Brazilian Journal of Probability and Statistics
2019, Vol. 33, No. 3, 558–585
https://doi.org/10.1214/18-BJPS403

Stochastic monotonicity from an Eulerian viewpoint

Davide Gabrielli and Ida Germana Minelli
University of L'Aquila

Abstract. Stochastic monotonicity is a well-known partial order relation between probability measures defined on the same partially ordered set. Strassen theorem establishes equivalence between stochastic monotonicity and the existence of a coupling compatible with respect to the partial order. We consider the case of a countable set and introduce the class of *finitely decomposable flows* on a directed acyclic graph associated to the partial order. We show that a probability measure stochastically dominates another probability measure if and only if there exists a finitely decomposable flow having divergence given by the difference of the two measures. We illustrate the result with some examples.

1 Introduction

Given a partially ordered set (from now on, *poset*) V there is a naturally induced partial order relation on the set of probability measures on V, usually called *stochastic monotonicity*. Given two probability measures μ_1, μ_2, we say that μ_2 stochastically dominates μ_1, and write $\mu_1 \preceq \mu_2$, if the expectation of any bounded increasing function with respect to the measure μ_1 is less or equal than the expectation with respect to μ_2.

Strassen theorem (Lindvall (1999, 1992), Strassen (1965)) is an important and powerful result in Probability Theory. In the case of a countable set V claims that μ_2 stochastically dominates μ_1 if and only if there exists a coupling between the two measures that gives zero weight to pairs of elements not increasingly ordered.

We consider the case of a countable poset V and show a new equivalent statement for stochastic domination, based on a graph structure associated to V. Indeed, the partial order structure of a countable poset can be described in terms of an acyclic directed graph (V, E). On such a graph, it is possible to define a *flow*, which is an assignment of a positive weight, representing the amount of mass flown, to each directed edge. The *divergence* of the flow on a vertex is defined as the difference between the amount of mass flown outside and the amount of mass flown into the vertex. We will define the class of *finitely decomposable* flows on (V, E), that is, flows that can be decomposed as a summable superposition of "elementary flows" associated to finite self avoiding paths on the graph (see Section 2 for the definition), and we will prove that μ_2 stochastically dominates μ_1 if and

Key words and phrases. Stochastic monotonicity, couplings, flows on networks.
Received July 2017; accepted May 2018.

only if there exists a finitely decomposable flow having as divergence the difference between the two measures. This statement may be reformulated intuitively by saying that one measure stochastically dominates another one when it is possible to transform this second measure into the first one by moving mass according to the partial order structure. We emphasize that with this new formulation stochastic domination is shown to be equivalent to the existence of a flow on a directed graph encoding all the information about the partial order. This allows to connect directly monotonicity results to the geometry of the underlying partial order. In particular, in Section 5, we will discuss a dual problem for which the homological structure of the directed graph turns out to be a relevant characteristic. The motivation of the title is the following. In fluid theory the Lagrangian description follows the trajectories of the particles while the Eulerian one observes the local flow. A coupling gives a Lagrangian description of the transference plan of mass while a flow gives an Eulerian one.

We have therefore the 3 equivalent statements: (1) $\mu_1 \preceq \mu_2$; (2) there exists a finite decomposable flow having divergence $\mu_1 - \mu_2$: (3) there exists a compatible coupling. The content of Strassen theorem is that (1) $\Leftrightarrow$ (3). We show how to prove all the remaining implications. We present all the proofs since they are interesting in themselves and reflect the geometric structure behind. In particular it is interesting to note that the proof of the implication (2) $\Rightarrow$ (3) is constructive. The following mathematical structures are involved in the proofs.

To prove the two implications of (1) $\Leftrightarrow$ (2) we use Farkas lemma Schrijver (2003) and a suitable infinite dimensional version of such lemma. To prove the two implications (2) $\Leftrightarrow$ (3) we use ideas from the theory of mass transportation (Rachev and Rüschendorf (1998), Santambrogio (2015)) where an equivalence with a continuous problem of flows also appears (Beckmann (1952), Santambrogio (2015)). In particular, we give a constructive proof obtained by an algorithmic construction that associates a coupling to any finite acyclic flow (Paolini and Stepanov (2012)) and is a discrete version of a construction due to S.K. Smirnov on bounded domains of $\mathbb{R}^n$ (Smirnov (1993)). By a limiting argument, we extend this result to the class of finitely decomposable flows.

There is a kind of hierarchical structure between the statements (1), (2) and (3) and the proof of any implication (i) $\Rightarrow$ (j) require a difficult argument when $i < j$ while a simple construction is enough when $i > j$. The proof of the equivalence between statements (1), (2) and (3) can be obtained by one of the two cycles of implications: (1) $\Rightarrow$ (3) $\Rightarrow$ (2) $\Rightarrow$ (1) or (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (1).

Our result is similar in spirit to the equivalence between the Monge-Kantorovich problem with cost $|x - y|$ and a minimal flow problem proposed by Beckmann (1952). This equivalence in illustrated in Chapter 4 of Santambrogio (2015). The reason of this equivalence is in the fact that the cost in the Monge-Kantorovich problem does not depend on the details of the transference plan but only on the flow of mass locally observed. The same happens for stochastic order. In particular our result is the counterpart for stochastic monotonicity of Theorem 4.6 in Santambrogio (2015) for mass transportation.

In the finite case the partial order structure can be encoded by a minimal directed acyclic graph called *Hasse diagram*. The result in the finite case is implicit in Hwang (1979) where it is proved using the theory of convex games. There are also similar ideas and statements in Müller (2013) (and references therein) obtained by duality. A clear formulation in terms of a flow on network problem is however missing and moreover we extend the result to the countable infinite case. Item (v) of Theorem 1 of Kamae, Krengel and O'Brien (1977) can be seen as a very special case of our formulation.

At the beginning of Section 2 we give a more detailed overview of the basic ideas and constructions behind our results. In Section 5, we discuss some examples. Even if the examples are simple they are important to point out the change of perspective with respect to the usual approach. The proofs of monotonicity are indeed obtained with computations that are different from the usual ones.

The structure of the paper is the following. In Section 2, we fix notation and state the main result of the paper. In Section 3, we prove our main Theorem 2.1. In Section 4, we prove some auxiliary Propositions 2.2 and 2.3. In Section 5, we discuss some examples.

2 Preliminaries, notation and main results

In this section, we discuss the general framework, introduce notation and state our main results. We start with a short introductive illustration of our results. We discuss informally the intuitive idea and the novelty with respect to the classic statement.

2.1 Preliminaries

We start recalling again the Strassen theorem (see, for example, Section 1 of Chapter IV of Lindvall (1992)). It says that the statements (1)$\mu_1 \preceq \mu_2$, and (3) there exists a compatible coupling (see Section 2.3 for a precise definition) ρ between μ_1 and μ_2, are indeed equivalent.

A coupling ρ determines a transportation of mass according to which the initial distribution of mass μ_1 is transformed into the final distribution of mass μ_2. The value $\rho(x, y)$ fix the amount of mass that has to be moved from site x to site y. Another possible description of the transportation is obtained looking at the channels through which the mass can flow and recording the amount of mass flown, without taking care of origin and destination. This is obtained giving a flow Q for which $Q(x, x')$ is the amount of mass flown across the channel (x, x'). This second description is less detailed since you have not a complete view of the transference plan and several couplings can correspond to the same flow. These two perspectives are similar to the Lagrangian and the Eulerian point of view in fluid theory.

In the case of a countable partial order the channels are naturally individuated by the edges of an acyclic directed graph (V, E) determining the partial order. For a flow on a graph, there is a natural definition of discrete divergence. If initially we have a distribution of mass μ_1 and we let flow mass according to a flow such that $\operatorname{div} Q = \mu_1 - \mu_2$ at the end, we obtain the distribution of mass μ_2. Note that the existence of a flow satisfying $\operatorname{div} Q = \mu_1 - \mu_2$ is not obvious since the edges of the graph are oriented and the mass can flow only in one direction. We can formulate now a natural statement: (2) there exists a flow on a directed acyclic graph determining the partial order such that $\operatorname{div} Q = \mu_1 - \mu_2$. The main result of this paper is that the statements (1), (2) and (3) are all equivalent.

To prove that $\mu_1 \preceq \mu_2$ we can then use the statement (3) constructing a compatible coupling. This is a collection of $|V| \times |V|$ numbers with $2|V| + 1$ constraints. We can however also use statement (2) exhibiting a flow on (V, E) such that $\operatorname{div} Q = \mu_1 - \mu_2$. This is a simpler object determined by a collection of $|E|$ numbers with $|V|$ constraints.

2.2 Digraphs and posets

We consider a countable set V. A directed graph, called shortly a *digraph*, with vertices set V is a pair (V, E) where $E \subset V \times V$ is a collection of directed edges. We assume that there are not edges of the type (x, x). A directed path γ from $x \in V$ to $y \in V$ is a sequence of vertices $\gamma := (x_0, \dots, x_n)$ such that $x_0 = x$, $x_n = y$ and $(x_i, x_{i+1}) \in E$, for $i = 0, \dots, n-1$. The integer n is the length of the directed path and is denoted also by $|\gamma|$. If there exists an i such that $(u, v) = (x_i, x_{i+1})$ we write $(u, v) \in \gamma$. Given a subset $S \subseteq V$, if $x_i \in S$ for any i we write $\gamma \subseteq S$. We call $\gamma^- := x_0$ the starting point of the path and $\gamma^+ := x_n$ its final point. A directed cycle is a directed path for which $x_0 = x_n$. Given two paths $\gamma = (x_0, \dots, x_n)$ and $\gamma' = (x'_0, \dots, x'_k)$ such that $x_n = x'_0$ we denote by

$$\gamma \star \gamma' := \big(x_0, \dots, x_n, x'_1, \dots, x'_k\big), \tag{2.1}$$

the path given by their concatenation. A path is called self-avoiding if $x_i \neq x_j$ when $i \neq j$. A digraph containing no directed cycles is called a directed acyclic graph. Given a digraph (V, E) we call $(V, \mathcal{E})$ the un-directed graph with edges

$$\mathcal{E} := \big\{\{x, y\} : (x, y) \in E \text{ or } (y, x) \in E\big\}.$$

Given a digraph (V, E) we can construct a new digraph $(V, \overline{E})$ called its *transitive closure*. A pair $(x, y) \in \overline{E}$ if and only if there exists a directed path form x to y. When $|V| < +\infty$ and (V, E) is an acyclic digraph we can define also a new directed acyclic graph $(V, \underline{E})$ that is called its *transitive reduction*: it is the minimal acyclic digraph having the same transitive closure as (V, E), that is, such that for any digraph (V, F) with $(V, \overline{E}) = (V, \overline{F})$ we have $\underline{E} \subseteq F$. When the original digraph (V, E) is acyclic and $|V| < +\infty$, it can be shown that $(V, \underline{E})$ is uniquely determined (see Bang-Jensen and Gutin (2001, Section 4.3)).

A partial order relation $\leq$ on V is a subset $S \subseteq V \times V$ satisfying the properties of *reflexivity, antisymmetry and transitivity*. When $(x, y) \in S$ we write $x \leq y$ and the pair $(V, \leq)$ is called a partially ordered set or simply a *poset*. Then, as can be easily checked, if we set $\overline{E} = S \setminus \{(x, x); x \in V\}$, the pair $(V, \overline{E})$ gives an acyclic digraph whose transitive closure coincides with itself. On the other hand, any acyclic digraph (V, E) induces a partial order on V through the relation $x \leq y \Leftrightarrow x = y \text{or} (x, y) \in \overline{E}$. So, a poset can be described with an acyclic digraph.

Note that such a description is not unique, since different digraphs can have the same transitive closure. However, when V is finite, it is uniquely identified as the transitive reduction $(V, \underline{E})$ and it is called the *Hasse diagram* of the poset.

When $|V| = +\infty$, it is not always possible to define the transitive reduction (think, as an example, to the set of rationals); nevertheless, any acyclic digraph has a well defined transitive closure and consequently it determines a partial order on V. Any countable infinite poset can be described in terms of an acyclic digraph.

2.3 Couplings and flows

Let μ_1 and μ_2 be two probability measures on a poset $(V, \leq)$.

A *coupling* between μ_1 and μ_2 is a probability measure ρ on $V \times V$ such that

$$\begin{cases} \sum_{y \in V} \rho(x, y) = \mu_1(x), & \forall x \in V, \\ \sum_{x \in V} \rho(x, y) = \mu_2(y), & \forall y \in V. \end{cases}$$

We say that a coupling ρ is *compatible* with the partial order $\leq$ if

$$\rho\big\{(x, y) : x \leq y\big\} = 1.$$

We say that μ_2 *stochastically dominates* μ_1 with respect to the partial order $\leq$ and write $\mu_1 \preceq \mu_2$ if, for any bounded increasing function $f : V \longrightarrow \mathbb{R}$ (i.e., a function such that $f(x) \leq f(y)$ whenever $x \leq y$ in V) we have

$$\mu_1(f) \leq \mu_2(f),$$

where $\mu(f) = \mathbb{E}_\mu(f)$ denotes expectation with respect to μ.

Let (V, E) be a digraph.

A *flow* on (V, E) is a map $Q : E \to \mathbb{R}^+$. The *divergence* of Q at $x \in V$ is defined by

$$\operatorname{div} Q(x) := \sum_{y:(x,y) \in E} Q(x, y) - \sum_{y:(y,x) \in E} Q(y, x). \tag{2.2}$$

When $|V| = +\infty$ the divergence is not always well defined. In this case we say that the divergence of a flow Q exists and is given by (2.2) if both series appearing in the r.h.s. of (2.2) are convergent for any $x \in V$.

We denote by $E(Q)$ the elements $(x, y) \in E$ such that $Q(x, y) > 0$. We say that a flow is *acyclic* if the digraph $(V, E(Q))$ is acyclic. Given a directed path $\gamma = (x_0, \dots, x_n)$ on (V, E), we associate to it the flow Q_γ defined by

$$Q_\gamma(x, y) := \begin{cases} 1 & \text{if } (x, y) \in \gamma, \\ 0 & \text{otherwise.} \end{cases} \tag{2.3}$$

On the set of flows on a fixed digraph there is a natural partial order structure defined by $Q \leq Q'$ if $Q(x, y) \leq Q'(x, y)$ for any $(x, y) \in E$.

2.4 Finitely decomposable flows

We say that a flow Q on the digraph (V, E) is *finitely decomposable* if there exists a countable family of finite self avoiding directed paths $\{\gamma_n\}_{n\in\mathbb{N}}$ and a sequence $\{q_n\}_{n\in\mathbb{N}}$ of weights $q_n \geq 0$ with $\sum_n q_n < +\infty$, such that

$$Q = \sum_n q_n Q_{\gamma_n}. \tag{2.4}$$

Note that in an acyclic digraph any path is self avoiding. If $|V| < +\infty$, then any flow is finitely decomposable since we have for example,

$$Q = \sum_{(x,y)\in E} Q(x, y) Q_{(x,y)}.$$

A finitely decomposable flow is not necessarily summable since we have

$$\sum_{(x,y)\in E} Q(x, y) = \sum_n q_n |\gamma_n| \tag{2.5}$$

and the r.h.s. of (2.5) can be infinite. Note that a finite decomposition (2.4) of a finitely decomposable flow induces naturally a finite measure on Γ, the countable set of all finite self-avoiding paths on (V, E). This is simply

$$\sum_n q_n \delta_{\gamma_n}, \tag{2.6}$$

where δ is the delta Dirac measure.

Since the paths in (2.4) are self avoiding, every single path γ_n may contribute just once to the outgoing or ingoing flux at a single site. This implies that the divergence of a finitely decomposable flow is well defined and it is given by

$$\operatorname{div} Q(x) = \sum_{\{n:\gamma_n^- = x\}} q_n - \sum_{\{n:\gamma_n^+ = x\}} q_n. \tag{2.7}$$

2.5 A third equivalent statement in Strassen theorem

Our main result is the following.

Theorem 2.1. *Let* $(V, \leq)$ *be a countable partial order and let* (V, E) *be a directed acyclic graph such that its transitive closure* $(V, \overline{E})$ *induces the partial order* $\leq$. *The following statements are equivalent.*

1. $\mu_1 \preceq \mu_2$,
2. *there exists a finitely decomposable flow* Q *on* (V, E) *such that* $\operatorname{div} Q = \mu_1 - \mu_2$.
3. *there exists a compatible coupling between* μ_1 *and* μ_2.

We recall that Strasseen theorem states the equivalence between stochastic domination $\mu_1 \preceq \mu_2$ and the existence of a compatible coupling between μ_1 and μ_2. Statement (2) shows that finding such a coupling is equivalent to solve a flow on network problem.

If V is finite then any flow is finitely decomposable and moreover the digraph (V, E) can be conveniently fixed as the Hasse diagram $(V, \underline{E})$ of the poset.

In general it is not easy to verify if a flow Q is finitely decomposable, so we state a sufficient and a necessary condition.

Let (V, E) be an infinite digraph. An invading sequence of vertices $\{V_n\}_{n\in\mathbb{N}}$ is a sequence of subsets $V_n \subseteq V$ such that $|V_n| < +\infty$, $V_n \subseteq V_{n+1}$ and $\cup_n V_n = V$. Given a flow Q we say that it has *zero flux towards infinity* (see Bertini, Faggionato and Gabrielli (2015) for the original definition and related results) if there exists an invading sequence of vertices such that

$$\lim_{n\to+\infty}\Bigg(\sum_{x\in V_n, y\notin V_n} Q(x,y)\Bigg) = 0.$$

We have the following sufficient condition.

Proposition 2.2. *Let* Q *be a flow on an infinite acyclic digraph* (V, E) *such that* $\operatorname{div} Q = \mu_1 - \mu_2$. *If* Q *has zero flux towards infinity, then it is finitely decomposable.*

Next, the proposition gives a necessary condition.

Proposition 2.3. *Let* Q *be a flow on an infinite digraph* (V, E). *If* Q *is finitely decomposable then for all invading sequences*

$$\lim_{n\to+\infty} \sup\big\{Q(x,y) : \{x,y\} \cap \{V \setminus V_n\} \neq \varnothing\big\} = 0. \tag{2.8}$$

Propositions 2.2 and 2.3 are proved in Section 4.

3 Proof of Theorem 2.1

We discuss the proofs of all the implications apart (1) $\Leftrightarrow$ (3) that is the content of the classic Strassen theorem. The proofs of our theorem can be obtained using one of the two cycles of implications: (1) $\Rightarrow$ (3) $\Rightarrow$ (2) $\Rightarrow$ (1) or (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (1). We give the proofs of all the implications since they are interesting in themselves and give insight to the geometric structures involved. Remarkably the proof (2) $\Rightarrow$ (3) is constructive, uses ideas from mass transportation theory and shows how to construct a coupling starting from an acyclic finite decomposable flow.

We discuss before the arguments needed to prove the theorem in the finite case $|V| < +\infty$ and then show how to extend them to the infinite countable case.

3.1 The finite case

In this section, we consider the case $|V| < +\infty$.

3.1.1 (2) $\Rightarrow$ (1). Let Q be a finite flow such that $\operatorname{div} Q = \mu_1 - \mu_2$. Then by a discrete integration by parts we have

$$\mu_2(f) - \mu_1(f) = -\sum_{x\in V} f(x)\operatorname{div} Q(x) = \sum_{(x,y)\in E} Q(x,y)\big(f(y) - f(x)\big) \geq 0,$$

for any increasing function f.

3.1.2 (1) $\Rightarrow$ (2). Farkas lemma (see Schrijver (2003) volume A Section 5.4) states that, given an $n \times m$ matrix A and $b \in \mathbb{R}^n$, there exists $x \in (\mathbb{R}^+)^m$ such that $Ax = b$ if and only if for any $y \in \mathbb{R}^n$ such that $A^T y \in (\mathbb{R}^+)^m$ the inequality $y \cdot b \geq 0$ holds (where $\cdot$ denotes the Euclidean scalar product). Let us consider the adjacency matrix A of (V, E). It is a $|V| \times |E|$ matrix whose rows and columns are labeled respectively with the vertices of V and the edges of E and it is defined by fixing equal to $+1$ the element corresponding to the row x and the column (x, y), and by fixing equal to -1 the element corresponding to the row y and the column (x, y). All the remaining elements in the column (x, y) are set equal to 0. With this definition, given a flow Q we have

$$\operatorname{div} Q(x) = AQ(x).$$

Moreover, given a function $f : V \to \mathbb{R}$ we have that

$$-A^T f(x,y) = f(y) - f(x).$$

The function f is increasing if and only if $A^T f(x, y) \leq 0$ for any $(x, y) \in \underline{E}$. The result now follows applying Farkas lemma with the matrix A coinciding with the adjacency matrix and taking the vector $b = \mu_1 - \mu_2$.

3.1.3 (3) $\Rightarrow$ (2). Suppose that there exists a compatible coupling ρ between μ_1 and μ_2. If $x \le y$ and $x \ne y$ there exists at least one directed path in (V, E) going from x to y. Fix one of them arbitrarily and call it $\gamma_{(x,y)}$. Recalling definition (2.3), we construct the flow

$$Q := \sum_{\{x,y \in V : x \ne y\}} \rho(x,y) Q_{\gamma_{(x,y)}}. \tag{3.1}$$

This flow is finitely decomposable by definition and it satisfies div $Q = \mu_1 - \mu_2$. Indeed, using (2.7), the fact that ρ is a compatible coupling between μ_1 and μ_2 and the fixed paths, we have

$$\text{div}\, Q(x) = \sum_{\gamma : \gamma^- = x} \rho(\gamma^-, \gamma^+) - \sum_{\gamma : \gamma^+ = x} \rho(\gamma^-, \gamma^+) \tag{3.2}$$

$$= \sum_{y : x \le y} \rho(x,y) - \sum_{y : y \le x} \rho(y,x) = \mu_1(x) - \mu_2(x). \tag{3.3}$$

3.1.4 (2) $\Rightarrow$ (3). Let Q be a flow on a finite acyclic digraph (V, E) such that div $Q = \mu_1 - \mu_2$. In order to generate a compatible coupling ρ between μ_1 and μ_2, we use a variation of the algorithmic construction in Paolini and Stepanov (2012) that is a discrete version of the original decomposition due to S.K. Smirnov on bounded domains of $\mathbb{R}^n$ (Smirnov (1993)) and associates a coupling to a finite acyclic flow.

Define $V_- := \{x \in V : \mu_1(x) > \mu_2(x)\}$ and $V_+ := \{x \in V : \mu_2(x) > \mu_1(x)\}$. First of all, we show that it is possible to decompose the flow like

$$Q = \sum_n q_n Q_{\gamma_n}, \tag{3.4}$$

where the paths γ_n are such that $\gamma_n^- \in V_-$ and $\gamma_n^+ \in V_+$ for any n. Consider any finite decomposition of Q and suppose that for example, there exists a site $x \in V_-$ and a m such that $\gamma_m^+ = x$ (the other cases can be handled similarly). Since by definition $\mu_1(x) > \mu_2(x)$ there exist necessarily some paths $\{\gamma_n\}_{n \in \mathcal{N}}$ of the decomposition such that $\gamma_n^- = x$ for any $n \in \mathcal{N}$ and moreover $\sum_{n \in \mathcal{N}} q_n > q_m$. We can then find some weights $\{q_n'\}_{n \in \mathcal{N}}$ such that $\sum_{n \in \mathcal{N}} q_n' = q_m$ and $q_n' \le q_n$. With these weights, we construct the new decomposition

$$\sum_{n \in \mathcal{N}} [q_n' Q_{\gamma_m \star \gamma_n} + (q_n - q_n') Q_{\gamma_n}] + \sum_{n \notin \mathcal{N} \cup m} q_n Q_{\gamma_n}. \tag{3.5}$$

Since (V, E) is acyclic the paths obtained by concatenation are still self avoiding. Performing a finite number of times a procedure of this type the final decomposition will have the required property.

Consider now a decomposition such that $\gamma_n^- \in V_-$ and $\gamma_n^+ \in V_+$ for any n. This condition immediately implies that

$$\sum_{\{n : \gamma_n^- = x\}} q_n = \mu_1(x) - \mu_2(x), \qquad x \in V_-. \tag{3.6}$$

By (2.7), we get

$$\begin{aligned}\sum_n q_n &= \sum_{x\in V_-}\sum_{\{n:\gamma_n^-=x\}} q_n\\ &= \sum_{x\in V_-}\left[\mu_1(x)-\mu_2(x)\right] \qquad (3.7)\\ &= \frac{1}{2}\sum_x\left|\mu_1(x)-\mu_2(x)\right|.\end{aligned}$$

In particular, we deduce that the l.h.s. of (3.7) is smaller or equal than 1.

Using this special decomposition, we can construct the coupling. We define

$$\rho(x,y) := \begin{cases}\min\{\mu_1(x),\mu_2(x)\} & \text{if } x=y,\\ \displaystyle\sum_{\{n:\gamma_n^-=x,\gamma_n^+=y\}} q_n & \text{if } x\neq y.\end{cases} \qquad (3.8)$$

Using (3.6) and the analogous formula for V_+ it is easy to verify that ρ defined in (3.8) is a coupling between μ_1 and μ_2. This coupling is clearly compatible since if there exists a γ_n such that $\gamma_n^- = x$ and $\gamma_n^+ = y$ then necessarily $x \le y$. This completes the proof.

Note that for the coupling given above, we have

$$\sum_{x\neq y}\rho(x,y) = \sum_n q_n = \frac{1}{2}\sum_x\left|\mu_1(x)-\mu_2(x)\right|, \qquad (3.9)$$

so that any coupling constructed in this way is an optimal one.

3.2 The infinite case

In this section, we extend the proof to the infinite case.

3.2.1 (2) $\Rightarrow$ (1). Let Q be a finitely decomposable flow such that $\operatorname{div} Q = \mu_1 - \mu_2$. Then, recalling (2.7) and using the summability of the weights q_n, we have for any increasing function $f \in L^\infty(V)$

$$\mu_2(f)-\mu_1(f) = -\sum_{x\in V} f(x)\operatorname{div} Q(x) = \sum_n q_n\big(f(\gamma_n^+)-f(\gamma_n^-)\big) \ge 0.$$

3.2.2 (1) $\Rightarrow$ (2). We start with a preliminary result. Let $D \subseteq L^1(V)$ be the subset of functions that can be obtained as divergence of a finitely decomposable flow. The subset D is clearly convex and we have the following result.

Lemma 3.1. *The subset D is closed in $L^1(V)$.*

Proof. Let $\{f^{(n)}\}_n \subset D$ be a sequence which converges to f in $L^1(V)$. Since $f^{(n)} \in D$ then there exists a sequence of finitely decomposable flows $Q^{(n)}$ such that $\operatorname{div} Q^{(n)} = f^{(n)}$. We need to show that there exists a finitely decomposable flow Q such that $\operatorname{div} Q = f$. Let us write the finite decomposition of $Q^{(n)}$ as $Q^{(n)} = \sum_k q_k^{(n)} Q_{\gamma_k^{(n)}}$. Consider ε_n a sequence of positive numbers converging to 0 when $n \to +\infty$. Fix k_n^* as the minimal integer such that $\sum_{k>k_n^*} q_k^{(n)} \leq \varepsilon_n$ and consider the finite flows $\widetilde{Q}^{(n)} := \sum_{k \leq k_n^*} q_k^{(n)} Q_{\gamma_k^{(n)}}$. Let us call also $\widetilde{f}^{(n)} := \operatorname{div} \widetilde{Q}^{(n)}$. By construction, using (2.7), we have $\sum_x |\widetilde{f}^{(n)}(x) - f^{(n)}(x)| \leq 2\varepsilon_n$ so that $\widetilde{f}^{(n)}$ also converges to f in $L^1(V)$. Let $W_-^{(n)} := \{x \in V : \widetilde{f}^{(n)}(x) > 0\}$ and $W_+^{(n)} := \{x \in V : \widetilde{f}^{(n)}(x) < 0\}$. Since $\widetilde{Q}^{(n)}$ are finite flows we can construct finite decompositions like in Section 3.1.4 such that each path $\widetilde{\gamma}_k^{(n)}$ of the decompositions satisfies $\widetilde{\gamma}_k^{(n),-} \in W_-^{(n)}$ and $\widetilde{\gamma}_k^{(n),+} \in W_+^{(n)}$. Let $\widetilde{q}_k^{(n)}$ be the corresponding weights. We fix, arbitrary and once for all, for any pair of different vertices $x \leq y$ one path $\gamma_{x,y}$ going from x to y. The final flows that we construct are $\widehat{Q}^{(n)} := \sum_k \widetilde{q}_k^{(n)} Q_{\gamma_{\widetilde{\gamma}_k^{(n),-},\widetilde{\gamma}_k^{(n),+}}}$. Clearly we have $\operatorname{div} \widehat{Q}^{(n)} = \operatorname{div} \widetilde{Q}^{(n)} = \widetilde{f}^{(n)}$.

By (2.6) (using the suitable weights and paths), the sequence of flows $\{\widehat{Q}^{(n)}\}_{n\in\mathbb{N}}$ induces a sequence $\{M^{(n)}\}_{n\in\mathbb{N}}$ of finite measures on the set Γ of finite self avoiding paths. Let us show that this sequence is tight and $\{M^{(n)}(\Gamma)\}_n$ is uniformly bounded, so that Prohorov theorem applies. Since the paths of the decomposition exit from vertices in $W_-^{(n)}$ and end in vertices in $W_+^{(n)}$, for each n, we have that relation (3.7) holds and consequently we have

$$\sum_\gamma M^{(n)}(\gamma) = \sum_k \widetilde{q}_k^{(n)} = \frac{1}{2}\sum_x |\widetilde{f}^{(n)}(x)|. \tag{3.10}$$

Since $\{\widetilde{f}^{(n)}\}$ converges to f in $L^1(V)$, we have that the r.h.s. of (3.10) converges to $\frac{1}{2}\sum_x |f(x)| < +\infty$ and this implies that the l.h.s. of (3.10) is uniformly bounded. Now, let $\{V_n\}_n$ be an invading sequence of vertices and define

$$\widetilde{V}_n := \{z : \exists x, y \in V_n \text{ with } z \in \gamma_{x,y}\}.$$

We define also $\Gamma_n := \{\gamma \in \Gamma : \gamma \subseteq \widetilde{V}_n\}$. We have

$$M^{(n)}(\Gamma_k^c) \leq \sum_{\{\gamma : \{\gamma^-,\gamma^+\} \cap V_k^c \neq \emptyset\}} M^{(n)}(\gamma) = \sum_{x \notin V_k} |\widetilde{f}^{(n)}(x)|.$$

Tightness follows now directly from (2.3), the convergence of $\widetilde{f}^{(n)}$ to f and the summability of f. By Prohorov theorem, the sequence is relatively compact. Let $M = \sum_k q_k \delta_{\gamma_k}$ be the weak limit of a subsequence of $\{M^{(n)}\}_n$ and let us consider the finitely decomposable flow $Q := \sum_k q_k Q_{\gamma_k}$. Then, at least along a subsequence

$\{n'\}$, using the weak convergence of $\{M^{(n')}\}_{n'}$ to M and the (point-wise) convergence of $\widetilde{f}^{(n')}$ to f, we have

$$\begin{aligned}\operatorname{div} Q(x) &= \sum_{k:\gamma_k^-=x} q_k - \sum_{k,:\gamma_k^+=x} q_k \\ &= \lim_{n'\to+\infty}\Big(\sum_{\gamma:\gamma^-=x} M^{(n')}(\gamma) - \sum_{\gamma:\gamma^+=x} M^{(n')}(\gamma)\Big) \\ &= \lim_{n'\to+\infty} \widetilde{f}^{(n')}(x) = f(x).\end{aligned}$$

This means that Q is a finite decomposable flow such that $\operatorname{div} Q = f$ and consequently $f \in D$ □

Let us suppose that $\mu_2(f)-\mu_1(f) \geq 0$ for any increasing function $f \in L^\infty(V)$. This can be written as $\langle f, \mu_2 - \mu_1\rangle_V \geq 0$ where $\mu_2 - \mu_1 \in L^1(V)$ and $\langle\cdot,\cdot\rangle_V$ is the $L^\infty(V)$, $L^1(V)$ dual paring defined by

$$\langle f, g\rangle_V := \sum_{x\in V} f(x)g(x), \qquad f \in L^\infty(V), g \in L^1(V). \tag{3.11}$$

Let Q_e for $e \in E$ be the flow defined by (2.3) for the elementary path γ given by the single edge e. A function $f \in L^\infty(V)$ is increasing if and only if $\nabla f \in L^\infty(E)$, defined for $(x, y) \in E$ by $\nabla f(x, y) = f(y) - f(x)$, is such that

$$\langle \nabla f, Q_e\rangle_E \geq 0, \qquad \forall e \in E, \tag{3.12}$$

where $\langle\cdot,\cdot\rangle_E$ is the dual pairing for functions on edges.

We need to show that $\mu_1 - \mu_2 \in D$. Let us suppose by contradiction that this is not the case. Since D is convex and closed, by Hahn–Banach theorem we deduce that there exists an $f^* \in L^\infty(V)$ such that

$$\begin{cases}\langle f^*, \operatorname{div} Q\rangle_V < 0, & \forall Q \text{ finitely decomposable,}\\ \langle f^*, \mu_1 - \mu_2\rangle_V > 0.\end{cases} \tag{3.13}$$

In particular by the first inequality, we have that

$$-\langle f^*, \operatorname{div} Q_e\rangle_V = \langle \nabla f^*, Q_e\rangle_E > 0, \qquad \forall e \in E,$$

which means that f^* is increasing. This fact together with the second inequality in (3.13) gives a contradiction.

3.2.3 (3) ⇒ (2). The proof of this implication is equal to the finite case.

3.2.4 (2) $\Rightarrow$ (3). We need to extend the construction done in the finite case to the infinite one. Let Q be a finitely decomposable flow such that div $Q=\mu_1-\mu_2$. Let $\{V_n\}_n$ be an invading sequence of vertices such that $\cup_{k\leq n}\gamma_k\subseteq V_n$ where the paths γ_k are the ones involved in the finite decomposition (2.4) of Q.

For each n, we consider the finite digraph having vertices $V_n\cup\{g\}$ where g is a ghost site. The set of edges E_n contains all edges $(x,y)\in E$ such that $x,y\in V_n$, moreover it contains the edges (g,z) or (z,g) with $z\in V_n$ if respectively there exists an $(x,z)\in E$ such that $x\notin V_n$ or there exists an $(z,y)\in E$ such that $y\notin V_n$. The digraph $(V_n\cup\{g\},E_n)$ is not necessarily acyclic. Starting from the flow Q on (V,E) we construct a flow Q_n on $(V_n\cup\{g\},E_n)$ as follows

$$Q_n(x,y):=\begin{cases} Q(x,y) & \text{if } x,y\in V_n,\\ \sum_{z\notin V_n} Q(x,z) & \text{if } y=g, x\in V_n,\\ \sum_{z\notin V_n} Q(z,y) & \text{if } x=g, y\in V_n.\end{cases} \tag{3.14}$$

The series appearing in (3.14) are convergent since Q has a well-defined divergence. We have

$$\text{div}\, Q_n(x)=\begin{cases} \text{div}\, Q(x)=\mu_1(x)-\mu_2(x), & \text{if } x\in V_n,\\ \sum_{y\notin V_n}\big(\mu_1(y)-\mu_2(y)\big), & \text{if } x=g.\end{cases}$$

In general the flow Q_n will not be acyclic, but by removing cycles we can obtain an acyclic flow Q_n^* having the same divergence as Q_n. This is done in two steps. The first one is as follows. Consider a path γ_k of the finite decomposition of Q such that it exits from and enters in V_n several times. This is possible only if $k>n$. After the identification of all the sites outside V_n with the single ghost site g the path will not be anymore self-avoiding (see Figure 1). If we remove the cycles that

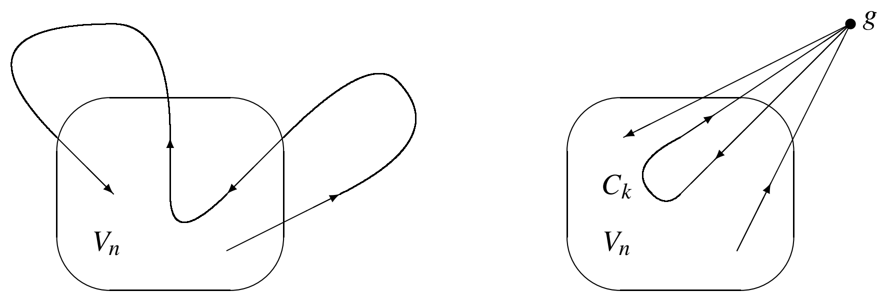

Figure 1 *A path γ_k entering and exiting several times in and from V_n (left). The same path after the identification of the sites outside V_n with the ghost site g (right). The cycle C_k containing g has to be removed.*

have been created (all of which will contain the ghost site) the self-avoiding path that we obtain will exit from V_n or enter in V_n at most once. The corresponding transformation on the flow Q_n is the following. Let us consider the example of Figure 1 and call C_k the cycle in $(V_n \cup \{g\}, E_n)$ created after the identification of all the sites outside V_n with the single ghost site g. By construction, we have $q_k Q_{C_k} \subseteq Q_n$ so that $Q_n - q_k Q_{C_k}$ is still a flow on $(V_n \cup \{g\}, E_n)$ having the same divergence as Q_n since closed paths do not contribute to the divergence. We consider iteratively each path γ_k of the original cyclic decomposition with $k > n$ and remove the cycles as illustrated above. After this deletion procedure the flow Q_n^1 obtained is still not necessarily acyclic.

The second step is as follows. Since the digraph $(V_n \cup \{g\}, E_n)$ is finite we can consider a finite number of cycles C_i' and weights q_i' such that $Q_n^* := Q_n^1 - \sum_i q_i' Q_{C_i'}$ is acyclic (see, for example, the construction in Gabrielli and Valente (2012)). The choice of the cycles and weights in this last step is arbitrary.

To the values of the flow Q_n^* on edges entering or exiting from the ghost site g can contribute only the paths γ_k with $k > n$. Moreover, the deletion procedure outlined above guarantees that every single path γ_k with $k > n$ may contribute no more than once to the total flux entering in g (that is, $\sum_{x \in V_n} Q_n^*(x, g)$) or to the total flux exiting from g (that is $\sum_{x \in V_n} Q_n^*(g, x)$). This means that we have the bounds

$$\begin{cases} \sum_{x \in V_n} Q_n^*(x, g) \leq \sum_{k>n} q_k, \\ \sum_{x \in V_n} Q_n^*(g, x) \leq \sum_{k>n} q_k. \end{cases} \tag{3.15}$$

Let us now consider the flow $\widetilde{Q}_n^*$ such that $\widetilde{Q}_n^*(x, y) = Q_n^*(x, y)$ when both x and y belong to V_n and $\widetilde{Q}_n^*(x, y) = 0$ otherwise. The flow $\widetilde{Q}_n^*$ can be naturally interpreted as a flow on the original digraph (V, E) and by construction $\widetilde{Q}_n^* \leq Q$. We have also

$$\operatorname{div} \widetilde{Q}_n^*(x) = \mu_1(x) - \mu_2(x) + \delta_n(x), \qquad x \in V_n, \tag{3.16}$$

where by (3.15) we have

$$\sum_x |\delta_n(x)| \leq 2 \sum_{k>n} q_k. \tag{3.17}$$

We define the following sequences of measures on V

$$\mu_1^{(n)}(x) := \begin{cases} \mu_1(x) + \delta_n(x) & \text{if } x \in V_n, \delta_n(x) > 0, \\ \mu_1(x) & \text{if } x \in V_n, \delta_n(x) \leq 0, \\ 0 & \text{if } x \notin V_n, \end{cases} \tag{3.18}$$

$$\mu_2^{(n)}(x) := \begin{cases} \mu_2(x) - \delta_n(x) & \text{if } x \in V_n, \delta_n(x) < 0, \\ \mu_2(x) & \text{if } x \in V_n, \delta_n(x) \geq 0, \\ 0 & \text{if } x \notin V_n. \end{cases} \tag{3.19}$$

We have div $\widetilde{Q}_n^* = \mu_1^{(n)} - \mu_2^{(n)}$ and $\sum_x(\mu_1^{(n)}(x) - \mu_2^{(n)}(x)) = 0$. Since $|E(\widetilde{Q}_n^*)| < +\infty$ we can apply the finite algorithmic construction used in the proof of Theorem 2.1 (which works also for pairs of finite measures having the same total mass) obtaining from the acyclic flow $\widetilde{Q}_n^*$ a measure $\rho^{(n)}$ on $V \times V$ such that $\sum_x \rho^{(n)}(x, y) = \mu_2^{(n)}(y)$ and $\sum_y \rho^{(n)}(x, y) = \mu_1^{(n)}(x)$. Since the mass is transported along edges of the original digraph (V, E) we deduce that $\rho^{(n)}(x, y) = 0$ if $x \npreceq y$. Now, let us show that the measures $\{\rho^{(n)}\}_n$ have total mass uniformly bounded and form a tight family. The bound on the mass follows by

$$\sum_x \sum_y \rho^{(n)}(x, y) = \sum_x \mu_1^{(n)}(x) \leq \sum_x \bigl[\mu_1(x) + |\delta_n(x)|\bigr] \leq 1 + 2\sum_{k=1}^{+\infty} q_k.$$

The tightness follows by the following argument. Fix an arbitrary $\varepsilon > 0$ and let m^* be an integer number such that

$$\max\bigl\{\mu_1(V_{m^*}^c), \mu_2(V_{m^*}^c)\bigr\} < \varepsilon,$$

where the upper index c denotes the complementary set. Fix also n^* such that $2\sum_{k=n^*}^{+\infty} q_k < \varepsilon$. Then we have for any $n > n^*$ and $m > m^*$

$$\begin{aligned}
\rho^{(n)}\bigl((V_m \times V_m)^c\bigr) &= \rho^{(n)}\bigl(V_m^c \times V_m\bigr) + \rho^{(n)}\bigl(V_m \times V_m^c\bigr) + \rho^{(n)}\bigl(V_m^c \times V_m^c\bigr) \\
&\leq \rho^{(n)}\bigl(V \times V_m^c\bigr) + \rho^{(n)}\bigl(V_m^c \times V\bigr) \\
&= \mu_2^{(n)}\bigl(V_m^c\bigr) + \mu_1^{(n)}\bigl(V_m^c\bigr) \\
&\leq \mu_2\bigl(V_m^c\bigr) + \mu_1\bigl(V_m^c\bigr) + \sum_x \bigl|\delta_n(x)\bigr| \leq 3\varepsilon.
\end{aligned}$$

By Prohorov theorem there exists a subsequence, that we still call $\{\rho^{(n)}\}_n$, that is weakly convergent. Let us call ρ its weak limit. Since by (3.17), (3.18) and (3.19) $\mu_i^{(n)}(x) \to \mu_i(x)$ for any x we immediately obtain

$$\sum_y \rho(x, y) = \lim_{n \to +\infty} \sum_y \rho^{(n)}(x, y) = \lim_{n \to +\infty} \mu_1^{(n)}(x) = \mu_1(x).$$

A similar result holds for μ_2. This means that ρ is a coupling between μ_1 and μ_2. Since $\rho^{(n)}(x, y) = 0$ when $x \npreceq y$ this will be true also for the limiting measure ρ. This completes the proof.

4 Proof of Propositions 2.2 and 2.3

In this section, we give the proofs of the auxiliary Propositions 2.2 and 2.3 that are useful to identify finitely decomposable flows.

Proof of Proposition 2.2. Consider the invading sequence $\{V_n\}_n$ for which the outgoing flux towards infinity $\sum_{x\in V_n, y\notin V_n} Q(x,y) =: \phi_n^+$ is converging to zero when n diverges. Since

$$\sum_{x\in V_n, y\notin V_n} Q(x,y) - \sum_{x\notin V_n, y\in V_n} Q(x,y) = \mu_1(V_n) - \mu_2(V_n), \tag{4.1}$$

then also the incoming flux from infinity $\sum_{x\notin V_n, y\in V_n} Q(x,y) := \phi_n^-$ is converging to zero when n diverges. All the series in (4.1) are convergent since $|V_n| < +\infty$ and the series appearing in the definition of div Q (2.2) are supposed to be summable.

For each n we consider the finite digraph having vertices $V_n \cup g_- \cup g_+$ where $g_\pm$ are ghost sites. The set of edges E_n contains all edges $(x,y) \in E$ such that $x, y \in V_n$, moreover it contains edges of type (g_-, z) or (z, g_+) with $z \in V_n$ if respectively, there exists an $(x,z) \in E$ such that $x \notin V_n$ or there exists an $(z,y) \in E$ such that $y \notin V_n$. Since the original graph is acyclic also this new finite digraph is acyclic. Starting from the flow Q on (V, E), we associate to it a flow Q_n on $(V_n \cup g_- \cup g_+, E_n)$ as follows

$$Q_n(x,y) := \begin{cases} Q(x,y) & \text{if } x, y \in V_n, \\ \sum_{z\notin V_n} Q(x,z) & \text{if } y = g_+, x \in V_n, \\ \sum_{z\notin V_n} Q(z,y) & \text{if } x = g_-, y \in V_n. \end{cases}$$

We have

$$\operatorname{div} Q_n(x) = \operatorname{div} Q(x) = \mu_1(x) - \mu_2(x), \qquad x \in V_n.$$

We have also $\operatorname{div} Q_n(g_-) = \phi_n^-$ and $\operatorname{div} Q_n(g_+) = -\phi_n^+$. Let us introduce two sequences of measures on $V_n \cup g_- \cup g_+$ defined as

$$\mu_1^{(n)}(x) := \begin{cases} \mu_1(x) & \text{if } x \in V_n, \\ \phi_n^- & \text{if } x = g_-, \\ 0 & \text{if } x = g_+, \end{cases} \tag{4.2}$$

$$\mu_2^{(n)}(x) := \begin{cases} \mu_2(x) & \text{if } x \in V_n, \\ \phi_n^+ & \text{if } x = g_+, \\ 0 & \text{if } x = g_-. \end{cases} \tag{4.3}$$

Then Q_n is a flow on a finite acyclic digraph and moreover $\operatorname{div} Q_n = \mu_1^{(n)} - \mu_2^{(n)}$. Applying the finite algorithmic construction of Section 3.1.4, we obtain a finite decomposition

$$Q_n = \sum_m q_m^{(n)} Q_{\gamma_m^{(n)}} \tag{4.4}$$

for suitable weights $q_m^{(n)}$ and paths $\gamma_m^{(n)}$. The paths $\gamma_m^{(n)}$ are self-avoiding paths on the digraph $(V_n \cup g_- \cup g_+, E_n)$ but to every path γ on this digraph it can be easily associated a self-avoiding path $\widetilde{\gamma}$ on the original digraph (V, E). This is done simply transforming any edge $(g_-, x) \in \gamma$ into an arbitrary edge $(y, x) \in E(Q)$ with $y \notin V_n$ and any edge $(x, g_+) \in \gamma$ into an arbitrary edge $(x, y) \in E(Q)$ with $y \notin V_n$. After this identification, we obtain an acyclic finitely decomposable flow on (V, E)

$$\widetilde{Q}_n := \sum_m q_m^{(n)} Q_{\widetilde{\gamma}_m^{(n)}}. \tag{4.5}$$

By construction, we have

$$\widetilde{Q}_n(x, y) = Q(x, y), \tag{4.6}$$

for any n big enough so that $x, y \in V_n$.

Recall that Γ is the countable set of all finite self-avoiding paths on the digraph (V, E). Let also $\Gamma_n \subseteq \Gamma$ be the subset of all the paths $\gamma \subseteq V_n$. To the decomposition (4.5) we associate by (2.6) a finite measure on Γ given by

$$M^{(n)} := \sum_m q_m^{(n)} \delta_{\widetilde{\gamma}_m^{(n)}}. \tag{4.7}$$

We now show that $\{M^{(n)}\}_n$ is a tight sequence of measures with total mass uniformly bounded. Recall that the coefficients $q_m^{(n)}$ in (4.7) are the same of (4.4) so that by (3.9) they satisfy

$$\sum_{\gamma \in \Gamma} M^{(n)}(\gamma) = \sum_m q_m^{(n)} = \frac{1}{2}\Big[\phi_n^+ + \phi_n^- + \sum_{x \in V_n} \big|\mu_1(x) - \mu_2(x)\big|\Big].$$

Since $\phi_n^\pm$ are converging to zero we have an uniform bound on the total mass. Moreover, we have

$$M^{(n)}(\Gamma_k^c) = \sum_{\{m : \widetilde{\gamma}_m^{(n)} \subseteq V_k^c\}} q_m^{(n)} + \sum_{\{m : \widetilde{\gamma}_m^{(n)} \cap V_k \neq \emptyset, \widetilde{\gamma}_m^{(n)} \cap V_k^c \neq \emptyset\}} q_m^{(n)}. \tag{4.8}$$

The first term on the right-hand side of (4.8) is 0 when $n \leq k$. When $n > k$ can be estimated using (3.6) as

$$\begin{aligned} \sum_{\{m : \widetilde{\gamma}_m^{(n)} \subseteq V_k^c\}} q_m^{(n)} &\leq \sum_{x \notin V_k} \sum_{\{m : \widetilde{\gamma}_m^{(n)-} = x\}} q_m^{(n)} \\ &\leq \sum_{x \notin V_k} \big|\mu_1^{(n)}(x) - \mu_2^{(n)}(x)\big| \\ &\leq \phi_n^+ + \phi_n^- + \sum_{x \notin V_k} \big|\mu_1(x) - \mu_2(x)\big|. \end{aligned} \tag{4.9}$$

The second term in (4.8) can be directly estimated by $\phi_k^+ + \phi_k^-$ independently of n. With these bounds the tightness of the sequence of measures $\{M^{(n)}\}_n$ can be easily established using the condition of zero flux towards infinity. By Prohorov theorem, we can then extract a subsequence that we still call $\{M^{(n)}\}_n$ which weakly converges to a finite measure $M := \sum_m q_m \delta_{\tilde{\gamma}_m}$ on Γ. The function that associate to any path γ the value 1 if $(x, y) \in \gamma$ and zero otherwise is continuous and bounded on Γ endowed of the discrete topology. By (4.6), we deduce that if we construct the flow $\widetilde{Q} := \sum_m q_m Q_{\tilde{\gamma}_m}$ then we have that $\widetilde{Q}(x, y) = \lim_{n \to +\infty} \widetilde{Q}_n(x, y) = Q(x, y)$. This means that $\widetilde{Q}$ coincides with the original flow Q. Since $\widetilde{Q}$ is clearly finitely decomposable we are done. □

Proof of Proposition 2.3. Let us suppose by contradiction that $\sum_n q_n Q_{\gamma_n}$ is a finite decomposition of Q and that (2.8) does not converge to zero for an invading sequence. This means that there exists an ε and an infinite sequence of edges $\{e_i\}_{i \in \mathbb{N}}$ such that $Q(e_i) > \varepsilon$ for any i. Let n^* be such that $\sum_{n > n^*} q_n < \varepsilon$. Let e_{i^*} such that $e_{i^*} \notin \cup_{n \le n^*} \gamma_n$. Then we have

$$\varepsilon < Q(e_{i^*}) = \sum_{n > n^*} q_n Q_{\gamma_n}(e_{i^*}) < \varepsilon,$$

a contradiction. □

5 Examples

In this section, we discuss some examples of applications of Theorem 2.1. Even if simple they are conceptually important since we use arguments that are different from the usual ones. In Example 5.1, we obtain the classic condition for stochastic monotonicity on $\mathbb{Z}$. Instead of construct a coupling we need just to perform a discrete integration. The same happens in example 5.2. In Example 5.3, we show that the problem of stochastic monotonicity has a dual problem coinciding with the nonemptiness of a polyhedron in a space whose dimension is the number of independent cycles of the Hasse diagram. A discrete Poisson equation can be relevant in this dual problem. Example 5.4 is a special issue of 5.3. In Example 5.5, we generalize a classic construction. A coupling of two random variables can be constructed writing them as functions of a common random variable. If the functions satisfy a monotonicity property then the coupling is a monotone one. We show that there is a similar construction for flows that works under less restrictive conditions on the functions. In Example 5.6, we show that there is a natural construction for flows that is the counterpart of the product coupling. The mechanism is a bit tricky and works due to the presence of a telescopic sum. In Example 5.7, we obtain very shortly the result of Holley (1974).

5.1 The one dimensional case

We discuss the simplest countable poset, that is $\mathbb{Z}$ with the usual partial order relation. We want to get the well known Lindvall (1992) necessary and sufficient conditions to have $\mu_1 \preceq \mu_2$, using item (2) of Theorem 2.1. In this case the partial order can be described by the Hasse diagram corresponding to the acyclic digraph $(\mathbb{Z}, E)$ where $E = \{(x, x+1)\}_{x\in\mathbb{Z}}$. The condition div $Q = \mu_1 - \mu_2$ reads

$$Q(x, x+1) - Q(x-1, x) = \mu_1(x) - \mu_2(x),$$

and with a finite telescopic sum for any $y < x$ we get

$$Q(x, x+1) - Q(y, y+1) = \sum_{z=y+1}^{x} \big(\mu_1(z) - \mu_2(z)\big). \tag{5.1}$$

By Proposition 2.3, a necessary condition to have that Q is finitely decomposable is that $\lim_{y\to-\infty} Q(y, y+1) = 0$. Taking the limit $y \to -\infty$ in (5.1), we then get

$$Q(x, x+1) = \sum_{z=-\infty}^{x} \big(\mu_1(z) - \mu_2(z)\big). \tag{5.2}$$

This means that there is at most one finitely decomposable flow having divergence equal to $\mu_1 - \mu_2$ that is (5.2). Consider the invading sequence $V_n := \{-n, \dots, n\}$. The flux exiting from V_n coincides with $Q(n, n+1)$ that by (5.2) is converging to zero when $n \to +\infty$. By Proposition 2.2, Q is finitely decomposable. The last condition that Q has to satisfy to be a flow is $Q(x, x+1) \geq 0$ for any $x \in \mathbb{Z}$. This condition reads

$$\sum_{z=-\infty}^{x} \big(\mu_1(z) - \mu_2(z)\big) = F_1(x) - F_2(x) \geq 0, \qquad \forall x \in \mathbb{Z}, \tag{5.3}$$

where $F_i(x) := \sum_{z=-\infty}^{x} \mu_i(z)$ is the distribution function of the measure μ_i. This is the classic condition to have $\mu_1 \preceq \mu_2$ in this case.

5.2 Finite and infinite trees

We consider the case of posets described by digraphs (V, E) such that the associated graph $(V, \mathcal{E})$ is a tree. We discuss both the finite and the infinite case.

Let us start with the finite case. Removing one edge of $\mathcal{E}$ the graph is divided into two connected components. If the edge that has been removed is $\{x, y\}$ and $x \leq y$ we call $T_-^{\{x,y\}}$ the connected component containing x and $T_+^{\{x,y\}}$ the connected component containing y. Using a discrete Gauss Green identity, we get that there is a unique solution to the equation div $Q = \mu_1 - \mu_2$ that is

$$Q(x, y) = \sum_{z\in T_-^{\{x,y\}}} \big(\mu_1(z) - \mu_2(z)\big). \tag{5.4}$$

The left-hand side of (5.4) is the flux from $T_-^{\{x,y\}}$ to $T_+^{\{x,y\}}$ while the right-hand side is the sum of the divergences in $T_-^{\{x,y\}}$. Since Q has to be a flow on (V, E) it must be positive and this gives

$$\sum_{z\in T_-^e} \big(\mu_1(z)-\mu_2(z)\big) \geq 0, \qquad \forall e \in \mathcal{E} \tag{5.5}$$

that is the necessary and sufficient condition to have $\mu_1 \preceq \mu_2$.

If $(V, \mathcal{E})$ is an infinite tree, then the equation div $Q = \mu_1 - \mu_2$ has not an unique solution. However, if Q is a finitely decomposable flow we have

$$\begin{aligned} Q(x,y) &= \sum_{\{n:\gamma_n^- \in T_-^{\{x,y\}}\}} q_n - \sum_{\{n:\gamma_n^+ \in T_-^{\{x,y\}}\}} q_n \\ &= \sum_{\{z\in T_-^{\{x,y\}}\}} \operatorname{div} Q(z) \\ &= \sum_{\{z\in T_-^{\{x,y\}}\}} \big(\mu_1(z)-\mu_2(z)\big). \end{aligned} \tag{5.6}$$

This means that there is at most one finitely decomposable solution to the equation div $Q = \mu_1 - \mu_2$ that is still given by (5.4). Indeed, as in Section 5.1, using Proposition 2.2 it can be easily shown that this solution is finitely decomposable. The positivity Q gives the same condition (5.5) of the finite case.

5.3 A dual problem

We consider the case $|V| < +\infty$. A discrete vector field on $(V, \mathcal{E})$ is a map ϕ on pairs of ordered vertices (x, y) with $\{x, y\} \in \mathcal{E}$ satisfying the condition $\phi(x, y) = -\phi(y, x)$.

Let $\Lambda(\mathcal{E})$ be the vector space of discrete vector fields on $(V, \mathcal{E})$. This is a $|\mathcal{E}|$ dimensional vector space. The following are classic results (see Biggs (1974) or Gabrielli and Valente (2012) for a short introduction) and we give just a short informal overview. A discrete vector field ϕ is a gradient if there exists a function $f : V \to \mathbb{R}$ such that $\phi(x, y) = f(y) - f(x)$. The divergence at $x \in V$ of a discrete vector field ϕ is defined by $\operatorname{div} \phi(x) := \sum_{y:\{x,y\}\in\mathcal{E}} \phi(x, y)$. We have the orthogonal decomposition

$$\Lambda(\mathcal{E}) = \Lambda_g(\mathcal{E}) \oplus \Lambda_d(\mathcal{E}),$$

where $\Lambda_g(\mathcal{E})$ is the $|V| - 1$ dimensional subspace of gradient discrete vector fields and $\Lambda_d(\mathcal{E})$ is the $|\mathcal{E}| - |V| + 1$ dimensional subspace of divergence free discrete vector fields. The orthogonality is with respect to the scalar product

$$\sum_{\{x,y\}\in\mathcal{E}} \phi(x,y)\psi(x,y), \qquad \phi, \psi \in \Lambda(\mathcal{E}).$$

A basis for $\Lambda_d(\mathcal{E})$ is obtained choosing a suitable collection of divergence free discrete vector fields naturally associated to elementary independent cycles. Fix $(V, \mathcal{T})$ a spanning tree of $(V, \mathcal{E})$, in particular $|\mathcal{T}| = |V| - 1$. For any $e \in \mathcal{E} \setminus \mathcal{T}$ the graph $(V, \mathcal{T} \cup \{e\})$ contains an unique cycle with distinct vertices. Let us fix an arbitrary orientation on this cycle. On the graph $(V, \mathcal{T} \cup \{e\})$ there exists a unique, up to a multiplicative factor, divergence free discrete vector field ϕ_e. This is defined by fixing $\phi_e(x, y) = 1$ if (x, y) belongs to the oriented cycle, $\phi_e(x, y) = -1$ if (y, x) belongs to the oriented cycle and $\phi_e(x, y) = 0$ otherwise. The collection $\{\phi_e\}_{e \in \mathcal{E} \setminus \mathcal{T}}$ is a basis of $\Lambda_d(\mathcal{E})$.

All the discrete vector fields satisfying

$$\operatorname{div} \phi = \mu_1 - \mu_2 \tag{5.7}$$

are given by

$$\phi^* + \sum_{e \in \mathcal{E} \setminus \mathcal{T}} \alpha_e \phi_e, \tag{5.8}$$

where the α_e are arbitrary real numbers and ϕ^* is an arbitrary solution to (5.7), for example, of gradient type.

Let $(V, \underline{E})$ be the Hasse diagram of a finite poset and let $(V, \underline{\mathcal{E}})$ be the corresponding undirected graph. Consider also μ_1 and μ_2 two probability measures on V. The flows on $(V, \underline{E})$ having divergence coinciding with $\mu_1 - \mu_2$ are in bijection with the discrete vector fields on $(V, \underline{\mathcal{E}})$ having the same divergence and such that $\phi(x, y) \geq 0$ when $(x, y) \in \underline{E}$. The bijection is through the natural identification $Q(x, y) = \phi(x, y)$ when $(x, y) \in \underline{E}$. The remaining values of the discrete vector field are fixed by the antisymmetry condition.

Using the above construction, we obtain that there exists a flow on $(V, \underline{E})$ having divergence $\mu_1 - \mu_2$ if and only if the following conditions are satisfied. For edges $\{x, y\} \in \underline{\mathcal{E}}$ that do not belong to any cycle of the basis, we have to impose $\phi^*(x, y) \geq 0$ when $(x, y) \in \underline{E}$. If we call $\underline{E}'$ the set of remaining edges, we have to impose that there exists a collection of real numbers $\{\alpha_e\}_{e \in \underline{\mathcal{E}} \setminus \mathcal{T}}$ such that

$$\phi^*(x, y) + \sum_e \alpha_e \phi_e(x, y) \geq 0, \qquad \forall (x, y) \in \underline{E}'. \tag{5.9}$$

Recall that $\phi_e(x, y)$ is taking just the values -1, 0, $+1$. Conditions (5.9) in the α variables is equivalent to the statement that a polyhedron on $\mathbb{R}^{|\underline{E}|-|V|+1}$ obtained as the intersection of $|\underline{E}'|$ half-spaces (one for each $(x, y) \in \underline{E}'$) is not empty. The interesting feature is that it is a geometric problem on a space of dimension equal to the number of independent cycles of the Hasse diagram.

Consider for example the Hasse diagram of Figure 2 (left) having one single cycle and such that $\underline{E}' = \underline{E}$. Since the Hasse diagram has only one independent cycle the stochastic monotonicity condition will reduce to a one dimensional problem. Choosing arbitrarily one orientation we can label vertices as $V := \{1, 2, \ldots, n\}$

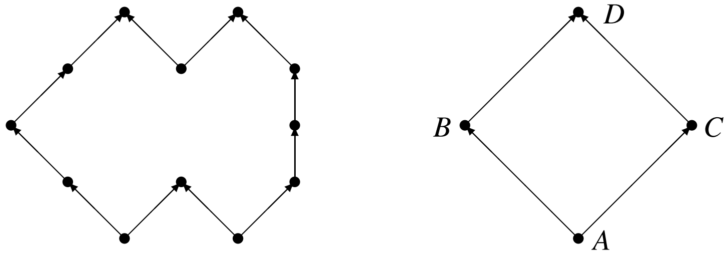

Figure 2 *An Hasse diagram with one single cycle such that* $\underline{E}' = \underline{E}$ (*left*). *The special case of the Hasse diagram of an elementary lattice* (*right*).

and the edges as $\mathcal{E} := \{\{x, x+1\}\}_{x=1}^{n}$ where the sum is modulo n. Equation (5.8) reduces to

$$\phi^{\alpha} := \phi^{*} + \alpha\phi, \tag{5.10}$$

where $\phi(x, x+1) = 1$ for any x, α is an arbitrary real number and ϕ^* is any given discrete vector field such that $\operatorname{div}\phi^* = \mu_1 - \mu_2$. We can fix for example

$$\phi^{*}(x, x+1) = \sum_{y=1}^{x}\bigl(\mu_1(y) - \mu_2(y)\bigr), \qquad x = 1, \dots, n. \tag{5.11}$$

Let $\underline{E}^{+} := \{(x, y) \in \underline{E} : y = x + 1\}$ and $\underline{E}^{-}$ the complementary set. Conditions (5.9) become

$$\begin{cases} \phi^{*}(x, y) + \alpha \geq 0, & (x, y) \in \underline{E}^{+}, \\ \phi^{*}(x, y) - \alpha \geq 0, & (x, y) \in \underline{E}^{-}, \end{cases} \tag{5.12}$$

that are equivalent to the single inequality

$$\max_{(x,y)\in\underline{E}^{+}}\left\{\sum_{z=1}^{x}\bigl(\mu_2(z) - \mu_1(z)\bigr)\right\} \leq \min_{(x,y)\in\underline{E}^{-}}\left\{\sum_{z=1}^{y}\bigl(\mu_2(z) - \mu_1(z)\bigr)\right\}. \tag{5.13}$$

Condition (5.13) is a necessary and sufficient condition to have $\mu_1 \preceq \mu_2$ on a poset like the one on the left of Figure 2. If we consider the special case on the right-hand side of Figure 2 (5.13) becomes

$$\begin{aligned} &\bigl|\mu_1(B) - \mu_2(B)\bigr| + \bigl|\mu_1(C) - \mu_2(C)\bigr| \\ &\quad \leq \bigl(\mu_1(A) - \mu_2(A)\bigr) - \bigl(\mu_1(D) - \mu_2(D)\bigr). \end{aligned} \tag{5.14}$$

It is not immediate to get the single inequality (5.14) without condition (2) of Theorem 2.1.

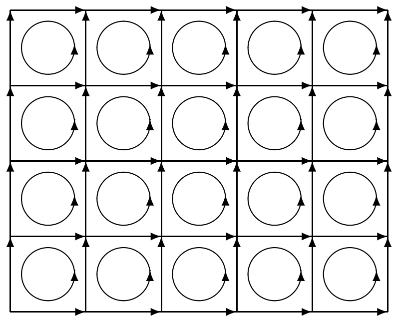

Figure 3 *The Hasse diagram for a finite bi-dimensional grid and the oriented independent elementary cycles associated to the elementary faces of the lattice.*

5.4 The two dimensional case

Let us consider for simplicity two probability measures μ_1 and μ_2 on the set $V = \mathbb{Z}^2 \cap ([0, N] \times [0, M])$. We denote by $\leq$ the usual partial order relation on $\mathbb{Z}^2$, i.e., $(x_1, x_2) \leq (y_1, y_2)$ if $x_i \leq y_i$ for $i = 1, 2$. Then $(V, \leq)$ is a poset with Hasse diagram as in Figure 3. We apply the general framework of Section 5.3. The number of independent cycles is NM and in Figure 3 it is shown a choice of a basis of cycles one for each face of the squared lattice. The problem of establishing wether $\mu_1 \preceq \mu_2$ is equivalent to the problem of determining if a polyhedron in dimension NM identified by $2NM + M + N$ inequalities is empty or not. The inequalities are one for each edge of the Hasse diagram. Let $\alpha(x_1, x_2)$ be the real variable associated to the elementary cycle centered in $(x_1 + \frac{1}{2}, x_2 + \frac{1}{2})$. Let also ϕ^* be a solution of (5.7). We can for example consider

$$\begin{cases} \phi^*\big((x_1, x_2), (x_1 + 1, x_2)\big) \\ \quad = \frac{1}{2}\big([F_1 - F_2](x_1, x_2) - [F_1 - F_2](x_1, x_2 - 1)\big), \\ \phi^*\big((x_1, x_2), (x_1, x_2 + 1)\big) \\ \quad = \frac{1}{2}\big([F_1 - F_2](x_1, x_2) - [F_1 - F_2](x_1 - 1, x_2)\big), \end{cases} \tag{5.15}$$

where for $i = 1, 2$, $F_i(x_1, x_2) := \sum_{(y_1, y_2) \leq (x_1, x_2)} \mu_i(y_1, y_2)$ is the distribution function of μ_i. Recalling (5.9), the inequalities can be summarized by

$$\begin{cases} \dfrac{[F_1 - F_2](x_1, x_2) - [F_1 - F_2](x_1 - 1, x_2)}{2} \\ \quad \geq \alpha(x_1, x_2) - \alpha(x_1 - 1, x_2), \\ \dfrac{[F_1 - F_2](x_1, x_2) - [F_1 - F_2](x_1, x_2 - 1)}{2} \\ \quad \geq \alpha(x_1, x_2 - 1) - \alpha(x_1, x_2), \end{cases} \tag{5.16}$$

that have to be satisfied for any vertex (x_1, x_2) of the grid. Clearly, when in (5.16) it appears a variable α associated to an elementary face outside of the grid we mean that its value is zero.

A complete characterization of when inequalities (5.16) determine a nonempty polyhedron is difficult but it can be given in some special cases like for example a strip ($M = 1$). However it is easy to find sufficient conditions to have $\mu_1 \preceq \mu_2$. For example, choosing $\alpha = \pm\frac{1}{2}[F_1 - F_2]$ we deduce that if $F_1 - F_2$ is increasing in one of the two coordinates then $\mu_1 \preceq \mu_2$.

A similar scheme can be developed for planar posets.

5.5 A generalized construction

A very general and much used construction of a compatible coupling is obtained considering two functions G_1 and G_2 defined on a set Ω, taking values on the poset $(V, \leq)$ and such that $G_1(\omega) \leq G_2(\omega)$ for any $\omega \in \Omega$. For simplicity of notation, we consider the case Ω countable but the general case can be handled similarly. Given a random variable U taking values on Ω the joint law ρ of the random variables $(X_1, X_2) = (G_1(U), G_2(U))$ is a compatible coupling between the distribution μ_1 of X_1 and the distribution μ_2 of X_2 so that $\mu_1 \preceq \mu_2$. Using flows the argument is as well elementary. For any $x \leq y$ fix a path $\gamma_{x,y}$ on (V, E). Then

$$\sum_{\omega\in\Omega} \mathbb{P}(U = \omega) Q_{\gamma_{G_1(\omega),G_2(\omega)}} = \sum_{x,y} \rho(x, y) Q_{\gamma_{x,y}}$$

is a flow with divergence coinciding with $\mu_1 - \mu_2$. The coupling argument does not work if $G_1 \not\leq G_2$ while the flow argument can still work generalizing this classic construction.

We illustrate the simplest possible approach that can be generalized in several ways. Recall that $(V, \mathcal{E})$ is the un-oriented graph associated to (V, E). To any pair $x, y \in V$ (not necessarily ordered) we associate a fixed path $\gamma_{x,y}$ in $(V, \mathcal{E})$ going from x to y. This is a sequence $(x_0, x_1, \dots x_n)$ such that $x_0 = x$, $x_n = y$ and $\{x_i, x_{i+1}\} \in \mathcal{E}$. To any path γ on $(V, \mathcal{E})$, we associate the discrete vector field ϕ_γ defined by

$$\phi_\gamma(u, v) := \begin{cases} 1 & \text{if } (u, v) \in \gamma, \\ -1 & \text{if } (v, u) \in \gamma, \\ 0 & \text{otherwise.} \end{cases} \tag{5.17}$$

We consider the discrete vector field

$$\phi = \sum_{\omega\in\Omega} \mathbb{P}(U = \omega)\phi_{\gamma_{G_1(\omega),G_2(\omega)}} = \sum_{x,y} \rho(x, y)\phi_{\gamma_{x,y}}. \tag{5.18}$$

If $\phi(x, y) \geq 0$ for any $(x, y) \in E$, then the flow on (V, E) defined by $Q(x, y) = \phi(x, y)$ has divergence $\mu_1 - \mu_2$ and we deduce $\mu_1 \preceq \mu_2$. The basic idea is the following. It may happen that $G_1(U) \not\leq G_2(U)$ that corresponds to negative flows across some edges in (V, E). Nevertheless the total net flow across each edge is positive and this is enough to prove $\mu_1 \preceq \mu_2$.

A simple illustrative case is the following. Let U_1 be a random variable taking values on $V = \mathbb{Z}^2 \cap ([0, N] \times [0, M])$ and having distribution μ. The random variable U_2 is obtained moving the random lattice point U_1 uniformly at random on one of its 4 nearest neighbors vertices of $\mathbb{Z}^d$. If this point is outside the rectangle $[0, N] \times [0, M]$, then $U_2 = U_1$. We fix $U = (U_1, U_2)$ and $X_1 = G_1(U_1, U_2) = U_1$ and $X_2 = G_2(U_1, U_2) = U_2$. We have that the law of $(G_1(U_1, U_2), G_2(U_1, U_2))$ is not a monotone coupling of $\mu_1 = \mu$ and μ_2 the law of X_2. The discrete vector field (5.18) is however

$$\begin{cases} \phi\big(x, x + (1, 0)\big) = \dfrac{1}{4}\big[\mu(x) - \mu\big(x + (1, 0)\big)\big] \\ \qquad x_1 = 0, \dots, N - 1; x_2 = 0, \dots, N, \\ \phi\big(x, x + (0, 1)\big) = \dfrac{1}{4}\big[\mu(x) - \mu\big(x + (0, 1)\big)\big] \\ \qquad x_1 = 0, \dots, N; x_2 = 0, \dots, N - 1. \end{cases} \tag{5.19}$$

We deduce immediately that if μ is decreasing (i.e. $\mu(x) \geq \mu(x + (1, 0))$ and $\mu(x) \geq \mu(x + (0, 1))$ for edges belonging to the rectangle) then the vector field is positive along increasing directions and $\mu_1 \preceq \mu_2$.

5.6 Product couplings and flows

Let $(V, \leq)$ be a finite poset with associated the Hasse diagram $(V, \underline{E})$ and consider the product partial order on V^N. An element $\eta \in V^N$ is written as $\eta = (\eta(1), \dots, \eta(N))$ and for $\eta, \xi \in V^N$ we write $\eta \leq \xi$ if $\eta(i) \leq \xi(i)$ for any i. Given $\eta \in V^N$, $x \in V$ and $i = 1, \dots, N$ we denote by

$$\eta_x^i = \big(\eta(1), \dots, \eta(i - 1), x, \eta(i + 1), \dots, \eta(N)\big)$$

the element of V^N with $\eta(i)$ replaced by x. The Hasse diagram $(V^N, \underline{E}^N)$ for the product poset has a directed edge (η, ξ) if and only if $\xi = \eta_x^i$ for some i, x and $(\eta(i), x) \in \underline{E}$.

Let μ_1^i and μ_2^i $i = 1, \dots, N$ be a collection of probability measures on V such that for any i we have $\mu_1^i \preceq \mu_2^i$.

Then for the product measures we have $\bigotimes_{i=1}^N \mu_1^i \preceq \bigotimes_{i=1}^N \mu_2^i$: indeed, if ρ^i is a monotone coupling between μ_1^i and μ_2^i then $\bigotimes_{i=1}^N \rho^i$ is a monotone coupling between $\bigotimes_{i=1}^N \mu_1^i$ and $\bigotimes_{i=1}^N \mu_2^i$.

Let us illustrate that there is an equivalent construction with flows. Let Q^i be a flow on $(V, \underline{E})$ such that $\operatorname{div} Q^i(x) = \mu_1^i(x) - \mu_2^i(x)$. Let us define

$$\gamma^i(\eta) = \bigg[\prod_{j<i} \mu_2^j\big(\eta(j)\big)\bigg]\bigg[\prod_{j>i} \mu_1^j\big(\eta(j)\big)\bigg] \tag{5.20}$$

and observe that (5.20) does not depend on $\eta(i)$. We define the flow Q on $(V^N, \underline{E}^N)$ as

$$Q(\eta, \eta_x^i) = \gamma^i(\eta) Q^i(\eta(i), x). \tag{5.21}$$

Then we have

$$\begin{aligned} \operatorname{div} Q(\eta) &= \sum_{i=1}^{N} \gamma^i(\eta) \sum_{y \in V} [Q^i(\eta(i), y) - Q^i(y, \eta(i))] \\ &= \sum_{i=1}^{N} \gamma^i(\eta)[\mu_1^i(\eta(i)) - \mu_2^i(\eta(i))] \\ &= \prod_{i=1}^{N} \mu_1^i(\eta(i)) - \prod_{i=1}^{N} \mu_2^i(\eta(i)), \end{aligned} \tag{5.22}$$

where the last equality follows from the special form (5.20) since the sum in the second line of (5.22) is telescopic and only the initial and final terms survive.

5.7 Lattices

For an integer $N \geq 2$, consider the lattice $V := \{0, 1\}^N$ with the usual partial order. An element $\eta \in V$ has the form $\eta = (\eta(1), \dots, \eta(N))$ with $\eta(i) \in \{0, 1\}$. The Hasse diagram associated is $(V, \underline{E})$ where $(\eta, \eta') \in \underline{E}$ if and only if η' is obtained by η changing one coordinate of η from 0 to 1. The digraph on the right of Figure 2 is the Hasse diagram for this poset when $N = 2$.

A classic sufficient condition to have stochastic monotonicity is the Holley condition Holley (1974) that is,

$$\mu_2(\eta \vee \xi)\mu_1(\eta \wedge \xi) \geq \mu_2(\eta)\mu_1(\xi) \qquad \forall \eta, \xi \in V. \tag{5.23}$$

In (5.23), given $\eta, \xi \in V$ we call $\eta \vee \xi$ and $\eta \wedge \xi$ the elements of V defined by

$$(\eta \vee \xi)(i) := \max\{\eta(i), \xi(i)\}, \qquad (\eta \wedge \xi)(i) := \min\{\eta(i), \xi(i)\}.$$

An alternative sufficient condition to have $\mu_1 \preceq \mu_2$ is discussed in Hosaka (2009). This condition is very simple and natural and is the following.

Hosaka condition: if $\mu_1 - \mu_2$ is a not increasing function then $\mu_1 \preceq \mu_2$.

Differently from the Holley condition depends just on the difference between the two measures and can be easily proved using flows. The proof in Hosaka (2009) is also elementary but it assumes Holley result while our proof is completely independent. The result is strictly related to the geometry of the poset. It is possible indeed to construct posets (with Hasse diagram being a tree for example) for which $\mu_1 - \mu_2$ is not increasing but nevertheless $\mu_1 \not\preceq \mu_2$.

Our proof is by induction. For $N = 2$ a necessary and sufficient condition to have $\mu_1 \preceq \mu_2$ is (5.14). It is easy by a direct inspection to check that if $\mu_1 - \mu_2$

is non increasing then (5.14) holds. Let us now assume that if $\mu_1 - \mu_2$ is not increasing then $\mu_1 \preceq \mu_2$ for a fixed N and show that we can deduce that the same holds also for $N+1$. Any $\eta \in \{0,1\}^{N+1}$ is of the form $(\widetilde{\eta}, 0)$ or $(\widetilde{\eta}, 1)$ with $\widetilde{\eta} \in \{0,1\}^N$. We define

$$(\widetilde{\mu}_1 - \widetilde{\mu}_2)(\widetilde{\eta}) := \frac{[\mu_1(\widetilde{\eta}, 1) - \mu_2(\widetilde{\eta}, 1)] + [\mu_1(\widetilde{\eta}, 0) - \mu_2(\widetilde{\eta}, 0)]}{2}. \tag{5.24}$$

Since $\mu_1 - \mu_2$ is nonincreasing on $\{0,1\}^{N+1}$ then also $\widetilde{\mu}_1 - \widetilde{\mu}_2$ is nonincreasing on $\{0,1\}^N$. By induction, there exists a flow $\widetilde{Q}$ on the Hasse diagram of the poset of order N such that $\operatorname{div} \widetilde{Q} = \widetilde{\mu}_1 - \widetilde{\mu}_2$. Now, consider the flow Q on the Hasse diagram of the poset of order $N+1$ defined as follows: for any directed edge $(\widetilde{\eta}, \widetilde{\xi})$ of the Hasse diagram of order N we pose

$$Q\big((\widetilde{\eta}, 1), (\widetilde{\xi}, 1)\big) = Q\big((\widetilde{\eta}, 0), (\widetilde{\xi}, 0)\big) = \widetilde{Q}(\widetilde{\eta}, \widetilde{\xi}),$$

while, for all the remaining edges we fix $Q(\eta, \eta') = 0$. We have

$$\operatorname{div} Q(\widetilde{\eta}, 1) = \operatorname{div} Q(\widetilde{\eta}, 0) = \operatorname{div} \widetilde{Q}(\widetilde{\eta}) = (\widetilde{\mu}_1 - \widetilde{\mu}_2)(\widetilde{\eta}). \tag{5.25}$$

We now define another flow Q^a on the Hasse diagram of order $N+1$ as follows: for any $\widetilde{\eta} \in \{0,1\}^N$ we pose

$$Q^a\big((\widetilde{\eta}, 0), (\widetilde{\eta}, 1)\big) := \frac{[\mu_1(\widetilde{\eta}, 0) - \mu_2(\widetilde{\eta}, 0)] - [\mu_1(\widetilde{\eta}, 1) - \mu_2(\widetilde{\eta}, 1)]}{2}. \tag{5.26}$$

We fix then $Q^a(\eta, \eta') = 0$ for all the remaining edges. This is a well-defined flow since the right-hand side of (5.26) is nonnegative being $\mu_1 - \mu_2$ not increasing. Using (5.25) and (5.26), we obtain that the flow $Q + Q^a$ satisfies $\operatorname{div}(Q + Q^a) = \mu_1 - \mu_2$ and the proof is complete.

References

Bang-Jensen, J. and Gutin, G. (2001). *Digraphs. Theory, Algorithms and Applications. Sprint Sinceger Monographs in Mathematics*. London, Ltd., London: Springer. MR1798170

Beckmann, M. (1952). A continuous model of transportation. *Econometrica* **20**, 643–660. MR0068196

Bertini, L., Faggionato, A. and Gabrielli, D. (2015). Large deviations of the empirical flow for continuous time Markov chains. *Annales de L'Institut Henri Poincaré Probabilités et Statistiques* **51**, 867–900. MR3365965

Biggs, N. (1974). *Algebraic Graph Theory. Cambridge Tracts in Mathematics* **67**. London: Cambridge University Press. MR0347649

Gabrielli, D. and Valente, C. (2012). Which random walks are cyclic?. *Latin American Journal of Probability and Mathematical Statistics* **9**, 231–267. MR2923192

Holley, R. (1974). Remarks on the FKG inequalities. *Communications in Mathematical Physics* **36**, 227–231. MR0341552

Hosaka, K. (2009). An alternative condition for stochastic domination. *Journal of Applied Probability* **46**, 1198–1200. MR2582717

Hwang, F. K. (1979). Majorization on a partially ordered set. *Proceedings of the American Mathematical Society* **76**, 199–203. MR0537073

Kamae, T., Krengel, U. and O'Brien, G. L. (1977). Stochastic inequalities on partially ordered spaces. *Annals of Probability* **5**, 899–912. MR0494447

Lindvall, T. (1992). Lectures on the coupling method. *Wiley Series in Probability and Mathematical Statistics: Probability and Mathematical Statistics*. MR1180522

Lindvall, T. (1999). On Strassen's theorem on stochastic domination. *Electronic Communications in Probability* **4**, 51–59. MR1711599

Müller, A. (2013). Duality theory and transfers for stochastic order relations in Stochastic orders in reliability and risk. In *Lecture Notes in Statist., Vol. 208*, 41–57. New York: Springer. MR3156868

Paolini, E. and Stepanov, E. (2012). Decomposition of acyclic normal currents in a metric space. *Journal of Functional Analysis* **263**, 3358–3390. MR2984069

Rachev, S. T. and Rüschendorf, L. (1998). *Mass Transportation Problems Vol. I. Theory*. *Probability and Its Applications (New York)*. New York: Springer. MR1619171

Santambrogio, F. (2015). *Optimal Transport for Applied Mathematicians Calculus of Variations, PDEs, and Modeling*. *Progress in Nonlinear Differential Equations and Their Applications* **87**. Cham: Springer. MR3409718

Schrijver, A. (2003). Combinatorial optimization polyhedra and efficiency. In *Algorithms and Combinatorics, Vol. 24*. Springer. MR1956925

Smirnov, S. K. (1993). Decomposition of solenoidal vector charges into elementary solenoids, and the structure of normal one-dimensional flows. *(Russian) Algebra i Analiz* **5**, 206–238. Translation in St. Petersburg Math. J. 5 (1994), no. 4, 841–867. MR1246427

Strassen, V. (1965). The existence of probability measures with given marginals. *The Annals of Mathematical Statistics* **36**, 423–439. MR0177430

DISIM
University of L'Aquila
Via Vetoio
67100 Coppito
L'Aquila
Italy
E-mail: gabriell@univaq.it
ida.minelli@univaq.it