Brazilian Journal of Probability and Statistics
2019, Vol. 33, No. 4, 894–914
https://doi.org/10.1214/18-BJPS428

Bayesian inference on power Lindley distribution based on different loss functions

Abbas Pak[a], M. E. Ghitany[b] and Mohammad Reza Mahmoudi[c]

[a]*Shahrekord University*
[b]*Kuwait University*
[c]*Fasa University*

Abstract. This paper focuses on Bayesian estimation of the parameters and reliability function of the power Lindley distribution by using various symmetric and asymmetric loss functions. Assuming suitable priors on the parameters, Bayes estimates are derived by using squared error, linear exponential (linex) and general entropy loss functions. Since, under these loss functions, Bayes estimates of the parameters do not have closed forms we use lindley's approximation technique to calculate the Bayes estimates. Moreover, we obtain the Bayes estimates of the parameters using a Markov Chain Monte Carlo (MCMC) method. Simulation studies are conducted in order to evaluate the performances of the proposed estimators under the considered loss functions. Finally, analysis of a real data set is presented for illustrative purposes.

1 Introduction

The power Lindley distribution specified by the probability density function (p.d.f.)

$$f(t;\gamma,\delta)=\frac{\gamma\delta^2}{\delta+1}(1+t^\gamma)t^{\gamma-1}e^{-\delta t^\gamma},\quad t>0,\gamma,\delta>0, \tag{1.1}$$

and survival function

$$R(t;\gamma,\delta)=\left(1+\frac{\delta}{\delta+1}t^\gamma\right)e^{-\delta t^\gamma},\quad t>0,\gamma,\delta>0, \tag{1.2}$$

was introduced by Ghitany et al. (2013) as a new distribution useful to analyze lifetime data. They studied many characteristics of this model and demonstrated its use in modeling real data as compared to the calassical Gamma and Weibull distributions. Recently, Ghitany, Al-Mutairi and Aboukhamseen (2015) developed inference procedures for the stress-strength power Lindley models using the maximum likelihood and bootstrap methods. From now on, power Lindley distribution with the parameters γ and δ will be denoted as PL(γ,δ).

Key words and phrases. Power Lindley distribution, Bayesian estimation, maximum likelihood estimation, squared error loss function, asymmetric loss function.

Received June 2017; accepted December 2018.

In the Bayesian setting, the observer combine subjective opinion based on insight or experience with the available observations to get balanced values and to update the estimates as more information and data become accessible. So, the parameters of interest are assumed to be random variables with some prior probability distributions. Then, in order to conduct a Bayesian analysis, usually quadratic loss function is considered. A very popular quadratic loss is the squared error (SE) loss function given by

$$\Delta_{\mathrm{SE}}\big(g(\theta), \hat{g}(\theta)\big) = \big(g(\theta) - \hat{g}(\theta)\big)^2, \tag{1.3}$$

where $\hat{g}(\theta)$ is an estimate of the parametric function $g(\theta)$. The Bayes estimate of $g(\theta)$ against SE loss function is the posterior mean given by

$$\hat{g}_{\mathrm{SE}}(\theta) = E_\theta\big[g(\theta) \mid \text{data}\big]. \tag{1.4}$$

Assuming SE loss function, Bayesian inference for the lifetime distributions is considered by several authors. Among others, Banerjee and Kundu (2008) addressed Bayesian inference for Weibull distribution using a type II hybrid censored sample. Lin, Chou and Huang (2012) studied Bayesian estimation of Weibull parameters by using an adaptive progressively hybrid censored sample.

However, using SE loss function in the Bayesian approach leads to equal penalization for underestimation and overestimation which is inappropriate in practical purposes. For instance, in estimating the reliability characteristics, overestimation is more serious than the underestimation. Therefore, different asymmetric loss functions are considered by researchers such as linex and general entropy (GE) loss functions for implementing Bayesian method in various field of reliability inference. The linex loss function given by

$$\Delta_{\mathrm{LE}}\big(g(\theta), \hat{g}(\theta)\big) = \exp\big[\nu\big(g(\theta) - \hat{g}(\theta)\big)\big] - \nu\big(g(\theta) - \hat{g}(\theta)\big) - 1, \quad \nu \neq 0. \tag{1.5}$$

is a popular asymmetric loss function that penalizes underestimation and overestimation for negative and positive ν, respectively. Under this loss function, the Bayesian estimate of $g(\theta)$ is given by

$$\hat{g}_{\mathrm{LE}}(\theta) = -\frac{1}{\nu} \log E_\theta\big[\exp\big(-\nu g(\theta)\big) \mid \text{data}\big]. \tag{1.6}$$

Recently, estimating the parameters of Bathtub-shaped distribution based on this loss function is discussed by Ahmed (2014).

Another important asymmetric loss is the GE loss function defined as

$$\Delta_{\mathrm{GE}}\big(g(\theta), \hat{g}(\theta)\big) \propto \left(\frac{\hat{g}(\theta)}{g(\theta)}\right)^{\omega} - \omega \log\left(\frac{\hat{g}(\theta)}{g(\theta)}\right) - 1, \quad \omega \neq 0. \tag{1.7}$$

Based on this loss function, the Bayes estimate of $g(\theta)$ is given by

$$\hat{g}_{\mathrm{GE}}(\theta) = \big\{E_\theta\big[\big(g(\theta)\big)^{-\omega} \mid \text{data}\big]\big\}^{-1/\omega}. \tag{1.8}$$

The main purpose of this paper is to discuss on Bayesian estimation of the parameters and reliability function of the PL(γ, δ) distribution by using the above mentioned loss functions. In Section 2, considering suitable priors on the parameters, some expressions are provided for the Bayes estimates of the parameters γ, δ and the reliability $R(t)$. Since these expressions cannot be simplified to closed forms, we employ Lindley approach to calculate the approximate Bayes estimates in Section 3. In Section 4, we employ a Markov Chain Monte Carlo (MCMC) procedure to draw random samples from the posterior distributions and use them to derive the Bayes estimates. In Section 5, simulation studies are conducted to evaluate the accuracy of the different estimators. Finally, for illustrative purposes, analysis of a real data set is presented in Section 6.

2 Bayesian analyses

Let $X_1, \dots, X_n$ be a random sample of size n from the PL(γ, δ) distribution. For given data from this sample, $\mathbf{x} = (x_1, \dots, x_n)$, the likelihood function is given by

$$\mathcal{L}(\gamma, \delta; \mathbf{x}) = \frac{\gamma^n \delta^{2n}}{(\delta+1)^n} e^{-\delta \sum_{i=1}^n x_i^\gamma} \prod_{i=1}^n (1 + x_i^\gamma) x_i^{\gamma-1}. \tag{2.1}$$

In order to implement a Bayesian analysis, some prior distributions on the parameters are required. Here, we assume that γ and δ are independent random variables from the gamma models with respective densities

$$\pi_1(\gamma; a_1, b_1) \propto \gamma^{a_1-1} e^{-b_1\gamma}, \quad \gamma > 0, \tag{2.2}$$

$$\pi_2(\delta; a_2, b_2) \propto \delta^{a_2-1} e^{-b_2\delta}, \quad \delta > 0, \tag{2.3}$$

where the hyperparameters a_i, b_i, $i = 1, 2$, are positive. By combining the likelihood function (2.1) with the prior densities in (2.2) and (2.3), the posterior joint probability density function of γ and δ given the data can be written as

$$\pi^*(\gamma, \delta \mid \mathbf{x}) = \frac{\gamma^{n+a_1-1} e^{-\gamma b_1} \delta^{2n+a_2-1}}{C(\delta+1)^n} e^{-\delta(b_2 + \sum_{i=1}^n x_i^\gamma)} \prod_{i=1}^n (1 + x_i^\gamma) x_i^{\gamma-1} \tag{2.4}$$

in which

$$C = \int_0^\infty \int_0^\infty \frac{\gamma^{n+a_1-1} e^{-\gamma b_1} \delta^{2n+a_2-1}}{(\delta+1)^n} e^{-\delta(b_2 + \sum_{i=1}^n x_i^\gamma)} \prod_{i=1}^n (1 + x_i^\gamma) x_i^{\gamma-1} \, d\gamma \, d\delta.$$

In the following, considering the loss functions Δ_{SE}, Δ_{LE} and Δ_{GE} explained in the Introduction, we derive the Bayes estimates of γ, δ and reliability parameter $R(t)$. First, assuming SE loss function, the estimates of γ and δ become

$$\hat{\gamma}_{\mathrm{SE}} = E[\gamma \mid \mathbf{x}]$$

$$= \frac{1}{C}\int_0^\infty \int_0^\infty \frac{\gamma^{n+a_1} e^{-\gamma b_1} \delta^{2n+a_2-1}}{(\delta+1)^n} e^{-\delta(b_2+\sum_{i=1}^n x_i^\gamma)} \times \prod_{i=1}^n (1+x_i^\gamma) x_i^{\gamma-1}\, d\gamma\, d\delta \tag{2.5}$$

and

$$\begin{aligned}\hat{\delta}_{\mathrm{SE}} &= E[\delta \mid \mathbf{x}] \\ &= \frac{1}{C}\int_0^\infty \int_0^\infty \frac{\gamma^{n+a_1-1} e^{-\gamma b_1} \delta^{2n+a_2}}{(\delta+1)^n} e^{-\delta(b_2+\sum_{i=1}^n x_i^\gamma)} \\ &\quad \times \prod_{i=1}^n (1+x_i^\gamma) x_i^{\gamma-1}\, d\gamma\, d\delta,\end{aligned} \tag{2.6}$$

respectively. Also, the Bayes estimate of $R(t)$ against the loss Δ_{SE} can be derived as

$$\begin{aligned}\hat{R}_{\mathrm{SE}}(t) &= E\left[\left(1+\frac{\delta}{\delta+1}t^\gamma\right)e^{-\delta t^\gamma} \mid \mathbf{x}\right] \\ &= \frac{1}{C}\int_0^\infty \int_0^\infty \frac{\gamma^{n+a_1-1} e^{-\gamma b_1} \delta^{2n+a_2-1}}{(\delta+1)^n} e^{-\delta(b_2+\sum_{i=1}^n x_i^\gamma)} \\ &\quad \times \left(1+\frac{\delta}{\delta+1}t^\gamma\right)e^{-\delta t^\gamma} \prod_{i=1}^n (1+x_i^\gamma) x_i^{\gamma-1}\, d\gamma\, d\delta.\end{aligned} \tag{2.7}$$

By using the loss function Δ_{LE}, the Bayes estimates of γ, δ and the reliability parameter become, respectively,

$$\begin{aligned}\hat{\gamma}_{\mathrm{LE}} &= -\frac{1}{\nu}\log\bigl(E[\exp(-\nu\gamma) \mid \mathbf{x}]\bigr) \\ &= -\frac{1}{\nu}\log\left\{\frac{1}{C}\int_0^\infty \int_0^\infty \frac{\gamma^{n+a_1-1} e^{-\gamma(b_1+\nu)} \delta^{2n+a_2-1}}{(\delta+1)^n} e^{-\delta(b_2+\sum_{i=1}^n x_i^\gamma)} \right. \\ &\quad \left. \times \prod_{i=1}^n (1+x_i^\gamma) x_i^{\gamma-1}\, d\gamma\, d\delta\right\},\end{aligned} \tag{2.8}$$

$$\begin{aligned}\hat{\delta}_{\mathrm{LE}} &= -\frac{1}{\nu}\log\bigl(E[\exp(-\nu\delta) \mid \mathbf{x}]\bigr) \\ &= -\frac{1}{\nu}\log\left\{\frac{1}{C}\int_0^\infty \int_0^\infty \frac{\gamma^{n+a_1-1} e^{-\gamma b_1} \delta^{2n+a_2-1}}{(\delta+1)^n} e^{-\delta(b_2+\sum_{i=1}^n x_i^\gamma+\nu)} \right. \\ &\quad \left. \times \prod_{i=1}^n (1+x_i^\gamma) x_i^{\gamma-1}\, d\gamma\, d\delta\right\},\end{aligned} \tag{2.9}$$

and

$$\hat{R}_{\mathrm{LE}}(t) = -\frac{1}{\nu}\log\big(E\big[\exp\left(-\nu R(t)\right) \mid \mathbf{x}\big]\big)$$
$$= -\frac{1}{\nu}\log\frac{1}{C}\int_0^\infty\int_0^\infty \frac{\gamma^{n+a_1-1}e^{-\gamma b_1}\delta^{2n+a_2-1}}{(\delta+1)^n}e^{-\delta(b_2+\sum_{i=1}^n x_i^\gamma)}$$
$$\times \exp\left(-\nu\left(1+\frac{\delta}{\delta+1}t^\gamma\right)e^{-\delta t^\gamma}\right)\prod_{i=1}^n(1+x_i^\gamma)x_i^{\gamma-1}\,d\gamma\,d\delta. \quad (2.10)$$

Next, considering the entropy loss function Δ_{GE}, we obtain the Bayes estimates of the parameters as

$$\hat{\gamma}_{\mathrm{GE}} = E\big[\gamma^{-w} \mid \mathbf{x}\big]^{-\frac{1}{w}}, \quad (2.11)$$

$$\hat{\delta}_{\mathrm{GE}} = E\big[\delta^{-w} \mid \mathbf{x}\big]^{-\frac{1}{w}}, \quad (2.12)$$

and

$$\hat{R}_{\mathrm{GE}}(t) = E\left[\left(1+\frac{\delta}{\delta+1}t^\gamma\right)^{-w} e^{\delta w t^\gamma} \,\middle|\, \mathbf{x}\right]^{-\frac{1}{w}}, \quad (2.13)$$

in which the required conditional expectations can be calculated as

$$E\big[\gamma^{-w} \mid \mathbf{x}\big] = \frac{1}{C}\int_0^\infty\int_0^\infty \frac{\gamma^{n+a_1-w-1}e^{-\gamma b_1}\delta^{2n+a_2-1}}{(\delta+1)^n}e^{-\delta(b_2+\sum_{i=1}^n x_i^\gamma)}$$
$$\times\prod_{i=1}^n(1+x_i^\gamma)x_i^{\gamma-1}\,d\gamma\,d\delta,$$

$$E\big[\delta^{-w} \mid \mathbf{x}\big] = \frac{1}{C}\int_0^\infty\int_0^\infty \frac{\gamma^{n+a_1-1}e^{-\gamma b_1}\delta^{2n+a_2-w-1}}{(\delta+1)^n}e^{-\delta(b_2+\sum_{i=1}^n x_i^\gamma)}$$
$$\times\prod_{i=1}^n(1+x_i^\gamma)x_i^{\gamma-1}\,d\gamma\,d\delta,$$

$$E\left[\left(1+\frac{\delta}{\delta+1}t^\gamma\right)^{-w} e^{\delta w t^\gamma} \,\middle|\, \mathbf{x}\right]$$
$$= \frac{1}{C}\int_0^\infty\int_0^\infty \frac{\gamma^{n+a_1-1}e^{-\gamma b_1}\delta^{2n+a_2-1}}{(\delta+1)^n}e^{-\delta(b_2+\sum_{i=1}^n x_i^\gamma)}$$
$$\times\left(1+\frac{\delta}{\delta+1}t^\gamma\right)^{-w} e^{\delta w t^\gamma}\prod_{i=1}^n(1+x_i^\gamma)x_i^{\gamma-1}\,d\gamma\,d\delta.$$

It is clear that all the above Bayes estimators are involved the double integrals for which simple closed forms cannot be obtained. Therefore, in the next sections, we employ two popular approximation procedures to calculate the approximate Bayes estimates of the parameters.

3 Lindley approximation

In this section, we use Lindley technique for computing the Bayes estimates studied in Section 2. Let $h(\gamma, \delta)$ be any function of the parameters. Then, we have

$$\begin{aligned} E\big(h(\gamma,\delta) \mid \mathbf{x}\big) &= \int_0^\infty \int_0^\infty h(\gamma,\delta)\pi^*(\gamma,\delta \mid \mathbf{x})\, d\gamma\, d\delta \\ &= \frac{\int_0^\infty \int_0^\infty h(\gamma,\delta)\exp(L(\gamma,\delta;\mathbf{x}) + \eta(\gamma,\delta))\, d\gamma\, d\delta}{\int_0^\infty \int_0^\infty \exp(L(\gamma,\delta;\mathbf{x}) + \eta(\gamma,\delta))\, d\gamma\, d\delta}, \end{aligned} \tag{3.1}$$

where $\eta(\gamma,\delta) = \log \pi_1(\gamma; a_1, b_1) + \log \pi_2(\delta; a_2, b_2)$ and $L(\gamma,\delta;\mathbf{x})$ is the log-likelihood function of the parameters γ and δ given by

$$\begin{aligned} L(\gamma,\delta;\mathbf{x}) &= = \log \mathcal{L}(\gamma,\delta;\mathbf{x}) \\ &= n\log\gamma + 2n\log\delta - n\log(\delta+1) - \delta\sum_{i=1}^n x_i^\gamma \\ &\quad + \sum_{i=1}^n \big[\log\big(1 + x_i^\gamma\big) + (\gamma - 1)\log x_i\big]. \end{aligned} \tag{3.2}$$

According to Lindley (1980), the Bayes estimate of $h(\gamma,\delta)$ is approximated by

$$\begin{aligned} E\big[h(\gamma,\delta) \mid \mathbf{x}\big] &\approx h(\gamma,\delta) + \frac{1}{2}\sum_{i=1}^2\sum_{j=1}^2 h_{ij}\tau_{ij} + \sum_{i=1}^2 \eta_i V_i + \frac{1}{2}\sum_{i=1}^2 L_{iii}\tau_{ii}V_i + \\ &\quad + \frac{1}{2}\big[L_{112}(2\tau_{12}V_1 + \tau_{11}V_2) + L_{122}(\tau_{22}V_1 + 2\tau_{12}V_2)\big], \end{aligned} \tag{3.3}$$

where $h_1 = \partial h/\partial\gamma$, $h_2 = \partial h/\partial\delta$, $h_{11} = \partial^2 h/\partial\gamma^2$, $h_{22} = \partial^2 h/\partial\delta^2$, $h_{12} = h_{21} = \partial^2 h/\partial\gamma\partial\delta$ and $V_r = \sum_j h_j \tau_{rj}$ with τ_{ij} being the (i, j)th elements of the inverse of the matrix $[-\partial^2 L/\partial\gamma\, \partial\delta]$. Calculating all the expressions in (3.3) at the MLEs $\hat{\gamma}$ and $\hat{\delta}$ of the parameters γ and δ, the approximate Bayes estimate of $h(\gamma,\delta)$ is obtained. For more details about the MLEs and their standard errors, see Ghitany et al. (2013).

In our case, the required quantities in expression (3.3) are derived as

$$L_{11} = \frac{\partial^2 L(\gamma,\delta)}{\partial\gamma^2}\Big|_{\hat{\gamma},\hat{\delta}} = -\frac{n}{\hat{\gamma}^2} - \hat{\delta}\sum_{i=1}^n x_i^{\hat{\gamma}}(\log x_i)^2 + \sum_{i=1}^n \frac{x_i^{\hat{\gamma}}(\log x_i)^2}{(1 + x_i^{\hat{\gamma}})^2},$$

$$L_{12} = L_{21} = \frac{\partial^2 L(\gamma,\delta)}{\partial\gamma\partial\delta}\Big|_{\hat{\gamma},\hat{\delta}} = -\sum_{i=1}^n x_i^{\hat{\gamma}} \log x_i,$$

$$L_{22} = \frac{\partial^2 L(\gamma,\delta)}{\partial\delta^2}\Big|_{\hat{\gamma},\hat{\delta}} = -\frac{2n}{\hat{\delta}^2} + \frac{n}{(\hat{\delta}+1)^2},$$

$$L_{112} = \frac{\partial^3 L(\gamma,\delta)}{\partial\gamma^2\partial\delta}\Big|_{\hat\gamma,\hat\delta} = -\sum_{i=1}^{n} x_i^{\hat\gamma}(\log x_i)^2, \qquad L_{122} = \frac{\partial^3 L(\gamma,\delta)}{\partial\gamma\partial\delta^2}\Big|_{\hat\gamma,\hat\delta} = 0,$$

$$L_{111} = \frac{\partial^3 L(\gamma,\delta)}{\partial\gamma^3}\Big|_{\hat\gamma,\hat\delta} = \frac{2n}{\hat\gamma^3} - \hat\delta\sum_{i=1}^{n} x_i^{\hat\gamma}(\log x_i)^3 + \sum_{i=1}^{n}\frac{x_i^{\hat\gamma}(\log x_i)^3(1-x_i^{2\hat\gamma})}{(1+x_i^{\hat\gamma})^4},$$

$$L_{222} = \frac{\partial^3 \ell(\gamma,\delta)}{\partial\delta^3}\Big|_{\hat\gamma,\hat\delta} = \frac{4n}{\hat\delta^3} - \frac{2n}{(\hat\delta+1)^3},$$

$$\eta_1 = \frac{a_1-1}{\hat\gamma} - b_1, \qquad \eta_2 = \frac{a_2-1}{\hat\delta} - b_2.$$

In the following, we derive the approximate Bayes estimates of γ, δ and $R(t)$ under the considered loss functions (1.3), (1.5) and (1.7).

3.1 Lindley's approximate Bayes estimates using squared error loss function

For computing the estimate of γ under SE loss function, let $h(\gamma,\delta)=\gamma$. Hence, $h_1=1$, $h_2=h_{11}=h_{21}=h_{12}=h_{22}=0$ and we have

$$\hat\gamma_{\mathrm{SE}} \approx \hat\gamma + \tau_{11}\eta_1 + \tau_{21}\eta_2 + \frac{1}{2}\big(L_{111}\tau_{11}^2 + L_{222}\tau_{22}\tau_{12} + 3L_{112}\tau_{12}\tau_{11}\big). \tag{3.4}$$

Similarly, setting $h(\gamma,\delta)=\delta$, we have $h_2=1$, $h_1=h_{11}=h_{21}=h_{12}=h_{22}=0$. Therefore, the Bayes estimate of δ under SE loss function becomes

$$\hat\delta_{\mathrm{SE}} \approx \hat\delta + \tau_{22}\eta_2 + \tau_{12}\eta_1 + \frac{1}{2}\big(L_{111}\tau_{11}\tau_{12} + L_{222}\tau_{22}^2 + L_{112}\big(\tau_{11}\tau_{22} + 2\tau_{12}^2\big)\big). \tag{3.5}$$

Next, considering $h(\gamma,\delta)=R(t)$, the approximate Bayes estimate of $R(t)$ against loss function Δ_{SE} is obtained as

$$\begin{aligned}\hat R_{\mathrm{SE}}(t) \approx{}& \hat R(t;\gamma,\delta) + 0.5(R_{11}\tau_{11} + R_{22}\tau_{22} + 2R_{12}\tau_{12}) + \eta_1 V_1 + \eta_2 V_2\\ &+ 0.5\big[L_{111}\tau_{11}V_1 + L_{222}\tau_{22}V_2 + L_{112}(2\tau_{12}V_1 + \tau_{11}V_2)\big],\end{aligned} \tag{3.6}$$

where

$$V_1 = R_1\tau_{11} + R_2\tau_{12}, \qquad V_2 = R_1\tau_{12} + R_2\tau_{22},$$

$$R_1 = -\frac{\hat\delta^2}{\hat\delta+1} t^{\hat\gamma}(\log t)e^{-\hat\delta t^{\hat\gamma}}\big(1+t^{\hat\gamma}\big), \tag{3.7}$$

$$R_2 = -t^{\hat\gamma}e^{-\hat\delta t^{\hat\gamma}}\left[1 - \frac{1}{(\hat\delta+1)^2} + \frac{\hat\delta}{\hat\delta+1}t^{\hat\gamma}\right], \tag{3.8}$$

$$R_{11} = \hat\delta t^{\hat\gamma}(\log t)^2 e^{-\hat\delta t^{\hat\gamma}}\left[-1 - t^{\hat\gamma} + \frac{\hat\delta}{(\hat\delta+1)}\big(1 - 3t^{\hat\gamma} + \hat\delta t^{2\hat\gamma}\big)\right], \tag{3.9}$$

$$R_{22} = t^{\hat{\gamma}} e^{-\hat{\delta} t^{\hat{\gamma}}} \left[t^{\hat{\gamma}} - \frac{2}{(\hat{\delta}+1)^3} - \frac{\hat{\delta}}{(\hat{\delta}+1)} t^{2\hat{\gamma}} - \frac{2}{(\hat{\delta}+1)^2} t^{\hat{\gamma}} \right], \tag{3.10}$$

$$R_{12} = -t^{\hat{\gamma}} (\log t) e^{-\hat{\delta} t^{\hat{\gamma}}} \left[1 - \hat{\delta} + \frac{1}{(\hat{\delta}+1)^2} (1 - \hat{\delta} t^{\hat{\gamma}}) + \frac{\hat{\delta}}{(\hat{\delta}+1)} (2t^{\hat{\gamma}} - t^{2\hat{\gamma}}) \right]. \tag{3.11}$$

3.2 Lindley's approximate Bayes estimates using Linex loss function

Under LE loss function, the approximate Bayes estimates of γ is obtained by choosing $h(\gamma, \delta) = e^{-\nu\gamma}$. Thus, we have

$$\hat{\gamma}_{\text{LE}} \approx -\frac{1}{\nu} \log\{ e^{-\nu\hat{\gamma}} + 0.5 h_{11} \tau_{11} + h_1 (\tau_{11}\eta_1 + \tau_{21}\eta_2) + 0.5 h_1 (L_{111}\tau_{11}^2 + L_{222}\tau_{22}\tau_{12} + 3L_{112}\tau_{12}\tau_{11}) \},$$

where $h_1 = -\nu e^{-\nu\hat{\gamma}}$ and $h_{11} = \nu^2 e^{-\nu\hat{\gamma}}$.

Similarly, by choosing $h(\gamma, \delta) = e^{-\nu\delta}$ in (24), it is seen that

$$\hat{\delta}_{\text{LE}} \approx -\frac{1}{\nu} \log\{ e^{-\nu\hat{\delta}} + 0.5 h_{22} \tau_{22} + h_2 (\tau_{22}\eta_2 + \tau_{12}\eta_1) + 0.5 h_2 [L_{111}\tau_{11}\tau_{12} + L_{222}\tau_{22}^2 + L_{112} (\tau_{11}\tau_{22} + 2\tau_{12}^2)] \},$$

where $h_2 = -\nu e^{-\nu\hat{\delta}}$ and $h_{22} = \nu^2 e^{-\nu\hat{\delta}}$.

To evaluate the estimate of $R(t)$ under linex loss, let $h(\gamma, \delta) = e^{-\nu R(t)}$ and obtain $h_1 = -\nu e^{-\nu\hat{R}(t;\gamma,\delta)} R_1$, $h_2 = -\nu e^{-\nu R(t;\gamma,\delta)} R_2$, $h_{11} = \nu e^{-\nu\hat{R}(t;\gamma,\delta)} (\nu R_1^2 - R_{11})$, $h_{22} = \nu e^{-\nu\hat{R}(t;\gamma,\delta)} (\nu R_2^2 - R_{22})$ and $h_{12} = \nu e^{-\nu\hat{R}(t;\gamma,\delta)} (\nu R_1 R_2 - R_{12})$. Then,

$$\hat{R}_{\text{LE}}(t; \gamma, \delta) \approx -\frac{1}{\nu} \log\{ e^{-\nu\hat{R}(t;\gamma,\delta)} + 0.5 (h_{11}\tau_{11} + h_{22}\tau_{22} + 2h_{12}\tau_{12}) + V_1\eta_1 + V_2\eta_2 + 0.5 [L_{111}\tau_{11}V_1 + L_{222}\tau_{22}V_2 + L_{112}(2\tau_{12}V_1 + \tau_{11}V_2)] \},$$

where R_1, R_2, R_{11}, R_{22}, R_{12} are given by (3.7)–(3.11), respectively.

3.3 Lindley's approximate Bayes estimates using general entropy loss function

Setting $h(\gamma, \delta) = \gamma^{-w}$, the approximate Bayes estimates of γ under GE loss function is obtained as

$$\hat{\gamma}_{\text{GE}} \approx \{ \hat{\gamma}^{-\omega} + 0.5 h_{11}\tau_{11} + h_1 (\tau_{11}\eta_1 + \tau_{12}\eta_2) + 0.5 h_1 [L_{111}\tau_{11}^2 + L_{222}\tau_{12}\tau_{22} + 3L_{112}\tau_{12}\tau_{11})] \}^{-1/\omega},$$

where $h_1 = -\omega\gamma^{-\omega-1}$ and $h_{11} = \omega(\omega+1)\gamma^{-\omega-2}$.

Similarly, for the parameter δ, we use $h(\gamma,\delta)=\delta^{-w}$ and obtain

$$\hat{\delta}_{\mathrm{GE}} \approx \{\hat{\delta}^{-\omega} + 0.5h_{22}\tau_{22} + h_2(\tau_{22}\eta_2 + \tau_{12}\eta_1) + 0.5h_2[L_{111}\tau_{11}\tau_{12} + L_{222}\tau_{22}^2 + L_{112}(\tau_{11}\tau_{22} + \tau_{12}^2)]\}^{-1/\omega},$$

where $h_2 = -\omega\delta^{-\omega-1}$ and $h_{22} = \omega(\omega+1)\delta^{-\omega-2}$.

Finally, the reliability $R(t)$ can be estimated by considering $h(\gamma,\delta) = R(t)^{-w}$. Then, we have

$$\hat{R}_{\mathrm{GE}}(t;\gamma,\delta) \approx \{\hat{R}^{-w}(t;\gamma,\delta) + 0.5(h_{11}\tau_{11} + h_{22}\tau_{22} + 2h_{12}\tau_{12}) + V_1\eta_1 + V_2\eta_2 + 0.5[L_{111}\tau_{11}V_1 + L_{222}\tau_{22}V_2 + L_{112}(2\tau_{12}V_1 + \tau_{11}V_2)]\}^{-1/\omega},$$

where $h_1 = -\omega R(t)^{-\omega-1}R_1$, $h_2 = -\omega R(t)^{-\omega-1}R_2$, $h_{11} = \omega(\omega+1) \times R(t)^{-\omega-2}R_1^2 - \omega R(t)^{-\omega-1}R_{11}$, $h_{22} = \omega(\omega+1)R(t)^{-\omega-2}R_2^2 - \omega R(t)^{-\omega-1}R_{22}$ and $h_{12} = \omega(\omega+1)R(t)^{-\omega-2}R_1R_2 - \omega R(t)^{-\omega-1}R_{12}$.

4 MCMC

In this section, we adopt Gibbs sampling method to extract random samples from the conditional densities of the parameters and use them to calculate the Bayes estimates. From (2.4), the conditional posterior densities of γ and δ, respectively, can be extracted as

$$\pi_1^*(\gamma \mid \delta, \mathbf{x}) \propto \mathrm{Gamma}(n+a_1, b_1)e^{-\delta\sum_{i=1}^n x_i^\gamma}\prod_{i=1}^n(1+x_i^\gamma)x_i^{\gamma-1} \tag{4.1}$$

and

$$\pi_2^*(\delta \mid \gamma, \mathbf{x}) \propto \mathrm{Gamma}\left(2n+a_2, b_2+\sum_{i=1}^n x_i^\gamma\right)\frac{1}{(\delta+1)^n}. \tag{4.2}$$

Note that the conditional densities in (4.1) and (4.2) are not in the form of known distributions. We observed by experimentation that they are similar to normal distribution. Therefore, in the following algorithm we employ the well-known Metropolis–Hastings (M–H) with normal proposal distribution to generate samples from these distributions.

(1) Let the initial values of the parameters to be (γ^0, δ^0) and set $l = 1$.

(2) Considering the proposal distribution $q(\gamma) \equiv I(\gamma > 0)N(\gamma^{l-1}, 1)$ for the M–H method, generate γ^l, from $\pi_1^*(\gamma \mid \delta^{l-1}, \mathbf{x})$.

(3) Generate δ^l, from $\pi_2^*(\delta \mid \gamma^l, \mathbf{x})$ using M–H method with the proposal distribution $q(\delta) \equiv I(\delta > 0)N(\delta^{l-1}, 1)$.

(4) Compute $R(t;\gamma,\delta)$ from (2) and set $l = l+1$.

(5) Repeat Steps 2–4 for M times, and obtain γ^l, δ^l and $R^l(t;\gamma,\delta)$ for $l = 1, \dots, M$.

The steps of M–H technique used in the above algorithm can be described as follows:

- Set $\sigma = \mu^{l-1}$.
- Generate τ using the proposal distribution $q(\mu) \equiv I(\mu > 0)N(\mu^{l-1}, 1)$.
- Let $p(\sigma, \tau) = \min\{1, \pi_i^*(\tau)q(\sigma)/\pi_i^*(\sigma)q(\tau)\}$.
- Accept τ with probability $p(\sigma, \tau)$, or accept σ with probability $1 - p(\sigma, \tau)$.

By using the generated random samples from the above Gibbs technique, the Bayes estimates of the parameters γ, δ and reliability $R(t)$ against squared error loss function become $\tilde{\gamma}_{\mathrm{SE}} = \sum_{l=1}^{M} \gamma^l/M$, $\tilde{\delta}_{\mathrm{SE}} = \sum_{l=1}^{M} \delta^l/M$ and $\tilde{R}_{\mathrm{SE}}(t;\gamma,\delta) = \sum_{l=1}^{M} R^l(t;\gamma,\delta)/M$, respectively.

Next, considering LE loss function, the Bayes estimates of the parameters are obtained as

$$\tilde{\gamma}_{\mathrm{LE}} = -\frac{1}{\nu}\log\left(\frac{1}{M}\sum_{l=1}^{M} e^{-\nu\gamma^l}\right), \tag{4.3}$$

$$\tilde{\delta}_{\mathrm{LE}} = -\frac{1}{\nu}\log\left(\frac{1}{M}\sum_{l=1}^{M} e^{-\nu\delta^l}\right), \tag{4.4}$$

and

$$\tilde{R}_{\mathrm{LE}}(t;\gamma,\delta) = -\frac{1}{\nu}\log\left(\frac{1}{M}\sum_{l=1}^{M} e^{-\nu R^l(t;\gamma,\delta)}\right). \tag{4.5}$$

Finally, Bayes estimates of the parameters under GE loss function can be computed as

$$\tilde{\gamma}_{\mathrm{GE}} = \left(\frac{1}{M}\sum_{l=1}^{M}\left(\frac{1}{\gamma^l}\right)^{w}\right)^{-1/w}, \tag{4.6}$$

$$\tilde{\delta}_{\mathrm{GE}} = \left(\frac{1}{M}\sum_{l=1}^{M}\left(\frac{1}{\delta^l}\right)^{w}\right)^{-1/w}, \tag{4.7}$$

and

$$\tilde{R}_{\mathrm{GE}}(t;\gamma,\delta) = \left(\frac{1}{M}\sum_{l=1}^{M}\left(\frac{1}{R^l(t;\gamma,\delta)}\right)^{w}\right)^{-1/w}. \tag{4.8}$$

5 Simulation study

In the preceding sections, the estimates of the parameters γ, δ and reliability $R(t)$ are obtained by using Bayesian method. Considering different loss functions, all the Bayes estimators are derived and the corresponding approximate estimates are provided by applying Lindley and Gibbs sampling approaches. In this section, we performed Monte Carlo simulations to evaluate the behaviour of the proposed

Table 1 *Lindley approximation of the Bayes estimates of the parameters based on Prior I when* $(\gamma, \delta) = (2, 1)$

			L_{LE}			L_{GE}		
n		L_{SE}	$\nu = -0.5$	$\nu = 1$	$\nu = 1.5$	$w = -0.5$	$w = 1$	$w = 1.5$
15	γ	0.250032	0.295630	0.201307	0.187958	0.319410	0.298008	0.284657
	δ	0.061596	0.064693	0.056923	0.055508	0.069539	0.064841	0.064092
	$R(t)$	0.011503	0.011479	0.011605	0.011688	0.011875	0.012873	0.013256
20	γ	0.160382	0.168773	0.133702	0.126290	0.204023	0.173819	0.167660
	δ	0.042608	0.044210	0.040173	0.039401	0.046641	0.044369	0.043852
	$R(t)$	0.008236	0.008219	0.008297	0.008340	0.008404	0.009032	0.009258
30	γ	0.091785	0.098578	0.081668	0.078550	0.107706	0.095918	0.093657
	δ	0.028672	0.029398	0.027539	0.027150	0.030389	0.029351	0.029122
	$R(t)$	0.005680	0.005671	0.005710	0.005731	0.005760	0.006062	0.006176
50	γ	0.052990	0.055436	0.049155	0.047819	0.058290	0.053939	0.053114
	δ	0.016150	0.016749	0.016152	0.016028	0.017125	0.016834	0.016721
	$R(t)$	0.003369	0.003368	0.003379	0.003384	0.003396	0.003499	0.003540
70	γ	0.035529	0.036738	0.033613	0.032977	0.038196	0.035957	0.035546
	δ	0.011734	0.011854	0.011588	0.011542	0.012068	0.011993	0.011920
	$R(t)$	0.002439	0.002439	0.002440	0.002441	0.002449	0.002493	0.002512

Bayes estimators under various sample sizes. For comparison purposes, the ML estimates of the parameters are also computed. The performance of the competitive estimates has been compared in terms of their mean squared errors (MSE). All the calculations are conducted using R 2.14.0 (R Development Core Team (2011)).

We have considered two sets of parameter values as $(\gamma, \delta) = (2, 1), (1.5, 1.5)$ and three different choices of ν and w as $-0.5, 1, 1.5$ for both LE and GE loss functions. In each case, a random sample of size n is generated from PL model and the ML estimates of the unknown parameters are obtained from the system of equations provided by Ghitany et al. (2013). To evaluate the Bayes estimates, we take two different sets of hyper-parameter values as

Prior I: $a_1 = a_2 = b_1 = b_2 = 0.0001$.

Prior II: $a_1 = a_2 = b_1 = b_2 = 2$.

Then, approximate Bayes estimates of the parameters γ, δ and $R(t)$ are computed by using Lindley method. Tables 1–4 present the MSEs of the estimates obtained from 10,000 replications. Notice that all the ML and Bayes estimators of $R(t)$ are evaluated at $t = 1$. Further, the approximate Bayes estimates of the parameters are obtained by applying Gibbs sampling technique. To this end, Markov chains of size 7500 are generated and the first 2500 of the observations are removed to eliminate the effect of the starting distribution. Then, in order to reduce the dependence among the generated samples, we take every 5th sampled value which result in final chains of size 1000. The respective MSEs of the estimates based on 10,000 replications are reported in Tables 5–8. The following points are observed from the tabulated values.

Table 2 *Lindley approximation of the Bayes estimates of the parameters based on Prior I when* $(\gamma, \delta) = (1.5, 1.5)$

n		L_{SE}	L_{LE}			L_{GE}		
			$\nu = -0.5$	$\nu = 1$	$\nu = 1.5$	$w = -0.5$	$w = 1$	$w = 1.5$
15	γ	0.151673	0.178655	0.127650	0.118191	0.166916	0.142313	0.137388
	δ	0.124283	0.161149	0.119588	0.113283	0.133521	0.125747	0.123342
	$R(t)$	0.010421	0.010717	0.010773	0.010891	0.011160	0.012694	0.012751
20	γ	0.107981	0.118331	0.091376	0.085537	0.112504	0.098492	0.095672
	δ	0.097056	0.144670	0.085897	0.082564	0.092847	0.088994	0.088068
	$R(t)$	0.008274	0.008205	0.008220	0.008312	0.008458	0.009044	0.009199
30	γ	0.059570	0.063249	0.053536	0.051300	0.061126	0.055923	0.054897
	δ	0.055660	0.061490	0.054277	0.052834	0.057226	0.055561	0.055138
	$R(t)$	0.005154	0.005329	0.005334	0.005369	0.005438	0.005797	0.005933
50	γ	0.030904	0.032088	0.028920	0.028141	0.031473	0.029642	0.029289
	δ	0.031701	0.034571	0.032351	0.031892	0.033758	0.033275	0.032837
	$R(t)$	0.003040	0.003225	0.003229	0.003248	0.003383	0.003421	0.003644
70	γ	0.021194	0.021774	0.020211	0.019816	0.021464	0.020548	0.020371
	δ	0.022834	0.023264	0.022143	0.021901	0.022627	0.022370	0.022311
	$R(t)$	0.002242	0.002237	0.002238	0.002245	0.002256	0.002323	0.002362

(1) Under SE loss function, the Bayes estimates of γ obtained by using Lindley procedure have smaller MSEs than its ML estimates. Considering linex loss function, the choice $\nu = 1.5$ results in better performances, while for the GE

Table 3 *ML estimates and Lindley approximation of the Bayes estimates of the parameters based on Prior II when* $(\gamma, \delta) = (2, 1)$

n		MLE	L_{SE}	L_{LE}			L_{GE}		
				$\nu = -0.5$	$\nu = 1$	$\nu = 1.5$	$w = -0.5$	$w = 1$	$w = 1.5$
15	γ	0.263190	0.102527	0.139735	0.123660	0.094421	0.140526	0.139405	0.137784
	δ	0.064565	0.047698	0.049991	0.038613	0.036048	0.047953	0.041764	0.041513
	$R(t)$	0.012308	0.008072	0.007855	0.008540	0.008788	0.008647	0.010478	0.011062
20	γ	0.165813	0.087761	0.098601	0.094170	0.087842	0.105378	0.104510	0.103266
	δ	0.045399	0.035237	0.037563	0.031367	0.022931	0.037862	0.033631	0.033492
	$R(t)$	0.009147	0.006528	0.006417	0.006774	0.006907	0.006842	0.007863	0.008202
30	γ	0.102043	0.066959	0.068702	0.068290	0.067026	0.074987	0.072973	0.072768
	δ	0.028731	0.024575	0.026521	0.022801	0.022110	0.025699	0.023638	0.023557
	$R(t)$	0.005867	0.004733	0.004684	0.004843	0.004903	0.004877	0.005356	0.005529
50	γ	0.053796	0.042170	0.043047	0.041720	0.041565	0.044995	0.043904	0.043762
	δ	0.017306	0.015691	0.016051	0.015078	0.014835	0.016139	0.015423	0.015236
	$R(t)$	0.003580	0.003129	0.003114	0.003165	0.003183	0.003178	0.003348	0.003409
70	γ	0.037352	0.031564	0.032057	0.031149	0.031032	0.032971	0.032384	0.032298
	δ	0.011847	0.011043	0.011219	0.010744	0.010674	0.011526	0.011292	0.010991
	$R(t)$	0.002476	0.002247	0.002240	0.002264	0.002273	0.002271	0.002355	0.002387

Table 4 *ML estimates and Lindley approximation of the Bayes estimates of the parameters based on Prior II when* $(\gamma, \delta) = (1.5, 1.5)$

n		MLE	L_{SE}	L_{LE}			L_{GE}		
				$\nu = -0.5$	$\nu = 1$	$\nu = 1.5$	$w = -0.5$	$w = 1$	$w = 1.5$
15	γ	0.154112	0.078228	0.082565	0.074935	0.074017	0.084877	0.081068	0.080739
	δ	0.131162	0.063904	0.068775	0.067103	0.065495	0.072401	0.069529	0.061588
	$R(t)$	0.010533	0.006806	0.006494	0.006579	0.006944	0.006778	0.007734	0.008305
20	γ	0.105497	0.066505	0.071456	0.061337	0.061103	0.070091	0.066183	0.065980
	δ	0.100684	0.061576	0.064623	0.061214	0.060281	0.064320	0.062996	0.059798
	$R(t)$	0.008575	0.006243	0.006056	0.006109	0.006324	0.006257	0.006833	0.007158
30	γ	0.061084	0.046055	0.048515	0.042882	0.042077	0.047450	0.044926	0.044623
	δ	0.056740	0.042934	0.044383	0.041794	0.041631	0.043533	0.043098	0.042177
	$R(t)$	0.005250	0.004304	0.004215	0.004281	0.004328	0.004310	0.004529	0.004655
50	γ	0.032249	0.027738	0.028549	0.026571	0.026210	0.028102	0.027234	0.027115
	δ	0.032462	0.027605	0.028287	0.026909	0.026767	0.027553	0.027416	0.027384
	$R(t)$	0.003124	0.002773	0.002747	0.002775	0.002784	0.002783	0.002880	0.002931
70	γ	0.022409	0.020128	0.020560	0.019472	0.019246	0.020344	0.019822	0.019742
	δ	0.023140	0.020767	0.021047	0.020389	0.020251	0.020823	0.020763	0.020639
	$R(t)$	0.002291	0.002111	0.002095	0.002108	0.002118	0.002115	0.002157	0.002281

loss function, $w = 1.5$ provides reasonable MSEs. These conclusions are valid for both Lindley and MCMC techniques that were used for computing the Bayes estimates.

Table 5 *MCMC estimates of the parameters based on Prior I when* $(\gamma, \delta) = (2, 1)$

n		L_{SE}	L_{LE}			L_{GE}		
			$\nu = -0.5$	$\nu = 1$	$\nu = 1.5$	$w = -0.5$	$w = 1$	$w = 1.5$
15	γ	0.283689	0.217895	0.150149	0.129412	0.252481	0.187006	0.175455
	δ	0.065398	0.067693	0.059717	0.055318	0.0706561	0.065934	0.065019
	$R(t)$	0.011021	0.010940	0.011285	0.011470	0.011703	0.013340	0.014162
20	γ	0.165563	0.173438	0.110217	0.099419	0.152926	0.12662	0.122038
	δ	0.047357	0.048291	0.040972	0.038677	0.045312	0.043522	0.043017
	$R(t)$	0.007882	0.007839	0.008025	0.008136	0.008219	0.009907	0.010732
30	γ	0.091157	0.093495	0.073806	0.069715	0.087309	0.079440	0.078027
	δ	0.029914	0.030208	0.027447	0.026628	0.029181	0.028478	0.028341
	$R(t)$	0.005488	0.005470	0.005544	0.005584	0.005621	0.006248	0.006539
50	γ	0.052551	0.056618	0.046812	0.045081	0.051324	0.047615	0.047044
	δ	0.016759	0.017209	0.016085	0.015853	0.016575	0.016423	0.016397
	$R(t)$	0.003307	0.003304	0.003320	0.003331	0.003345	0.003517	0.003593
70	γ	0.035307	0.037116	0.032643	0.031772	0.034738	0.034434	0.033153
	δ	0.011776	0.011949	0.011523	0.011441	0.011786	0.011729	0.011723
	$R(t)$	0.002408	0.002408	0.002409	0.002410	0.002420	0.002483	0.002513

Table 6 *MCMC estimates of the parameters based on Prior I when* $(\gamma, \delta) = (1.5, 1.5)$

n		L_{SE}	L_{LLF}			L_{ELF}		
			$\nu = -0.5$	$\nu = 1$	$\nu = 1.5$	$w = -0.5$	$w = 1$	$w = 1.5$
15	γ	0.208389	0.220980	0.131726	0.104491	0.209078	0.146930	0.132507
	δ	0.265734	0.281878	0.185959	0.144508	0.212375	0.178872	0.145537
	$R(t)$	0.011917	0.011894	0.012057	0.012167	0.012263	0.012822	0.013187
20	γ	0.132869	0.163437	0.090127	0.076813	0.121572	0.094853	0.088547
	δ	0.154882	0.196609	0.104114	0.090274	0.142155	0.113509	0.107273
	$R(t)$	0.008771	0.008775	0.008812	0.008856	0.009403	0.012690	0.014178
30	γ	0.060585	0.063407	0.052336	0.048192	0.061701	0.053834	0.052011
	δ	0.062631	0.061897	0.059383	0.055117	0.065229	0.061756	0.060219
	$R(t)$	0.005486	0.005491	0.005495	0.005503	0.005697	0.005807	0.006334
50	γ	0.031949	0.034116	0.028517	0.027239	0.031004	0.028790	0.028265
	δ	0.036338	0.038434	0.033199	0.032125	0.035561	0.033907	0.033601
	$R(t)$	0.003255	0.003240	0.003244	0.003267	0.003303	0.003529	0.003765
70	γ	0.021599	0.022564	0.020032	0.019424	0.021168	0.020141	0.019873
	δ	0.023808	0.024699	0.022442	0.021960	0.023462	0.022734	0.022596
	$R(t)$	0.002243	0.002239	0.002240	0.002247	0.002266	0.002392	0.002454

(2) The Bayes estimates of δ based on prior II give better performances than the MLE in terms of minimum MSEs. Moreover, the choice $\nu = 1.5$ for the linex loss seems to be reasonable. For the GE loss function, $w = 1.5$ is a better value in computing the Bayes estimate based on the two approximation techniques. Therefore,

Table 7 *MCMC estimates of the parameters based on Prior II when* $(\gamma, \delta) = (2, 1)$

n		L_{SE}	L_{LE}			L_{GE}		
			$\nu = -0.5$	$\nu = 1$	$\nu = 1.5$	$w = -0.5$	$w = 1$	$w = 1.5$
15	γ	0.140984	0.188162	0.115760	0.109437	0.132786	0.131501	0.127385
	δ	0.048358	0.050570	0.045746	0.041123	0.055019	0.045710	0.044635
	$R(t)$	0.008317	0.008121	0.008792	0.009071	0.009078	0.010553	0.011561
20	γ	0.100028	0.118145	0.091329	0.087796	0.100842	0.097527	0.097102
	δ	0.037685	0.035132	0.034048	0.031794	0.038314	0.034191	0.033808
	$R(t)$	0.006531	0.006433	0.006773	0.006917	0.006919	0.008632	0.008910
30	γ	0.069861	0.066795	0.064508	0.063954	0.069315	0.068640	0.068299
	δ	0.026221	0.028013	0.023417	0.022389	0.025211	0.023337	0.023088
	$R(t)$	0.004637	0.004588	0.004754	0.004823	0.004813	0.0055543	0.005661
50	γ	0.042476	0.044328	0.040648	0.040590	0.042370	0.042126	0.042073
	δ	0.016098	0.016648	0.015213	0.014872	0.015796	0.015232	0.015158
	$R(t)$	0.003082	0.003068	0.003118	0.003179	0.003138	0.003363	0.003458
70	γ	0.031667	0.032547	0.030757	0.030711	0.031520	0.031597	0.031426
	δ	0.011198	0.011447	0.010791	0.010632	0.011061	0.010804	0.010775
	$R(t)$	0.002219	0.002213	0.002236	0.002245	0.002247	0.002349	0.002392

Table 8 *MCMC estimates of the parameters based on Prior II when* $(\gamma, \delta) = (1.5, 1.5)$

n		L_{SE}	L_{LLF}			L_{ELF}		
			$\nu=-0.5$	$\nu=1$	$\nu=1.5$	$w=-0.5$	$w=1$	$w=1.5$
15	γ	0.082743	0.097528	0.078449	0.066623	0.091364	0.084971	0.080027
	δ	0.080189	0.093071	0.089323	0.078350	0.095949	0.072730	0.082663
	$R(t)$	0.008013	0.007870	0.007950	0.008087	0.008536	0.008722	0.009178
20	γ	0.076814	0.084518	0.063228	0.056740	0.080369	0.066922	0.064531
	δ	0.075723	0.081501	0.071993	0.065924	0.079568	0.067502	0.061681
	$R(t)$	0.006861	0.006689	0.006799	0.006914	0.007045	0.007156	0.007328
30	γ	0.041272	0.047194	0.042802	0.040146	0.048976	0.044170	0.043290
	δ	0.043949	0.046620	0.045113	0.043294	0.050229	0.047251	0.047005
	$R(t)$	0.004502	0.004320	0.004449	0.004527	0.004573	0.004896	0.005127
50	γ	0.028712	0.028991	0.026394	0.025620	0.028104	0.026811	0.026647
	δ	0.029820	0.030142	0.027668	0.027014	0.029281	0.028317	0.028208
	$R(t)$	0.002810	0.002788	0.002793	0.002822	0.002837	0.003035	0.003104
70	γ	0.020483	0.020619	0.019342	0.018934	0.020187	0.019553	0.019429
	δ	0.020835	0.021085	0.020729	0.020422	0.021417	0.021151	0.021038
	$R(t)$	0.002131	0.002115	0.002189	0.002238	0.002159	0.002218	0.002262

$\hat{\delta}_{\text{LE}}$ and $\tilde{\delta}_{\text{LE}}$ with $\nu = 1.5$ and the estimator $\hat{\delta}_{\text{GE}}$ and $\tilde{\delta}_{\text{GE}}$ with $w = 1.5$ are superior than their respective competitors. But, the performances of these choices cannot be compared since for some cases $\hat{\delta}_{\text{LE}}$ and $\tilde{\delta}_{\text{LE}}$ are better while for other cases the opposite is true.

(3) For the reliability parameters $R(t)$, implementing the Bayesian procedure based on prior I, the MSE of the estimates are almost close to that of MLEs, however, prior II provides estimates with smaller MSEs compared to the ML method. Among estimators evaluated from the GE loss function, the choice $\nu = -0.5$ leads to better performances while $w = -0.5$ is a better value for the linex loss function.

(4) The ML estimates obtained based on larger sample size n have smaller MSEs as we expected. Similar improvements are observed for the Bayes estimates evaluated from different loss functions. Further, it is observed that in some cases, the MSEs of the Bayes estimates obtained from MCMC procedure are smaller than that of Bayesian Lindley estimates while for other cases the opposite is true. However, these MSE values are close to each other as the sample size increases.

Moreover, to observe the sensitivity of the Bayes estimators with respect to different informative priors, we have considered two different bivariate priors as follows:

Prior III: $a_1 = 5, a_2 = 2, b_1 = 1, b_2 = 4$.

Prior IV: $a_1 = 2.5, a_2 = 1.5, b_1 = 0.5, b_2 = 3$.

These priors are chosen with the same means but different variances. Notice that the variances of Prior III are smaller than that of Prior IV. For the two cases, the approximate Bayes estimates of parameters against the squared error, linex and general entropy loss functions are computed by applying Lindley and Gibbs

Table 9 *Lindley approximation of the Bayes estimates of the parameters based on Prior III when* $(\gamma, \delta) = (2, 1)$

n		L_{SE}	L_{LE}			L_{GE}		
			$\nu = -0.5$	$\nu = 1$	$\nu = 1.5$	$w = -0.5$	$w = 1$	$w = 1.5$
15	γ	0.107541	0.113278	0.109045	0.095538	0.151220	0.143967	0.135377
	δ	0.042733	0.048119	0.034729	0.031470	0.046932	0.041794	0.038621
	$R(t)$	0.007265	0.006946	0.007538	0.007820	0.008432	0.009351	0.009562
20	γ	0.092862	0.104752	0.091329	0.090729	0.122086	0.115242	0.108917
	δ	0.033807	0.035169	0.032531	0.026046	0.028737	0.034291	0.033654
	$R(t)$	0.006419	0.005172	0.005327	0.005418	0.006251	0.06946	0.007037
30	γ	0.061949	0.063719	0.062296	0.051371	0.064432	0.062128	0.058315
	δ	0.021752	0.025408	0.023621	0.023067	0.026721	0.024943	0.024509
	$R(t)$	0.004160	0.003807	0.004492	0.004813	0.004522	0.004965	0.005214
50	γ	0.042165	0.042939	0.041016	0.039237	0.041828	0.040719	0.040375
	δ	0.013812	0.014107	0.013354	0.012592	0.014931	0.012744	0.011662
	$R(t)$	0.002837	0.002671	0.002742	0.002931	0.002746	0.003197	0.003239
70	γ	0.021736	0.027798	0.026475	0.026133	0.029258	0.027503	0.027157
	δ	0.010891	0.011466	0.010235	0.009242	0.011770	0.010625	0.010391
	$R(t)$	0.001975	0.001839	0.002046	0.002091	0.002316	0.002471	0.002652

sampling techniques. The results based on priors III and IV are given in Tables 9–12. It is observed that MSEs of the Bayes estimates using Prior III in almost all cases are smaller than those using Prior IV.

6 Data analysis

To display the application of the different methods to real data, let us consider the following data set reported in Bader and Priest (1982) on the tensile strength of 69 carbon fibers tested under tension at gauge lengths of 20 mm:

1.312, 1.314, 1.479, 1.552, 1.700, 1.803, 1.861, 1.865, 1.944, 1.958, 1.966, 1.997, 2.006, 2.021, 2.027, 2.055, 2.063, 2.098, 2.14, 2.179, 2.224, 2.240, 2.253, 2.270, 2.272, 2.274, 2.301, 2.301, 2.359, 2.382, 2.382, 2.426, 2.434, 2.435, 2.478, 2.490, 2.511, 2.514, 2.535, 2.554, 2.566, 2.57, 2.586, 2.629, 2.633, 2.642, 2.648, 2.684, 2.697, 2.726, 2.770, 2.773, 2.800, 2.809, 2.818, 2.821, 2.848, 2.88, 2.954, 3.012, 3.067, 3.084, 3.090, 3.096, 3.128, 3.233, 3.433, 3.585, 3.585.

Ghitany et al. (2013) showed that the $\mathrm{PL}(\gamma, \delta)$ fits this data set very well. Here, considering various loss functions, we compute the approximate Bayes estimates of the unknown parameters. First, from the the above data set, the ML estimates of the parameters are obtained and the MLE of the reliability $R(t; \gamma, \delta)$ at $t = 1, 1.5$ are computed. To analyze the data under Bayesian perspective, all the hyper-parameters are considered to be 0.001 and three values -0.5, 1, 1.5 are assumed for both ν and w in the linex and entropy loss functions, respectively. Then, the Bayes estimates of the interested parameters under SE, LE and

Table 10 *Lindley approximation of the Bayes estimates of the parameters based on Prior IV when* $(\gamma, \delta) = (2, 1)$

n		L_{SE}	L_{LE} $\nu = -0.5$	$\nu = 1$	$\nu = 1.5$	L_{GE} $w = -0.5$	$w = 1$	$w = 1.5$
15	γ	0.113048	0.127291	0.121416	0.096081	0.152544	0.143718	0.136039
	δ	0.043165	0.048793	0.036529	0.032873	0.047172	0.044836	0.042297
	$R(t)$	0.008654	0.008137	0.008806	0.008413	0.009587	0.010968	0.011244
20	γ	0.095173	0.105884	0.096723	0.091845	0.123706	0.116394	0.111428
	δ	0.035075	0.037956	0.032691	0.026726	0.028394	0.034107	0.033956
	$R(t)$	0.006423	0.005605	0.006117	0.006792	0.006328	0.007831	0.008479
30	γ	0.062017	0.063991	0.062138	0.055826	0.070564	0.067719	0.061976
	δ	0.024328	0.026211	0.024703	0.023126	0.026947	0.024825	0.024493
	$R(t)$	0.004583	0.004130	0.004476	0.004795	0.004786	0.005213	0.005567
50	γ	0.041893	0.044728	0.041364	0.040329	0.045133	0.042794	0.042107
	δ	0.015139	0.015874	0.014737	0.013295	0.016076	0.014619	0.013942
	$R(t)$	0.002974	0.002643	0.002817	0.003139	0.002716	0.003328	0.003381
70	γ	0.024136	0.027860	0.027143	0.026528	0.029437	0.027821	0.027316
	δ	0.011239	0.011577	0.010852	0.010511	0.011963	0.011438	0.010758
	$R(t)$	0.002113	0.002128	0.002139	0.002278	0.002342	0.002574	0.002618

GE loss functions are calculated using Lindley's approximation. Moreover, we have used MCMC procedure developed in Section 4 to compute the Bayes estimates. Random samples of 200,000 realizations are generated from the posterior

Table 11 *MCMC estimates of the parameters based on Prior III when* $(\gamma, \delta) = (2, 1)$

n		L_{SE}	L_{LE} $\nu = -0.5$	$\nu = 1$	$\nu = 1.5$	L_{GE} $w = -0.5$	$w = 1$	$w = 1.5$
15	γ	0.098329	0.116507	0.112736	0.104971	0.142813	0.137669	0.135920
	δ	0.039617	0.041724	0.028293	0.027561	0.042319	0.040673	0.038893
	$R(t)$	0.007139	0.007514	0.007963	0.008107	0.007723	0.008349	0.008651
20	γ	0.091856	0.105273	0.103119	0.096347	0.114529	0.102653	0.098704
	δ	0.035916	0.036423	0.027819	0.026315	0.030798	0.029155	0.028243
	$R(t)$	0.005922	0.004719	0.005182	0.005207	0.006028	0.006717	0.007439
30	γ	0.061846	0.064953	0.064127	0.058511	0.070893	0.062214	0.057969
	δ	0.024716	0.028148	0.027064	0.025817	0.027633	0.025591	0.024108
	$R(t)$	0.004738	0.004206	0.004573	0.004724	0.004827	0.005036	0.005074
50	γ	0.040786	0.043615	0.039844	0.037027	0.044139	0.041532	0.041093
	δ	0.013827	0.014495	0.013218	0.012758	0.016843	0.014097	0.012718
	$R(t)$	0.003118	0.002967	0.003316	0.003428	0.003097	0.003586	0.004117
70	γ	0.023604	0.027153	0.024519	0.024389	0.027714	0.026548	0.025911
	δ	0.010427	0.009475	0.008537	0.008216	0.010594	0.009718	0.009357
	$R(t)$	0.002045	0.002079	0.002166	0.002341	0.002448	0.002635	0.002684

Table 12 *MCMC estimates of the parameters based on Prior IV when* $(\gamma, \delta) = (2, 1)$

n		L_{SE}	L_{LE}			L_{GE}		
			$\nu = -0.5$	$\nu = 1$	$\nu = 1.5$	$w = -0.5$	$w = 1$	$w = 1.5$
15	γ	0.108273	0.138966	0.132074	0.098451	0.163219	0.148705	0.142133
	δ	0.041428	0.045672	0.031971	0.030758	0.043986	0.042863	0.041791
	$R(t)$	0.007947	0.007693	0.008108	0.008327	0.008914	0.009469	0.010327
20	γ	0.091728	0.113945	0.109671	0.105138	0.124098	0.113876	0.105429
	δ	0.037265	0.038620	0.030477	0.029132	0.031564	0.030827	0.027359
	$R(t)$	0.006176	0.005539	0.005722	0.005804	0.006927	0.008516	0.009273
30	γ	0.062239	0.065718	0.064692	0.059117	0.071784	0.062938	0.058107
	δ	0.024967	0.028514	0.027177	0.026658	0.028804	0.025912	0.025041
	$R(t)$	0.004807	0.004223	0.004572	0.004761	0.005166	0.005397	0.005617
50	γ	0.040954	0.043871	0.040566	0.037248	0.044621	0.041795	0.041173
	δ	0.014328	0.014694	0.013507	0.012811	0.016944	0.014227	0.013316
	$R(t)$	0.003725	0.003178	0.003618	0.004493	0.003326	0.003717	0.004589
70	γ	0.023718	0.027951	0.025563	0.024819	0.028326	0.027182	0.026159
	δ	0.010748	0.011369	0.009633	0.009178	0.011246	0.010319	0.009475
	$R(t)$	0.002083	0.002146	0.002237	0.002411	0.002679	0.002813	0.002834

densities in (33)–(34) and the first 100,000 realizations are deleted to diminish the trace of initial samples. Then, one observation in every 10 iterations is saved to break the autocorrelation between generated samples. In order to check the convergence of MCMC chain, the scale reduction factor estimate $\sqrt{\mathrm{Var}(\Lambda)/U}$ is used in which Λ is the estimand of interest and $\mathrm{Var}(\Lambda) = (n-1)U/n + Q/n$ where n is the iteration number of each chain, and U and Q are the within and between sequence variances, respectively (see Gelman et al. (2003)). Here, the computed factor values of each estimands were less than 1.1, which is an acceptable value for their convergency. Further, the trace plots corresponding to Markov chains for the parameters are provided in Figures 1 and 2 which show the convergence of MCMC simulations. Tables 13 and 14 report different Bayes estimates of the parameters as well as their ML estimates. It is seen that in most of the cases, the Bayes estimates computed using Lindley and MCMC methods are very close to the corresponding MLEs. Applying MCMC approach for computing the approximate Bayes estimates, it is observed that the estimates of the parameter γ are slightly greater than its ML estimate while the opposite is true for the parameters δ and $R(t)$.

7 Conclusions

In most of the Bayesian studies, a well-known symmetric loss function, namely squared error loss, is employed for estimation purposes. But, to achieve more practical inferences under Bayesian viewpoint, it is also required to consider asymmetric loss functions. In this paper, we have discussed on Bayesian estimation of the

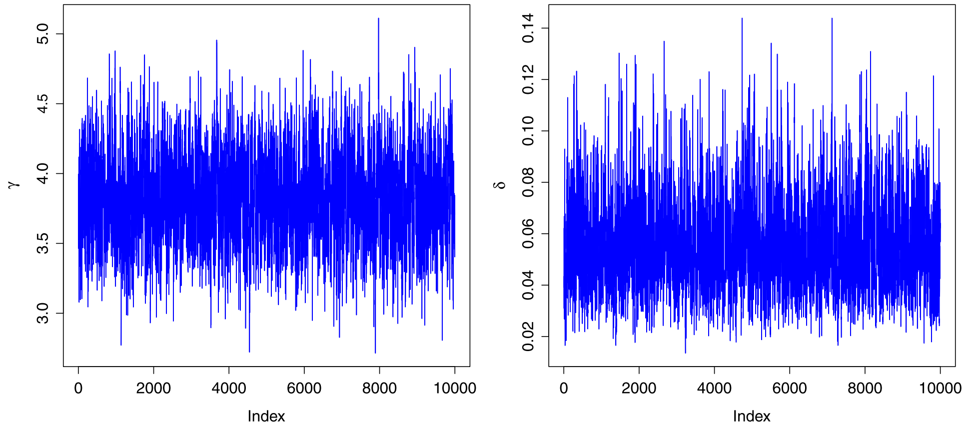

Figure 1 *Plots of Metropolis–Hastings Markov chains for the parameters γ and δ.*

unknown parameters and the reliability function from a two-parameter PL distribution by using these two types of loss functions. Indeed, considering squared error, linex and general entropy loss functions, all the Bayes estimates were computed by assuming gamma priors on the parameters. Since the Bayes estimates of the interested parameters could not be obtained analytically, we have provided Lindley's approach as well as the MCMC procedure to calculate the approximate Bayes estimates. Moreover, for comparison, the maximum likelihood estimates of the parameters are derived. In order to assess the accuracy of the different estimators, Monte Carlo simulations are conducted. It is observed that the performances of the Bayes estimators based on more informative priors are superior than the corresponding ML estimators. By increasing the sample size n, expected improvements are observed in the performances of all estimators. Further, the per-

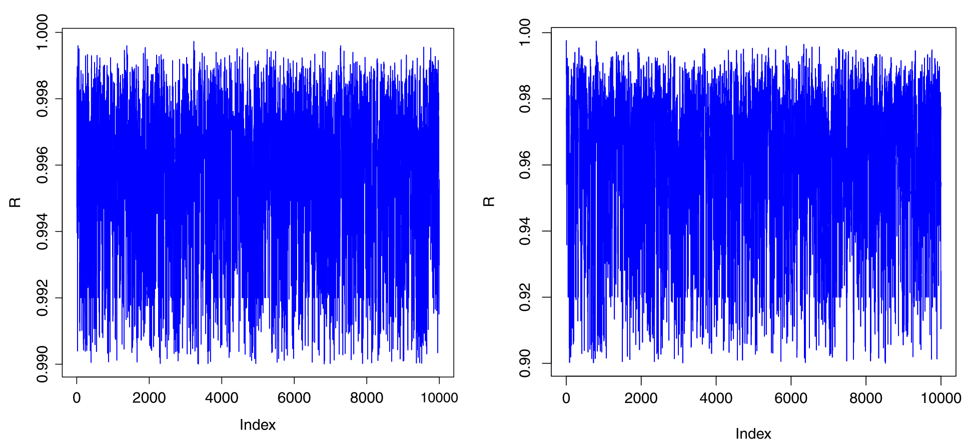

Figure 2 *Plots of Metropolis–Hastings Markov chains for $R(t)$ at $t = 1$ (left) and $t = 1.5$ (right).*

Table 13 *ML estimates and Lindley approximation of the Bayes estimates of the parameters*

Estimates		γ	δ	$R(t)$	
				$t=1$	$t=1.5$
MLE		3.867776	0.049670	0.996571	0.966846
L_{SE}		3.861718	0.052271	0.995905	0.955635
L_{LE}	$\nu=-0.5$	3.886501	0.052334	0.995906	0.955667
	$\nu=1$	3.52145	0.047246	0.995903	0.955612
	$\nu=1.5$	3.52081	0.048725	0.995902	0.955600
L_{GE}	$w=-0.5$	3.438702	0.046110	0.995904	0.955623
	$w=1$	3.417042	0.047805	0.995901	0.955588
	$w=1.5$	3.344282	0.049169	0.995900	0.955571

Table 14 *The Bayes estimates of the parameters from Gibbs sampling technique*

Estimates		γ	δ	$R(t)$	
				$t=1$	$t=1.5$
L_{SE}		3.901173	0.049406	0.996402	0.966835
L_{LE}	$\nu=-0.5$	3.917276	0.049450	0.996403	0.966827
	$\nu=1$	3.88792	0.049320	0.996400	0.966751
	$\nu=1.5$	3.870954	0.049277	0.996399	0.966743
L_{GE}	$w=-0.5$	3.896954	0.048583	0.996401	0.966816
	$w=1$	3.884054	0.046223	0.996398	0.966721
	$w=1.5$	3.879723	0.045575	0.996397	0.966690

formances of the MCMC and Lindley procedures cannot be compared since for some cases Lindley's approximation provides most precise parameter estimates while for other cases the opposite is true, but, the Lindley method is computationally slower.

Acknowledgments

The authors would like to thank the referees for their constructive comments and suggestions which improved and enriched the presentation of the paper.

References

Ahmed, E. A. (2014). Bayesian estimation based on progressive type II censoring from two-parameter bathtub-shaped lifetime model: An Markov chain Monte Carlo approach. *Journal of Applied Statistics* **41**, 752–768. MR3291784

Bader, M. G. and Priest, A. M. (1982). Statistical aspects of fibre and bundle strength in hyprid composites. In *International Conference on Composite Materials (ICCM-IV)* (T. Hayashi, K. Kawata and S. Umekawa, eds.), *Progress in Science and Engineering Composites* **4**, 1129–1136. *Tokyo, Japan*.

Banerjee, A. and Kundu, D. (2008). Inference based on type-II hybrid censored data from a Weibull distribution. *IEEE Transactions on Reliability* **57**, 369–378.

Gelman, A., Carlin, J. B., Stern, H. S. and Rubin, D. B. (2003). *Bayesian Data Analysis*, 2nd ed. London: Chapman and Hall. MR2027492

Ghitany, M. E., Al-Mutairi, D. K. and Aboukhamseen, S. M. (2015). Estimation of the reliability of a stress-strength system from power Lindley distributions. *Communications in Statistics Simulation and Computation* **44**, 118–136. MR3238554

Ghitany, M. E., Al-Mutairi, D. K., Balakrishnan, N. and Al-Enezi, L. J. (2013). Power lindley distribution and associated inference. *Computational Statistics & Data Analysis* **64**, 20–33. MR3061887

Lin, C. T., Chou, C. C. and Huang, Y. L. (2012). Inference for the Weibull distribution with progressive hybrid censoring. *Computational Statistics & Data Analysis* **56**, 451–467. MR2853746

Lindley, D. V. (1980). Approximate Bayesian method. *Trabajos de Estadistica* **31**, 223–237.

R Development Core Team (2011). *A Language and Environment for Statistical Computing*. Vienna, Austria: R Foundation for Statistical Computing.

A. Pak
Department of Computer Sciences
Shahrekord University
Shahrekord
Iran
E-mail: abbas.pak1982@gmail.com

M. E. Ghitany
Department of Statistics
and Operations Research
Faculty of Science
Kuwait University
Kuwait
E-mail: meghitany@yahoo.com

M. R. Mahmoudi
Department of Statistics
Fasa University
Fasa
Iran
E-mail: mahmoudi.m.r@fasau.ac.ir