*Brazilian Journal of Probability and Statistics*
2019, Vol. 33, No. 2, 397–401
https://doi.org/10.1214/18-BJPS394


# A temporal perspective on the rate of convergence in first-passage percolation under a moment condition

**Daniel Ahlberg**

*Stockholm University*

**Abstract.** We study the rate of convergence in the celebrated Shape Theorem in first-passage percolation, obtaining the precise asymptotic rate of decay for the probability of linear order deviations under a moment condition. Our results are presented from a temporal perspective and complement previous work by the same author, in which the rate of convergence was studied from the standard spatial perspective.

## 1 Introduction

Consider first-passage percolation on the $\mathbb{Z}^d$ nearest-neighbour lattice for $d \geq 2$. Large deviations were first studied in the context of first-passage percolation in the 1980s, in a pioneering work of Grimmett and Kesten (1984). In this work, together with the subsequent work of Kesten (1986), the authors investigate the rate of convergence of travel times to the so-called time constant, and provide necessary and sufficient conditions for exponential decay for the probability of linear order deviations. For the exponential decay to hold, one requires finite moment of exponential order on the passage times.

It was only recently, in Ahlberg (2015), that large deviations in the regime of polynomial decay of the probability tails were studied. This regime is highly interesting since it is in this regime that strong laws such as the Subadditive Ergodic theorem due to Kingman (1968), and the Shape theorem due to Richardson (1973) (whose precise conditions were determined by Cox and Durrett (1981)) cease to hold. In this paper, we complement earlier results by the same author (Ahlberg (2015)) by offering a temporal perspective on the rate of convergence in the Shape theorem. These results are sharp in the regime of polynomial decay on the probability tails.

We will assume throughout that the edges of the $\mathbb{Z}^d$ lattice are assigned independent non-negative random weights from some probability distribution satisfying $F(0) < p_c(d)$, where $p_c(d)$ denotes the critical probability for bond percolation on $\mathbb{Z}^d$. The resulting weighted graph induces a random (pseudo-)metric structure to $\mathbb{Z}^d$, where the distance $T(x, y)$ between two sites $x, y \in \mathbb{Z}^d$ is given by the minimal weight accumulated along paths connecting the two points. The existence of

*Key words and phrases.* First-passage percolation, shape theorem, large deviations.
Received December 2017; accepted January 2018.

a *time constant* follows from a simple application of the Subadditive Ergodic theorem (Kingman (1968)). In fact, without further assumptions on the edge weights, the limit

$$\mu(z) := \lim_{n\to\infty} \frac{T(0, nz)}{n}, \tag{1}$$

exists in probability, and is finite and nonzero for all $z \neq 0$, see Kesten (1986).

Richardson (1973) realized that the above convergence holds in all directions simultaneously, in that a large ball $\mathscr{B}_t := \{z \in \mathbb{Z}^d : T(0, z) \leq t\}$ in the metric $T$ is asymptotically comparable to the deterministic ball $\mathscr{B}_t^\mu := \{z \in \mathbb{Z}^d : \mu(z) \leq t\}$. Cox and Durrett (1981) provided the precise condition for Richardson's result to hold. Let $Y$ denote the minimum of the $2d$ weights assigned to the edges adjacent to the origin. The result of Cox and Durrett says that if $\mathbb{E}[Y^d] < \infty$, then, almost surely, for every $\varepsilon \in (0, 1)$

$$\mathscr{B}^\mu_{(1-\varepsilon)t} \subset \mathscr{B}_t \subset \mathscr{B}^\mu_{(1+\varepsilon)t} \tag{2}$$

for large enough $t$. The given moment condition is also necessary for this conclusion.

In this paper, we investigate in further detail the probability that (2) fails, and that $\mathscr{B}_t$ deviates significantly from its asymptotic rate of growth. Let

$$\mathscr{T}_\varepsilon := \{t \geq 0 : \text{either inclusion in (2) fails}\}.$$

We will study the behaviour of $\mathbb{P}(t \in \mathscr{T}_\varepsilon)$ for large $t$ and fixed $\varepsilon \in (0, 1)$.

The result of Cox and Durrett implies that $\mathscr{T}_\varepsilon$ is almost surely bounded if and only if $\mathbb{E}[Y^d] < \infty$. As it turns out, not only is $\mathscr{T}_\varepsilon$ unbounded unless $\mathbb{E}[Y^d] < \infty$, but $\mathbb{P}(t \in \mathscr{T}_\varepsilon)$ may in fact be bounded away from zero even in the case when $\mathbb{E}[Y^\alpha] < \infty$ for all $\alpha < d$. (See the remark at the end of the paper.) When $\mathbb{E}[Y^d] < \infty$, however, we show that in the regime of polynomial tails of the weight distribution, the decay of $\mathbb{P}(t \in \mathscr{T}_\varepsilon)$ is governed by the tails of $Y$.

**Theorem.** *Assume that $d \geq 2$, $F(0) < p_c(d)$ and $\mathbb{E}[Y^d] < \infty$. Then, for every $\varepsilon > 0$ and $q \geq 1$, there is $c = c(\varepsilon, d, q)$ such that for all $t \geq 1$*

$$ct^d\mathbb{P}(Y > t) \leq \mathbb{P}(t \in \mathscr{T}_\varepsilon) \leq \frac{1}{c}t^d\mathbb{P}(Y > ct) + \frac{1}{c}t^{-q}.$$

Using Markov's inequality we find that for each $\alpha > 0$, the condition $\mathbb{E}[Y^{d+\alpha}] < \infty$ implies that $\mathbb{P}(t \in \mathscr{T}_\varepsilon) = O(t^{-\alpha})$. Another consequence of the theorem is the following characterization.

**Corollary.** *Assume that $F(0) < p_c(d)$. For every $\alpha \geq 0$, $\varepsilon > 0$ and $d \geq 2$,*

$$\mathbb{E}[Y^{d+\alpha}] < \infty \quad \Leftrightarrow \quad \int_0^\infty t^{\alpha-1}\mathbb{P}(t \in \mathscr{T}_\varepsilon)\,dt < \infty.$$

Together with Fubini's theorem, we see that

$$\mathbb{E}\,|\mathscr{T}_\varepsilon| = \mathbb{E}\int_0^\infty 1_{\{t\in\mathscr{T}_\varepsilon\}}\,dt = \int_0^\infty \mathbb{P}(t\in\mathscr{T}_\varepsilon)\,dt,$$

where $|\cdot|$ denotes Lebesgue measure. Hence, the conclusion of the corollary was for $\alpha = 1$ known already in Ahlberg (2015, Theorem 2), but the statement for general $\alpha > 0$ was previously unknown. The proofs we present below will be heavily based on ideas and results from Ahlberg (2015).

## 2 Proof

We first recall some results that will be used in the analysis: For every $\varepsilon > 0$ and $d \geq 2$ there exist constants $M = M(\varepsilon, d)$ and $\gamma = \gamma(\varepsilon, d)$ such that

$$\mathbb{P}\big(T(0,z) - \mu(z) < -\varepsilon x\big) \leq Me^{-\gamma x} \qquad \text{for } z\in\mathbb{Z}^d \text{ and } x \geq |z|. \tag{3}$$

If, in addition, $\mathbb{E}[Y^\alpha] < \infty$ for some $\alpha > 0$ and $q \geq 1$, then we may choose the constant $M = M(\alpha, \varepsilon, d, q)$ so that for all $z \in \mathbb{Z}^d$ and $x \geq |z|$ we have

$$\mathbb{P}\big(T(0,z) - \mu(z) > \varepsilon x\big) \leq M\mathbb{P}(Y > x/M) + Mx^{-q}. \tag{4}$$

The former statement was first proved by Grimmett and Kesten, Kesten (1984, 1986) for coordinate directions, and later extended by Ahlberg (2015) to the present form. The latter statement is original from Ahlberg (2015).

Fix $\varepsilon > 0$ and $q \geq 1$. We want to estimate the decay of $\mathbb{P}(t \in \mathscr{T}_\varepsilon)$. We let

$$\begin{aligned} A(z) &:= \big\{t \geq 0 : T(0,z) > t \text{ and } \mu(z) \leq (1-\varepsilon)t\big\}, \\ B(z) &:= \big\{t \geq 0 : T(0,z) \leq t \text{ and } \mu(z) > (1+\varepsilon)t\big\}. \end{aligned}$$

Note that $t \in \mathscr{T}_\varepsilon$ if and only if $t \in A(z) \cup B(z)$ for some $z \in \mathbb{Z}^d$. Moreover, $t \in A(z)$ indicates that the time to reach $z$ is unusually long, whereas $t \in B(z)$ means that the time to reach $z$ is exceptionally short.

We proceed with the proof of the theorem. We begin with the lower bound, which is elementary and does not require the bounds in (3) and (4).

### 2.1 The lower bound

Let $D$ denote the set of all $z \in \mathbb{Z}^d$ such that $\mu(z) \leq (1-\varepsilon)t$ and whose $\ell^1$-distance from the origin is even. Let $Y(z)$ denote the minimum weight among the $2d$ edges adjacent to $z$, and note that the $Y(z)$'s are independent for $z \in D$, as points in $D$ are at $\ell^1$-distance at least 2. Since $A(z) \subset \mathscr{T}_\varepsilon$ and $T(0,z) \geq Y(z)$ for every $z \in \mathbb{Z}^d$, we obtain

$$\begin{aligned} \mathbb{P}(t\in\mathscr{T}_\varepsilon) &\geq \mathbb{P}\big(T(0,z) > t \text{ for some } z \in D\big) \\ &\geq \mathbb{P}\big(Y(z) > t \text{ for some } z \in D\big). \end{aligned} \tag{5}$$

An application of Cachy-Schwarz's inequality shows that any non-negative random variable $X$ satisfies $\mathbb{P}(X > 0) \geq \mathbb{E}[X]^2/\mathbb{E}[X^2]$, and if $X$ is binomially distributed with parameters $n$ and $p$, then a further lower bound is given by $np/(1+np)$. Applying this to (5) leaves us with

$$\mathbb{P}(t \in \mathscr{T}_\varepsilon) \geq \frac{|D|\mathbb{P}(Y > t)}{1 + |D|\mathbb{P}(Y > t)}. \tag{6}$$

By assumption $\mathbb{E}[Y^d] < \infty$, which implies that $t^d\mathbb{P}(Y > t) \leq \mathbb{E}[Y^d]$ via Markov's inequality. Since the set $D$ grows as $t^d$, we obtain the required lower bound on $\mathbb{P}(t \in \mathscr{T}_\varepsilon)$.

### 2.2 The upper bound

We now continue with the upper bound. The union bound leaves us with

$$\mathbb{P}(t \in \mathscr{T}_\varepsilon) \leq \sum_{\mu(z) \leq (1-\varepsilon)t} \mathbb{P}\big(T(0,z) > t\big) + \sum_{\mu(z) > (1+\varepsilon)t} \mathbb{P}\big(T(0,z) \leq t\big),$$

and we will treat the two sums separately.

Let us first note that it is immediate from subadditivity and (1) that $\mu(kz) = k\mu(z)$ and $\mu(z+y) \leq \mu(z) + \mu(y)$ for all $k \geq 1$ and $y, z \in \mathbb{Z}^d$. In particular, $\mu$ is comparable to Euclidean distance. We also notice that the condition $x \geq |z|$ in (3) and (4) is not essential. The bounds can be extended to $x \geq \delta|z|$ by adjusting the constants $\gamma$ and $M$. Based on these observations, we use (3) to find $M_1 = M_1(\varepsilon, d)$ and $\gamma_1 = \gamma_1(\varepsilon, d)$ so that

$$\begin{aligned}\sum_{\mu(z) > (1+\varepsilon)t} \mathbb{P}\big(T(0,z) \leq t\big) &\leq \sum_{\mu(z) \geq (1+\varepsilon)t} \mathbb{P}\Big(T(0,z) - \mu(z) \leq -\frac{\varepsilon}{1+\varepsilon}\mu(z)\Big) \\ &\leq \sum_{\mu(z) \geq t} M_1 e^{-\gamma_1 \mu(z)},\end{aligned}$$

which is at most $M_2 e^{-\gamma_2 t}$ for $t \geq 1$, and some constants $M_2$ and $\gamma_2$. Using (4) we may find $M_3 = M_3(\varepsilon, d, q)$ such that

$$\begin{aligned}\sum_{\mu(z) \leq (1-\varepsilon)t} \mathbb{P}\big(T(0,z) > t\big) &\leq \sum_{\mu(z) \leq (1-\varepsilon)t} \mathbb{P}\big(T(0,z) - \mu(z) > \varepsilon t\big) \\ &\leq \sum_{\mu(z) \leq t} \Big(M_3 \mathbb{P}(Y \geq t/M_3) + \frac{M_3}{t^{d+q}}\Big).\end{aligned}$$

Since the cardinality of the set $\mathscr{B}_t^\mu$ grows at the order of $t^d$ we obtain

$$\mathbb{P}(t \in \mathscr{T}_\varepsilon) \leq M_4 t^d \mathbb{P}(Y \geq t/M_3) + M_4 t^{-q} + M_2 e^{-\gamma_2 t},$$

for some $M_4$ and all $t \geq 1$, which gives the required upper bound.

**Remark.** We note that the condition $\mathbb{E}[Y^d] < \infty$ cannot be relaxed in general. Consider for instance the case when $\mathbb{P}(Y > t) = (d-1)t^{-d}$ for $t \geq 1$. Then $\mathbb{E}[Y^\alpha]$ is finite for all $\alpha < d$ and infinite for $\alpha = d$, but the bound in (6) shows that the probability $\mathbb{P}(t \in \mathscr{T}_\varepsilon)$ is bounded away from zero.

## Acknowledgments

The results of this paper were deduced upon reque by Tonći Antunović and Eviatar Procaccia, and was in part supported by Grant 237-2013-7302 from the Swedish Research Council.

## References

Ahlberg, D. (2015). A Hsu–Robbins–Erdős strong law in first-passage percolation. *Annals of Probability* **43**, 1992–2025. MR3353820

Cox, J. T. and Durrett, R. (1981). Some limit theorems for percolation processes with necessary and sufficient conditions. *Annals of Probability* **9**, 583–603.

Grimmett, G. and Kesten, H. (1984). First-passage percolation, network flows and electrical resistances. *Zeitschrift für Wahrscheinlichkeitstheorie und Verwandte Gebiete* **66**, 335–366.

Kesten, H. (1986). Aspects of first passage percolation. In *École d'été de probabilités de Saint-Flour, XIV—1984*. *Lecture Notes in Math.* **1180**, 125–264. Berlin: Springer. MR0876084

Kingman, J. F. C. (1968). The ergodic theory of subadditive stochastic processes. *Journal of the Royal Statistical Society, Series B* **30**, 499–510. MR0254907

Richardson, D. (1973). Random growth in a tessellation. *Proceedings of the Cambridge Philosophical Society* **74**, 515–528.

Department of Mathematics
Stockholm University
SE - 106 91 Stockholm
Sweden
E-mail: daniel.ahlberg@math.su.se
URL: http://staff.math.su.se/daniel.ahlberg/