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Approximating diffusion reflections at elastic boundaries

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Abstract

We show a probabilistic functional limit result for one-dimensional diffusion processes that are reflected at an elastic boundary which is a function of the reflection local time. Such processes are constructed as limits of a sequence of diffusions which are discretely reflected by small jumps at an elastic boundary, with reflection local times being approximated by ε-step processes. The construction yields the Laplace transform of the inverse local time for reflection. Processes and approximations of this type play a role in finite fuel problems of singular stochastic control.

Keywords: reflected diffusion; elastic boundary; inverse local time; Laplace transform.
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1 Introduction

The classical Skorokhod problem is that of reflecting a path at a boundary. It is a standard tool to construct solutions to SDEs with reflecting boundary conditions. The fundamental example is Brownian motion with values in $[0, \infty)$ being reflected at a constant boundary at zero, solved by Skorokhod [16]. Starting with Tanaka [17], well-known generalizations concern diffusions in multiple dimensions with normal or oblique reflection at the boundary of some given (time-invariant) domain in the Euclidean space of certain smoothness or other kinds of regularity, cf. e.g. [10, 3]. Other generalizations admit for an a-priori given but time-dependent boundary, see for instance [11].

Our contribution is a functional limit result for reflection at a boundary which is a function of the reflection local-time L, for general one-dimensional diffusions X. Because of the mutual interaction between boundary and diffusion, see Figure 1a, we call the boundary *elastic*. Such elastic boundaries appear typically in solutions to singular control problems of finite fuel type, where the optimal control is the reflection local time that keeps a diffusion process within a no-action region, cf. Karatzas and Shreve [5]. In order to explicitly construct the control (pathwise via Skorokhod's Lemma), finite fuel studies typically assume that the dynamics of the diffusion can be expressed without reference to the control (see e.g. [7, 4]). This is different to our setup, where the non-linear mutual interdependence between diffusion and control (local time) subverts direct construction by Skorokhod's lemma, already for OU processes [18, Remark 1]. We relate to a concrete application in context of optimal liquidation for a financial asset position in Remark 3.4.

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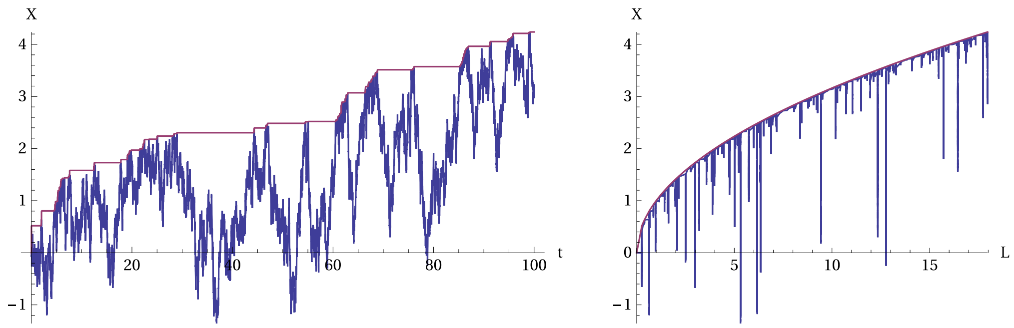

(a) X against real time t.

(b) X against local time L.

Figure 1: Example. Brownian motion X_t (blue) reflected at the elastic boundary $g(L) = \sqrt{L}$ (purple), where L is the reflection local time of X at boundary $g(L)$.

A natural idea for approximation is to proxy 'infinitesimal' reflections by small ε-jumps ΔL^ε, thereby inducing jumps of the elastic reflection boundary, see Figure 2. This allows to express excursion lengths of the approximating diffusion X^ε in terms of independent hitting times for continuous diffusions, what naturally leads to an explicit expression (3.9) for the Laplace transform of the inverse local time of X. In our singular control context, L^ε is asymptotically optimal at first order if L is optimal, see Remark 3.4. Our main result is Theorem 3.2. We prove ucp-convergence of $(X^\varepsilon, L^\varepsilon)$ to (X, L) by showing in Section 4 tightness of the approximation sequence $(X^\varepsilon, L^\varepsilon)_\varepsilon$ and using Kurtz–Protter's notion of uniformly controlled variations (UCV), introduced in [8].

2 Elastic reflection: model and notation

We consider a filtered probability space $\big(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P}\big)$ with one-dimensional $(\mathcal{F}_t)$-Brownian motion W and filtration $(\mathcal{F}_t)$ satisfying the usual conditions of right-continuity and completeness. Let $\sigma : \mathbb{R} \to (0, \infty)$ and $b : \mathbb{R} \to \mathbb{R}$ be Lipschitz-continuous and such that the continuous $\mathbb{R}$-valued (b, σ)-diffusion $\mathrm{d}Z_t = b(Z_t)\,\mathrm{d}t + \sigma(Z_t)\,\mathrm{d}W_t$ with generator $\mathcal{G} := \frac{1}{2}\sigma(x)^2 \frac{\mathrm{d}^2}{\mathrm{d}x^2} + b(x)\frac{\mathrm{d}}{\mathrm{d}x}$ is regular and recurrent. Moreover, let X be a (b, σ)-diffusion with reflection at an elastic boundary. This means that for a given non-decreasing $g \in C^1([0, \infty))$, the processes (X, L) satisfy

$$\mathrm{d}X_t = b(X_t)\,\mathrm{d}t + \sigma(X_t)\,\mathrm{d}W_t - \mathrm{d}L_t\,, \qquad X_0 = g(0)\,, \tag{2.1}$$

with the reflection local time L being a continuous non-decreasing process L that only grows when X is at the (local-time-dependent) boundary $g(L)$, i.e.

$$\mathrm{d}L_t = \mathbb{1}_{\{X_t = g(L_t)\}}\,\mathrm{d}L_t\,, \quad L_0 = 0\,, \quad \text{with } X_t \leq g(L_t) \text{ for all } t \geq 0. \tag{2.2}$$

Note that the reflecting boundary is not deterministic in real time and space coordinates. Instead, the boundary $g(L)$, at which the diffusion X is being reflected, is elastic in the sense that it is itself a stochastic process which retracts when being hit, cf. Figure 1b. Strong existence and uniqueness of (X, L) follow from classical results (cf. Remark 3.3) and are also an outcome of our explicit construction below, see Lemma 4.9.

We are particularly interested (see Remark 3.4) in the inverse local time

$$\tau_\ell := \inf\{t > 0 \mid L_t > \ell\}. \tag{2.3}$$

Remark 2.1. Note that $\{t \geq 0 \mid X_t = g(L_t)\}$ is a.s. of Lebesgue measure zero by [13, ex. VI.1.16]. For a constant boundary $g(\ell) \equiv a$, Tanaka's formula for symmetric local times [13, ex. VI.1.25] hence shows that the process L, that we obtain as a solution to the SDE with reflection (2.1) – (2.2), is the symmetric local time of the continuous semimartingale X at given level $a \in \mathbb{R}$, i.e. $L_t = \lim_{\varepsilon \searrow 0} \frac{1}{2\varepsilon} \int_0^t \mathbb{1}_{(a-\varepsilon,a+\varepsilon)}(X_s)\, \mathrm{d}\langle X, X\rangle_s$.

We denote by H^y the first hitting time of a point y by a (b,σ)-diffusion, and write $H^{x \to y}$ for the hitting time when the diffusion starts in x. Note that $\mathbb{P}[H^{x\to y} < \infty] = 1$ for all x, y by our assumption on the diffusion being regular and recurrent.

3 Approximation by small ε-reflections

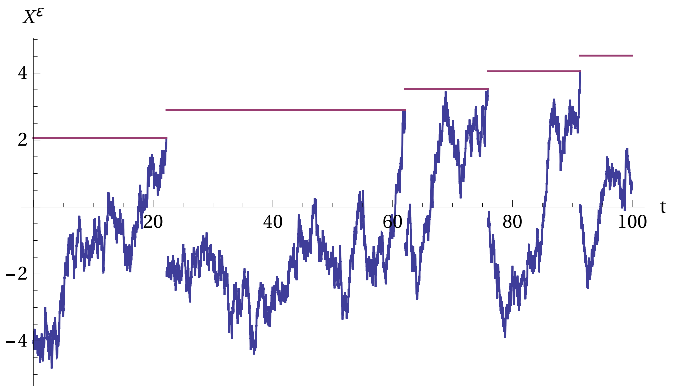

Figure 2: Approximating processes X^ε and $g(L^\varepsilon) = \sqrt{L^\varepsilon}$ for $\varepsilon = 4$.

We construct solutions to (2.1) – (2.2) and derive an explicit representation (3.9) of the Laplace transform of the inverse local time at boundary g by approximating reflection by jumps in the following system of SDEs:

$$\mathrm{d}X_t^\varepsilon = b(X_t^\varepsilon)\,\mathrm{d}t + \sigma(X_t^\varepsilon)\,\mathrm{d}W_t - \mathrm{d}L_t^\varepsilon\,, \qquad X_{0-}^\varepsilon := g(0)\,, \tag{3.1}$$

$$L_t^\varepsilon := \sum_{0\le s\le t} \Delta L_s^\varepsilon \quad \text{with } \Delta L_t^\varepsilon := \begin{cases} \varepsilon & \text{if } X_{t-}^\varepsilon = g(L_{t-}^\varepsilon), \\ 0 & \text{otherwise,} \end{cases} \qquad L_{0-}^\varepsilon := 0\,, \tag{3.2}$$

$$\tau_\ell^\varepsilon := \inf\{t > 0 \mid L_t^\varepsilon > \ell\} \quad \text{for } \ell \ge 0. \tag{3.3}$$

As soon as process X^ε hits the boundary, it is reflected by a jump of fixed size $\varepsilon > 0$. We will speak of L^ε as discrete local time, as it is approximating L in the sense of Theorem 3.2. Since the target reflected diffusion X starts at the boundary g, we now have $X_0^\varepsilon = g(0) - \varepsilon$ after an initial jump $\Delta L_0^\varepsilon = \varepsilon$ away from $X_{0-}^\varepsilon := g(0)$.

Lemma 3.1. *For any $\varepsilon > 0$, the SDE (3.1)–(3.2) has a unique (up to indistinguishability) strong global solution $(X_t^\varepsilon, L_t^\varepsilon)_{t\ge 0}$. Moreover, uniqueness in law holds.*

Proof. Indeed, one can argue by results [14, V.9–11, V.17] for classical diffusion SDEs with Lipschitz coefficients (b,σ) by inductive construction on $[\![0, \tau_n[\![$ where for $n \ge 1$, $\tau_n := \inf\{t > \tau_{n-1} \mid X_{t-}^\varepsilon = g(n\varepsilon)\} = \tau_{\varepsilon n}^\varepsilon$ with $\tau_0 := 0$. Clearly L_t^ε equals $L_{\tau_{n-1}}^\varepsilon$ for $t \in [\![\tau_{n-1}, \tau_n[\![$ and $L_{\tau_n}^\varepsilon = L_{\tau_{n-1}}^\varepsilon + \varepsilon$, while $X_u^\varepsilon = F(X_{\tau_{n-1}}^\varepsilon, (W_{\tau_{n-1}+s})_{s\ge 0})_{u-\tau_{n-1}}$ on $[\![\tau_{n-1}, \tau_n[\![$ holds for a suitable functional representation F of strong solutions to (b,σ)-diffusions [14, Theorem V.10.4]. Such construction extends to $[\![0, \tau_\infty[\![$ for $\tau_\infty := \lim_n \tau_n$.

It suffices to show $\tau_\infty = \infty$ (a.s.). To this end, let $g_\infty := \lim_n g(n\varepsilon) \in \mathbb{R} \cup \{\infty\}$. In the case $g_\infty < \infty$, one can find $x, y \in \mathbb{R}$ with $g_\infty - \varepsilon < x < y < g_\infty$. By recurrence of (b,σ)-diffusions, we have (a.s.) finite times $\tau_0^y := \inf\{t > 0 \mid X_t^\varepsilon = y\}$, $\tau_n^x := \inf\{t > \tau_{n-1}^y \mid X_t^\varepsilon = x\}$, $\tau_n^y := \inf\{t > \tau_n^x \mid X_t^\varepsilon = y\}$, for $n \in \mathbb{N}$. The durations $\tau_n^y - \tau_n^x$, $n \in \mathbb{N}$, for upcrossings of the interval $[x, y]$ are i.i.d., by the strong Markov property of the time-homogeneous diffusion. Moreover, X^ε is continuous on all $[\![\tau_n^x, \tau_n^y]\!]$. By the law of large numbers, $\frac{1}{n}\sum_{i=1}^n \exp(-\lambda(\tau_i^y - \tau_i^x))$ converges almost surely for $n \to \infty$

to the Laplace transform $\mathbb{E}_x[\exp(-\lambda H^y)]$, $\lambda \geq 0$, of the time H^y for hitting y by the (b,σ)-diffusion process (started at x). This expectation is strictly less than 1 for $\lambda > 0$, as $H^y > 0$ P_x-a.s. for $y > x$, whereas the limit of $\frac{1}{n}\sum_{i=1}^{n}\exp(-\lambda(\tau_i^y - \tau_i^x))$ equals 1 on $\{\tau_\infty < \infty\}$, where $\lim_{i\to\infty}(\tau_i^y - \tau_i^x) = 0$. Hence $P[\tau_\infty < \infty] = 0$.

If $g_\infty = \infty$, let $\tau_n' := \inf\{t > \tau_{n-1} \mid X_{t-}^\varepsilon = g((n-1)\varepsilon)\}$, for $n \geq 1$, so that $\tau_{n-1} < \tau_n' \leq \tau_n$ and $X_{\tau_n'-}^\varepsilon = g((n-1)\varepsilon) = X_{(\tau_{n-1})-}^\varepsilon$. Using the time change $\varphi_t := \int_0^t \sum_{n=1}^\infty 1_{[\![\tau_n',\tau_n[\![}\,\mathrm{d}u$ with inverse $s_t := \inf\{u \mid \varphi_u > t\}$, we get (cf. [14, IV.30.10]) that $X_t' := X_{s_t}^\varepsilon$, $t \geq 0$, solves the SDE $\mathrm{d}X_t' = b(X_t')\,\mathrm{d}t + \sigma(X_t')\,\mathrm{d}W_t'$, $X_0' = g(0)$, on $[\![0,\varphi_\infty[\![$ for $\varphi_\infty := \sup_t \varphi_t$, with respect to $W_t' = \int_0^{s_t}\sum_{n=1}^\infty 1_{[\![\tau_n',\tau_n[\![}\,\mathrm{d}W_u$. We have $W_t' = B_{t\wedge\varphi_\infty}$ for some Brownian motion B on $[0,\infty)$ by the Dambis-Dubins-Schwarz theorem, cf. [6, Thm. 3.4.6, Prob. 3.4.7]. So X' solves the (b,σ)-diffusion SDE w.r.t. B on $[\![0,\varphi_\infty[\![$. Consider a (b,σ)-diffusion $\tilde{X}$ w.r.t. B on $[0,\infty)$. By the usual Gronwall argument for uniqueness of SDE solutions, we get $X' = \tilde{X}$ on all $[\![0,\varphi_{\tau_n}]\!]$ and hence $X' = \tilde{X}$ on $[\![0,\varphi_\infty[\![$. In particular, X' remains a.s. bounded on any finite time interval $[\![0,T[\![$ with $T \leq \varphi_\infty$. However, in the event $\{\tau_\infty < \infty\} \subset \{\varphi_\infty < \infty\}$, we get from $X_{\varphi_{\tau_n}}' = g(n\varepsilon) \to \infty$ that $\sup_{t<\varphi_\infty} X_t' = \infty$. Hence, we must have $\mathbb{P}[\tau_\infty < \infty] = 0$. $\square$

By (3.1) – (3.3), we have $\tau_0^\varepsilon = \tau_{0-}^\varepsilon = 0$ and $\tau_\ell^\varepsilon = \tau_{(k-1)\varepsilon}^\varepsilon$ for $\ell \in [(k-1)\varepsilon, k\varepsilon)$ with $k \in \mathbb{N}$, and $\tau_{k\varepsilon}^\varepsilon$ is the k-th jump time of X^ε and L^ε within period $(0,\infty)$. For $\ell = k\varepsilon$, the approximating process X^ε is a continuous (b,σ)-diffusion on stochastic intervals $[\![\tau_{\ell-}^\varepsilon, \tau_\ell^\varepsilon[\![$, and $X_{\tau_\ell^\varepsilon}^\varepsilon = X_{\tau_{(\ell-)}^\varepsilon}^\varepsilon - \varepsilon = g(L_{\tau_{(\ell-)}^\varepsilon}^\varepsilon) - \varepsilon = g(\ell-\varepsilon) - \varepsilon$. For such $\ell = k\varepsilon$, we shall call $\tau_\ell^\varepsilon - \tau_{\ell-}^\varepsilon$ the length of the (k-th) excursion of X^ε away from the boundary. Note that this excursion length is independent of $\mathcal{F}_{\tau_{(\ell-)}^\varepsilon}^\varepsilon$ and its (conditional) distribution is

$$\tau_\ell^\varepsilon - \tau_{\ell-}^\varepsilon \sim H^{g(\ell)} \quad \text{under } \mathbb{P}_{g(\ell-\varepsilon)-\varepsilon}\,, \tag{3.4}$$

what is also denoted as $\tau_\ell^\varepsilon - \tau_{\ell-}^\varepsilon \overset{d}{=} H^{g(\ell-\varepsilon)-\varepsilon \,\to\, g(\ell)}$. The Laplace transform of first hitting times $H^{x\,\to\,z}$ is well-known, see e.g. [14, V.50]: for $x, z \in \mathbb{R}$ and $\lambda > 0$,

$$\mathbb{E}\big[e^{-\lambda H^{x\,\to\,z}}\big] \equiv \mathbb{E}_x\big[e^{-\lambda H^z}\big] = \begin{cases} \Phi_{\lambda,-}(x)/\Phi_{\lambda,-}(z) & \text{if } x < z, \\ \Phi_{\lambda,+}(x)/\Phi_{\lambda,+}(z) & \text{if } x > z, \end{cases} \tag{3.5}$$

where functions $\Phi_{\lambda,\pm}$ are uniquely determined up to a constant factor as the increasing ($\Phi_{\lambda,-}$) respectively decreasing ($\Phi_{\lambda,+}$) positive solutions Φ of the differential equation $\mathcal{G}\Phi = \lambda\Phi$ with generator $\mathcal{G} = \frac{1}{2}\sigma(x)^2\frac{\mathrm{d}^2}{\mathrm{d}x^2} + b(x)\frac{\mathrm{d}}{\mathrm{d}x}$ of the (b,σ)-diffusion. Since we assume the boundary function g to be non-decreasing, only $\Phi_{\lambda,-}$ is of interest for our purpose.

Due to independence of Brownian increments over disjoint time intervals, the Laplace transform of the inverse local time can be calculated from a sum of (independent) excursion lengths at (discrete) local times $\ell_n := \varepsilon n$ as

$$\begin{aligned}
\mathbb{E}\big[\exp\big(-\lambda\tau_\ell^\varepsilon\big)\big] &= \mathbb{E}\Big[\exp\Big(-\lambda\sum_{n=1}^{\lfloor \ell/\varepsilon\rfloor}\big(\tau_{\ell_n}^\varepsilon - \tau_{\ell_n-}^\varepsilon\big)\Big)\Big] = \prod_{n=1}^{\lfloor \ell/\varepsilon\rfloor}\mathbb{E}\Big[\exp\Big(-\lambda\big(\tau_{\ell_n}^\varepsilon - \tau_{\ell_n-}^\varepsilon\big)\Big)\Big] \\
&= \prod_{n=1}^{\lfloor \ell/\varepsilon\rfloor}\mathbb{E}_{g(\ell_n-\varepsilon)-\varepsilon}\big[\exp\big(-\lambda H^{g(\ell_n)}\big)\big] = \prod_{n=1}^{\lfloor \ell/\varepsilon\rfloor}\frac{\Phi_{\lambda,-}\big(g(\ell_n-\varepsilon)-\varepsilon\big)}{\Phi_{\lambda,-}\big(g(\ell_n)\big)} \\
&= \exp\Bigg(\sum_{n=1}^{\lfloor \ell/\varepsilon\rfloor}\log\Bigg(\frac{\Phi_{\lambda,-}\big(g(\ell_n-\varepsilon)-\varepsilon\big)}{\Phi_{\lambda,-}\big(g(\ell_n)\big)}\Bigg)\Bigg),
\end{aligned} \tag{3.6}$$

for $\ell \geq 0$ and $\lambda > 0$. With $h_n(\xi) := \Phi_{\lambda,-}\big(g(\ell_n-\xi)-\xi\big)$, each summand in (3.6) equals

$$\begin{aligned}
\log h_n(\varepsilon) - \log h_n(0) &= \int_0^\varepsilon \frac{h_n'(\xi)}{h_n(\xi)}\,\mathrm{d}\xi = -\int_0^\varepsilon \big(g'(\ell_n-\xi)+1\big)\frac{\Phi_{\lambda,-}'\big(g(\ell_n-\xi)-\xi\big)}{\Phi_{\lambda,-}\big(g(\ell_n-\xi)-\xi\big)}\,\mathrm{d}\xi \\
&= -\int_{\ell_{n-1}}^{\ell_n}\big(g'(a)+1\big)\frac{\Phi_{\lambda,-}'\big(g(a)+a-\ell_n\big)}{\Phi_{\lambda,-}\big(g(a)+a-\ell_n\big)}\,\mathrm{d}a\,.
\end{aligned} \tag{3.7}$$

Therefore, we obtain

$$\mathbb{E}\big[\exp\big(-\lambda\tau_\ell^\varepsilon\big)\big] = \exp\bigg(-\int_0^{\varepsilon\lfloor \ell/\varepsilon\rfloor} \big(g'(a)+1\big)\frac{\Phi'_{\lambda,-}\big(g(a)+a-\varepsilon\lceil a/\varepsilon\rceil\big)}{\Phi_{\lambda,-}\big(g(a)+a-\varepsilon\lceil a/\varepsilon\rceil\big)}\,\mathrm{d}a\bigg). \tag{3.8}$$

Intuitively, this already suggests the formula (3.9) when taking $\varepsilon \to 0$.

Theorem 3.2. *The approximations $(X_t^\varepsilon, L_t^\varepsilon)_{t\geq 0}$ from* (3.1)–(3.2) *converge uniformly in probability for $\varepsilon \to 0$ to a pair $(X_t, L_t)_{t\geq 0}$ of continuous adapted processes with non-decreasing L, which is the unique strong solution (globally on $[0,\infty)$) to the reflected SDE* (2.1)–(2.2)*. The inverse local time $\tau_\ell := \inf\{t>0 \mid L_t > \ell\}$ has the Laplace transform*

$$\mathbb{E}\big[e^{-\lambda\tau_\ell}\big] = \exp\bigg(-\int_0^{\ell} \big(g'(a)+1\big)\frac{\Phi'_{\lambda,-}\big(g(a)\big)}{\Phi_{\lambda,-}\big(g(a)\big)}\,\mathrm{d}a\bigg) \quad \text{for } \lambda>0,\ \ell\geq 0, \tag{3.9}$$

where $\Phi_{\lambda,-}$ is the (up to a constant factor) unique positive increasing solution of the differential equation $\mathcal{G}\Phi = \lambda\Phi$, for $\mathcal{G}$ denoting the generator of the (b,σ)-diffusion.

Proof. Existence and uniqueness of (X,L) is shown in Lemma 4.9 below. Corollary 4.10 gives uniform convergence in probability. Using dominated convergence for the right-hand side of equation (3.8), we find $\lim_{\varepsilon\to 0}\mathbb{E}[e^{-\lambda\tau_\ell^\varepsilon}] = \exp\big(-\int_0^\ell (g'(a)+1)\frac{\Phi'_{\lambda,-}(g(a))}{\Phi_{\lambda,-}(g(a))}\,\mathrm{d}a\big)$. For the left-hand side, it suffices to prove weak convergence $\tau_\ell^\varepsilon \Rightarrow \tau_\ell$ as $\varepsilon \to 0$ for all $\ell \geq 0$. This is done in Corollary 4.11 below. □

Remark 3.3. Existence and uniqueness for (X,L) can also be concluded from classical results, cf. [3, suitably extended to non-bounded domains], by considering the pair (X,L) as a degenerate diffusion in $\mathbb{R}^2$ with oblique reflection in direction $(-1,+1)$ at a smooth boundary, see Figure 1b. This uses an iteration argument involving the Skohorod-map and yields another approximation by a sequence of continuous processes. Yet, these do not satisfy the target diffusive dynamics inside the domain, except at the limiting fixed point (unless (b,σ) are constant). In contrast, $(X^\varepsilon, L^\varepsilon)$ adheres to the same dynamics as (X,L) between jump times, cf. (2.1) and (3.1), is Markovian and has a natural interpretation.

Remark 3.4. An application example for (3.9) and elastically reflected diffusions is the optimal execution for the sale of a financial asset position if liquidity is stochastic, see [1]. A large trader with adverse price impact seeks to maximize expected proceeds from selling θ risky assets in an illiquid market. His trading strategy A (predictable, càdlàg, non-decreasing) affects the asset price $S_t = f(Y_t^A)\bar{S}_t$ via a volume impact process $\mathrm{d}Y_t^A = -\beta Y_t^A\,\mathrm{d}t + \hat{\sigma}\,\mathrm{d}B_t - \mathrm{d}A_t$ with $\bar{S}_t = \mathcal{E}(\sigma W)_t$ for an increasing function f, and Brownian motions (B,W) with correlation ρ. The gains to maximize in expectation are

$$G_T(A) := \int_0^T e^{-\delta t} f(Y_t^A)\bar{S}_t\,\mathrm{d}A_t^c + \sum_{\substack{0\leq t\leq T\\ \Delta A_t\neq 0}} e^{-\delta t}\bar{S}_t \int_0^{\Delta A_t} f(Y_{t-}^A - x)\,\mathrm{d}x.$$

The optimal strategy turns out to be the local time L of a reflected Ornstein-Uhlenbeck process X (with $b(x) := \rho\sigma\hat{\sigma} - \beta x$ and $\sigma(x) = \sigma > 0$) at a suitable elastic boundary g, as in (2.1)–(2.2), see [1, Section 3]. After a change of measure argument, one can write the expected proceeds from such strategies as $\mathbb{E}[G_\infty(L)] = \int_0^\theta f\big(g(\ell)\big)\mathbb{E}\big[e^{-\delta\tau_\ell}\big]\,\mathrm{d}\ell$. To find the optimal free boundary g, one can then apply (3.9) to express the proceeds as a functional of the boundary g, and optimize over all possible boundaries by solving a calculus of variations problem. This is key to the proof in [1]. The discrete local time L^ε has a natural interpretation as the step process which approximates the continuous optimal strategy L by doing small block trades, as they would be realistic in an actual implementation, with identical (no-)action region. The approximation is asymptotically optimal for the control problem. Indeed, straightforward calculations similar to the derivation of (3.8) show that L^ε is asymptotically optimal in first order, i.e. $\mathbb{E}[G_\infty(L)] = \mathbb{E}[G_\infty(L^\varepsilon)] + \mathcal{O}(\varepsilon)$.

4 Tightness and convergence

To show convergence of $(\tau_\ell^\varepsilon)_\varepsilon$, we will prove that the pair of càdlàg processes $(X^\varepsilon, L^\varepsilon)$ forms a tight sequence in $\varepsilon \to 0$. Applying weak convergence theory for SDEs by Kurtz and Protter [9], we show that any limit point (for $\varepsilon \to 0$) satisfies (2.1) and (2.2). Uniqueness in law for solutions of (2.1) – (2.2) will then allow to conclude Theorem 3.2.

Let $(\varepsilon_n)_{n\in\mathbb{N}}$ be a sequence with $\varepsilon_n \to 0$ and consider the sequence $(X^{\varepsilon_n}, L^{\varepsilon_n})_n$. To show tightness, we will apply the following criterion due to Aldous.

Proposition 4.1 ([2, Cor. to Thm. 16.10])**.** *Let $(E, |\cdot|)$ be a separable Banach space. If a sequence $(Y^n)_{n\in\mathbb{N}}$ of adapted, E-valued càdlàg processes satisfies the following two conditions, then it is tight.*

(a) The sequences $\big(J_T(Y^n)\big)_n$ and $(Y_0^n)_n$ are tight (in $\mathbb{R}$, resp. E) for any $T \in (0,\infty)$, with $J_T(Y^n) := \sup_{0<t\le T}\big|Y_t^n - Y_{t-}^n\big|$ denoting the largest jump until time T.

(b) For any $T \in (0,\infty)$ and $\varepsilon_0, \eta > 0$ there exist $\delta_0 > 0$ and $n_0 \in \mathbb{N}$ such that for all $n \ge n_0$, all (discrete) Y^n-stopping times $\hat\tau \le T$ and all $\delta \in (0, \delta_0]$ we have

$$\mathbb{P}\big[|Y_{\hat\tau+\delta}^n - Y_{\hat\tau}^n| \ge \varepsilon_0\big] \le \eta\,.$$

To get tightness one needs to control both jump size and, regarding $(L_n^\varepsilon)_n$, the frequency of jumps simultaneously. As we are considering processes with jumps of size $\pm\varepsilon_n \to 0$, only the latter is not yet clear. To this end, the next lemma provides a technical bound on X^{ε_n}, L^{ε_n}, while a second lemma constricts the probability that X^{ε_n} (respectively L^{ε_n}) performs a number of N_n jumps in a time interval of fixed length.

Lemma 4.2 (Upper bound)**.** *Fix a time horizon $T \in (0,\infty)$ and $\eta \in (0,1]$. Then there exists a constant $M \in \mathbb{R}$ such that $\mathbb{P}[\exists n : g(L_T^{\varepsilon_n} - \varepsilon_n) > M] \le \eta$, with the domain of definition for the function g being extended by $g(-x) := g(0)$ for $-x < 0$.*

Proof. Consider a continuous (b,σ)-diffusion Y that starts at time $t = 0$ at $g(0)$. For $n \in \mathbb{N}$ and $k = 0, 1, 2, \ldots$, let $\ell(n,k) := k\varepsilon_n$. By induction over k, using comparison for diffusion SDEs, cf. [6, Theorem 5.2.18], one obtains that (a.s.) $X_t^{\varepsilon_n} \le Y_t$ for $t \in [\![0, \tau_{\ell(n,k)}^{\varepsilon_n}[\![$ for all $k \ge 1$, and hence $X^{\varepsilon_n} \le Y$ on $[0,\infty)$ (a.s.) because $\lim_{k\to\infty} \tau_{\ell(n,k)}^{\varepsilon_n} = \infty$ for any n by Lemma 3.1. Hence, on the event $\{\exists n : g(L_T^{\varepsilon_n} - \varepsilon_n) > M\}$ we have $\sup_{t\in[0,T]} Y_t \ge M$, and hence $H^{g(0)\to M} \le T$. Thus $\mathbb{P}[\exists n : g(L_T^{\varepsilon_n} - \varepsilon_n) > M] \le \mathbb{P}[H^{g(0)\to M} \le T]$. Now the claim follows since $\lim_{M\to\infty} \mathbb{P}[H^{g(0)\to M} \le T] = 0$. □

Lemma 4.3 (Frequency of jumps)**.** *Fix $T \in (0,\infty)$, $\varepsilon_0, \eta > 0$, and set $N_n := \lceil \varepsilon_0/\varepsilon_n \rceil$. Then there exists $\delta > 0$ and $n_0 \in \mathbb{N}$ such that for every bounded stopping time $\hat\tau \le T$ we have $\mathbb{P}\big[J_{\hat\tau,\delta}^{\varepsilon_n} \ge N_n\big] \le \eta$ for all $n \ge n_0$, where $J_{\hat\tau,\delta}^{\varepsilon_n} := \inf\{k \mid L_{\hat\tau}^{\varepsilon_n} + k\varepsilon_n \ge L_{\hat\tau+\delta}^{\varepsilon_n}\}$ is the number of jumps of X^{ε_n}, respectively L^{ε_n}, in time $]\!]\hat\tau, \hat\tau+\delta]\!]$.*

Proof. We will first find an estimate for the jump count probability for arbitrary but fixed $\delta > 0$, $n \in \mathbb{N}$, $N_n \in \mathbb{N}$ and $\hat\tau \le T$. Only in part 2) of the proof we will consider $(N_n)_{n\in\mathbb{N}}$ as stated, to study the limit $n \to \infty$. More precisely, we will show in part 1) that, given $\mathcal{F}_{\hat\tau}$, for every $\lambda > 0$ there exist $k_{n,\lambda} \in \{0, 1, \ldots, N_n - 1\}$ s.t. for $x_n := g(L_{\hat\tau}^{\varepsilon_n} + \varepsilon_n k_{n,\lambda})$,

$$\mathbb{P}\big[J_{\hat\tau,\delta}^{\varepsilon_n} \ge N_n \;\big|\; \mathcal{F}_{\hat\tau}\big] \le e^{\lambda\delta}\left(\frac{\Phi_{\lambda,-}(x_n - \varepsilon_n)}{\Phi_{\lambda,-}(x_n)}\right)^{N_n-1}. \tag{4.1}$$

1) In this part, fix arbitrary $\delta > 0$, $n \in \mathbb{N}$, $N_n \in \mathbb{N}$ and $\hat\tau \le T$. We enumerate the jumps and estimate the sum of excursion lengths by δ. Let $\ell_k := L^{\varepsilon_n}_{\hat\tau} + k\varepsilon_n$ be the (discrete) local time at the k-th jump after time $\hat\tau$. If X^{ε_n} has at least N_n jumps in the interval $\rrbracket \hat\tau, \hat\tau+\delta \rrbracket$, it is doing at least $N_n - 1$ complete excursions (cf. (3.4)), so that, noting that $\tau^{\varepsilon_n}_{L^{\varepsilon_n}_t - \varepsilon_n} \le t < \tau^{\varepsilon_n}_{L^{\varepsilon_n}_t}$ (for all $t \ge 0$) and $\ell_{N_n-1} + \varepsilon_n \le L^{\varepsilon_n}_{\hat\tau+\delta}$, we have

$$\delta = (\hat\tau + \delta) - \hat\tau \ge \tau^{\varepsilon_n}_{L^{\varepsilon_n}_{\hat\tau+\delta} - \varepsilon_n} - \tau^{\varepsilon_n}_{L^{\varepsilon_n}_{\hat\tau}} \ge \sum_{k=1}^{N_n-1} \big(\tau^{\varepsilon_n}_{\ell_k} - \tau^{\varepsilon_n}_{\ell_{k-1}}\big) \stackrel{d}{=} \sum_{k=1}^{N_n-1} H_k$$

with the last equality being in distribution conditionally on $\mathcal{F}_{\hat\tau}$, for H_k being conditionally independent and distributed as $H^{g(\ell_{k-1})-\varepsilon_n \to g(\ell_k)}$. Clearly, ℓ_k is $\mathcal{F}_{\hat\tau}$-measurable. By the Laplace transform (3.5) of H_k and the Markov inequality, we get for $\lambda > 0$

$$\begin{aligned}
\mathbb{P}\big[J^{\varepsilon_n}_{\hat\tau,\delta} \ge N_n \,\big|\, \mathcal{F}_{\hat\tau}\big] &\le \mathbb{P}\Bigg[\sum_{k=1}^{N_n-1} H_k \le \delta \,\Bigg|\, \mathcal{F}_{\hat\tau}\Bigg] \le e^{\lambda\delta}\,\mathbb{E}\Bigg[\exp\Bigg(-\lambda \sum_{k=1}^{N_n-1} H_k\Bigg) \,\Bigg|\, \mathcal{F}_{\hat\tau}\Bigg] \\
&= e^{\lambda\delta} \prod_{k=1}^{N_n-1} \mathbb{E}\Big[\exp\Big(-\lambda H^{g(\ell_{k-1})-\varepsilon_n \to g(\ell_k)}\Big) \,\Big|\, \mathcal{F}_{\hat\tau}\Big] \\
&= e^{\lambda\delta} \prod_{k=1}^{N_n-1} \frac{\Phi_{\lambda,-}\big(g(\ell_{k-1}) - \varepsilon_n\big)}{\Phi_{\lambda,-}\big(g(\ell_k)\big)} \le e^{\lambda\delta} \prod_{k=1}^{N_n-1} \frac{\Phi_{\lambda,-}\big(g(\ell_k) - \varepsilon_n\big)}{\Phi_{\lambda,-}\big(g(\ell_k)\big)} \\
&\le e^{\lambda\delta}\left(\max_{0 \le k < N_n} \frac{\Phi_{\lambda,-}\big(g(\ell_k) - \varepsilon_n\big)}{\Phi_{\lambda,-}\big(g(\ell_k)\big)}\right)^{N_n-1} = e^{\lambda\delta}\left(\frac{\Phi_{\lambda,-}(x_n - \varepsilon_n)}{\Phi_{\lambda,-}(x_n)}\right)^{N_n-1}
\end{aligned}$$

where $x_n := g(\ell_k)$ for the index $k = k_{n,\lambda}$ attaining the maximum.

2) For given $\delta > 0$ and $\hat\tau \le T$, let us now consider the sequence $N_n = \lceil \varepsilon_0/\varepsilon_n \rceil$, $n \in \mathbb{N}$. To investigate the limit $n \to \infty$, first observe that by Taylor expansion

$$\log \frac{\Phi_{\lambda,-}(x - \varepsilon_n)}{\Phi_{\lambda,-}(x)} = -\varepsilon_n \frac{\Phi'_{\lambda,-}(x)}{\Phi_{\lambda,-}(x)} + \varepsilon_n r(x, \varepsilon_n),$$

where $r(\cdot, \varepsilon_n) \to 0$ converges uniformly on compacts for $\varepsilon_n \to 0$. Since $\hat\tau + \delta \le T + \delta$ is bounded, Lemma 4.2 yields a constant $M \in \mathbb{R}$ such that $\mathbb{P}\big[\exists n : x_n > M\big] \le \frac{\eta}{2}$ for the x_n from above. On the event $\{\forall n : x_n \in I\}$ with compact $I := [g(0), M]$, we have uniform convergence of $r(x_n, \varepsilon_n)$ and thereby get

$$\begin{aligned}
\limsup_{n\to\infty} e^{\lambda\delta}\left(\frac{\Phi_{\lambda,-}(x_n - \varepsilon_n)}{\Phi_{\lambda,-}(x_n)}\right)^{N_n-1} &= \exp\left(\lambda\delta + \limsup_{n\to\infty}\,(N_n - 1)\log\frac{\Phi_{\lambda,-}(x_n - \varepsilon_n)}{\Phi_{\lambda,-}(x_n)}\right) \\
&= \exp\left(\lambda\delta + \limsup_{n\to\infty}\,(N_n\varepsilon_n - \varepsilon_n)\left(r(x_n, \varepsilon_n) - \frac{\Phi'_{\lambda,-}(x_n)}{\Phi_{\lambda,-}(x_n)}\right)\right) \\
&\le \exp\left(\lambda\delta - \varepsilon_0 \inf_{x\in I} \frac{\Phi'_{\lambda,-}(x)}{\Phi_{\lambda,-}(x)}\right) = \sup_{x\in I} \exp\left(\lambda\delta - \varepsilon_0 \frac{\Phi'_{\lambda,-}(x)}{\Phi_{\lambda,-}(x)}\right).
\end{aligned}$$

By [12, Theorem 1], $\psi^x(\lambda) := \frac{1}{2}\Phi'_{\lambda,-}(x)/\Phi_{\lambda,-}(x)$ is the Laplace exponent of $A^x(\kappa^x_\cdot)$, where κ^x_ℓ is the inverse local time at constant level x of a (b,σ)-diffusion Z^x starting at x, and $A^x(t)$ is the occupation time $A^x(t) := \int_0^t \mathbb{1}_{\{Z^x_s \le x\}}\,\mathrm{d}s$. So we get for $\lambda \to \infty$ that $\exp\big(-2\varepsilon_0\psi^x(\lambda)\big) = \mathbb{E}_x\big[\exp\big(-\lambda A^x(\kappa^x_{2\varepsilon_0})\big)\big] \to 0$. By compactness of I and Dini's theorem there exists $\lambda = \lambda_{\varepsilon_0,\eta,M}$ such that for $\delta := 1/\lambda$ we have

$$\limsup_{n\to\infty} e^{\lambda\delta}\left(\frac{\Phi_{\lambda,-}(x_n - \varepsilon_n)}{\Phi_{\lambda,-}(x_n)}\right)^{N_n-1} \le e^{\lambda\delta} \sup_{x\in I} \exp\big(-2\varepsilon_0\psi^x(\lambda)\big) \le \frac{\eta}{2} \tag{4.2}$$

on the event $\{x_n \le M \text{ for all } n\}$. By equation (4.1) and $\mathbb{P}[\exists n : x_n > M] \le \eta/2$, this completes the proof. □

Using the preceding two lemmas, we will first prove tightness of $(L^{\varepsilon_n})_n$ and of $(X^{\varepsilon_n})_n$ separately. Tightness of the pair $(X^{\varepsilon_n}, L^{\varepsilon_n})_n$ is handled afterwards.

Lemma 4.4 (Tightness of the local time approximations)**.** *The sequence $(L^{\varepsilon_n})_n$ of càdlàg processes defined by* (3.1) *and* (3.2) *satisfies Aldous' criterion and thus is tight.*

Proof. Part (a) of Proposition 4.1 is clear, as the initial value $L_0^{\varepsilon_n} = \varepsilon_n$ is deterministic and $J_T(L^{\varepsilon_n}) \le \varepsilon_n$. For part (b), consider $T, \eta, \varepsilon_0 > 0$ and any bounded L^{ε_n}-stopping time $\hat\tau \le T$. The event $|L^{\varepsilon_n}_{\hat\tau+\delta} - L^{\varepsilon_n}_{\hat\tau}| \ge \varepsilon_0$ means that L^{ε_n} performs at least $N_n := \lceil \varepsilon_0/\varepsilon_n \rceil$ jumps in the stochastic interval $\rrbracket \hat\tau, \hat\tau+\delta \rrbracket$. Lemma 4.3 yields some n_0 and $\delta_0 = \delta_0(\varepsilon_0)$ such that Aldous' criterion is satisfied for all $n \ge n_0$. Hence, $(L^{\varepsilon_n})_n$ is tight by Proposition 4.1. $\square$

Next we show boundedness of $(X^{\varepsilon_n})_n$, needed for Lemma 4.6 to prove tightness.

Lemma 4.5 (Bounding the diffusion approximations)**.** *Let $T \in (0,\infty)$ and $\eta > 0$. Then there exists $M \in \mathbb{R}$ such that $\mathbb{P}[\sup_{t\in[0,T]} |X_t^{\varepsilon_n}| > M] < \eta$ for all $n \in \mathbb{N}$.*

Proof. By Lemma 4.2, for every $n \in \mathbb{N}$ the process X^{ε_n} on $[0,T]$ is bounded from above by a constant M with probability at least $1-\eta/2$. It remains to show that it is also bounded from below with high probability. To this end, we will construct a process Y that is a lower bound for all X^{ε_n} and then argue for Y.

For $\hat\varepsilon := \sup_n \varepsilon_n$ consider a (b,σ)-diffusion Y which is discretely reflected by jumps of size $-\hat\varepsilon$ at a constant boundary $c := g(0) - \hat\varepsilon$, with $Y_0 = y := g(0) - 2\hat\varepsilon$. Such Y is a special case of (3.1)–(3.2), for a constant boundary function: $\mathrm{d}Y_t = b(Y_t)\,\mathrm{d}t + \sigma(Y_t)\,\mathrm{d}W_t - L_t^Y$ with $L_t^Y := \sum_{0\le s\le t} \Delta L_t^Y$ and $\Delta L_t^Y := \hat\varepsilon \mathbb{1}_{\{Y_{t-}=c\}}$. Let $\tau_k^Y := \inf\{t > 0 \mid L_t^Y > k\hat\varepsilon\}$ be the k-th hitting time of Y at the boundary c. So on all $\llbracket \tau_k^Y, \tau_{k+1}^Y \llbracket$, Y is a continuous (b,σ)-diffusion starting in y. Now for fixed n, $\varepsilon := \varepsilon_n$, note that $X^\varepsilon_{\tau^\varepsilon_{m\varepsilon}} = g((m-1)\varepsilon) - \varepsilon \ge c \ge Y_{\tau^\varepsilon_{m\varepsilon}}$ by monotonicity of g. As $\tau^\varepsilon_{m\varepsilon} \to \infty$ for $m \to \infty$ by Lemma 3.1, induction over the inverse (discrete) local times $\tau^\varepsilon_{m\varepsilon}$, $m \in \mathbb{N}$, yields $X^\varepsilon \ge Y$ on $\llbracket \tau_k^Y, \tau_{k+1}^Y \rrbracket$ if $X^\varepsilon_{\tau_k^Y} \ge Y_{\tau_k^Y}$ by comparison results [6, Thm. 5.2.18]. Since $X_0^\varepsilon \ge Y_0$, the latter follows by induction over k. As $\tau_k^Y \to \infty$ for $k \to \infty$ by Lemma 3.1, we get $X^{\varepsilon_n} \ge Y$ on $[0,\infty)$ for all n. So it suffices to show $\mathbb{P}[\inf_{t\in[0,T]} Y_t < -M] < \eta/2$ for some M, which directly follows from the càdlàg property of Y. $\square$

Lemma 4.6 (Tightness of the reflected diffusion approximations)**.** *The sequence $(X^{\varepsilon_n})_n$ of càdlàg processes from* (3.1) *and* (3.2) *satisfies Aldous' criterion and thus is tight.*

Proof. Condition (a) of Proposition 4.1 holds. To verify part (b), let $\eta > 0$, $T \in (0,\infty)$, and $\hat\tau \le T$ be a stopping time. By Lemma 4.5, $|X^{\varepsilon_n}_{\hat\tau}|$ is with a probability of at least $1-\eta/4$ bounded by some constant M (not depending on n and $\hat\tau$). Let us consider the events $\{X^{\varepsilon_n}_{\hat\tau+\delta} \le X^{\varepsilon_n}_{\hat\tau} - \varepsilon_0\} \cup \{X^{\varepsilon_n}_{\hat\tau+\delta} \ge X^{\varepsilon_n}_{\hat\tau} + \varepsilon_0\} = \{|X^{\varepsilon_n}_{\hat\tau+\delta} - X^{\varepsilon_n}_{\hat\tau}| \ge \varepsilon_0\}$ separately.

1) First consider $\{X^{\varepsilon_n}_{\hat\tau+\delta} \le X^{\varepsilon_n}_{\hat\tau} - \varepsilon_0\}$. For $\xi := X^{\varepsilon_n}_{\hat\tau}$ we construct a reflected process Y^ξ such that $Y^\xi_t \le X^{\varepsilon_n}_{\hat\tau+t}$ for all $t \ge 0$. We then estimate $\mathbb{P}[X^{\varepsilon_n}_{\hat\tau+\delta} \le X^{\varepsilon_n}_{\hat\tau} - \varepsilon_0]$ by means of $\mathbb{P}[Y^x_\delta \le x - \varepsilon_0]$ in (4.3), uniformly for all n large enough. We estimate the latter in (4.4) using the probability of a down-crossing in time δ of intervals $[x-\varepsilon_0, x-2\hat\varepsilon]$ by a continuous diffusion. Covering $\bigcup_x [x-\varepsilon_0, x-2\hat\varepsilon]$ by finitely many intervals $[y_k, y_{k+1}]$ in (4.5) then allows us to choose $\delta > 0$ sufficiently small.

To this end, choose $\hat\varepsilon \le \varepsilon_0/4$ and n large enough such that $\varepsilon_n \le \hat\varepsilon$, and let $(Y^\xi_t)_{t\ge0}$ be the (b,σ)-diffusion w.r.t. the Brownian motion $(W_{\hat\tau+t} - W_{\hat\tau})_{t\ge0}$ with $Y_0^\xi = \xi - 2\hat\varepsilon$, which is discretely reflected by jumps of size $-\hat\varepsilon$ at a constant boundary at level $\xi - \hat\varepsilon$. More precisely, $\mathrm{d}Y^\xi_t = b(Y^\xi_t)\,\mathrm{d}t + \sigma(Y^\xi_t)\,\mathrm{d}W_{\hat\tau+t} - K^\xi_t$ with (discrete) local time $K^\xi_t := \sum_{0\le s\le t} \Delta K^\xi_s$ for $\Delta K^\xi_t := \hat\varepsilon \mathbb{1}_{\{Y^\xi_{t-} = \xi - \hat\varepsilon\}}$. Global existence and uniqueness of (Y^ξ, K^ξ) follows from the proof of Lemma 3.1. By comparison arguments and induction as in the proof of Lemma 4.5, one verifies $Y^\xi_t \le X^{\varepsilon_n}_{\hat\tau+t}$ for $t \in [0,\infty)$. Indeed, [6, Theorem 5.2.18] gives

$Y^{\xi}_{\cdot} \le X^{\varepsilon_n}_{\hat\tau+\cdot}$ on $[\![0,\tau_1[\![$ until the first jump of either $Y^{\xi}_{\cdot}$ or $X^{\varepsilon_n}_{\hat\tau+\cdot}$ at time $\tau_1 > 0$. If only Y^{ξ} jumps, we have $Y^{\xi}_{\tau_1} = Y^{\xi}_{(\tau_1)-} - \hat\varepsilon \le X^{\varepsilon_n}_{(\tau_1)-} - \hat\varepsilon = X^{\varepsilon_n}_{\tau_1} - \hat\varepsilon$, but if $X^{\varepsilon_n}_{\hat\tau+\cdot}$ jumps, we have $X^{\varepsilon_n}_{\hat\tau+\tau_1} = g(L^{\varepsilon_n}_{(\hat\tau+\tau_1)-}) - \varepsilon_n \ge g(L^{\varepsilon_n}_{\hat\tau}) - \varepsilon_n = \xi \ge Y^{\xi}_{\tau_1}$. Now $Y^{\xi}_{\tau_1} \le X^{\varepsilon_n}_{\hat\tau+\tau_1}$, so we get $Y^{\xi}_{\cdot} \le X^{\varepsilon_n}_{\hat\tau+\cdot}$ on $[\![\tau_k,\tau_{k+1}[\![$ by induction for all jump times τ_k of $(Y^{\xi}_{\cdot}, X^{\varepsilon_n}_{\hat\tau+\cdot})$.

Using $Y^{\xi}_{\delta} \le X^{\varepsilon_n}_{\hat\tau+\delta}$ and the strong Markov property of Y^{ξ} w.r.t. $(\mathcal{F}_{\hat\tau+t})_{t\ge0}$, we get

$$\mathbb{P}\big[X^{\varepsilon_n}_{\hat\tau+\delta} \le X^{\varepsilon_n}_{\hat\tau} - \varepsilon_0, |X^{\varepsilon_n}_{\hat\tau}| \le M\big] \le \sup_{-M\le x\le M} \mathbb{P}[Y^{x}_{\delta} \le x - \varepsilon_0]\,. \tag{4.3}$$

By construction Y^{ξ} depends on n and τ (through ξ), while the right-hand side of (4.3) does not. Thus one only needs to bound the probability of an $(\varepsilon_0 - 2\hat\varepsilon)$-displacement of diffusions Y^x with starting points $x - 2\hat\varepsilon$ from a compact set, which are reflected (by $(-\hat\varepsilon)$-jumps) at constant boundaries $x - \hat\varepsilon$. By the arguments in the proof of Lemma 4.3 (here applied for Y^x which is reflected at a constant boundary), for $\delta = \delta_0 > 0$ there exists $N \in \mathbb{N}$ with the following property: for every $x \in [-M, M]$, the number $J^x_\delta := \inf\{k \mid k\hat\varepsilon \ge K^x_\delta\}$ of jumps of Y^x until time δ is bounded by $N-1$ with probability at least $1 - \eta/8$.

Indeed, by (4.1), fixing $\delta > 0$, $\lambda := 1/\delta$, one gets for any x that $\mathbb{P}[J^x_\delta \ge \lceil N(x)\rceil] \le \eta/8$ where $N(x) := 1 + \big(\log(\eta/8) - 1\big)/\big(\log\Phi_{\lambda,-}(x-\hat\varepsilon) - \log\Phi_{\lambda,-}(x)\big) \in \mathbb{R}$. Compactness of $[-M, M]$ and continuity of $N(x)$ gives $N := \lceil \sup_{x\in[-M,M]} N(x)\rceil < \infty$. Hence,

$$\sup_{x\in[-M,M]} \mathbb{P}[Y^x_\delta \le x - \varepsilon_0, J^x_\delta \le N-1] \le N \sup_{x\in[-M,M]} \mathbb{P}[H^{x-2\hat\varepsilon\,\to\,x-\varepsilon_0} \le \delta], \tag{4.4}$$

since for the event under consideration, the process Y^x would have to move at least once (in at most N occasions) continuously from $x - 2\hat\varepsilon$ to $x - \varepsilon_0$. Let $d := (\varepsilon_0 - 2\hat\varepsilon)/2 \ge \varepsilon_0/4 > 0$, $K := \lfloor 2M/d\rfloor$ and $y_k := kd - M$. For $x \in [y_k, y_{k+1}]$, we have $H^{y_{k-2}\,\to\,y_{k-2}-d} \le H^{x-\varepsilon_0\,\to\,x-2\hat\varepsilon}$ since $[y_{k-2} - d, y_{k-2}] \subset [x - \varepsilon_0, x - 2\hat\varepsilon]$, so by $[-M, M] \subset [y_0, y_{K+1}]$ we get

$$\begin{aligned}
\mathbb{P}\big[H^{X^{\varepsilon_n}_{\hat\tau}-\varepsilon_n\,\to\,X^{\varepsilon_n}-\varepsilon_0} \le \delta, |X^{\varepsilon_n}_{\hat\tau}| \le M\big] &\le \eta/8 + N \sup_{x\in[-M,M]} \mathbb{P}[H^{x-2\hat\varepsilon\,\to\,x-\varepsilon_0} \le \delta] \\
= \eta/8 + N \max_{k=0,\dots,K} \sup_{x\in[kd-M,(k+1)d-M]} &\mathbb{P}\big[H^{x-2\hat\varepsilon\,\to\,x-\varepsilon_0} \le \delta\big] \\
\le \eta/8 + N \max_{k=-2,\dots,K} \mathbb{P}\big[H^{y_k\,\to\,y_k-d} \le \delta\big]\,. &
\end{aligned} \tag{4.5}$$

For a sufficiently small $\delta = \delta_1 \in (0, \delta_0]$ the right-hand side of (4.5) can be made smaller than $\eta/4$. The above holds for all n such that $\varepsilon_n \le \hat\varepsilon$, meaning that there is some n_0 such that is holds for all $n \ge n_0$. Note that δ_1 only depends on T (via M and K) and on n_0 but not on n. Hence, for all $\delta \in (0, \delta_1]$, all $n \ge n_0$ and all $\hat\tau \le T$ we have

$$\mathbb{P}[X^{\varepsilon_n}_{\hat\tau+\delta} \le X^{\varepsilon_n}_{\hat\tau} - \varepsilon_0] \le \frac{\eta}{2}\,. \tag{4.6}$$

2) For the alternative second case $X^{\varepsilon_n}_{\hat\tau+\delta} \ge X^{\varepsilon_n}_{\hat\tau} + \varepsilon_0$, consider the solution $(Y_t)_{t\ge\hat\tau}$ on $[\![\hat\tau,\infty[\![$ of $\mathrm{d}Y_t = b(Y_t)\,\mathrm{d}t + \sigma(Y_t)\,\mathrm{d}W_t$ with $Y_{\hat\tau} = X^{\varepsilon_n}_{\hat\tau}$. Using comparison results for continuous diffusions [6, Theorem 5.2.18] inductively over times $[\![\tau^{\varepsilon_n}_{(k-1)\varepsilon_n}, \tau^{\varepsilon_n}_{k\varepsilon_n}[\![$, we find $Y_t \ge X^{\varepsilon_n}_t$ for all $t \in [\![\hat\tau,\infty[\![$, a.s. Hence, arguing like in the previous case

$$\begin{aligned}
\mathbb{P}\big[X^{\varepsilon_n}_{\hat\tau+\delta} \ge X^{\varepsilon_n}_{\hat\tau} + \varepsilon_0, |X^{\varepsilon_n}_{\hat\tau}| \le M\big] &\le \mathbb{P}\big[Y_{\hat\tau+\delta} \ge Y_{\hat\tau} + \varepsilon_0, |Y_{\hat\tau}| \le M\big] \\
&\le \sup_{-M\le y\le M} \mathbb{P}\big[H^{y\,\to\,y+\varepsilon_0} \le \delta\big].
\end{aligned} \tag{4.7}$$

As in (4.5) we find a $\delta_2 > 0$ such that for all $\delta \in (0, \delta_2]$ the right side of (4.7) is bounded by $\eta/4$. Hence we have $\mathbb{P}[X^{\varepsilon_n}_{\hat\tau+\delta} \ge X^{\varepsilon_n}_{\hat\tau} + \varepsilon_0] \le \eta/2$, so with (4.6), Proposition 4.1 applies. $\square$

Now, to prove joint tightness of $(X^{\varepsilon_n}, L^{\varepsilon_n})_n$, we can utilize the fact that both processes satisfy Aldous' criterion and that their jump times and jump magnitudes are identical.

Lemma 4.7 (Tightness of joint approximations)**.** *The sequence* $(X^{\varepsilon_n}, L^{\varepsilon_n})_n$ *of càdlàg* $\mathbb{R}^2$*-valued processes defined by* (3.1) *and* (3.2) *is tight.*

Proof. In view of Proposition 4.1, choose the space $E := \mathbb{R}^2$ equipped with Euclidean norm $|\cdot|$ and let $Y^n := (X^{\varepsilon_n}, L^{\varepsilon_n}) \in D\big([0,\infty), E\big)$. Then $Y_0^n = (g(0) - \varepsilon_n, \varepsilon_n)$ and $J_T(Y^n) = \sqrt{2}\varepsilon_n$ form tight sequences in E and $\mathbb{R}$, respectively. Furthermore,

$$\mathbb{P}\big[|Y^n_{\hat\tau+\delta} - Y^n_{\hat\tau}| \geq \varepsilon_0\big] \leq \mathbb{P}\Big[|X^{\varepsilon_n}_{\hat\tau+\delta} - X^{\varepsilon_n}_{\hat\tau}| \geq \frac{\varepsilon_0}{2}\Big] + \mathbb{P}\Big[|L^{\varepsilon_n}_{\hat\tau+\delta} - L^{\varepsilon_n}_{\hat\tau}| \geq \frac{\varepsilon_0}{2}\Big].$$

Hence Y^n also satisfies Aldous's criterion and therefore is tight. ☐

Tightness only implies weak convergence of a subsequence. It remains to show (in Lemma 4.9) that every limit point satisfies (2.1) and (2.2) and that uniqueness in law holds. The latter will follow from pathwise uniqueness results for SDEs with reflection, while for the former we apply results from [9] on weak converges of SDEs. For that purpose, note that the approximated local times form a *good* sequence of semimartingales (cf. [9, Definition 7.3]), as shown in the following lemma.

Lemma 4.8. *The sequence* $(L^{\varepsilon_n})_n$ *is of uniformly controlled variation and thus* good*.*

Proof. Let $\delta := \sup_n \varepsilon_n$. Then all processes L^{ε_n} have jumps of size at most $\delta < \infty$. Fix some $\alpha > 0$. By tightness, there exists some $C \in \mathbb{R}$ such that $\mathbb{P}[L^{\varepsilon_n}_\alpha > C] \leq 1/\alpha$. So the stopping time $\tau_{n,\alpha} := \inf\{t \geq 0 \mid L^{\varepsilon_n}_t > C\}$ satisfies $\mathbb{P}[\tau_{n,\alpha} \leq \alpha] = \mathbb{P}[L^{\varepsilon_n}_\alpha > C] \leq 1/\alpha$. Moreover, by monotonicity of L^{ε_n} we have $\mathbb{E}\Big[\int_0^{t\wedge\tau_{n,\alpha}} \mathrm{d}|L^{\varepsilon_n}|_s\Big] = \mathbb{E}[L^{\varepsilon_n}_{t\wedge\tau_{n,\alpha}}] \leq C < \infty$. Hence (L^{ε_n}) is of uniformly controlled variation in the sense of [9, Definition 7.5]. So by [9, Theorem 7.10] it is a *good* sequence of semimartingales. ☐

We have gathered all necessary results to prove convergence of our approximating diffusions and local times to the continuous counterpart.

Lemma 4.9 (Weak convergence of the approximations)**.** *The sequence* $(X^{\varepsilon_n}, L^{\varepsilon_n})_n$ *of càdlàg processes defined by* (3.1) – (3.2) *converges weakly to the unique continuous strong solution* (X, L) *of* (2.1) – (2.2)*.*

Proof. By Prokhorov's theorem, tightness of $(X^{\varepsilon_n}, L^{\varepsilon_n}, W)_n$ implies weak convergence of a subsequence to some limit point, $(X^{\varepsilon_{n_k}}, L^{\varepsilon_{n_k}}, W)_k \Rightarrow (\tilde X, \tilde L, \tilde W) \in D\big([0,\infty), \mathbb{R}^3\big)$. Continuity of $(\tilde X, \tilde L)$ is clear since $\varepsilon_n \to 0$ is the maximum jump size. First we prove that $(\tilde X, \tilde L)$ satisfies the asserted SDEs. Afterwards, we will prove uniqueness of the limit point. To ease notation, let w.l.o.g. the subsequence (n_k) be (n).

By [9, Theorem 8.1] we get that $(\tilde X, \tilde L)$ satisfy (2.1) for the semimartingale $\tilde W$. That $\tilde W$ is a Brownian motion follows from standard arguments, cf. [11, proof of Theorem 1.9]. As $D\big([0,\infty), \mathbb{R}^3\big)$ is separable we find, by an application of the Skorokhod representation theorem, that $\tilde L$ is non-decreasing and $\tilde X_t \leq g(\tilde L_t)$ for all $t \geq 0$, $\mathbb{P}$-a.s. because these properties already hold for $(X^{\varepsilon_n}, L^{\varepsilon_n})$.

To prove that $\tilde L$ grows only at times t with $\tilde X_t = g(\tilde L_t)$, we have to approximate the indicator function by continuous functions. For $\delta > 0$ define

$$h_\delta(x,\ell) := \begin{cases} \big(x - g(\ell)\big)/\delta + 1 & \text{for } g(\ell) - \delta \leq x \leq g(\ell), \\ 1 - \big(x - g(\ell)\big)/\delta & \text{for } g(\ell) \leq x \leq g(\ell) + \delta, \\ 0 & \text{otherwise,} \end{cases}$$

$$h_0(x,\ell) := \mathbb{1}_{\{x = g(\ell)\}} \text{ and } H^{\delta,n}_t := h_\delta(X^{\varepsilon_n}_t, L^{\varepsilon_n}_t) \text{ and } \tilde H^\delta_t := h_\delta(\tilde X_t, \tilde L_t).$$

For $\delta \searrow 0$ the functions $h_\delta \searrow h_0$ converge pointwise monotonically. Continuity of h_δ implies weak convergence $(H^{\delta,n}, L^{\varepsilon_n}) \Rightarrow (\tilde{H}^\delta, \tilde{L})$. By Lemma 4.8, (L^{ε_n}) is a good sequence. So for every $\delta > 0$ the stochastic integrals $\int_0^\cdot H^{\delta,n}_{s-}\,\mathrm{d}L^{\varepsilon_n}_s \Rightarrow \int_0^\cdot \tilde{H}^\delta_{s-}\,\mathrm{d}\tilde{L}_s$ converge weakly. Note that $\mathrm{d}L^{\varepsilon_n}_t = H^{0,n}_{t-}\,\mathrm{d}L^{\varepsilon_n}_t$. Hence, for every $\delta > 0$ we have

$$\int_0^\cdot H^{\delta,n}_{s-}\,\mathrm{d}L^{\varepsilon_n}_s = \int_0^\cdot H^{\delta,n}_{s-}H^{0,n}_{s-}\,\mathrm{d}L^{\varepsilon_n}_s = \int_0^\cdot H^{0,n}_{s-}\,\mathrm{d}L^{\varepsilon_n}_s = L^{\varepsilon_n}\,.$$

By the weak convergence $L^{\varepsilon_n} \Rightarrow \tilde{L}$ it follows for every $\delta > 0$ that $\tilde{L}_t = \int_0^t \tilde{H}^\delta_{s-}\,\mathrm{d}\tilde{L}_s$. By monotonicity of $\tilde{L}$, $\mathrm{d}\tilde{L}_t$ defines a random measure on $[0,\infty)$. Hence monotone convergence of $\tilde{H}^\delta_t \searrow \tilde{H}^0_t$ yields $\mathrm{d}\tilde{L}_t = h_0(\tilde{X}_t, \tilde{L}_t)\,\mathrm{d}\tilde{L}_t$.

Thus, we showed that $(X^\varepsilon, L^\varepsilon)$ converges in distribution to a weak solution $(\tilde{X}, \tilde{L})$ of the reflected SDE, i.e. it might be defined on a different probability space with its own Brownian motion. Note that $(\tilde{X}, \tilde{L})$ is continuous on $[0,\infty)$ and that $\tilde{\tau}_\infty := \sup_k \tilde{\tau}_k = \infty$ a.s., where $\tilde{\tau}_k := \inf\{t > 0 \mid |\tilde{X}_t| \vee \tilde{L}_t > k\}$. To show the existence and uniqueness of a strong solution as stated in the theorem, we will use the results from [3]. Consider the domain $\bar{G} := \{(x,\ell) \in \mathbb{R}^2 \mid x \le g(\ell), \ell \ge 0\}$. We may interpret the process (X_t, L_t) as a continuous diffusion in $\bar{G}$ with oblique reflection in direction $(-1,+1)$ at the boundary, although the notion of a two-dimensional reflection seems unusual here, because (X, L) only varies in one dimension in the interior of G. The unbounded domain G can be exhausted by bounded domains $G_k := \big\{(x,\ell) \in G \bigm| |x|, |\ell| < k\big\}$, which might have a non-smooth boundary especially at $(g(0), 0)$, but still satisfy [3, Cond. (3.2)]. Hence, by [3, Cor. 5.2] the process (X, L) exists (up to explosion time) on the initial probability space and is (strongly) unique on $[\![0, \tau_k[\![$ with exit time $\tau_k := \inf\{t > 0 \mid |X_t| \vee L_t > k\}$, for all $k \in \mathbb{N}$. So (X, L) is unique until explosion time $\tau_\infty := \sup_k \tau_k$. Moreover, by [3, Theorem 5.1] we have the following pathwise uniqueness result: for any two continuous solutions (X^1, L^1) and (X^2, L^2) with explosion times τ^1_∞ and τ^2_∞, respectively defined on the same probability space with the same Brownian motion and the same initial condition, we have that $X^1 = X^2$ and $L^1 = L^2$ on $[\![0, \tau^1_k \wedge \tau^2_k]\!]$ for every $k \in \mathbb{N}$ a.s. Using a known argument due to Yamada and Watanabe, ideas being as in [6, Ch. 5.3.D], one can bring the two (weak) solutions $(\tilde{X}, \tilde{L}, \tilde{W})$ and (X, L, W) to a canonical space with a common Brownian motion. By pathwise uniqueness there, one concludes that $\tau_\infty = \infty$ a.s. (as $\tilde{\tau}_\infty = \infty$). Hence the strong solution (X, L) does not explode in finite time. In addition, we conclude uniqueness in law like in [6, Prop. 5.3.20] and thus any weak limit point of the approximating sequence $(X^\varepsilon, L^\varepsilon)$ will have the same law as (X, L). □

This convergence result can be strengthened as follows.

Corollary 4.10 (Convergence in probability)**.** *The sequence* $(X^{\varepsilon_n}, L^{\varepsilon_n})_n$ *of càdlàg processes defined by* (3.1)–(3.2) *converges in probability to* (X, L) *defined by* (2.1)–(2.2).

Proof. Following the proof of [8, Cor. 5.6], we will strengthen weak convergence $(X^{\varepsilon_n}, L^{\varepsilon_n}) \Rightarrow (X, L)$ to convergence in probability. First, note that Lemma 4.9 implies weak convergence of the triple $(X^{\varepsilon_n}, L^{\varepsilon_n}, W) \Rightarrow (X, L, W)$ by e.g. [15, Corollary 3.1]. Hence, for every bounded continuous $F : D([0,\infty); \mathbb{R}^2) \to \mathbb{R}$ and every bounded continuous $G : C([0,\infty); \mathbb{R}) \to \mathbb{R}$, we have $\lim_{n\to\infty} \mathbb{E}[F(X^{\varepsilon_n}, L^{\varepsilon_n})G(W)] = \mathbb{E}[F(X, L)G(W)]$. Now, the previous equation even holds for all bounded measurable G by L^1-approximation of measurable functions by continuous functions. By strong uniqueness of (X, L), there exists a measurable function $H : C([0,\infty); \mathbb{R}) \to C([0,\infty); \mathbb{R}^2)$ such that $(X, L) = H(W)$. In particular, $G(W) := F(H(W)) = F(X, L)$ is bounded and measurable, so we conclude

$$\begin{aligned}
&\lim_{n\to\infty} \mathbb{E}\big[(F(X^{\varepsilon_n}, L^{\varepsilon_n}) - F(X, L))^2\big] \\
&\quad= \lim_{n\to\infty} \big(\mathbb{E}\big[F(X^{\varepsilon_n}, L^{\varepsilon_n})^2\big] - 2\mathbb{E}\big[F(X^{\varepsilon_n}, L^{\varepsilon_n})F(X, L)\big] + \mathbb{E}\big[F(X, L)^2\big]\big) = 0
\end{aligned}$$

and hence convergence in probability follows. □

Corollary 4.11 (Weak convergence of the inverse local times)**.** *For any $\ell > 0$, the sequence $(\tau_\ell^{\varepsilon_n})_n$ from* (3.3) *converges in law to the inverse local time τ_ℓ defined by* (2.3)*.*

Proof. Convergence $L^{\varepsilon_n} \Rightarrow L$ implies $L_t^{\varepsilon_n} \Rightarrow L_t$ at all continuity points of L, i.e. at all points, hence $\mathbb{P}\left[\tau_\ell^{\varepsilon_n} \le t\right] = \mathbb{P}\left[L_t^{\varepsilon_n} \ge \ell\right] \to \mathbb{P}[L_t \ge \ell] = \mathbb{P}[\tau_\ell \le t]$. □

This completes the proof of Theorem 3.2.

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