

Stein characterizations for linear combinations of gamma random variables

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Abstract. In this paper we propose a new, simple and explicit mechanism allowing to derive Stein operators for random variables whose characteristic function satisfies a simple ODE. We apply this to study random variables which can be represented as linear combinations of (not necessarily independent) gamma distributed random variables. The connection with Malliavin calculus for random variables in the second Wiener chaos is detailed. An application to McKay Type I random variables is also outlined.

1 Introduction and overview

1.1 On Stein’s method

Stein’s method is a popular and versatile probabilistic toolkit for stochastic approximation. Presented originally in the context of Gaussian CLTs with dependent summands (see Stein (1972)) it has now been extended to cater for a wide variety of quantitative asymptotic results, see Chen, Goldstein and Shao (2010) for a thorough overview in the context of Gaussian approximation or <https://sites.google.com/site/steinsmethod> for an up-to-date list of references on non-Gaussian and non-Poisson Stein-type results.

Given two random objects F, F_∞ , Stein’s method allows to compute fine bounds on quantities of the form

$$\sup_{h \in \mathcal{H}} |\mathbb{E}[h(F)] - \mathbb{E}[h(F_\infty)]|$$

with \mathcal{H} some meaningful class of functions with respect to which both F and F_∞ are integrable (Zolotarev’s integral probability metrics Zolotarev (1983), which include for example, the total variation distance and the Kolmogorov distance, are of the above form). The method rests on three pins:

- A. a “Stein pair”, that is, a linear operator and a class of functions $(\mathcal{A}_\infty, \mathcal{F}(\mathcal{A}_\infty))$ such that $\mathbb{E}[\mathcal{A}_\infty(f(F_\infty))] = 0$ for all test functions $f \in \mathcal{F}(\mathcal{A}_\infty)$;
- B. a “Stein equation and its magic factors”, that is, a contractive inverse operator \mathcal{A}_∞^{-1} acting on the centered functions $\bar{h} = h - \mathbb{E}h(F_\infty)$ in \mathcal{H} and tight bounds on $\mathcal{A}_\infty^{-1}(\bar{h})$ and its derivatives;
- C. handles on the structure of F (such as $F = F_n = T(X_1, \dots, X_n)$ a U -statistic, $F = F(X)$ a functional of an isonormal Gaussian process, F a statistic on a random graph, etc.).

Given the conjunction of these three elements one can then apply some form of transfer principle:

$$\sup_{h \in \mathcal{H}} |\mathbb{E}[h(F)] - \mathbb{E}[h(F_\infty)]| = \sup_{h \in \mathcal{H}} |\mathbb{E}[\mathcal{A}_\infty(\mathcal{A}_\infty^{-1}(\bar{h}(F)))]|; \quad (1.1)$$

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remarkably the right-hand side of the above is often much more amenable to computations than the left-hand side, even in particularly unfavourable circumstances. This has resulted in Stein's method delivering several striking successes (see Barbour, Holst and Janson (1992), Chen, Goldstein and Shao (2010), Nourdin and Peccati (2012)) which have led the method to becoming the recognised and acclaimed tool it is today.

Given a target F_∞ , the identification of an appropriate Stein operator \mathcal{A}_∞ is the cornerstone of Stein's method. While historically most practical implementations relied on ad hoc arguments, several general tools exist, including Stein's *density approach* (Stein (1986)) and Barbour's *generator approach* (Barbour (1990)). A general theory for Stein operators is available in Ley, Reinert and Swan (2017). It is easy to see that, given any sufficiently regular target F_∞ , there are infinitely many admissible choices of operator \mathcal{A}_∞ and the difficulty is to identify those that shall lead to quantities useful for tackling (1.1). In many important cases, particularly Pearson or Ord random variables, these "useful" operators are first order differential operators (see Döbler (2015)) or difference operators (see Ley and Swan (2013)). Higher order differential operators are sometimes necessary to characterize more complex distributions, see Gaunt (2014), Peköz, Röllin and Ross (2013) for random variables with densities satisfying second order differential equations and Gaunt (2017, 2018), Gaunt, Mijoule and Swan (2019) for random variables which can be written as the product of independent Pearson variables satisfying certain conditions.

The purpose of this paper is to add to the literature on Stein's method by proposing a new, simple and explicit mechanism allowing to derive Stein operators for random variables whose characteristic function satisfies a simple ODE. We apply this to study random variables which can be represented as linear combinations of (not necessarily independent) gamma distributed random variables. The connection with Malliavin calculus for random variables in the second Wiener chaos is detailed. An application to the study of McKay Type I random variables is also outlined.

1.2 The Malliavin–Stein method and its extensions

If F_∞ is standard Gaussian random variable, then the Stein operator is $\mathcal{A}_\infty f(x) = f'(x) - xf(x)$ with $\mathcal{F}(\mathcal{A}_\infty)$ the class of all differentiable functions such that $\mathbb{E}|f'(F_\infty)| < \infty$. The simple structure of both the operator and the class, as well as the wide variety of possible choices for F , entail that all stars align beautifully well for a Gaussian target and that many paths are open for exploration. A particularly fruitful path was opened by Ivan Nourdin and Giovanni Peccati who, in Nourdin and Peccati (2009b), identified the possibility of intertwining Stein's method with Malliavin calculus. Given a sufficiently regular centered random variable F with finite variance and smooth density, the first step in this direction is to define its Stein kernel $\tau_F(F)$ through the integration by parts formula

$$\mathbb{E}[\tau_F(F)f'(F)] = \mathbb{E}[Ff(F)] \quad \text{for all absolutely continuous } f, \quad (1.2)$$

(see Stein's monograph Stein (1986) for the origins of this concept and for a detailed study when F is Pearson distributed). Then, for f_h a solution to $f'_h(x) - xf_h(x) = h(x) - \mathbb{E}[h(F_\infty)]$ (i.e. $f_h = \mathcal{A}_\infty^{-1}(\bar{h})$), we can write

$$\mathbb{E}[h(F)] - \mathbb{E}[h(F_\infty)] = \mathbb{E}[f'_h(F) - Ff_h(F)] = \mathbb{E}[(1 - \tau_F(F))f'_h(F)].$$

By the Cauchy–Schwarz inequality, we have

$$|\mathbb{E}[h(F)] - \mathbb{E}[h(F_\infty)]| \leq \|f'_h\| \sqrt{\mathbb{E}[(1 - \tau_F(F))^2]}$$

and at this stage two good things happen: (i) the constant $\sup_{h \in \mathcal{H}} \|f'_h\|$ (which is intrinsically Gaussian and does not depend on the law of F) is bounded for wide and relevant classes \mathcal{H} ; (ii) the quantity

$$S(F \parallel F_\infty) = \mathbb{E}[(1 - \tau_F(F))^2] \tag{1.3}$$

(called the *Stein discrepancy*) is tractable, via Malliavin calculus, as soon as F is a sufficiently regular functional of a Gaussian process because, in this case, the Stein kernel is $\tau_F(F) = \langle DF, -DL^{-1}F \rangle_{\mathcal{H}}$, where D and L^{-1} stand for Malliavin derivative and pseudo-inverse Ornstein–Uhlenbeck operators. These two realizations spawned an entire new field of research known as “Malliavin–Stein method” or as “Nourdin–Peccati” method, see Nourdin and Peccati (2009b), Nourdin and Peccati (2012) or the dedicated webpage <https://sites.google.com/site/malliavinstein>.

Extensions of the Malliavin–Stein method outside of the Gaussian framework have been studied as well. The first natural target to tackle is $F_\infty = 2G - d$, $d > 0$ where G has gamma law with parameter $d/2$ (i.e., F_∞ is centered gamma) with operator

$$\mathcal{A}_\infty f(x) = 2(x + d)f'(x) - xf(x), \tag{1.4}$$

see Luk (1994), Pickett (2004), Döbler and Peccati (2018) as well as Nourdin and Peccati (2009a), Azmoodeh, Campese and Poly (2014), Azmoodeh et al. (2016) for applications of the Malliavin–Stein method. Mimicking the Gaussian approach outlined above, one captures the difference in law between F_∞ and some arbitrary F by considering solutions to the ODE $2(x + d)f'_h(x) - xf_h(x) = h(x) - \mathbb{E}[h(F_\infty)]$ and

$$\begin{aligned} |\mathbb{E}[h(F)] - \mathbb{E}[h(F_\infty)]| &= \mathbb{E}[2(F + d)f'_h(F) - Ff_h(F)] \\ &\leq \|f'_h\| \mathbb{E}|2(F + d) - \tau_F(F)|. \end{aligned}$$

Again it is necessary, for the last bound to be of interest, that f'_h be uniformly bounded in h (see, for example, Luk (1994), Pickett (2004), Döbler and Peccati (2018)) and that $\tau_F(F)$ have good properties; see Döbler and Peccati (2018) and Nourdin and Peccati (2009b), Section 3.3, for an illustration as well as Kusuoka and Tudor (2012), Eden and Viquez (2015) for further explorations for targets F_∞ belonging to the Pearson family.

Important progress in this direction is due to Gaunt (2014, 2017). In Gaunt (2017), he shows that if $F_\infty = N_1 \times N_2$ where N_1 and N_2 are two independent $\mathcal{N}(0, 1)$ random variables then its law is characterized by a *second order* Stein equation

$$xf''(x) + f'(x) - xf(x) = h(x) - \mathbb{E}[h(F_\infty)] \tag{1.5}$$

and in Gaunt (2013, 2014) he studies the entire family of Variance-Gamma distributions (see Example 2.3 below), obtains Stein operators \mathcal{A}_∞ and also bounds on the solutions on the resulting Stein equations $\mathcal{A}_\infty f = h$ under smoothness assumptions on h . These results are used in Eichelsbacher and Thäle (2014), where Gaunt’s estimates are combined with higher order Stein kernels firstly introduced in Nourdin and Peccati (2010) (see below, Definition 1.1) in order to extend the scope of the Nourdin–Peccati approach to targets of the form

$$F_\infty = \sum_{i=1}^d \alpha_{\infty,i} (N_i^2 - 1) \tag{1.6}$$

where the coefficients $\{\alpha_{\infty,i} : i = 1, \dots, d\}$ are non-zero and distinct and the N_i are i.i.d. standard Gaussian (actually $d = 2$ in Eichelsbacher and Thäle (2014), but we shall consider the general case from here onwards).

As we shall see (e.g., in (2.3)), random variables of the form (1.6) are characterized by Stein operators \mathcal{A}_∞ which are differential operators of order d . In order for Nourdin and

Peccati's approach to function for such operators one needs to introduce higher order versions of the Stein kernel (1.2), one for each degree of the operator. This is exactly the purpose of Section 2.1.

Definition 1.1 (see [Nourdin and Peccati \(2012\)](#)). Let $F \in \mathbb{D}^\infty$ the class of infinitely many times Malliavin differentiable random variables (see [Nourdin and Peccati \(2012\)](#), Chapter 2, for a detailed discussion). The sequence of random variables $\{\Gamma_i(F)\}_{i \geq 0} \subset \mathbb{D}^\infty$ is recursively defined as follows. Set $\Gamma_0(F) = F$ and, for every $i \geq 1$,

$$\Gamma_i(F) = \langle DF, -DL^{-1}\Gamma_{i-1}(F) \rangle_{\mathcal{H}}.$$

Iterated gammas from Definition 1.1 are higher order versions of the Stein kernel (1.2); by definition we have $\Gamma_1(F) = \tau_F(F)$. Also note how $\mathbb{E}[\tau_F(F)] = \text{Var}(F)$ and (see again [Nourdin and Peccati \(2012\)](#)) the cumulants of the random element F and the iterated Malliavin Γ -operators are linked by the relation $\kappa_{r+1}(F) = r! \mathbb{E}[\Gamma_r(F)]$ for $r = 0, 1, \dots$.

Targets F_∞ of the form (1.6) admit operators $\mathcal{A}_\infty f = \sum_{j=0}^d a_j f^{(j)}$ with a_j polynomials and $d \geq 1$. Mimicking the Gaussian and gamma cases, a direct extension of the Nourdin–Peccati approach then consists, in principle, in writing out

$$\begin{aligned} \mathbb{E}[h(F)] - \mathbb{E}[h(F_\infty)] &= \mathbb{E} \left[\sum_{j=0}^d a_j(F) f_h^{(j)}(F) \right] \\ &= \mathbb{E} \left[\sum_{j=0}^d (\tilde{a}_j(F) - \Gamma_j(F)) f_h^{(j)}(F) \right]. \end{aligned}$$

for f_h a solution to the ODE $\mathcal{A}_\infty f(x) = h(x) - \mathbb{E}[h(F_\infty)]$. In order for this approach to be useful it is necessary that both $\Gamma_j(F)$ and $f_h^{(j)}$ be tractable. So far the question of finding tight bounds on solutions to such higher order Stein equations is open; this seems to be a difficult problem to tackle in all generality.

Estimates on the derivatives of solutions to Stein equation are, however, not crucial for a version of Stein's method to apply to variables of the form (1.6), see the paragraph after Proposition 2.1, and [Arras et al. \(2019\)](#) for more details.

2 Stein-type characterization and main results

2.1 Stein operators for the second Wiener chaos

The aim of this section is to use the recent findings in [Azmoodeh, Peccati and Poly \(2014\)](#) to derive “appropriate” Stein equations, i.e. differential operators of finite order with polynomial coefficients, for random elements in the second Wiener chaos. Following [Nourdin and Poly \(2012\)](#), [Azmoodeh, Peccati and Poly \(2014\)](#), first we define two crucial polynomials P and Q as follows:

$$Q(x) = (P(x))^2 = \left(x \prod_{i=1}^d (x - \alpha_{\infty,i}) \right)^2. \quad (2.1)$$

Next, for some random element F living in a finite sum of Wiener chaoses, we consider the following quantity (whose first occurrence is in [Azmoodeh, Peccati and Poly \(2014\)](#))

$$\Delta(F, F_\infty) := \sum_{r=2}^{\deg(Q)} \frac{Q^{(r)}(0)}{r!} \frac{\kappa_r(F)}{2^{r-1}(r-1)!}. \quad (2.2)$$

Then the following result holds.

Proposition 2.1 (Azmoodeh, Peccati and Poly (2014), Proposition 3.2). *Let F be a centered random variable living in a finite sum of Wiener chaoses. Moreover, assume that*

- (i) $\kappa_r(F) = \kappa_r(F_\infty)$, for all $2 \leq r \leq d + 1 = \text{deg}(P)$, and
- (ii)

$$\mathbb{E} \left[\sum_{r=1}^{d+1} \frac{P^{(r)}(0)}{r!2^{r-1}} (\Gamma_{r-1}(F) - \mathbb{E}[\Gamma_{r-1}(F)]) \right]^2 = 0.$$

Then, $F \stackrel{\text{law}}{=} F_\infty$, and F belongs to the second Wiener chaos.

As we shall see in Section 2.1, Proposition 2.1 leads to Stein operators for random variables belonging to the second Wiener chaos. By analogy with the case of a Gaussian target, it appears that the quantity $\Delta(F, F_\infty)$ is the second-chaos equivalent of the (first-Wiener chaos) Stein discrepancy $S(F \parallel F_\infty)$ (1.3). Moreover we have shown, in a separate publication Arras et al. (2019), that estimating this quantity directly (without requiring any bounds on solutions to Stein equations) leads to bounds on the Wasserstein-2 distance between the law of F and the law of F_∞ .

We now show how item (ii) of Proposition 2.1 can be used to derive a Stein operator for F_∞ . To this end, set

$$a_l = \frac{P^{(l)}(0)}{l!2^{l-1}}, \quad 1 \leq l \leq d + 1,$$

$$b_l = \sum_{r=l}^{d+1} a_r \mathbb{E}[\Gamma_{r-l+1}(F_\infty)] = \sum_{r=l}^{d+1} \frac{a_r}{(r-l+1)!} \kappa_{r-l+2}(F_\infty), \quad 2 \leq l \leq d + 1.$$

Now, we introduce the following differential operator of order d (acting on functions $f \in C^d(\mathbb{R})$):

$$A_\infty f(x) := \sum_{l=2}^{d+1} (b_l - a_{l-1}x) f^{(d+2-l)}(x) - a_{d+1}x f(x). \tag{2.3}$$

Then, we have the following result (see Section 3 for a proof).

Theorem 2.1 (Stein characterization). *Assume that F is a general centered random variable living in a finite sum of Wiener chaoses (and hence smooth in the sense of Malliavin calculus) and let A_∞ be as in (2.3). Then $F=F_\infty$ (equality in distribution) if and only if $\mathbb{E}[A_\infty f(F)] = 0$ for all polynomials $f : \mathbb{R} \rightarrow \mathbb{R}$.*

Example 2.1. Consider the special case of only two non-zero distinct eigenvalues λ_1 and λ_2 , that is,

$$F_\infty = \lambda_1(N_1^2 - 1) + \lambda_2(N_2^2 - 1), \tag{2.4}$$

where $N_1, N_2 \sim \mathcal{N}(0, 1)$ are independent. In this case, the polynomial P takes the form $P(x) = x(x - \lambda_1)(x - \lambda_2)$. Simple calculations reveal that $P'(0) = \lambda_1\lambda_2$, $P''(0) = -2(\lambda_1 + \lambda_2)$, and $P^{(3)}(0) = 3!$. Also, $\kappa_2(F_\infty) = \mathbb{E}[\gamma_1(F_\infty)] = 2(\lambda_1^2 + \lambda_2^2)$, and $\kappa_3(F_\infty) = 2\mathbb{E}[\gamma_2(F_\infty)] = 4(\lambda_1^3 + \lambda_2^3)$. Then, the Stein equation (2.3) reduces to

$$A_\infty f(x) = -4(\lambda_1\lambda_2x + (\lambda_1 + \lambda_2)\lambda_1\lambda_2) f''(x) + 2(\lambda_1^2 + \lambda_2^2 + (\lambda_1 + \lambda_2)x) f'(x) - x f(x). \tag{2.5}$$

We also remark that when $\lambda_1 = -\lambda_2 = \frac{1}{2}$, and hence $F_\infty \stackrel{\text{law}}{=} N_1 \times N_2$, the Stein equation (2.5) coincides with that in Gaunt (2017), equation (1.9). One has to note that for general λ_1 and λ_2 , the random variables of the form (2.4) lie outside the *Variance-Gamma* class, see also Example 2.3.

2.2 A Fourier approach to Stein characterizations

The characteristic functions $\phi_F(\xi) = \mathbb{E}[e^{i\xi F}]$ (we drop the indexation in ∞ and write F instead of F_∞ , α_i instead of $\alpha_{\infty,i}$ etc. from now on) of random variables of the form (1.6) satisfy a simple ODE with polynomial coefficients, namely

$$\prod_{j=1}^d (1 - 2i\xi\alpha_j)\phi'_F(\xi) = -2\xi \sum_{j=1}^d \alpha_j^2 \prod_{k \neq j} (1 - 2i\xi\alpha_k)\phi_F(\xi). \quad (2.6)$$

Such are but particular cases of a wide family of variables to which the following simple lemma applies (see Section 3 for a proof).

Lemma 2.1. *Let $(a_k)_{0 \leq k \leq d}$ and $(b_k)_{0 \leq k \leq d'}$ be real numbers and consider the polynomials $A_d(\xi) = \sum_{k=0}^d a_k \xi^k$ and $B_{d'}(\xi) = \sum_{k=0}^{d'} b_k \xi^k$ with $d, d' \in \mathbb{N}$. Assume that the random variable F has a continuously differentiable characteristic function ϕ_F on \mathbb{R} such that*

$$A_d(i\xi)\phi'_F(\xi) = iB_{d'}(i\xi)\phi_F(\xi), \quad \xi \in \mathbb{R}. \quad (2.7)$$

Let Y be a real valued random variable such that $\mathbb{E}[|Y|] < +\infty$. Then $Y \stackrel{\text{law}}{=} F$ if and only if

$$\mathbb{E}[Y A_d(f)(Y) - B_{d'}(f)(Y)] = 0 \quad (2.8)$$

for all test functions $f \in \mathcal{S}(\mathbb{R}) := \{f \in C^\infty(\mathbb{R}) \mid \sup_{x \in \mathbb{R}} |x^\alpha f^{(\beta)}(x)| < \infty, \text{ for all } \alpha, \beta \in \mathbb{N}_0\}$ the Schwartz space of smooth functions with rapid decrease, where

$$A_d = \sum_{k=0}^d a_k \frac{d^k}{dx^k} \quad \text{and} \quad B_{d'} = \sum_{k=0}^{d'} b_k \frac{d^k}{dx^k}. \quad (2.9)$$

The differential operator $f \mapsto \mathcal{A}_\infty(f)(x) = xA_d(f)(x) - B_{d'}(f)(x)$ is a Stein operator for F with Stein class $\mathcal{S}(\mathbb{R})$.

From here onwards all test functions f are supposed to belong to $\mathcal{S}(\mathbb{R})$.

Example 2.2. If F is a normal random variable with mean μ and variance σ^2 , then $\phi_F(\xi) = e^{i\mu\xi - \sigma^2\xi^2/2}$ so that $\phi'_F(\xi) = i(\mu + \sigma^2(i\xi))\phi_F(\xi)$ and, in the notations of Lemma 2.1: $d = 0$, $a_0 = 1$, $d' = 1$, $b_0 = \mu$, and $b_1 = \sigma^2$ so that F is characterized by the identity

$$\mathbb{E}[Ff(F) - (\mu f(F) + \sigma^2 f'(F))] = 0$$

as expected (see Chen, Goldstein and Shao (2010)).

Example 2.3. If F is Variance-Gamma distributed, then its cumulant generating function is, in the classical parameterization,

$$\log \phi_F(\xi) = \mu i\xi + 2\lambda \log \gamma - \lambda \log(\alpha^2 - (\beta + i\xi)^2)$$

so that Lemma 2.1 applies with $d = 2$, $a_0 = \alpha^2 - \beta^2$, $a_1 = -2\beta$, and $a_2 = -1$, $d' = 2$, $b_0 = \mu(\alpha^2 - \beta^2) + 2\lambda\beta$, $b_1 = 2(\lambda - \mu\beta)$, and $b_2 = -\mu$ so that F is characterized by

$$\begin{aligned} & \mathbb{E}[F((\alpha^2 - \beta^2)f(F) - 2\beta f'(F) - f''(F)) \\ & - ((\mu(\alpha^2 - \beta^2) + 2\lambda\beta)f(F) + 2(\lambda - \mu\beta)f'(F) - \mu f''(F))] = 0 \end{aligned}$$

or, after simplifications,

$$\begin{aligned} &\mathbb{E}[(F - \mu)f''(F) + (2\beta(F - \mu) + 2\lambda)f'(F) \\ &\quad + ((\alpha^2 - \beta^2)(F - \mu) + 2\lambda\beta)f(F)] = 0. \end{aligned}$$

This is the result obtained by Gaunt (2014), Lemma 3.1.

Example 2.4. Take $\alpha_i = 1$ for all $i \geq 1$ in (1.6), i.e. $F = \sum_{i=1}^d (N_i^2 - 1) \sim \chi_{(d)}^2$ is a centered chi-squared random variable with d degree of freedom. The CF of a chi-squared distributed random variable is $\phi(\xi) = (1 - 2i\xi)^{-d/2}$, and so

$$(1 - 2i\xi)\phi'_F(\xi) = -2d\xi\phi_F(\xi).$$

Again, Lemma 2.1 applies, this time with $d = 1$, $a_0 = 1$, $a_1 = -2$, $d' = 1$, $b_0 = 0$, and $b_1 = 2d$ so that

$$\mathbb{E}[Ff(F)] = 2\mathbb{E}[(F + d)f'(F)].$$

This is the same as (1.4).

Example 2.5. A random variable F follows the type I McKay distribution with parameters $a > -(1/2)$, $b > 0$ and $c > 1$ when its PDF is proportional to the function

$$\forall x \in \mathbb{R}_+, \quad f_I(x) = x^a e^{-xc/b} I_a(x/b), \tag{2.10}$$

where $I_a(\cdot)$ denotes the modified Bessel function of the first kind and of order a , see Holm and Alouini (2004) for context and further references. Direct computations lead to

$$(\log \phi_F)'(\xi) = -i \frac{(1 + 2a)bc - (1 + 2a)b^2(i\xi)}{1 - c^2 + 2cb(i\xi) - b^2(i\xi)^2}. \tag{2.11}$$

Lemma 2.1 applies and we deduce that if F is type I McKay then

$$\begin{aligned} &\mathbb{E}[(1 - c^2)F + (1 + 2a)bc]f(F) \\ &\quad + (2cbF - (1 + 2a)b^2)f'(F) - b^2Ff''(F) = 0 \end{aligned} \tag{2.12}$$

for all $f \in \mathcal{S}(\mathbb{R})$.

2.3 Stein operators for sums of independent gamma

Before stating the next theorem, we need to introduce some notations. For any d -tuple $(\lambda_1, \dots, \lambda_d)$ of real numbers, we define the symmetric elementary polynomial of order $k \in \{1, \dots, d\}$ evaluated at $(\lambda_1, \dots, \lambda_d)$ by

$$e_k(\lambda_1, \dots, \lambda_d) = \sum_{1 \leq i_1 < i_2 < \dots < i_k \leq d} \lambda_{i_1} \cdots \lambda_{i_k}.$$

We set, by convention, $e_0(\lambda_1, \dots, \lambda_d) = 1$. Moreover, for any $(c_1, \dots, c_d) \in \mathbb{R}^*$ and any $k \in \{1, \dots, d\}$, we denote by (λc) the d tuple $(\lambda_1 c_1, \dots, \lambda_d c_d)$ and by $(\lambda c)_k$ the $d - 1$ tuple $(\lambda_1 c_1, \dots, \lambda_{k-1} c_{k-1}, \lambda_{k+1} c_{k+1}, \dots, \lambda_d c_d)$

The objects of interest in this section are the following generalizations of (1.6): for $d \geq 1$, $(m_1, \dots, m_d) \in \mathbb{N}^d$, $(\lambda_1, \dots, \lambda_d) \in \mathbb{R} \setminus \{0\}$ all distinct we consider

$$F = \sum_{i=1}^d \lambda_i (\gamma_i(m_i \alpha_i, c_i) - m_i \alpha_i c_i), \tag{2.13}$$

where, for any $(\alpha, c) \in \mathbb{R}_+^*$, we denote by $\gamma(\alpha, c)$ a gamma random variable with parameters (α, c) with density

$$\forall x \in \mathbb{R}_+ \setminus \{0\}, \quad \gamma_{\alpha,c}(x) = \frac{1}{c\Gamma(\alpha)} \left(\frac{x}{c}\right)^{\alpha-1} e^{-\frac{x}{c}}$$

and CF

$$\phi_{\gamma(\alpha,c)}(\xi) = (1 - ic\xi)^{-\alpha}.$$

The family $\{\gamma_i(m_i\alpha_i, c_i), i = 1, \dots, d\}$ is a collection of independent random variables. Applying Lemma 2.1 we obtain the following (proof in Section 3).

Theorem 2.2. *Let F be as in (2.13) and let Y be a real valued random variable such that $\mathbb{E}[|Y|] < +\infty$. Then $Y \stackrel{\text{law}}{=} F$ if and only if*

$$\begin{aligned} \mathbb{E} \left[\left(Y + \sum_{i=1}^d \lambda_i m_i \alpha_i c_i \right) (-1)^d \left(\prod_{j=1}^d \lambda_j c_j \right) f^{(d)}(Y) + \sum_{l=1}^{d-1} (-1)^l \left(Y e_l((\lambda c)) \right) \right. \\ \left. + \sum_{k=1}^d \lambda_k m_k \alpha_k c_k (e_l((\lambda c)) - e_l((\lambda c)_k)) \right) f^{(l)}(Y) + Y f(Y) \Big] = 0 \end{aligned} \quad (2.14)$$

for all $f \in \mathcal{S}(\mathbb{R})$.

Taking $\alpha_k = 1/2$ and $c_k = 2$ in the previous theorem implies the following straightforward corollary.

Corollary 2.1. *Let $d \geq 1, q \geq 1$ and $(m_1, \dots, m_d) \in \mathbb{N}^d$ such that $m_1 + \dots + m_d = q$. Let $(\lambda_1, \dots, \lambda_d) \in \mathbb{R} \setminus \{0\}$ pairwise distinct and consider*

$$F = \sum_{i=1}^{m_1} \lambda_1 (N_i^2 - 1) + \sum_{i=m_1+1}^{m_1+m_2} \lambda_2 (N_i^2 - 1) + \dots + \sum_{i=m_1+\dots+m_{d-1}+1}^q \lambda_d (N_i^2 - 1).$$

Let Y be a real valued random variable such that $\mathbb{E}[|Y|] < +\infty$. Then $Y \stackrel{\text{law}}{=} F$ if and only if

$$\begin{aligned} \mathbb{E} \left[\left(Y + \sum_{i=1}^d \lambda_i m_i \right) (-1)^d 2^d \left(\prod_{j=1}^d \lambda_j \right) f^{(d)}(Y) + \sum_{l=1}^{d-1} 2^l (-1)^l \left(Y e_l(\lambda_1, \dots, \lambda_d) \right) \right. \\ \left. + \sum_{k=1}^d \lambda_k m_k (e_l(\lambda_1, \dots, \lambda_d) - e_l((\underline{\lambda}_k))) \right) f^{(l)}(Y) + Y f(Y) \Big] = 0, \end{aligned} \quad (2.15)$$

for all $f \in \mathcal{S}(\mathbb{R})$.

Example 2.6. Let $d = 1, m_1 = q \geq 1$ and $\lambda_1 = \lambda > 0$. The differential operator reduces to (on smooth test functions), for all $x \in \mathbb{R}$

$$-2\lambda(x + q\lambda)f'(x) + xf(x).$$

This differential operator is similar to the one characterizing the gamma distribution of parameters $(q/2, 1/(2\lambda))$. Indeed, we have, for $F \stackrel{\text{law}}{=} \gamma(q/2, 1/(2\lambda))$, on smooth test function f :

$$\mathbb{E} \left[F f'(F) + \left(\frac{q}{2} - \frac{F}{2\lambda} \right) f(F) \right] = 0.$$

We can move from the first differential operator to the second one by performing a scaling of parameter $-1/(2\lambda)$ and the change of variable $x = y - q\lambda$.

Example 2.7. Let $d = 2, q = 2, \lambda_1 = -\lambda_2 = 1/2$ and $m_1 = m_2 = 1$. The differential operator (on smooth test functions) reduces to:

$$\begin{aligned} \mathcal{A}(f)(x) &= 4(x + \langle m, \lambda \rangle)\lambda_1\lambda_2 f''(x) - 2[xe_1(\lambda_1, \lambda_2) + \lambda_1 m_1(e_1(\lambda_1, \lambda_2) - e_1(\lambda_2)) \\ &\quad + \lambda_2 m_2(e_1(\lambda_1, \lambda_2) - e_1(\lambda_1))]f'(x) + xf(x), \\ &= -xf''(x) - f'(x) + xf(x), \end{aligned}$$

where we have used the fact that $e_1(\lambda_1, \lambda_2) = \lambda_1 + \lambda_2 = 0, e_1(\lambda_2) = \lambda_2 = -1/2, e_1(\lambda_1) = \lambda_1 = 1/2$. Therefore, up to a minus sign factor, we retrieve the differential operator associated with the random variable $F = N_1 \times N_2$.

We conclude this section by comparing the Stein-type operators defined by the Fourier approach with those obtained by the Malliavin calculus tools in (2.3) (see Section 3 for a proof).

Proposition 2.2. *The Stein-type operators defined in Corollary 2.1 and in (2.3) coincide, up to some normalizing constant.*

2.4 Stein operators for projections of multivariate gamma

Independence of the contributions, as required in (2.13), is not crucial. Indeed, consider all random variables of the form

$$F = \langle \gamma - K, \lambda \rangle = \sum_{i=1}^d \lambda_i(\gamma_i - k_i) \tag{2.16}$$

with $K = (k_1, \dots, k_d) \in \mathbb{R}^d$ and $\gamma = (\gamma_1, \dots, \gamma_d)$ a d -variate gamma distributed random variable defined as follows.

Definition 2.1 (Krishnamoorthy and Parthasarathy (1951)). A random vector $\gamma = (\gamma_1, \dots, \gamma_d)$ has a d -variate gamma distribution in the sense of Krishnamoorthy and Parthasarathy (1951) with degree of freedom $\nu = 2\alpha$ and covariance matrix C if its characteristic function is

$$\phi_\gamma(t_1, \dots, t_d) = |I_d - iCT|^{-\alpha} \tag{2.17}$$

with $t_j \geq 0$ for all $j, |\cdot|$ the determinant operator, I_d the $(d \times d)$ -identity, $\alpha > 0, T = \text{diag}(t_1, \dots, t_d)$, and C a symmetric positive definite $d \times d$ matrix.

Conditions on C and α under which (2.17) is a bona fide characteristic function have been thoroughly addressed in the literature, see Vere-Jones (1997), Eisenbaum and Kaspi (2009), Royen (2016) and references therein. In the sequel we suppose that these conditions are satisfied.

Lemma 2.2. *Let $\Lambda = \text{diag}(\lambda_1, \dots, \lambda_d)$ and $C = (c_{ij})_{1 \leq i, j \leq d}$ a symmetric positive definite matrix. For all $\xi \in \mathbb{R}$, we have*

$$|I_d - i\xi C \Lambda| = \sum_{j=0}^d (-1)^j r_j (i\xi)^j \tag{2.18}$$

with $r_0 = 1$ and

$$r_j = \sum_{\substack{S \subseteq [d] \\ \#(S)=j}} |C|_S \prod_{j \in S} \lambda_j, \tag{2.19}$$

(the summation in (2.19) is over all collections S of indices in $[d] = \{1, \dots, d\}$ with cardinality $\#(S) = j$, and $|C|_S$ is the determinant of the matrix $(C_{ij})_{i,j \in S}$).

Example 2.8. If $d = 3$ and $C = (c_{i,j})_{1 \leq i,j \leq 3}$, then

$$\begin{aligned} r_0 &= 1, \\ r_1 &= c_1 \lambda_1 + c_2 \lambda_2 + c_3 \lambda_3, \\ r_2 &= (c_1 c_2 - c_{12}^2) \lambda_1 \lambda_2 + (c_1 c_3 - c_{13}^2) \lambda_1 \lambda_3 + (c_2 c_3 - c_{23}^2) \lambda_2 \lambda_3, \\ r_3 &= |C| \lambda_1 \lambda_2 \lambda_3 \end{aligned}$$

(we also write c_j instead of c_{jj} for $j = 1, 2, 3$).

From Lemma 2.2, we deduce the CF of linear combinations of marginals of multivariate gamma random vectors: if γ has marginals $\gamma_j \sim \gamma(\alpha, c_j)$ and F is as in (2.16) then, letting $\kappa = \sum_{j=1}^d \lambda_j k_j$:

$$\begin{aligned} \phi_F(\xi) &= \mathbb{E}[e^{i\xi F}] = e^{-i\alpha\xi \sum_{j=1}^d \lambda_j k_j} \mathbb{E}[e^{i \sum_{j=1}^d (\xi \lambda_j) \gamma_j}] \\ &= e^{-i\alpha\kappa\xi} \phi_\gamma(\xi \lambda_1, \dots, \xi \lambda_d) \\ &= e^{-i\alpha\kappa\xi} \left(\sum_{j=0}^d (-1)^j r_j (i\xi)^j \right)^{-\alpha} \end{aligned}$$

with $(r_j)_{0 \leq j \leq d}$ given in Lemma 2.2. Taking derivatives, we obtain

$$\begin{aligned} &\left(\sum_{j=0}^d (-1)^j r_j (i\xi)^j \right) \phi'_F(\xi) \\ &= -i\alpha \left(\kappa \sum_{j=0}^d (-1)^j r_j (i\xi)^j + \sum_{j=1}^d (-1)^j j r_j (i\xi)^{j-1} \right) \phi_F(\xi). \end{aligned}$$

Applying Lemma 2.1 we deduce, after straightforward simplifications.

Theorem 2.3. Let F be defined in (2.16) and $(r_j)_{j=1,\dots,d}$ as in (2.19). Set $r_{d+1} = 0$. Let Y be a real valued random variable such that $\mathbb{E}[|Y|] < +\infty$. Then $Y \stackrel{\text{law}}{=} F$ if and only if

$$\begin{aligned} &\mathbb{E} \left[(F + \alpha(\kappa - r_1)) f(F) \right. \\ &\quad \left. + \sum_{j=1}^d (-1)^j (r_j (F + \alpha\kappa) - \alpha(j+1)r_{j+1}) f^{(j)}(F) \right] = 0 \end{aligned} \tag{2.20}$$

for all test functions $f \in \mathcal{S}(\mathbb{R})$.

Remark 2.1. If F is of the form (2.13) with all shape coefficients identical then $F = \sum_{i=1}^d \lambda_i \sum_{j=1}^{m_i} (\gamma_j(\alpha, c_i) - \alpha c_i)$. Letting $m = \sum_{j=1}^d m_j$, then F is of the form (2.16) for γ a m -variate gamma random variable with $m \times m$ diagonal correlation matrix $C =$

$\text{diag}((c_1)_{m_1}, \dots, (c_d)_{m_d})$ (we write $(x)_q = (x, \dots, x)$ a vector of length q). Applying Theorem 2.3 will lead, via (2.20), to an operator of order $m > d$ which coincides with (2.15) (and thus (2.3)) only when $m_i = 1$ for all i .

Example 2.9. If $d = 2$, then $F = \langle \gamma - K, \lambda \rangle$ has second-order differential Stein operator

$$\begin{aligned} \mathcal{A}f(x) &= (x + \alpha(\kappa - r_1))f(x) - \{r_1x + \alpha(r_1\kappa - 2(c_1c_2 - c_{12}^2)\lambda_1\lambda_2)\}f'(x) \\ &\quad + \lambda_1\lambda_2(c_1c_2 - c_{12}^2)(x + \alpha\kappa)f''(x), \end{aligned} \tag{2.21}$$

(recall that $\kappa = \sum_{j=1}^d \lambda_j k_j$ and $r_1 = \sum_{j=1}^d \lambda_j c_j$).

2.5 Application: McKay type I and combinations of two gamma variates

We continue the paper with applications of the identities in the case $d = 2$. There is interest, even in this simple situation, in obtaining handles on law of sums and differences of correlated gamma variates as these have applications in performance analysis, see, for example, Holm and Alouini (2004). Recall Example 2.5 and the corresponding operator

$$\begin{aligned} \mathcal{A}_{\text{McKay}}f(x) &= \left(x + \frac{(1 + 2a)bc}{1 - c^2}\right)f(x) + \frac{2cbx - (1 + 2a)b^2}{1 - c^2}f'(x) \\ &\quad - \frac{b^2}{1 - c^2}xf''(x) \end{aligned} \tag{2.22}$$

for type I McKay random variables with parameters a, b, c (see its pdf defined in (2.10)). From (2.12) (applied to functions of the form $f(x) = x^n$, along with a continuity argument for extending the identity to functions not in $\mathcal{S}(\mathbb{R})$) we immediately deduce

$$\begin{aligned} \mathbb{E}[F] &= \frac{(1 + 2a)bc}{c^2 - 1}, \\ \mathbb{E}[F^2] &= \frac{(2a + 1)b^2(2(a + 1)c^2 + 1)}{(c^2 - 1)^2} \end{aligned}$$

(see Holm and Alouini (2004), Equation (6)), as well as the formula

$$\begin{aligned} (1 - c^2)\mathbb{E}[F^{n+1}] + bc(1 + 2(a + n))\mathbb{E}[F^n] \\ - nb^2(1 + 2a + n - 1)\mathbb{E}[F^{n-1}] = 0 \end{aligned} \tag{2.23}$$

for all $n \geq 2$.

Corollary 2.2. *McKay Type I random variables can be represented as projections of bivariate gamma random variables with degree of freedom 2α and covariance matrix $C = \begin{pmatrix} c_1 & c_{12} \\ c_{12} & c_2 \end{pmatrix}$ whenever*

$$\begin{aligned} a &= \alpha - 1/2, \\ b &= 2 \frac{c_1c_2 - c_{12}^2}{\sqrt{(c_1 + c_2)^2 - 4(c_1c_2 - c_{12}^2)}}, \\ c &= \frac{c_1 + c_2}{\sqrt{(c_1 + c_2)^2 - 4(c_1c_2 - c_{12}^2)}}. \end{aligned}$$

Remark 2.2. Corollary 2.2 contains Theorems 3, 4 and 5 from Holm and Alouini (2004). In that paper they consider also the so-called McKay Type II distribution for which our method also applies; we do not perform the computations here.

Proof. Taking $K = 0$ and $\lambda_1 = \lambda_2 = 1$ in Example 2.9 we obtain that combinations of dependent gamma random variables $G_1 \sim \text{gamma}(\alpha, c_1)$ and $G_2 \sim \text{gamma}(\alpha, c_2)$ with identical shape parameter and covariance C have operator

$$\begin{aligned} A_{G_1+G_2}f(x) &= (x - \alpha(c_1 + c_2))f(x) \\ &\quad - ((c_1 + c_2)x - 2\alpha(c_1c_2 - c_{12}^2))f'(x) \\ &\quad + (c_1c_2 - c_{12}^2)xf''(x). \end{aligned} \quad (2.24)$$

We identify the coefficients in (2.24) and (2.22) to get the system of 4 equations:

$$\begin{aligned} \frac{bc}{1-c^2}(1+2a) &= -\alpha(c_1+c_2), & 2\frac{bc}{1-c^2} &= -(c_1+c_2), \\ \frac{b^2}{1-c^2}(1+2a) &= -2\alpha(c_1c_2-c_{12}^2), & \frac{b^2}{1-c^2} &= -(c_1c_2-c_{12}^2). \end{aligned}$$

Solving for a, b, c in terms of α, c_1, c_2 and c_{12} we immediately deduce that $a = \alpha - 1/2$ is necessary, so that the system reduces to

$$b^2 = (c_1c_2 - c_{12}^2)(c^2 - 1), \quad 2bc = (c_1 + c_2)(c^2 - 1)$$

and the result follows. \square

We end this subsection by discussing infinite divisibility of the law of projections of multivariate gamma distribution. Infinite divisibility of the multivariate gamma distribution has been addressed thoroughly in the literature (see Griffiths (1984), Bapat (1989), Eisenbaum and Kaspi (2006, 2009)). Thanks to the previous corollary, we are able to explicit the Lévy measure of the sum of two dependent gamma random variables using the parametrization (a, b, c) with $a > -(1/2)$, $b > 0$ and $c > 1$. We have the following straightforward corollary.

Corollary 2.3. *Let (G_1, G_2) be a 2-dimensional gamma random vector of parameters $2\alpha > 0$ and covariance matrix C such that $c_1c_2 > c_{12}^2$ and $c_1 + c_2 > 1$. Then, the law of $G_1 + G_2$ is infinitely divisible and its characteristic function is given by, for all $t \in \mathbb{R}$*

$$\phi_{G_1+G_2}(t) = \exp\left(\int_0^{+\infty} (e^{itx} - 1)\left(\frac{1}{2} + a\right)\left(e^{-\frac{c-1}{b}x} + e^{-\frac{c+1}{b}x}\right)\frac{dx}{x}\right), \quad (2.25)$$

with

$$\begin{aligned} a &= \alpha - \frac{1}{2}, \\ b &= 2\frac{c_1c_2 - c_{12}^2}{\sqrt{(c_1 + c_2)^2 - 4(c_1c_2 - c_{12}^2)}}, \\ c &= \frac{c_1 + c_2}{\sqrt{(c_1 + c_2)^2 - 4(c_1c_2 - c_{12}^2)}}. \end{aligned}$$

Moreover, the following identity in law holds true

$$G_1 + G_2 = \gamma_1 + \gamma_2 \quad (2.26)$$

where γ_1 and γ_2 are independent gamma random variables with parameters $(a + 1/2, (c - 1)/b)$ and $(a + 1/2, (c + 1)/b)$, respectively.

Proof. Let a, b and c be as in the statement of the corollary. By Corollary 2.2, we know that $G_1 + G_2$ has the same law as a McKay type I random variable with parameters (a, b, c) . Then, by (2.11),

$$(\log \phi_{G_1+G_2})'(\xi) = -i \frac{(1 + 2a)bc - (1 + 2a)b^2(i\xi)}{1 - c^2 + 2cb(i\xi) - b^2(i\xi)^2}.$$

Performing a partial fraction decomposition, we obtain straightforwardly

$$(\log \phi_{G_1+G_2})'(\xi) = ib \left(\frac{1}{2} + a \right) \left(\frac{1}{c - 1 - ib\xi} + \frac{1}{c + 1 - ib\xi} \right).$$

Now,

$$\frac{1}{c - 1 - ib\xi} = \int_0^{+\infty} \exp(-(c - 1 - ib\xi)x) dx$$

and similarly for the other term. By standard computations, we obtain formula (2.25). The identity (2.26) follows trivially. □

3 Proofs

Proof of Theorem 2.1. Repeatedly using the Malliavin integration by parts formulae Nourdin and Peccati (2012), Theorem 2.9.1, we obtain for any $2 \leq l \leq d + 2$ that

$$\begin{aligned} \mathbb{E}[F f^{(d-l+2)}(F)] &= \mathbb{E}[f^{(d)}(F) \Gamma_{l-2}(F)] \\ &\quad + \sum_{r=d-l+3}^{d-1} \mathbb{E}[f^{(r)}(F)] \mathbb{E}[\Gamma_{r+l-d-2}(F)]. \end{aligned} \tag{3.1}$$

For indices $l = 2, 3$, the second term in the right-hand side of (3.1) is understood to be 0. Summing from $l = 2$ up to $l = d + 2$, we obtain that

$$\begin{aligned} &\sum_{l=2}^{d+2} a_{l-1} \mathbb{E}[F f^{(d-l+2)}(F)] \\ &= \sum_{l=2}^{d+2} a_{l-1} \mathbb{E}[f^{(d)}(F) \Gamma_{l-2}(F)] \\ &\quad + \sum_{l=4}^{d+2} a_{l-1} \sum_{r=d-l+3}^{d-1} \mathbb{E}[f^{(r)}(F)] \mathbb{E}[\Gamma_{r+l-d-2}(F)] \\ &= \sum_{l=1}^{d+1} a_l \mathbb{E}[f^{(d)}(F) \Gamma_{l-1}(F)] \\ &\quad + \sum_{l=3}^{d+1} a_l \sum_{r=d-l+2}^{q-2} \mathbb{E}[f^{(r)}(F)] \mathbb{E}[\Gamma_{r+l-d-1}(F)] \\ &= \sum_{l=1}^{d+1} a_l \mathbb{E}[f^{(d)}(F) \Gamma_{l-1}(F)] \\ &\quad + \sum_{l=2}^{d+1} a_l \sum_{r=1}^{l-2} \mathbb{E}[f^{(d-r)}(F)] \mathbb{E}[\Gamma_{l-r-1}(F)]. \end{aligned} \tag{3.2}$$

On the other hand,

$$\begin{aligned}
 \sum_{l=2}^{d+1} b_l \mathbb{E}[f^{(d+2-l)}(F)] &= \sum_{l=0}^{d-1} b_{l+2} \mathbb{E}[f^{(d-l)}(F)] \\
 &= \sum_{l=0}^{d-1} \left[\sum_{r=l+2}^{d+1} a_r \mathbb{E}[\Gamma_{r-l-1}(F_\infty)] \right] \mathbb{E}[f^{(d-l)}(F)] \\
 &= \sum_{r=2}^{d+1} a_r \sum_{l=0}^{r-2} \mathbb{E}[\Gamma_{r-l-1}(F_\infty)] \times \mathbb{E}[f^{(d-l)}(F)]. \tag{3.3}
 \end{aligned}$$

Wrapping up, we finally arrive at

$$\begin{aligned}
 &\mathbb{E}[\mathcal{A}_\infty f(F)] \\
 &= -\mathbb{E} \left[f^{(d)}(F) \times \left(\sum_{r=1}^{d+1} a_r [\Gamma_{r-1}(F) - \mathbb{E}[\Gamma_{r-1}(F)]] \right) \right] \\
 &\quad + \sum_{r=2}^{d+1} a_r \sum_{l=0}^{r-2} \{ \mathbb{E}[f^{(d-l)}(F)] \times (\mathbb{E}[\Gamma_{r-l-1}(F_\infty)] - \mathbb{E}[\Gamma_{r-l-1}(F)]) \} \\
 &= -\mathbb{E} \left[f^{(d)}(F) \times \left(\sum_{r=1}^{d+1} a_r [\Gamma_{r-1}(F) - \mathbb{E}[\Gamma_{r-1}(F)]] \right) \right] \\
 &\quad + \sum_{r=2}^{d+1} a_r \sum_{l=0}^{r-2} \frac{\mathbb{E}[f^{(d-l)}(F)]}{(r-l-1)!} \times (\kappa_{r-l}(F_\infty) - \kappa_{r-l}(F)). \tag{3.4}
 \end{aligned}$$

We are now in a position to prove the claim. First, we assume that $F \stackrel{\text{law}}{=} F_\infty$. Then obviously $\kappa_r(F) = \kappa_r(F_\infty)$ for $r = 2, \dots, 2d + 2$, and moreover, random variable F belongs to the second Wiener chaos. Hence, according to [Azmoodeh, Peccati and Poly \(2014\)](#), Lemma 3, the Cauchy–Schwarz inequality, and the hypercontractivity property of the Wiener chaoses [Nourdin and Peccati \(2012\)](#), Theorem 2.7.2, we obtain that

$$\begin{aligned}
 |\mathbb{E}[\mathcal{A}_\infty f(F)]| &\leq \sqrt{\mathbb{E}[f^{(d)}(F)]^2} \times \sqrt{\mathbb{E} \left[\sum_{r=1}^{d+1} a_r (\Gamma_{r-1}(F) - \mathbb{E}[\Gamma_{r-1}(F)]) \right]^2} \\
 &= \sqrt{\mathbb{E}[f^{(d)}(F)]^2} \times \sqrt{\Delta(F, F_\infty)} \\
 &= \sqrt{\mathbb{E}[f^{(d)}(F)]^2} \times \sqrt{\Delta(F_\infty, F_\infty)} = 0.
 \end{aligned}$$

Conversely, assume that $\mathbb{E}[\mathcal{A}_\infty f(F)] = 0$ for all polynomial functions f . Then relation (3.4) implies that, by choosing appropriate polynomials f , we have $\kappa_r(F) = \kappa_r(F_\infty)$ for $r = 2, \dots, d + 1$. Now, combining this observation together with relation (3.4), we infer that

$$\mathbb{E} \left[F^n \sum_{r=1}^{d+1} a_r (\Gamma_{r-1}(F) - \mathbb{E}[\Gamma_{r-1}(F)]) \right] = 0, \quad n \geq 2.$$

Using for example, the Malliavin integrations by parts, and a similar argument as in the proof of [Azmoodeh, Peccati and Poly \(2014\)](#), Proposition 5, the latter equation can be turned into a linear recurrent relation between the cumulants of F of order up to $d + 1$. Combining this with the knowledge that the $d + 1$ first cumulants characterise all the cumulants of F and hence

the distribution F . Indeed, all the distributions in the second Wiener chaos are determined by their moments/cumulants [Nourdin and Peccati \(2012\)](#), Proposition 2.7.13, item 3. \square

Proof of Lemma 2.1. (\Rightarrow). Let us introduce two differential operators characterized by their symbols in Fourier domain. For smooth enough test functions, f ,

$$\begin{aligned} \mathcal{A}_d(f)(x) &= \frac{1}{2\pi} \int_{\mathbb{R}} \mathcal{F}(f)(\xi)(A_d(i\xi)) \exp(ix\xi) d\xi, \\ \mathcal{B}_{d'}(f)(x) &= \frac{1}{2\pi} \int_{\mathbb{R}} \mathcal{F}(f)(\xi)(B_{d'}(i\xi)) \exp(ix\xi) d\xi, \end{aligned}$$

with $\mathcal{F}(f)(\xi) = \int_{\mathbb{R}} f(x) \exp(-ix\xi) dx$. Integrating against smooth test functions the differential equation satisfied by the characteristic function ϕ_F , we have, for the left-hand side

$$\begin{aligned} \int_{\mathbb{R}} \mathcal{F}(\phi)(\xi) A_d(i\xi) \frac{d}{d\xi}(\phi_F(\xi)) d\xi &= \int_{\mathbb{R}} \mathcal{F}(\mathcal{A}_d(f))(\xi) \frac{d}{d\xi}(\phi_F(\xi)) d\xi \\ &= - \int_{\mathbb{R}} \frac{d}{d\xi}(\mathcal{F}(\mathcal{A}_d(f))(\xi)) \phi_F(\xi) d\xi \\ &= i \int_{\mathbb{R}} \mathcal{F}(x \mathcal{A}_d(f))(\xi) \phi_F(\xi) d\xi, \end{aligned}$$

where we have used the standard fact $d/d\xi(\mathcal{F}(f)(\xi)) = -i\mathcal{F}(xf)(\xi)$. Similarly, for the right-hand side,

$$\text{RHS} = i \int_{\mathbb{R}} \mathcal{F}(f)(\xi)(B_{d'}(i\xi)) \phi_F(\xi) d\xi = i \int_{\mathbb{R}} \mathcal{F}(\mathcal{B}_{d'}(f))(\xi) \phi_F(\xi) d\xi.$$

Thus,

$$\int_{\mathbb{R}} \mathcal{F}(x \mathcal{A}_d(f) - \mathcal{B}_{d'}(f))(\xi) \phi_F(\xi) d\xi = 0$$

for all $f \in \mathcal{S}(\mathbb{R})$. Going back in the space domain, we obtain the claim.

(\Leftarrow). We denote by $\mathcal{S}'(\mathbb{R})$ the space of tempered distributions, that is, the topological dual space of the Schwarz space. Let Y be a real valued random variable such that $\mathbb{E}[|Y|] < +\infty$ and

$$\mathbb{E}[Y \mathcal{A}_d(f)(Y) - \mathcal{B}_{d'}(f)(Y)] = 0, \quad f \in \mathcal{S}(\mathbb{R}). \tag{3.5}$$

Since $\mathbb{E}[|Y|] < +\infty$, the characteristic function of Y is continuously differentiable on the whole real line. Working similarly as in the first part of the proof (from space domain to Fourier domain), identity (3.5) leads to

$$A_d(i\xi) \frac{d}{d\xi}(\phi_Y)(\xi) = i B_{d'}(i\xi) \phi_Y(\xi), \quad \xi \in \mathbb{R}. \tag{3.6}$$

We also have $\phi_Y(0) = 1$. Without loss of generality, one can assume that $A_d(0) \neq 0$. Indeed, if $A_d(0) = 0$, then, thanks to (3.6), one has $B_{d'}(0) = 0$. Then, 0 is a root with the same multiplicity for the real polynomials A_d and $B_{d'}$. Therefore, the previous differential equation boils down to

$$\tilde{A}_d(i\xi) \frac{d}{d\xi}(\phi_Y)(\xi) = i \tilde{B}_{d'}(i\xi) \phi_Y(\xi), \quad \xi \in \mathbb{R}.$$

with $\tilde{A}_d(\cdot)$ and $\tilde{B}_{d'}(\cdot)$ real polynomials such that $\tilde{A}_d(0) \neq 0$. Thus, by Cauchy–Lipschitz theorem in a neighborhood I_0 of 0 [Teschl \(2012\)](#), Theorem 2.2, $\phi_Y(\xi) = \phi_F(\xi)$, for all $\xi \in I_0$. Now, the unique solution of the first order linear differential equation (3.6) in I_0 is analytic

in I_0 which implies that ϕ_Y and ϕ_F are analytic in I_0 . Theorem B, page 225 of Loeve (1977) concludes the proof of the lemma. \square

Proof of Theorem 2.2. Let $r_1 = \sum_{k=1}^d \lambda_k m_k \alpha_k c_k$. The CF of random variables as in (2.13) is

$$\phi_F(\xi) = e^{-i\xi r_1} \prod_{j=1}^d (1 - i\xi \lambda_j c_j)^{-m_j \alpha_j}.$$

Taking derivatives with respect to ξ one sees that

$$\phi'_F(\xi) = -i \left(r_1 + \sum_{j=1}^d \frac{\lambda_j m_j \alpha_j c_j}{1 - i\xi \lambda_j c_j} \right) \phi_F(\xi)$$

which, after straightforward simplifications, becomes (we denote $v_j = 1/(c_j \lambda_j)$ and $m\alpha = (m_1 \alpha_1, \dots, m_d \alpha_d)$)

$$\prod_{k=1}^d (v_k - i\xi) \phi'_F(\xi) = -i \left\{ r_1 \prod_{k=1}^d (v_k - i\xi) - \sum_{k=1}^d m_k \alpha_k \prod_{l=1, l \neq k}^d (v_l - i\xi) \right\} \phi_F(\xi).$$

We may apply Lemma 2.1 and all that remains is to compute explicitly the coefficients of the polynomials on either side of the above, that is, in \mathcal{A}_d and \mathcal{B}_d . First of all, let us consider the following polynomial in $\mathbb{R}[X]$:

$$P(x) = \prod_{j=1}^d (v_j - x) = (-1)^d \prod_{j=1}^d (x - v_j).$$

We denote by p_0, \dots, p_d the coefficients of $\prod_{j=1}^d (X - v_j)$ in the basis $\{1, X, \dots, X^d\}$. Vieta formula readily give:

$$\forall k \in \{0, \dots, d\}, \quad p_k = (-1)^{d+k} e_{d-k}(v_1, \dots, v_d).$$

It follows that the Fourier symbol of \mathcal{A}_d is given by:

$$\prod_{k=1}^d (v_k - i\xi) = P(i\xi) = \sum_{k=0}^d (-1)^k e_{d-k}(v_1, \dots, v_d) (i\xi)^k.$$

Thus, we have, for f smooth enough:

$$\mathcal{A}_d(f)(x) = \sum_{k=0}^d (-1)^k e_{d-k}(v_1, \dots, v_d) f^{(k)}(x).$$

Let us proceed similarly for the operator $B_{d,m,v}$. We denote by P_k the following polynomial in $\mathbb{R}[X]$ (for any $k \in \{1, \dots, d\}$):

$$P_k(x) = (-1)^{d-1} \prod_{l=1, l \neq k}^d (x - v_l).$$

A similar argument provides the following expression:

$$P_k(x) = \sum_{l=0}^{d-1} (-1)^l e_{d-1-l}(\underline{v}_k) x^l,$$

where $\underline{v}_k = (v_1, \dots, v_{k-1}, v_{k+1}, \dots, v_d)$. Thus, the symbol of the differential operator B_d is given by:

$$\sum_{k=1}^d m_k \alpha_k \prod_{l=1, l \neq k}^d (v_l - i \xi) = \sum_{l=0}^{d-1} (-1)^l \left(\sum_{k=1}^d m_k \alpha_k e_{d-1-l}(\underline{v}_k) \right) (i \xi)^l.$$

Thus, we have:

$$B_d(f)(x) = \sum_{l=0}^{d-1} (-1)^l \left(\sum_{k=1}^d m_k \alpha_k e_{d-1-l}(\underline{v}_k) \right) f^{(k)}(x).$$

Consequently, we obtain:

$$\mathbb{E} \left[(F + r_1) \sum_{k=0}^d (-1)^k e_{d-k}(v_1, \dots, v_d) f^{(k)}(F) - \sum_{l=0}^{d-1} (-1)^l \left(\sum_{k=1}^d m_k \alpha_k e_{d-1-l}(\underline{v}_k) \right) f^{(k)}(F) \right] = 0.$$

Finally, it is easy to see that

$$\forall k \in \{0, \dots, d\}, \quad \left(\prod_{j=1}^d c_j \lambda_j \right) e_k(v_1, \dots, v_d) = e_{d-k}(\lambda_1 c_1, \dots, \lambda_d c_d)$$

and the conclusion follows. □

Proof of Proposition 2.2. In order to lighten the notations, we consider the target law represented by $F = \sum_{i=1}^d \lambda_i (N_i^2 - 1)$, with $\lambda_j \neq \lambda_i$ if $i \neq j$ and $\{N_1, \dots, N_d\}$ is a collection of i.i.d. standard normal random variables. By (2.3), we have, for any smooth functions:

$$\mathcal{A}_\infty f(x) := \sum_{l=2}^{d+1} (b_l - a_{l-1}x) f^{(d+2-l)}(x) - a_{d+1}x f(x).$$

By a re-indexing argument, we have:

$$\mathcal{A}_\infty f(x) := \sum_{k=1}^d (b_{d+2-k} - a_{d-k+1}x) f^{(k)}(x) - a_{d+1}x f(x).$$

As a warm up, we start by computing a_{d+1} and a_{d-k+1} . We have, by definition:

$$a_{d+1} = \frac{P^{(d+1)}(0)}{(d+1)!2^d} = \frac{1}{2^d},$$

where we have used the definition of the polynomial $P(X)$. Moreover, we have:

$$a_{d-k+1} = \frac{P^{(d+1-k)}(0)}{(d+1-k)!2^{d-k}} = \frac{(-1)^k}{2^{d-k}} e_k(\lambda_1, \dots, \lambda_d),$$

where we have used the fact that $P^{(d+1-k)}(0)$ is equal to $(d+1-k)!$ times the $(d-k)$ th coefficient of the polynomial $\prod (X - \lambda_j)$. Now, let us compute b_{d+2-k} . We have, for $k \in \{1, \dots, d\}$:

$$b_{d+2-k} = \sum_{r=d+2-k}^{d+1} \frac{a_r}{(r+k-d-1)!} \kappa_{r+k-d}(F_\infty)$$

$$\begin{aligned}
&= 2^{k-d} \sum_{r=d+2-k}^{d+1} \frac{P^{(r)}(0)}{r!} \sum_{j=1}^d \lambda_j^{r+k-d} \\
&= 2^{k-d} \sum_{r=d+2-k}^{d+1} (-1)^{d+r-1} e_{d-r+1}(\lambda_1, \dots, \lambda_d) \sum_{j=1}^d \lambda_j^{r+k-d} \\
&= (-1)^{k+1} e_{k-1}(\lambda_1, \dots, \lambda_d) \sum_{j=1}^d \lambda_j^2 + \dots + (-1) e_1(\lambda_1, \dots, \lambda_d) \sum_{j=1}^d \lambda_j^k \\
&\quad + e_0(\lambda_1, \dots, \lambda_d) \sum_{j=1}^d \lambda_j^{k+1}.
\end{aligned}$$

Now the trick is to note that $\lambda_j e_{l-1}((\underline{\lambda}_j)) = e_l(\lambda_1, \dots, \lambda_d) - e_l((\underline{\lambda}_j))$. Thus, we have:

$$(-1) e_1(\lambda_1, \dots, \lambda_d) \sum_{j=1}^d \lambda_j^k + e_0(\lambda_1, \dots, \lambda_d) \sum_{j=1}^d \lambda_j^{k+1} = - \sum_{j=1}^d \lambda_j^k e_1((\underline{\lambda}_j)).$$

Using the previous equality recursively, we obtain:

$$\begin{aligned}
b_{d+2-k} &= 2^{k-d} \left[(-1)^{k+1} e_{k-1}(\lambda_1, \dots, \lambda_d) \sum_{j=1}^d \lambda_j^2 + (-1)^k \sum_{j=1}^d \lambda_j^3 e_{k-2}((\underline{\lambda}_j)) \right] \\
&= 2^{k-d} (-1)^k \left[\sum_{j=1}^d \lambda_j^2 (-e_{k-1}(\lambda_1, \dots, \lambda_d) \right. \\
&\quad \left. + e_{k-1}(\lambda_1, \dots, \lambda_d) - e_{k-1}((\underline{\lambda}_j))) \right] \\
&= 2^{k-d} (-1)^{k+1} \sum_{j=1}^d \lambda_j^2 e_{k-1}((\underline{\lambda}_j)) \\
&= 2^{k-d} (-1)^{k+1} \sum_{j=1}^d \lambda_j (e_k(\lambda_1, \dots, \lambda_d) - e_k((\underline{\lambda}_j))).
\end{aligned}$$

Wrapping everything up together, this ends the proof of the proposition. \square

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