desired bound holds by taking the second bound for $C > P(T+S)$ and P sufficiently large, and the first bound for $C \le P(T+S)$.

The statement about the infimum may be proved in an identical fashion by reformulating it as an equivalent bound on the supremum of $-M_\lambda$. We would then use the submartingale $e^{-\xi M_\lambda}$ and use $[M_\lambda]_t = [-M_\lambda]_t$. □

3 Main theorem

The one–point upper bound

Using Proposition 2.2 with $\eta = 0$, we can give the upper bound in (1.3).

Proposition 3.1. *For any $\delta > 0$*

$$\lim_{x\to\infty} \mathbb{P}\left(\max_{0\le\lambda\le x} M_{\lambda,T_\lambda} > \left(\frac{4}{\sqrt{\beta}}+\delta\right)\log x\right) = 0$$

Proof. As commented, it suffices to bound the probability for natural numbers λ and x. By Proposition 2.2 for any $\delta > 0$ sufficiently small there is an $\epsilon > 0$ and an x_0 sufficiently large so that for all $x > x_0$ and all $x > \lambda > \exp((\log x)^{3/4})$

$$\mathbb{P}\left(M_{\lambda,T_\lambda} > \left(\frac{4}{\sqrt{\beta}}+\delta\right)\log x\right) \le \exp\left(-(\log x)^2 \frac{\left(\frac{4}{\sqrt{\beta}}+2\delta\right)^2}{\frac{16}{\beta}\log\lambda}\right) \le \exp(-(\log x)(1+\epsilon)).$$

For smaller λ, we have, taking the $4/3$–power bound in Proposition 2.2, that for some $C_{\beta,\delta}$

$$\mathbb{P}\left(M_{\lambda,T_\lambda} > \left(\frac{4}{\sqrt{\beta}}+\delta\right)\log x\right) \le \exp\left(-(\log x)^{13/12} C_{\beta,\delta}\right)$$

Hence, taking a union bound over all natural numbers λ less than x gives the desired bound. □

Remark 3.2. In fact, the proof is easily modified to give

$$\limsup_{\lambda\to\infty}\left(\frac{M_{\lambda,T_\lambda}}{\log\lambda}\right) \le \frac{4}{\sqrt{\beta}}, \quad \text{a.s.}$$

The tube event and the lower bound

Let x be a natural number, and let R be a large parameter to be chosen later. Let $T'_\lambda = T_\lambda - R^2\sqrt{\log\lambda}$. Define an event $\mathcal{A}_\lambda$ given by

$$\mathcal{A}_\lambda = \begin{Bmatrix} |M_{\lambda,t} - \sqrt{\beta}t| \le R\sqrt{\log x},\ \forall\, 0\le t\le T'_x; \\ |[M_\lambda]_t - 2t| \le R,\ \forall\, 0\le t\le T'_x \end{Bmatrix}.$$

Let x be a natural number, and define

$$S_x = \sum_{\lambda=x}^{2x} \mathcal{E}(\sqrt{\beta} M_{\lambda,T'_x}) \mathbf{1}\{\mathcal{A}_\lambda\} \tag{3.1}$$

Notice that with this definition of S_x we will have that $S_x > 0$ if and only if the event $\mathcal{A}_\lambda$ occurs for some integer $\lambda \le x$. Using the Cauchy-Schwarz inequality for non-negative random variables, we arrive at the Paley-Zygmund inequality

$$\mathbb{P}(S_x > 0)\mathbb{E}S_x^2 \ge (\mathbb{E}S_x)^2. \tag{3.2}$$

We wish to show that this has probability going to 1 as $\lambda \to \infty$ for any $\delta > 0$. Hence, we need to produce a lower bound of the form

$$\mathbb{E}[\mathcal{E}(\sqrt{\beta} M_{\lambda,T'_x}) \mathbf{1}\{\mathcal{A}_\lambda\}] = Q_{\sqrt{\beta},\lambda}(\mathcal{A}_\lambda) \ge 1 - C_\beta e^{-R^{4/3}/C_\beta},$$

and we need to produce a similar upper bound on

$$\mathbb{E}[\mathcal{E}(\sqrt{\beta} M_{\lambda_1,T'_x}) \mathbf{1}\{\mathcal{A}_{\lambda_1}\} \mathcal{E}(\sqrt{\beta} M_{\lambda_2,T'_x}) \mathbf{1}\{\mathcal{A}_{\lambda_2}\}].$$

From these bounds we will be able to show that as $x \to \infty$

$$(\operatorname{Var} S_x)/x^2 \to 0 \qquad \text{and} \qquad \mathbb{E}S_x \ge x(1 - C_\beta e^{-R^{4/3}/C_\beta}). \tag{3.3}$$

Hence, we conclude (3.2) that for any $\epsilon > 0$ there is an R sufficiently large and an x_0 sufficiently large so that for all $x > x_0$

$$\mathbb{P}(S_x > 0) \ge \frac{(\mathbb{E}S_x)^2}{\mathbb{E}S_x^2} \ge 1 - \epsilon.$$

We have therefore shown that by letting R_x tend arbitrarily slowly to infinity

$$\max_{x \le \lambda \le 2x} \left\{ M_{\lambda,T'_x} \right\} \ge \sqrt{\beta} T'_x - R_x \sqrt{\log x}, \tag{3.4}$$

with probability going to 1 as $x \to \infty$.

One point lower bound

We need to find a lower bound on

$$\mathbb{E}[\mathcal{E}(\sqrt{\beta} M_{\lambda,T'_x}) \mathbf{1}\{\mathcal{A}_\lambda\}] = Q_{\sqrt{\beta},\lambda}(\mathcal{A}_\lambda),$$

which is on the order of unity. Recall that under $Q_{\sqrt{\beta},\lambda}$ the process $\alpha_{\lambda,\cdot} - \alpha_{-\lambda,\cdot}$ follows the accelerated stochastic sine equation (2.9) with $\xi = \sqrt{\beta}$. The process $M_{\lambda,t}$ referenced in the event $\mathcal{A}_\lambda$ can be expressed as

$$M_{\lambda,t} = u_{\lambda,\xi,t} - 2\lambda(1 - \tfrac{4}{\beta}\mathfrak{f}(t)).$$

Meanwhile, the performing the Doob decomposition on $u_{\lambda,\xi,t}$, we have

$$M_{\lambda,\xi,t} = u_{\lambda,\xi,t} - 2\lambda(1 - \tfrac{4}{\beta}\mathfrak{f}(t)) - \int_0^t 2\xi \sin\left(\tfrac{u_{\lambda,\xi,s}}{2}\right)^2 ds$$

The bracket process $[M_{\lambda,\xi}]_t$ is given as before by

$$[M_{\lambda,\xi}]_t = \int_0^t 4\sin\left(\tfrac{u_{\lambda,\xi,s}}{2}\right)^2 \, ds = 2t - \int_0^t 2\cos\left(u_{\lambda,\xi,s}\right) \, ds.$$

Hence we can write

$$\begin{aligned}Q_{\xi,\lambda}(\mathcal{A}_\lambda) \geq 1 - Q_{\xi,\lambda}\left(\sup_{0\leq t\leq T_x'}\left|M_{\lambda,\xi,t} + \int_0^t \xi\cos(u_{\lambda,\xi,s})\,ds\right| > R\sqrt{\log x}\right)\\ - Q_{\xi,\lambda}\left(\sup_{0\leq t\leq T_x'}\left|\int_0^t 2\cos\left(u_{\lambda,\xi,s}\right)\,ds\right| > R\right).\end{aligned}$$

By Propositions 2.1 and 2.2, we conclude that

$$Q_{\xi,\lambda}(\mathcal{A}_\lambda) \geq 1 - C_\beta e^{-R^{4/3}/C_\beta} \tag{3.5}$$

for some C_β sufficiently large and all λ sufficiently large.

Two point bound

Following the heuristic (1.6), we treat $M_{\lambda_1,t}$ and $M_{\lambda_2,t}$ as uncorrelated until $T_* = \frac{4}{\beta}\log_+|\lambda_1 - \lambda_2|$. Without loss of generality, suppose that $\lambda_2 \geq \lambda_1$. On the event $\mathcal{A}_{\lambda_2}$, we can estimate

$$\begin{aligned}\mathcal{E}(\sqrt{\beta}M_{\lambda_2,T_x'}) &= \mathcal{E}(\sqrt{\beta}M_{\lambda_2,T_*})\exp\left(\sqrt{\beta}(M_{\lambda_2,T_x'} - M_{\lambda_2,T_*}) - \tfrac{\beta}{2}([M_{\lambda_2}]_{T_x'} - [M_{\lambda_2}]_{T_*})\right)\\ &\leq \mathcal{E}(\sqrt{\beta}M_{\lambda_2,T_*})\exp\left(2\sqrt{\beta}R\sqrt{\log x} + \beta R\right).\end{aligned}$$

Hence, we have the estimate

$$\begin{aligned}&\mathbb{E}[\mathcal{E}(\sqrt{\beta}M_{\lambda_1,T_x'})\mathbf{1}\{\mathcal{A}_{\lambda_1}\}\,\mathcal{E}(\sqrt{\beta}M_{\lambda_2,T_x'})\mathbf{1}\{\mathcal{A}_{\lambda_2}\}]\\ \leq&\mathbb{E}\left[\mathcal{E}(\sqrt{\beta}M_{\lambda_1,T_x'})\mathcal{E}(\sqrt{\beta}M_{\lambda_2,T_*})\exp\left(2\sqrt{\beta}R\sqrt{\log x} + \beta R\right)\right].\end{aligned} \tag{3.6}$$

We now observe that

$$\mathcal{E}(\sqrt{\beta}M_{\lambda_1,T_x'})\mathcal{E}(\sqrt{\beta}M_{\lambda_2,T_*}) = \mathcal{E}(\sqrt{\beta}(M_{\lambda_1,T_x'} + M_{\lambda_2,T_*}))\exp\left(\beta[M_{\lambda_1},M_{\lambda_2}]_{T_*\wedge T_x'}\right). \tag{3.7}$$

By the Girsanov theorem, under the measure $\mathbb{S}$ with Radon–Nikodym derivative

$$\frac{d\mathbb{S}}{d\mathbb{P}} = \mathcal{E}(\sqrt{\beta}(M_{\lambda_1,T_x'} + M_{\lambda_2,T_*})),$$

we have that there is a finite variation process A_t bounded almost surely by an absolute constant so that

$$dU_t = dZ_t - \sqrt{\beta}A_t\,dt$$

is a standard complex $\mathbb{S}$–Brownian motion. Here Z_t is the standard complex Brownian motion used in equation (1.1) under the measure $\mathbb{P}$. Meanwhile (1.1) (also c.f. (1.5)) shows that $[M_{\lambda_1},M_{\lambda_2}]_t$ is a sum of integrals of $e^{i\sigma_1(\sigma_1\alpha_{\sigma_2\lambda_1,t}+\sigma_3\alpha_{\sigma_4\lambda_2,t})}$ with $\sigma_j \in \{1,-1\}$. Applying Proposition 2.1 to each of these integrals, we can conclude

$$\mathbb{P}\left([M_{\lambda_1},M_{\lambda_2}]_{T_*\wedge T_x'} > t + C\right) \leq e^{-t^2/C}$$

for sufficiently large C. Hence we conclude using (3.7) and (3.6) that there is some constant C_β so that for any $R > 0$

$$\mathbb{E}[\mathcal{E}(\sqrt{\beta}M_{\lambda_1,T_x'})\mathbf{1}\{\mathcal{A}_{\lambda_1}\}\,\mathcal{E}(\sqrt{\beta}M_{\lambda_2,T_x'})\mathbf{1}\{\mathcal{A}_{\lambda_2}\}] \leq e^{C_\beta + 2R\sqrt{\beta\log x} + \beta R}. \tag{3.8}$$

Fine estimate

We also need an estimate that improves when λ_1 and λ_2 are well separated. Once more, we estimate by dropping the indicators and writing

$$\mathbb{E}[\mathcal{E}(\sqrt{\beta}M_{\lambda_1,T_x'})\mathbf{1}\{\mathcal{A}_{\lambda_1}\}\mathcal{E}(\sqrt{\beta}M_{\lambda_2,T_x'})\mathbf{1}\{\mathcal{A}_{\lambda_2}\}] \leq \$\left(\exp\left(\beta[M_{\lambda_1}, M_{\lambda_2}]_{T_x'}\right)\right), \tag{3.9}$$

where

$$\frac{d\$}{d\mathbb{P}} = \mathcal{E}(\sqrt{\beta}(M_{\lambda_1,T_x'} + M_{\lambda_2,T_x'})).$$

Now, on applying Proposition 2.1, we have a tail bound of the form

$$\mathbb{P}\left([M_{\lambda_1}, M_{\lambda_2}]_{T_x'} > t + C_\beta/\Delta\right) \leq e^{-t^2\Delta^2/C_\beta}$$

where $\Delta = |\lambda_1 - \lambda_2|\mathfrak{f}(T_x')$ and $C_\beta > 0$ is a constant. This leads to an estimate of the form

$$\mathbb{E}[\mathcal{E}(\sqrt{\beta}M_{\lambda_1,T_x'})\mathbf{1}\{\mathcal{A}_{\lambda_1}\}\mathcal{E}(\sqrt{\beta}M_{\lambda_2,T_x'})\mathbf{1}\{\mathcal{A}_{\lambda_2}\}] \leq \exp\left(C_\beta/\Delta\right). \tag{3.10}$$

for some other C_β and all $\Delta \geq 1$.

The second moment

Here we estimate $\mathbb{E}S_x^2$. Recalling (3.1), we can write

$$\mathbb{E}S_x^2 = \sum_{\lambda_1=x}^{2x}\sum_{\lambda_2=x}^{2x}\mathbb{E}\left[\mathcal{E}(\sqrt{\beta}M_{\lambda_1,T_{\lambda_1}})\mathbf{1}\{\mathcal{A}_{\lambda_1}\}\mathcal{E}(\sqrt{\beta}M_{\lambda_2,T_{\lambda_2}})\mathbf{1}\{\mathcal{A}_{\lambda_2}\}\right]. \tag{3.11}$$

We partition this sum according to the magnitude of $|\lambda_1 - \lambda_2|$. Let S_0 be all those pairs (λ_1, λ_2) so that $|\lambda_1 - \lambda_2| \geq xe^{-\frac{1}{2}R^2\sqrt{\log x}}$. Let S_1 be the remaining pairs. Observe that the cardinality of S_1 is at most $2x^2e^{-\frac{1}{2}R^2\sqrt{\log x}}$.

For terms in S_0, we apply the fine bound (3.10). The term Δ that appears for such terms can be estimated uniformly by

$$\Delta \geq xe^{-\frac{1}{2}R^2\sqrt{\log x}} \cdot \tfrac{\beta}{4}e^{-\log x + R^2\sqrt{\log x}},$$

which tends to ∞ with x. In particular, we can estimate

$$\sum_{S_0}\mathbb{E}\left[\mathcal{E}(\sqrt{\beta}M_{\lambda_1,T_x'})\mathbf{1}\{\mathcal{A}_{\lambda_1}\}\mathcal{E}(\sqrt{\beta}M_{\lambda_2,T_x'})\mathbf{1}\{\mathcal{A}_{\lambda_2}\}\right] \leq x^2 \cdot (1 + O(e^{-\frac{1}{2}R^2\sqrt{\log x}})). \tag{3.12}$$

For the remaining terms, we apply the coarse bound (3.8), using which we conclude that

$$\sum_{S_1}\mathbb{E}\left[\mathcal{E}(\sqrt{\beta}M_{\lambda_1,T_x'})\mathbf{1}\{\mathcal{A}_{\lambda_1}\}\mathcal{E}(\sqrt{\beta}M_{\lambda_2,T_x'})\mathbf{1}\{\mathcal{A}_{\lambda_2}\}\right] \leq x^2e^{C_\beta - \frac{1}{2}R^2\sqrt{\log x} + 2R\sqrt{\beta\log x} + \beta R}. \tag{3.13}$$

Hence picking R sufficiently large (anything larger than $4\sqrt{\beta}$ will do), we have combining (3.11), (3.12) and (3.13) that

$$(\operatorname{Var} S_x)/x^2 \to 0$$

as $x \to \infty$, hence establishing (3.3).

Proof of main theorem

As in the proofs of Propositions 1.2 and 1.3, we get $\left(\alpha_{\lambda,t} - \alpha_{\lambda,T'_x} - 4\pi : t \geq T'_x, \lambda > 0\right)$ is stochastically dominated by $\left(\alpha_{\lambda\frac{4}{\beta}\mathfrak{f}(T'_x),t} : t \geq 0, \lambda > 0\right)$. Therefore we have by Proposition 2.2 that there is a $\gamma > 0$ so that for all $C > 0$,

$$\max_{x\leq\lambda\leq 2x} \mathbb{P}\left(\alpha_{\lambda,T_x} - \alpha_{\lambda,T'_x} - 2\lambda(\tfrac{4}{\beta})(\mathfrak{f}(T_x) - \mathfrak{f}(T'_x)) \leq -C + 4\pi\right) \leq e^{-\gamma C^2/(T_x - T'_x)}.$$

In particular we conclude that

$$\max_{x\leq\lambda\leq 2x} \left\{-M_{\lambda,T_x} + M_{\lambda,T'_x}\right\} \leq C_\beta R_x (\log x)^{3/4} \tag{3.14}$$

with probability going to 1.

Finally, we observe that for $0 \leq \lambda \leq 2x$,

$$0 \leq \alpha_{\lambda,\infty} - \alpha_{\lambda,T_x} = M_{\lambda,\infty} - M_{\lambda,T_x} + 2\lambda(\tfrac{4}{\beta}\mathfrak{f}(T_x)) \leq M_{\lambda,\infty} - M_{\lambda,T_x} + \tfrac{16}{\beta}.$$

Therefore, we conclude that

$$\max_{x\leq\lambda\leq 2x} \{M_{\lambda,\infty}\} \geq \max_{x\leq\lambda\leq 2x} \{M_{\lambda,T_x}\} - \frac{16}{\beta} \tag{3.15}$$

Combining (3.4), (3.14) and (3.15), we conclude that

$$\max_{x\leq\lambda\leq 2x} \{M_{\lambda,\infty}\} \geq \frac{4}{\sqrt{\beta}} \log(x) - C_\beta R_x (\log x)^{3/4} - (R_x^2 + R_x)\sqrt{\log x} - \frac{16}{\beta}$$

with probability going to 1 as $x \to \infty$.

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