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HEAT KERNEL ESTIMATES FOR SYMMETRIC JUMP PROCESSES WITH MIXED POLYNOMIAL GROWTHS[1]

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In this paper, we study the transition densities of pure-jump symmetric Markov processes in $\mathbb{R}^d$, whose jumping kernels are comparable to radially symmetric functions with mixed polynomial growths. Under some mild assumptions on their scale functions, we establish sharp two-sided estimates of the transition densities (heat kernel estimates) for such processes. This is the first study on global heat kernel estimates of jump processes (including non-Lévy processes) whose weak scaling index is not necessarily strictly less than 2. As an application, we proved that the finite second moment condition on such symmetric Markov process is equivalent to the Khintchine-type law of iterated logarithm at infinity.

1. Introduction and main results. The heat kernel provides an important link between probability theory and partial differential equation. In probability theory, the heat kernel of an operator $\mathcal{L}$ is the transition density $p(t, x, y)$ (if it exists) of the Markov process X, which possesses $\mathcal{L}$ as its infinitesimal generator. In the field of of partial differential equation, it is called the fundamental solution of the heat equation $\partial_t u = \mathcal{L}u$. However, except in a few special cases, obtaining an explicit expression of $p(t, x, y)$ is usually impossible. Thus finding sharp estimates of $p(t, x, y)$ is a fundamental issue both in probability theory and partial differential equation.

Although heat kernels for diffusion processes have been studied for over a century, heat kernel estimates for discontinuous Markov processes have only been studied in recent years. After pioneering works such as [3, 11, 32], obtaining sharp two-sided estimates of heat kernels for various classes of discontinuous Markov processes has become an active topic in modern probability theory (see [1, 2, 5–10, 12, 14–16, 18–22, 24–26, 29–31, 33, 35–37, 43, 44] and references therein). In [12], the authors investigated heat kernel estimates for symmetric discontinuous Markov processes (on a large class of metric measure spaces) whose jumping intensities are comparable to radially symmetric functions of variable order. In particular, the heat kernel estimates therein cover the class of symmetric Markov

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processes $X=(X_t,\mathbb{P}^x,x\in\mathbb{R}^d,t\geq 0)$, without diffusion part, whose jumping kernels $J(x,y)$ satisfy the following conditions:

$$\frac{c^{-1}}{|x-y|^d\phi_1(|x-y|)}\leq J(x,y)\leq\frac{c}{|x-y|^d\phi_1(|x-y|)},\quad x,y\in\mathbb{R}^d, \tag{1.1}$$

where ϕ_1 is a nondecreasing function on $[0,\infty)$ satisfying

$$c_1(R/r)^{\alpha_1}\leq\phi_1(R)/\phi_1(r)\leq c_2(R/r)^{\alpha_2},\quad 0<r<R<\infty \tag{1.2}$$

with $\alpha_1,\alpha_2\in(0,2)$. Under the assumptions (1.1) and (1.2), the transition density $p(t,x,y)$ of X has the following estimates: for any $t>0$ and $x,y\in\mathbb{R}^d$,

$$p(t,x,y)\asymp\left(\phi_1^{-1}(t)^{-d}\wedge\frac{t}{|x-y|^d\phi_1(|x-y|)}\right). \tag{1.3}$$

(See [12], Theorem 1.2.) Here and below, we denote $a\wedge b:=\min\{a,b\}$ and $f\asymp g$ if the quotient f/g remains bounded between two positive constants. Thus, ϕ_1 is *the scale function*, that is, $\phi_1(|x-y|)=t$ provides the borderline for $p(t,x,y)$ to have either near-diagonal estimates or off-diagonal estimates. Moreover, it is not difficult to show from (1.3) that

$$c^{-1}\phi_1(r)\leq\mathbb{E}^z[\tau_{B(z,r)}]\leq c\phi_1(r)\quad\text{for all } z\in\mathbb{R}^d, r>0, \tag{1.4}$$

where τ_A is the first exit time from A for the process X. (See [2] and [10], Section 4.3.) Here, the function ϕ_1 commonly appears throughout (1.1), (1.3) and (1.4). Thus, under the assumptions (1.1) and (1.2), for all $r>0$ and $x,y,z\in\mathbb{R}^d$ with $|x-y|=r$,

$$\frac{c^{-1}}{J(x,y)r^d}\leq\mathbb{E}^z[\tau_{B(z,r)}]\leq\frac{c}{J(x,y)r^d}. \tag{1.5}$$

In this paper, we investigate estimates of transition densities of pure-jump symmetric Markov processes in $\mathbb{R}^d$, whose jumping kernels satisfy (1.1) with general mixed polynomial growths, that is, ϕ_1 satisfies (1.2) with $\alpha_1,\alpha_2\in(0,\infty)$. As a corollary of one of the main results, we obtain a global sharp two-sided estimate of the Green function (see Corollary 1.3). Unlike the heat kernel estimates in (1.3), ϕ_1 may not be the scale function for the heat kernel in general (see (1.10) and Theorem 1.4). For instance, when the process X is a subordinate Brownian motion, Ante Mimica [35] established the heat kernel estimates for the case that the scaling order of characteristic exponent of X may not be strictly below 2 (see [44] for some partial generalization to Lévy processes). We are strongly motivated by the research done in [35] and consider the case when Φ in (1.10), which is a scale function for the heat kernel, satisfies a (local) lower weak scaling condition with scaling index greater than 1. Under this assumption, we establish two-sided heat kernel estimates of symmetric jump processes in $\mathbb{R}^d$. Our results provide the first sharp heat kernel estimates covering non-Lévy processes whose weak scaling index is not necessarily strictly less than 2. This has been a major open problem in this area (cf. [25, 44]).

In our setting, (1.5) does not hold in general and we only have

$$\mathbb{E}^z[\tau_{B(z,r)}] \lessapprox \frac{c}{J(x,y)r^d} \quad \text{for all } r > 0 \text{ and } x, y, z \in \mathbb{R}^d \text{ with } |x-y| = r.$$

(See (2.1) and Lemma 3.12 below.)

In [10], the authors considered heat kernel estimates for mixed-type symmetric jump processes on metric measure spaces under a general volume doubling condition. Using variants of the cut-off Sobolev inequalities and the Faber–Krahn inequalities, they established stability of heat kernel estimates. In particular, they established heat kernel estimates for α-stable-like processes even with $\alpha \geq 2$ when the underlying spaces have walk dimensions larger than 2 (see [19, 23, 36, 37] also). Note that Euclidean space has the walk dimension 2; thus, the results in [10] does not cover our results and, in fact, a general version of (1.5) does hold in [10]. By contrast, some results in [10, 13] are applicable to our study and we will use several main results in [10, 13] to show the parabolic Harnack inequality and the near-diagonal lower bound of $p(t,x,y)$.

Before we give the main results of this paper, we first describe our setup.

DEFINITION 1.1. Let $g : (0,\infty) \to (0,\infty)$, $a \in (0,\infty]$, $\beta_1, \beta_2, c, C > 0$.

(1) For $a < \infty$, we say that g satisfies $L_a(\beta_1, c)$ (resp. $L^a(\beta_1, c)$) if $g(R)/g(r) \geq c(R/r)^{\beta_1}$ for all $r \leq R < a$ (resp. $a \leq r \leq R$). We also say that g satisfies the weak lower scaling condition near 0 (resp. near ∞) with index β_1.

(2) We say that g satisfies $U_a(\beta_2, C)$ (resp. $U^a(\beta_2, C)$) if $g(R)/g(r) \leq C(R/r)^{\beta_2}$ for all $r \leq R < a$ (resp. $a \leq r \leq R$). We also say that g satisfies the weak upper scaling condition near 0 (resp. near ∞) with index β_2.

(3) When g satisfies $U_a(\beta, C)$ (resp. $L_a(\beta, c)$) with $a = \infty$, we say that g satisfies the global weak upper scaling condition $U(\beta, C)$ (resp. the global weak lower scaling condition $L(\beta, c)$).

Throughout this paper, except Sections 2.1 and 2.2, we will assume that $\psi : (0,\infty) \to (0,\infty)$ is a nondecreasing function satisfying $L(\beta_1, C_L)$, $U(\beta_2, C_U)$, and

$$\int_0^1 \frac{s}{\psi(s)}\,ds < \infty. \tag{1.6}$$

Denote $diag = \{(x,x) : x \in \mathbb{R}^d\}$. Assume that $J : \mathbb{R}^d \times \mathbb{R}^d \setminus diag \to [0,\infty)$ is a symmetric function satisfying

$$\frac{\bar{C}^{-1}}{|x-y|^d\psi(|x-y|)} \leq J(x,y) \leq \frac{\bar{C}}{|x-y|^d\psi(|x-y|)} \tag{1.7}$$

for all $(x, y) \in \mathbb{R}^d \times \mathbb{R}^d \setminus diag$, with some $\bar{C} \geq 1$. Note that (1.6) combined with (1.7) and $L(\beta_1, C_L)$ on ψ is a natural assumption to ensure that

$$\text{(1.8)} \qquad \sup_{x\in\mathbb{R}^d} \int_{\mathbb{R}^d} (|x-y|^2 \wedge 1) J(x, y)\, dy \leq c\left(\int_0^1 \frac{s\, ds}{\psi(s)} + \int_1^\infty \frac{ds}{s\psi(s)}\right) < \infty.$$

For $u, v \in L^2(\mathbb{R}^d, dx)$, define

$$\text{(1.9)} \qquad \mathcal{E}(u, v) := \int_{\mathbb{R}^d\times\mathbb{R}^d} \big(u(x) - u(y)\big)\big(v(x) - v(y)\big) J(x, y)\, dx\, dy$$

and $\mathcal{F} = \{f \in L^2(\mathbb{R}^d) : \mathcal{E}(f, f) < \infty\}$. By applying the lower scaling assumption $L(\beta_1, C_L)$ on ψ, (1.7) and (1.8) to [40], Theorem 2.1, and [41], Theorem 2.4, we observe that $(\mathcal{E}, \mathcal{F})$ is a regular Dirichlet form on $L^2(\mathbb{R}^d, dx)$. Thus, there is a Hunt process X associated with $(\mathcal{E}, \mathcal{F})$, starting from quasi-everywhere point in $\mathbb{R}^d$. Moreover, by (1.8) and [34], Theorem 3.1, X is conservative.

We define our scale function by

$$\text{(1.10)} \qquad \Phi(r) := \frac{r^2}{2\int_0^r \frac{s}{\psi(s)}\, ds}.$$

In general, the function Φ is strictly increasing, and is less than ψ (see (2.1)–(2.3) below). However, these two functions may not be comparable unless $\beta_2 < 2$. We remark here that the function Φ has been observed as the correct scale function recently (see [21, 22, 28, 35, 38]).

THEOREM 1.2. *Let ψ be a nondecreasing function satisfying $L(\beta_1, C_L)$ and $U(\beta_2, C_U)$. Assume that conditions (1.6) and (1.7) hold. Then, there is a conservative Feller process $X = (X_t, \mathbb{P}^x, x \in \mathbb{R}^d, t \geq 0)$ associated with $(\mathcal{E}, \mathcal{F})$ that can starts from every point in $\mathbb{R}^d$. Moreover, X has a jointly continuous transition density function $p(t, x, y)$ on $(0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d$ with the following estimates: there exist $a_U, c, C, \delta_1 > 0$ such that*

$$\text{(1.11)} \qquad p(t, x, y) \leq \frac{C}{\Phi^{-1}(t)^d} \wedge \left(\frac{Ct}{|x-y|^d \psi(|x-y|)} + \frac{C}{\Phi^{-1}(t)^d} e^{-\frac{a_U |x-y|^2}{\Phi^{-1}(t)^2}}\right)$$

and

$$\text{(1.12)} \quad p(t, x, y) \geq \frac{C^{-1}\mathbf{1}_{\{|x-y|\leq\delta_1\Phi^{-1}(t)\}}}{\Phi^{-1}(t)^d} + \frac{C^{-1}t}{|x-y|^d\psi(|x-y|)}\mathbf{1}_{\{|x-y|\geq\delta_1\Phi^{-1}(t)\}}.$$

The proofs of (1.11) and (1.12) are given in Section 4.1 and Proposition 4.6.

Let $G(x, y) = \int_0^\infty p(t, x, y)\, dt$ be the Green function for X. As a corollary of Theorem 1.2, we get sharp two-sided estimates for the Green function.

COROLLARY 1.3. *Suppose that the assumptions in Theorem 1.2 hold and $d > \beta_2 \wedge 2$. Then for any $x, y \in \mathbb{R}^d$, $G(x, y) \asymp \Phi(|x-y|)|x-y|^{-d}$.*

Using our scale function Φ, we define for $a > 0$,

$$(1.13)\qquad \mathscr{K}(s) := \sup_{b \le s} \frac{\Phi(b)}{b} \quad \text{and} \quad \mathscr{K}_\infty(s) := \begin{cases} \sup_{a \le b \le s} \dfrac{\Phi(b)}{b}, & s \ge a, \\ a^{-2}\Phi(a)s, & 0 < s < a. \end{cases}$$

If Φ satisfies $L_a(\delta, \widetilde{C}_L)$ with $\delta > 1$, then $\mathscr{K}(0) = 0$ and $\mathscr{K}$ is nondecreasing. Thus, the generalized inverse $\mathscr{K}^{-1}(t) := \inf\{s \ge 0 : \mathscr{K}(s) > t\}$ is well defined on $[0, \sup_{b<\infty} \frac{\Phi(b)}{b})$.

If Φ satisfies $L^a(\delta, \widetilde{C}_L)$ with $\delta > 1$, $\mathscr{K}_\infty$ and the generalized inverse $\mathscr{K}_\infty^{-1}$ are well defined and nondecreasing on $[0, \infty)$. Some properties of $\mathscr{K}$ and $\mathscr{K}_\infty$ are shown in Section 2.2. Here is the main result of this paper.

THEOREM 1.4. *Let ψ be a nondecreasing function satisfying $L(\beta_1, C_L)$ and $U(\beta_2, C_U)$. Assume that conditions* (1.6) *and* (1.7) *hold, and Φ satisfies $L_a(\delta, \widetilde{C}_L)$ or $L^a(\delta, \widetilde{C}_L)$ for some $a > 0$ and $\delta > 1$. Then, the following estimates hold*:

(1) *When Φ satisfies $L_a(\delta, \widetilde{C}_L)$: For every $T > 0$, there exist positive constants $c_1 = c_1(T, a, \delta, \beta_1, \beta_2, \widetilde{C}_L, C_L, C_U) \ge 1$ and $a_U \le a_L$ such that for any $(t, x, y) \in (0, T) \times \mathbb{R}^d \times \mathbb{R}^d$,*

$$\begin{aligned} &c_1^{-1}\left(\frac{1}{\Phi^{-1}(t)^d} \wedge \left(\frac{t}{|x-y|^d\psi(|x-y|)} + \frac{1}{\Phi^{-1}(t)^d} e^{-\frac{a_L|x-y|}{\mathscr{K}^{-1}(t/|x-y|)}}\right)\right) \\ &\quad \le p(t, x, y) \\ (1.14)\qquad &\quad \le c_1\left(\frac{1}{\Phi^{-1}(t)^d} \wedge \left(\frac{t}{|x-y|^d\psi(|x-y|)} + \frac{1}{\Phi^{-1}(t)^d} e^{-\frac{a_U|x-y|}{\mathscr{K}^{-1}(t/|x-y|)}}\right)\right). \end{aligned}$$

Moreover, if Φ satisfies $L(\delta, \tilde{C}_L)$, then (1.14) *holds for all $t \in (0, \infty)$.*

(2) *When Φ satisfies $L^a(\delta, \widetilde{C}_L)$: For every $T > 0$, there exist positive constants $c_2 = c_2(T, a, \delta, \beta_1, \beta_2, \widetilde{C}_L, C_L, C_U) \ge 1$ and $a'_U \le a'_L$ such that for any $(t, x, y) \in [T, \infty) \times \mathbb{R}^d \times \mathbb{R}^d$,*

$$\begin{aligned} &c_2^{-1}\left(\frac{1}{\Phi^{-1}(t)^d} \wedge \left(\frac{t}{|x-y|^d\psi(|x-y|)} + \frac{1}{\Phi^{-1}(t)^d} e^{-\frac{a'_L|x-y|}{\mathscr{K}_\infty^{-1}(t/|x-y|)}}\right)\right) \\ &\quad \le p(t, x, y) \\ &\quad \le c_2\left(\frac{1}{\Phi^{-1}(t)^d} \wedge \left(\frac{t}{|x-y|^d\psi(|x-y|)} + \frac{1}{\Phi^{-1}(t)^d} e^{-\frac{a'_U|x-y|}{\mathscr{K}_\infty^{-1}(t/|x-y|)}}\right)\right). \end{aligned}$$

In particular, if $\delta = 2$, then $\mathscr{K}_\infty^{-1}(t) \asymp t$ for $t \ge T$.

Theorem 1.4(2) covers [42], Corollary 3.11, where $\psi(r) = r^{2+\varepsilon}$, $r > 1$ and $\varepsilon > 0$, is considered. Note that if ψ satisfies $L_a(\delta, \widetilde{C}_L)$, then $\delta < 2$ and Φ satisfies $L_a(\delta, \widetilde{C}_L)$ by Lemma 2.4.

Using Theorems 1.2 and 1.4(2), we will show in Section 5 that the finite second moment condition is equivalent to the Khintchine-type law of iterated logarithm at infinity. In [17], Gnedenko proved this result for Lévy processes (see also [39], Proposition 48.9). The equivalence between the law of iterated logarithm and the finite second moment condition for non-Lévy processes has been a long standing open problem since the work done in [17].

A nonnegative C^∞ function ϕ on $(0,\infty)$ is called a Bernstein function if $(-1)^n\phi^{(n)}(\lambda) \le 0$ for every $n \in \mathbb{N}$ and $\lambda > 0$. The exponent $(r/\Phi^{-1}(t))^2$ in (1.11) is not comparable to $r/\mathscr{K}^{-1}(t/r)$ in general (see Lemma 2.8 and Corollary 6.1 below). However, the following corollary indicates that we can replace $r/\mathscr{K}^{-1}(t/r)$ with a simpler function $(r/\Phi^{-1}(t))^2$ if we additionally assume that $r \mapsto \Phi(r^{-1/2})^{-1}$ is a Bernstein function.

COROLLARY 1.5. *Let ψ be a nondecreasing function satisfying $L(\beta_1, C_L)$ and $U(\beta_2, C_U)$. Assume that conditions* (1.6) *and* (1.7) *hold*, Φ *satisfies* $L_a(\delta, \widetilde{C}_L)$ *some $a > 0$ and $\delta > 1$, and $r \mapsto \Phi(r^{-1/2})^{-1}$ is a Bernstein function. Then, for any $T > 0$, there exist positive constants $c \ge 1$ and $a_U \le a_L$ such that for all $(t,x,y) \in (0,T) \times \mathbb{R}^d \times \mathbb{R}^d$,*

$$\begin{aligned} &c^{-1}\left(\frac{1}{\Phi^{-1}(t)^d} \wedge \left(\frac{t}{|x-y|^d\psi(|x-y|)} + \frac{1}{\Phi^{-1}(t)^d} e^{-a_L \frac{|x-y|^2}{\Phi^{-1}(t)^2}}\right)\right) \\ &\quad \le p(t,x,y) \qquad\qquad (1.15) \\ &\quad \le c\left(\frac{1}{\Phi^{-1}(t)^d} \wedge \left(\frac{t}{|x-y|^d\psi(|x-y|)} + \frac{1}{\Phi^{-1}(t)^d} e^{-a_U \frac{|x-y|^2}{\Phi^{-1}(t)^2}}\right)\right). \end{aligned}$$

Moreover, if Φ satisfies $L(\delta, C_L)$ with $\delta > 1$, (1.15) *holds for all $t \in (0,\infty)$.*

The remainder of the paper is organized as follows. Section 2 describes some properties of ψ, Φ, $\mathscr{K}$ and $\mathscr{K}_\infty$, and verifies some relationships among them. Section 3 proves a preliminary upper bound and near-diagonal estimates of the transition density. Section 3.1 presents the Poincaré inequality, which is the first step to find a correct scale function. Section 3.3 uses scaled versions of X to obtain an upper bound of the transition density function (see Theorem 3.8). Although this upper bound is not sharp, it is good enough to get the lower bound on the survival probability and the CSJ(Φ) condition defined in [10]. Section 3.4 shows the near-diagonal lower bound of the transition function, parabolic Harnack inequality, and parabolic Hölder regularity by applying the results in [10, 13]. Section 4 describes the proof of off-diagonal estimates of the transition density function. Section 4.1 and Section 4.2 prove the off-diagonal upper bound and lower bound of the transition density function, respectively. As an application of the main result, in Section 5 we show that the finite second moment condition is equivalent to the Khintchine-type law of iterated logarithm at infinity. Section 6 provides examples covered by the main results.

Notation. Throughout this paper, the constants C_1, C_2, C_3, C_4, C_L, C_U, $\widetilde{C}_L$, β_1, β_2, δ, δ_1 will remain the same, whereas C, c, and $c_0, a_0, c_1, a_1, c_2, a_2, \dots$ represent constants having insignificant values that may be changed from one appearance to another. All these constants are positive finite. The labeling of the constants $c_1, c_2, \dots$ begins anew in the proof of each result. $c_i = c_i(a, b, c, \dots)$, $i = 0, 1, 2, \dots$, denote generic constants depending on $a, b, c, \dots$. The dependence on the dimension $d \geq 1$ and the constant $\bar{C}$ in (1.7) may not be explicitly mentioned. Recall that we use the notation $a \wedge b = \min\{a, b\}$ and $f \asymp g$ if the quotient f/g remains bounded between two positive constants. We denote $a \vee b := \max\{a, b\}$, $\mathbb{R}_+ := \{r \in \mathbb{R} : r > 0\}$, and $B(x, r) := \{y \in \mathbb{R}^d : |x - y| < r\}$.

2. Preliminary.

2.1. *Basic properties of ψ and Φ.* In this subsection, we will observe some elementary properties of ψ and Φ. Since ψ is nondecreasing and $\lim_{r\to 0} \psi(r) = 0$ by $L(\beta_1, C_L)$ for ψ, we have that

$$\Phi(r) = \frac{r^2}{2\int_0^r \frac{s}{\psi(s)}\,ds} < \frac{r^2}{2\int_0^r \frac{s}{\psi(r)}\,ds} = \psi(r). \tag{2.1}$$

Thus, under (1.7), we obtain that for any $x, y \in \mathbb{R}^d$,

$$J(x, y) \leq \frac{\bar{C}}{|x - y|^d \Phi(|x - y|)}. \tag{2.2}$$

Since $(1/\Phi(r))' = \frac{4}{r\psi(r)} - \frac{4}{r\Phi(r)} < 0$, $r \mapsto \Phi(r)$ is strictly increasing. Note that, since $r^2/\Phi(r)$ is increasing in r, we have that for any $0 < r \leq R$,

$$\Phi(R)/\Phi(r) \leq (R/r)^2. \tag{2.3}$$

From this, we see that if Φ satisfies $L^a(\beta, c)$, then $\beta \leq 2$.

REMARK 2.1. Suppose $g : (0, \infty) \to (0, \infty)$ is nondecreasing. If g satisfies $L_a(\beta, c)$, then g satisfies $L_b(\beta, c(ab^{-1})^\beta)$ for any $b > a$. Similarly, if g satisfies $L^a(\beta, c)$, then g satisfies $L^b(\beta, c(a^{-1}b)^\beta)$ for any $b < a$.

The next three results are straightforward. We skip their proofs.

LEMMA 2.2. *Let $g : (0, \infty) \to (0, \infty)$ be a nondecreasing function with $g(\infty) = \infty$. (1) If g satisfies $L_a(\beta, c)$ (resp. $U_a(\beta, C)$), then g^{-1} satisfies $U_{g(a)}(1/\beta, c^{-1/\beta})$ (resp. $L_{g(a)}(1/\beta, C^{-1/\beta})$). (2) If g satisfies $L^a(\beta, c)$ (resp. $U^a(\beta, C)$), then g^{-1} satisfies $U^{g(a)}(1/\beta, c^{-1/\beta})$ (resp. $L^{g(a)}(1/\beta, C^{-1/\beta})$).*

The following lemma will be used in the paper several times.

LEMMA 2.3. *Assume that* ψ *satisfies* $L(\beta, c)$ *and* $U(\widehat{\beta}, C)$. *Then*, *for any* $x \in \mathbb{R}^d$ *and* $r > 0$, $\int_r^\infty (s\psi(s))^{-1}\,ds \asymp 1/\psi(r)$.

The next lemma shows that the index in the weak scaling conditions for Φ is always in $(0, 2]$.

LEMMA 2.4. *Let* $a \in (0, \infty]$, $0 < \beta \le \widehat{\beta}$, $0 < c \le 1 \le C$.

(1) *If* ψ *satisfies* $U_a(\widehat{\beta}, C)$, *then* Φ *satisfies* $U_a(\widehat{\beta} \wedge 2, C)$.
(2) *If* ψ *satisfies* (1.6) *and* $L_a(\beta, c)$, *then* $\beta < 2$ *and* Φ *satisfies* $L_a(\beta, c)$.

We remark here that the comparability of ψ and Φ is equivalent to that the index of the weak upper scaling condition is strictly less than 2 (see [4], Corollaries 2.6.2 and 2.6.4).

2.2. *Basic properties of* $\mathscr{K}$ *and* $\mathscr{K}_\infty$. In this subsection, under the assumption that Φ satisfies $L_a(\delta, \widetilde{C}_L)$ or $L^a(\delta, \widetilde{C}_L)$ with $\delta > 1$, we establish some basic properties of $\mathscr{K}$ and $\mathscr{K}_\infty$ defined in (1.13).

The next lemma immediately follows from the definition of $\mathscr{K}$, (2.3) and assumption that Φ satisfies $L_a(\delta, \widetilde{C}_L)$ with $\delta > 1$.

LEMMA 2.5. *If* Φ *satisfies* $L_a(\delta, \widetilde{C}_L)$ *with* $\delta > 1$ *and* $a \in (0, \infty]$, *then* $\Phi(t)/t \le \mathscr{K}(t) \le \widetilde{C}_L^{-1}\Phi(t)/t$ *for* $t < a$, *and*

$$\widetilde{C}_L^2 (t/s)^{\delta-1} \le \mathscr{K}(t)/\mathscr{K}(s) \le \widetilde{C}_L^{-1} t/s \quad \text{for } s \le t < a. \tag{2.4}$$

For notational convenience, we introduce an auxiliary function

$$\widetilde{\Phi}_a(s) := \frac{\Phi(a)}{a^2} s^2 \mathbf{1}_{\{0<s<a\}} + \Phi(s)\mathbf{1}_{\{s \ge a\}},$$

so that we have $\mathscr{K}_\infty(s) = \sup_{b \le s} \frac{\widetilde{\Phi}_a(b)}{b}$.

The following lemma shows the relation between Φ and $\widetilde{\Phi}_a$. Since the proof is elementary, we skip the proof.

LEMMA 2.6. (1) *For any* $t > 0$, $\widetilde{\Phi}_a(t) \le \Phi(t)$ *and for* $t \ge c > 0$, $\widetilde{\Phi}_a(t) \ge ((c/a)^2 \wedge 1)\Phi(t)$. (2) *For* $0 < s < t$, $\widetilde{\Phi}_a(t)/\widetilde{\Phi}_a(s) \le t^2/s^2$. (3) *Suppose* Φ *satisfies* $L^a(\delta, \widetilde{C}_L)$ *with some* $\delta \le 2$. *Then*, $\widetilde{\Phi}_a$ *satisfies* $L(\delta, \widetilde{C}_L)$.

By Lemma 2.6(1) and (2.1), $\widetilde{\Phi}_a(t) \le \psi(t)$ for all $t > 0$ and $a > 0$. In the following lemma, we will see some properties of $\mathscr{K}_{\infty,a}$ which is similar to Lemma 2.5.

LEMMA 2.7. *Let* $a \in (0, \infty)$. *If* Φ *satisfies* $L^a(\delta, \widetilde{C}_L)$ *with* $\delta > 1$, *then* $\widetilde{\Phi}_a(t)/t \le \mathscr{K}_{\infty,a}(t) \le \widetilde{C}_L^{-1}\widetilde{\Phi}_a(t)/t$ *for* $t > 0$, *and*

$$\widetilde{C}_L^2 (t/s)^{\delta-1} \le \mathscr{K}_{\infty,a}(t)/\mathscr{K}_{\infty,a}(s) \le \widetilde{C}_L^{-1} t/s \quad \text{for } t > s > 0. \tag{2.5}$$

Moreover, for any $c_1 > 0$, *there exists* $c_2 = c_2(c_1, a, \delta, \widetilde{C}_L) \ge 1$ *such that for any* $t \ge c_1$,

$$c_2^{-1} \sup_{c_1 \le b \le t} \Phi(b)/b \le \mathscr{K}_{\infty,a}(t) \le c_2 \sup_{c_1 \le b \le t} \Phi(b)/b. \tag{2.6}$$

PROOF. The first claim and (2.5) follow from Lemmas 2.6(3) and 2.5. (2.6) follows from Remark 2.1, Lemma 2.6(1) and (2.6). $\square$

By Remark 2.1, if Φ satisfies the weak lower scaling condition at infinity, we will assume that Φ satisfies $L^1(\delta, \widetilde{C}_L)$ instead of $L^a(\delta, \widetilde{C}_L)$. Now we further assume that $\delta > 1$. Then, $\mathscr{K}_\infty = \mathscr{K}_{\infty,1}$ is nondecreasing function with $\mathscr{K}_\infty(0) = 0$ and $\lim_{t\to\infty} \mathscr{K}_\infty(t) = \infty$.

In the following lemma, we show some inequalities between Φ^{-1} and $\mathscr{K}^{-1}$, and between Φ^{-1} and $\mathscr{K}_\infty^{-1}$.

LEMMA 2.8. (1) *Suppose* Φ *satisfies* $L_a(\delta, \widetilde{C}_L)$ *with* $\delta > 1$ *and for some* $a > 0$. *For any* $T > 0$ *and* $b > 0$ *there exists a constant* $c_1 = c_1(b, \widetilde{C}_L, a, \delta, T) > 0$ *such that*

$$\Phi^{-1}(t) \le c_1 \mathscr{K}^{-1}\left(\frac{t}{b\Phi^{-1}(t)}\right) \quad \textit{for all } t \in (0, T), \tag{2.7}$$

and there exists a constant $c_2 = c_2(a, \widetilde{C}_L, \delta, T) \ge 1$ *such that for every* $t, r > 0$ *satisfying* $t < \Phi(r) \wedge T$,

$$\left(r/\Phi^{-1}(t)\right)^2 \le r/\mathscr{K}^{-1}(t/r) \le c_2\left(r/\Phi^{-1}(t)\right)^{\delta/(\delta-1)}. \tag{2.8}$$

Moreover, if $a = \infty$, *then* (2.7) *and* (2.8) *hold with* $T = \infty$. *In other words*, (2.7) *holds for all* $t < \infty$ *and* (2.8) *holds for* $t < \Phi(r)$.

(2) *Suppose* Φ *satisfies* $L^1(\delta, \widetilde{C}_L)$ *with* $\delta > 1$. *For any* $T > 0$ *and* $b > 0$ *there exists a constant* $c_3 = c_3(T, b, \widetilde{C}_L, \delta) \ge 1$ *such that for* $t \ge T$,

$$\Phi^{-1}(t) \le c_3 \mathscr{K}_\infty^{-1}\left(\frac{t}{b\Phi^{-1}(t)}\right), \tag{2.9}$$

and for any $T > 0$ *there exists a constant* $c_4 = c_4(a, \widetilde{C}_L, \delta, T) \ge 1$ *such that for every* $t, r > 0$ *satisfying* $T \le t \le \Phi(r)$,

$$c_4^{-1}\left(r/\Phi^{-1}(t)\right)^2 \le r/\mathscr{K}_\infty^{-1}(t/r) \le c_4\left(r/\Phi^{-1}(t)\right)^{\delta/(\delta-1)}. \tag{2.10}$$

PROOF. (1) (2.7) follows from Remark 2.1 and Lemma 2.5 and the first inequality in (2.8) follows from (2.3) and Lemma 2.5.

Let $c_2 := \widetilde{C}_L^{-2/(\delta-1)} \ge 1$. The second inequality in (2.8) follows from

$$\frac{t}{r} \ge \widetilde{C}_L^{-1} \frac{\Phi(c_2^{-1}\Phi^{-1}(t)^{\delta/(\delta-1)} r^{-1/(\delta-1)})}{c_2^{-1}\Phi^{-1}(t)^{\delta/(\delta-1)} r^{-1/(\delta-1)}} \ge \mathscr{K}\left(c_2^{-1}\Phi^{-1}(t)^{\delta/(\delta-1)} r^{-1/(\delta-1)}\right)$$

for $t < T \wedge \Phi(r)$, which uses the condition $L_a(\delta, \widetilde{C}_L)$ on Φ.

Since we only assumed $a \geq \Phi^{-1}(T)$ on T and c_1, C_1 are independent of T, (2.7) and (2.8) holds with $T = \infty$ when $a = \infty$.

(2) Fix $T_1 \in (0, \infty)$. By Lemma 2.6(3), $\widetilde{\Phi}$ satisfies $L(\delta, \widetilde{C}_L)$. Now the function $\widetilde{\Phi}$ satisfies the assumption of Lemma 2.8(1), thus (2.9) and (2.10) with functions $\widetilde{\Phi}$ and $\mathscr{K}_\infty$ hold with $T = \infty$. Now lemma follows from the fact that $\Phi^{-1}(t) \asymp \widetilde{\Phi}^{-1}(t)$ for $t \geq T_1$. $\square$

3. Near-diagonal estimates and preliminary upper bound.

3.1. *Functional inequalities.* Here we will prove (weak) Poincaré inequality with respect to our jumping kernel J. We start with a simple calculus.

LEMMA 3.1. *For $r > 0$, let $g : (0, r] \to \mathbb{R}$ be a continuous and nonincreasing function satisfying $\int_0^r sg(s)\,ds \geq 0$ and $h : [0, r] \to [0, \infty)$ be a subadditive measurable function with $h(0) = 0$, that is, $h(s_1) + h(s_2) \geq h(s_1 + s_2)$, for $0 < s_1, s_2 < r$ with $s_1 + s_2 < r$. Then, $\int_0^r h(s)g(s)\,ds \geq 0$.*

PROOF. Let $H(s) := s^{-2} \int_0^s h(t)\,dt$. Then, since $H'(s) \leq 0$ by the subadditivity, $H(s)$ is nonincreasing. Thus, we have that for any $0 < r_1 \leq r_2 < r_3 \leq r$,

$$\frac{1}{r_1^2} \int_0^{r_1} h(t)\,dt \geq \frac{1}{r_2^2} \int_0^{r_2} h(t)\,dt \geq \frac{1}{r_3^2 - r_2^2} \int_{r_2}^{r_3} h(t)\,dt. \tag{3.1}$$

If $g(r) \geq 0$, then the lemma is trivial since g is nonincreasing. Assume $g(r) < 0$ and let $r_0 := \inf\{s \leq r : g(s) < 0\}$. Let $0 < k := r_0^{-2} \int_0^{r_0} h(s)\,ds$. By using the continuity of g, $g(r_0) = 0$, and the integration by parts, we have

$$\int_0^{r_0} h(s)g(s)\,ds = -\int_0^{r_0} \int_0^s h(t)\,dt\,dg(s) \geq -k \int_0^{r_0} s^2\,dg(s) = k \int_0^{r_0} 2sg(s)\,ds$$

and

$$\int_{r_0}^r h(s)g(s)\,ds = \int_{r_0}^r \int_s^r h(t)\,dt\,dg(s) \geq k \int_{r_0}^r \left(r^2 - s^2\right) dg(s) = k \int_{r_0}^r 2sg(s)\,ds.$$

Thus, $\int_0^r h(s)g(s)\,ds = (\int_0^{r_0} + \int_{r_0}^r) h(s)g(s)\,ds \geq k \int_0^r 2sg(s)\,ds \geq 0$, which completes the proof. $\square$

By applying the above lemma, we have the following (weak) Poincaré inequality.

PROPOSITION 3.2. *There exists $C > 0$ such that for every bounded and measurable function f, $x_0 \in \mathbb{R}^d$ and $r > 0$,*

$$\begin{aligned} &\frac{C}{r^d \Phi(r)} \int_{B(x_0,r) \times B(x_0,r)} \left(f(y) - f(x)\right)^2 dx\,dy \\ &\quad \leq \int_{B(x_0,3r) \times B(x_0,3r)} \left(f(y) - f(x)\right)^2 J(x, y)\,dx\,dy. \end{aligned} \tag{3.2}$$

PROOF. Denote $B(r) := B(x_0, r)$. For $0 < s < 2r$, let

$$h(s) := s^{-d} \int_{B(3r-s)} \int_{|z|=s} \big(f(x+z) - f(x)\big)^2 \sigma(dz)\, dx,$$

where σ is surface measure of the ball. We observe that the left-hand side of (3.2) is bounded above by

$$\begin{aligned} &\frac{c_1}{r^d \Phi(r)} \int_{B(r)} \int_0^{2r} \int_{|z|=s} \big(f(x+z) - f(x)\big)^2 \sigma(dz)\, ds\, dx \\ &\quad \le \frac{c_1}{r^d \Phi(r)} \int_0^{2r} h(s) s^d\, ds \le \frac{2^{d+2} c_1}{\Phi(2r)} \int_0^{2r} h(s)\, ds, \end{aligned}$$

where the last inequality follows from (2.3).

On the other hand, the right-hand side of (3.2) is bounded below by

$$c_2 \int_{B(2r)} \int_{B(3r-|z|)} \big(f(x+z) - f(x)\big)^2 \frac{1}{|z|^d \psi(|z|)}\, dx\, dz = c_3 \int_0^{2r} h(s) \frac{1}{\psi(s)}\, ds.$$

Let $g(s) = \frac{1}{\psi(s)} - \frac{1}{\Phi(r)}$. Then, $g(s)$ is continuous, nonincreasing and $\int_0^r s g(s)\, ds = \int_0^r \frac{s}{\psi(s)} - \frac{s}{\Phi(r)}\, ds = 0$. Also, for $s_1, s_2 > 0$ with $s_1 + s_2 := s < 2r$,

$$\begin{aligned} h(s) &= \int_{|\xi|=1} \int_{B(3r-s)} \frac{(f(x+s\xi) - f(x))^2}{s}\, dx \sigma(d\xi) \\ &\le \int_{|\xi|=1} \int_{B(3r-s)} \frac{(f(x+s\xi) - f(x+s_2\xi))^2}{s_1} \\ &\qquad + \frac{(f(x+s_2\xi) - f(x))^2}{s_2}\, dx \sigma(d\xi) \\ &\le \int_{|\xi|=1} \int_{B(x_0+s_2\xi, 3r-s)} \frac{(f(x+s_1\xi) - f(x))^2}{s_1}\, dx \sigma(d\xi) + h(s_2) \\ &\le h(s_1) + h(s_2), \end{aligned}$$

where the first inequality follows from $\frac{(b_1+b_2)^2}{s} \le \frac{b_1^2}{s_1} + \frac{b_2^2}{s_2}$. Thus, the functions g and h satisfy the assertions of Lemma 3.1. Therefore, by Lemma 3.1 we have $\int_0^r h(s) \frac{1}{\Phi(r)}\, ds \le \int_0^r h(s) \frac{1}{\psi(s)}\, ds$, which implies (3.2). $\square$

COROLLARY 3.3. *There exists a constant $C > 0$ such that for any bounded $f \in \mathcal{F}$ and $r > 0$,*

$$\frac{1}{r^d} \int_{\mathbb{R}^d} \int_{B(x,r)} \big(f(x) - f(y)\big)^2\, dy\, dx \le C \Phi(r) \mathcal{E}(f, f). \tag{3.3}$$

PROOF. Fix $r > 0$ and let $\{x_n\}_{n\in\mathbb{N}}$ be a countable set in $\mathbb{R}^d$ satisfying $\bigcup_{n=1}^{\infty} B(x_n, r) = \mathbb{R}^d$ and $\sup_{y\in\mathbb{R}^d} |\{n : y \in B(x_n, 6r)\}| \le M$. Then by Proposition 3.2, the left-hand side of (3.3) is bounded above by

$$\begin{aligned}
&\sum_{n=1}^{\infty} \frac{1}{r^d} \int_{B(x_n,2r)\times B(x_n,2r)} \big(f(x) - f(y)\big)^2 \, dy\, dx \\
&\quad \le c_1 \sum_{n=1}^{\infty} \Phi(r) \int_{B(x_n,6r)\times B(x_n,6r)} \big(f(x) - f(y)\big)^2 J(x, y)\, dy\, dx \\
&\quad \le c_1 M\Phi(r) \int_{\mathbb{R}^d} \int_{B(x,12r)} \big(f(x) - f(y)\big)^2 J(x, y)\, dy\, dx \\
&\quad \le c_1 M\Phi(r)\mathcal{E}(f, f).
\end{aligned}$$

This finishes the proof. □

3.2. *Nash's inequality and near-diagonal upper bound in terms of* Φ. In this subsection, we observe that, using (2.3) and (3.3), Nash's inequality for $(\mathcal{E}, \mathcal{F})$ and the near-diagonal upper bound of $p(t, x, y)$ in terms of Φ hold. The proofs in this subsection are almost identical to the corresponding ones [12], Section 3. Thus, we skip the proofs.

THEOREM 3.4. *There is a positive constant* $c > 0$ *such that for every* $u \in \mathcal{F}$ *with* $\|u\|_1 = 1$, *we have* $\vartheta(\|u\|_2^2) \le c\mathcal{E}(u, u)$ *where* $\vartheta(r) := r/\Phi(r^{-1/d})$.

Recall that X is the Hunt process corresponding to our Dirichlet form $(\mathcal{E}, \mathcal{F})$ defined in (1.9) with jumping kernel J satisfying (1.7). By using our Nash's inequality (Theorem 3.4) and [1], Theorem 3.1, X has a density function $p(t, x, y)$ with respect to Lebesgue measure, which is quasi-continuous, and that the upper bound estimate holds quasi-everywhere.

THEOREM 3.5. *There is a properly exceptional set* $\mathcal{N}$ *of* X, *a positive symmetric kernel* $p(t, x, y)$ *defined on* $(0,\infty) \times (\mathbb{R}^d \setminus \mathcal{N}) \times (\mathbb{R}^d \setminus \mathcal{N})$, *and positive constants* C *depending on* $\bar{C}$ *in* (1.7) *and* β_1, C_L, *such that* $\mathbb{E}^x[f(X_t)] = \int_{\mathbb{R}^d} p(t, x, y)f(y)\, dy$, *and* $p(t, x, y) \le C\Phi^{-1}(t)^{-d}$ *for every* $x, y \in \mathbb{R}^d \setminus \mathcal{N}$ *and for every* $t > 0$. *Moreover, for every* $t > 0$, *and* $y \in \mathbb{R}^d \setminus \mathcal{N}$, $x \mapsto p(t, x, y)$ *is quasi-continuous on* $\mathbb{R}^d$.

3.3. *An upper bound of heat kernel using scaling*. In this section, we observe that the off-diagonal upper bound in [12], Sections 4.1–4.4, holds without the condition (1.14) in [12].

Fix $\rho > 0$ and define a bilinear form $(\mathcal{E}^\rho, \mathcal{F})$ by

$$\mathcal{E}^\rho(u, v) = \int_{\mathbb{R}^d\times\mathbb{R}^d} \big(u(x) - u(y)\big)\big(v(x) - v(y)\big)\mathbf{1}_{\{|x-y|\le\rho\}} J(x, y)\, dx\, dy.$$

Clearly, the form $\mathcal{E}^\rho(u,v)$ is well defined for $u, v \in \mathcal{F}$, and $\mathcal{E}^\rho(u,u) \le \mathcal{E}(u,u)$ for all $u \in \mathcal{F}$. Since ψ satisfies $L(\beta_1, C_L)$ and $U(\beta_2, C_U)$, for all $u \in \mathcal{F}$,

$$\mathcal{E}(u,u) - \mathcal{E}^\rho(u,u) \le 4\int_{\mathbb{R}^d} u^2(x)\,dx \int_{B(x,\rho)^c} J(x,y)\,dy \le \frac{c_0\|u\|_2^2}{\psi(\rho)}. \tag{3.4}$$

Thus, $\mathcal{E}_1(u,u) := \mathcal{E}(u,u) + \|u\|_2^2$ is equivalent to $\mathcal{E}_1^\rho(u,u) := \mathcal{E}^\rho(u,u) + \|u\|_2^2$ for every $u \in \mathcal{F}$, which implies that $(\mathcal{E}^\rho, \mathcal{F})$ is a regular Dirichlet form on $L^2(\mathbb{R}^d, dx)$. We call $(\mathcal{E}^\rho, \mathcal{F})$ the ρ-truncated Dirichlet form. The Hunt process associated with $(\mathcal{E}^\rho, \mathcal{F})$ which will be denoted by X^ρ can be identified in distribution with the Hunt process of the original Dirichlet form $(\mathcal{E}, \mathcal{F})$ by removing those jumps of size larger than ρ. We use $p^\rho(t,x,y)$ to denote the transition density function of X^ρ.

Note that although the function ψ may not be a correct scale function in our setting, we will still use ψ to define scaled processes. For $\eta > 0$, we define $(X^{(\eta)})_t := \eta^{-1} X_{\psi(\eta)t}$. Then, $X^{(\eta)}$ is a Hunt process in $\mathbb{R}^d$. We call $X^{(\eta)}$ the η-scaled process of X. Let

$$\psi^{(\eta)}(r) := \frac{\psi(\eta r)}{\psi(\eta)}, \qquad \Phi^{(\eta)}(r) := \frac{r^2}{2\int_0^r \frac{s}{\psi^{(\eta)}(s)}\,ds},$$
$$J^{(\eta)}(x,y) := \psi(\eta)\eta^d J(\eta x, \eta y).$$

We emphasize once more that ψ satisfies (1.6), $L(\beta_1, C_L)$ and $U(\beta_2, C_U)$. Furthermore, by Lemma 2.4 we have $\beta_1 < 2$. By definition, $\psi^{(\eta)}$ satisfies $L(\beta_1, C_L)$ and $U(\beta_2, C_U)$ for any $\eta > 0$. Also, $J^{(\eta)}$ satisfies that for $x, y \in \mathbb{R}^d$,

$$\frac{\bar{C}^{-1}}{|x-y|^d \psi^{(\eta)}(|x-y|)} \le J^{(\eta)}(x,y) \le \frac{\bar{C}}{|x-y|^d \psi^{(\eta)}(|x-y|)}, \tag{3.5}$$

where the constant $\bar{C} > 0$ is that of (1.7). Thus, Theorem 3.4 holds for η-scaled process $X^{(\eta)}$ with the same constants as X. that is, all constants are independent of η. Furthermore, since $\Phi^{(\eta)}(r) = \Phi(\eta r)/\psi(\eta)$, Lemma 2.4 enables that both Φ and $\Phi^{(\eta)}$ satisfies $L(\beta_1, C_L)$ and $U(2 \wedge \beta_2, C_U)$.

Since $J^{(\eta)}(x,y)$ is the jumping kernel of $X^{(\eta)}$, the Dirichlet form $(\mathcal{E}^{(\eta)}, \mathcal{F})$ associated with $X^{(\eta)}$ satisfies

$$\mathcal{E}^{(\eta)}(u,v) = \int_{\mathbb{R}^d \times \mathbb{R}^d} \big(u(x) - u(y)\big)\big(v(x) - v(y)\big) J^{(\eta)}(x,y)\,dx\,dy.$$

Also, since $\mathbb{P}^x(X^{(\eta)} \in A) = \mathbb{P}^x(\eta^{-1} X_{\psi(\eta)t} \in A) = \mathbb{P}^{\eta x}(X_{\psi(\eta)t} \in \eta A)$, we have $p^{(\eta)}(t,x,y) = \eta^d p(\psi(\eta)t, \eta x, \eta y)$, for a.e. $x, y \in \mathbb{R}^d$, where $p^{(\eta)}(t,x,y)$ is a transition density of $X^{(\eta)}$. For $\rho > 0$, let

$$J^{(\eta,\rho)}(x,y) = J^{(\eta)}(x,y)\mathbf{1}_{\{|x-y| \le \rho\}}, \qquad J^{(\eta)}_\rho(x,y) := J^{(\eta)}(x,y)\mathbf{1}_{\{|x-y| > \rho\}}.$$

Then,

$$\mathcal{E}^{(\eta,\rho)}(u,v) := \int_{\mathbb{R}^d\times\mathbb{R}^d} \big(u(x)-u(y)\big)\big(v(x)-v(y)\big) J^{(\eta,\rho)}(x,y)\,dx\,dy$$

is a ρ-truncated Dirichlet form for $X^{(\eta)}$. We use $X^{(\eta,\rho)}$ to denote a Hunt process corresponding to Dirichlet form $(\mathcal{E}^{(\eta,\rho)},\mathcal{F})$ and $p^{(\eta,\rho)}(t,x,y)$ to denote the transition density function of $X^{(\eta,\rho)}$. By the same argument as in (3.4), there exists $c>0$ such that any $u\in\mathcal{F}$,

$$c\big(\mathcal{E}^{(\eta)}(u,u)+\|u\|_2^2\big) \le \mathcal{E}^{(\eta,\rho)}(u,u)+\|u\|_2^2 \le \mathcal{E}^{(\eta)}(u,u)+\|u\|_2^2. \tag{3.6}$$

Without loss of generality, we assume that $\psi(1)=1$. Then $X^\rho = X^{(1,\rho)}$, $J^\rho = J^{(1,\rho)}$, $\mathcal{E}^\rho(u,v)=\mathcal{E}^{(1,\rho)}(u,v)$, and $p^\rho(t,x,y)=p^{(1,\rho)}(t,x,y)$.

Since the constants $\bar{C}$ in (1.7) and (3.5) are same, using [2], Lemma 3.1, we have the following.

LEMMA 3.6. *There exists $c>0$ such that for any $\rho>0$, $\eta>0$ and $x,y\in\mathbb{R}^d$, $p^{(\eta)}(t,x,y)\le p^{(\eta,\rho)}(t,x,y)+ct(\rho^d\psi^{(\eta)}(\rho))^{-1}$.*

In the following we give an upper estimate of $p^{(\eta,\rho)}(t,x,y)$. It is the counterpart of [12], Lemma 4.3.

LEMMA 3.7. *There exists a constant $C>0$, independent of $\eta,\lambda>0$, such that $p^{(\eta,\rho)}(t,x,y)\le \frac{Ct}{|x-y|^d\Phi^{(\eta)}(|x-y|)}$ for every $\eta>0$, $0<t\le\Phi^{(\eta)}(1)=\Phi(\eta)/\psi(\eta)$, $x,y\in\mathbb{R}^d\setminus\mathcal{N}$ with $|x-y|\ge 1$ and $\rho=\frac{\beta_1}{3(d+\beta_1)}|x-y|$.*

PROOF. Define $\gamma := \frac{\beta_1}{3(d+\beta_1)}$. We have by [1], Theorem 3.1, and the same way as that for [12], Theorem 3.2, using the above Nash-type inequality for $\mathcal{E}_1^{(\eta,\rho)}$, there exists constant $c_1>0$ such that for every $t\le\Phi^{(\eta)}(1)$, $x,y\in\mathbb{R}^d\setminus\mathcal{N}$, $\eta>0$ and $\rho\ge\gamma$, we have $p^{(\eta,\rho)}(t,x,y)\le c_1e^{c_2}(\Phi^{(\eta)})^{-1}(t)^{-d}$ since $\Phi^{(\eta)}(1)=\Phi(\eta)/\psi(\eta)\le c_2$. Using this and the equality $\int_0^\rho\frac{t}{\psi^{(\eta)}(t)}\,dt=\frac{\rho^2}{\Phi^{(\eta)}(\rho)}$, one can follow the proof of [12], Lemma 4.3, line by line to prove the lemma. We skip the details. □

Although we used ψ in scaled process, in the next theorem we are able to obtain an upper bound in terms of Φ.

THEOREM 3.8. *There exists a constant $C>0$ such that for any $t>0$ and $x,y\in\mathbb{R}^d\setminus\mathcal{N}$,*

$$p(t,x,y)\le C\bigg(\frac{1}{\Phi^{-1}(t)^d}\wedge\frac{t}{\Phi(|x-y|)|x-y|^d}\bigg). \tag{3.7}$$

PROOF. Note that (3.7) holds when t, x, y satisfies $t \geq \Phi(|x-y|)$ by Theorem 3.5. Thus, it suffices to show the case $t \leq \Phi(|x-y|)$. By [10], Lemma 7.2(1), for every $\eta > 0$, $0 < t \leq \Phi^{(\eta)}(1)$ and $x, y \in \mathbb{R}^d \setminus \mathcal{N}$ with $|x-y| \geq 1$,

$$p^{(\eta)}(t,x,y) \leq p^{(\eta,\rho)}(t,x,y) + t\|J_\rho^{(\eta)}\|_\infty \leq p^{(\eta,\rho)}(t,x,y) + \frac{c_1 t}{\psi^{(\eta)}(\rho)\rho^d}. \tag{3.8}$$

Applying Lemma 3.7 to (3.8), and using the condition $U(\beta_2, C_U)$ on $\psi^{(\eta)}$ and the inequality $\Phi^{(\eta)} \leq \psi^{(\eta)}$, we get

$$\begin{aligned} p^{(\eta)}(t,x,y) &\leq \frac{c_2 t}{|x-y|^d \Phi^{(\eta)}(|x-y|)} + \frac{c_1 t}{\psi^{(\eta)}(\gamma|x-y|)(\gamma|x-y|)^d} \\ &\leq \frac{c_3 t}{|x-y|^d \Phi^{(\eta)}(|x-y|)}, \end{aligned} \tag{3.9}$$

where $\gamma = \frac{\beta_1}{3(d+\beta_1)}$ is the constant in the proof of Lemma 3.7.

Let $\eta = |x-y|$. By (3.9) and $t/\psi(\eta) \leq \Phi(\eta)/\psi(\eta) = \Phi^{(\eta)}(1)$, we obtain

$$\begin{aligned} p(t,x,y) &= \eta^{-d} p^{(\eta)}\big(t/\psi(\eta), \eta^{-1}x, \eta^{-1}y\big) \\ &\leq c_3 \eta^{-d} \frac{t/\psi(\eta)}{(\eta^{-1}|x-y|)^d \Phi^{(\eta)}(\eta^{-1}|x-y|)} \\ &= c_3 \frac{t}{|x-y|^d \Phi(|x-y|)}, \end{aligned}$$

which concludes the proof. □

3.4. *Consequences of Poincaré inequality and Theorem* 3.8. The upper bound in (3.7) may not be sharp. However, there are several important consequences which are induced from (3.7). In this subsection, we will apply recent results in [10, 13] to (3.7).

LEMMA 3.9. *There exists a constant $C > 0$ such that $\mathbb{P}^x(\tau_{B(x,r)} \leq t) \leq Ct/\Phi(r)$ for any $r > 0$ and $x \in \mathbb{R}^d \setminus \mathcal{N}$.*

PROOF. Since we have the upper heat kernel estimates in (3.7), the condition $L(\beta_1, C_L)$ on Φ, and conservativeness of X, the lemma follows from the same argument as in the proof of [10], Lemma 2.7. □

For any open set $D \subset \mathbb{R}^d$, let $\mathcal{F}_D := \{u \in \mathcal{F} : u = 0 \text{ q.e. in } D^c\}$. Then, $(\mathcal{E}, \mathcal{F}_D)$ is also a regular Dirichlet form. We use $p^D(t,x,y)$ to denote the transition density function corresponding to $(\mathcal{E}, \mathcal{F}_D)$.

Recall that $(\mathcal{E}, \mathcal{F})$ is conservative. Thus, by Theorem 3.8 and [10], Theorem 1.15, we see that $\mathrm{CSJ}(\Phi)$ defined in [10] holds. Moreover, applying [9], Lemma 2.1, to (1.7), we have the (UJS) condition defined in [9]. By this, CSJ(Φ), (2.2) and Proposition 3.2, we have (7) in [13], Theorem 1.19.

Therefore, by [13], Theorem 1.19, following joint Hölder regularity holds for parabolic functions. Note that, by a standard argument, we now can take the continuous version of parabolic functions (for example, see [19], Lemma 5.12). We refer [13], Definition 1.13, for the definition of parabolic functions. Let $Q(t, x, r, R) := (t, t + r) \times B(x, R)$.

THEOREM 3.10. *There exist constants $c > 0$, $0 < \theta < 1$ and $0 < \epsilon < 1$ such that for all $x_0 \in \mathbb{R}^d$, $t_0 \geq 0$, $r > 0$ and for every bounded measurable function $u = u(t, x)$ that is parabolic in $Q(t_0, x_0, \Phi(r), r)$, the following parabolic Hölder regularity holds*:

$$\left|u(s, x) - u(t, y)\right| \leq c\left(r^{-1}\left[\Phi^{-1}\left(|s - t|\right) + |x - y|\right]\right)^{\theta} \sup_{[t_0, t_0+\Phi(r)]\times\mathbb{R}^d} |u|$$

for every $s, t \in (t_0, t_0 + \Phi(\epsilon r))$ and $x, y \in B(x_0, \epsilon r)$.

Since $p^D(t, x, y)$ is parabolic, from now on, we assume $\mathcal{N} = \varnothing$ and take the joint continuous versions of $p(t, x, y)$ and $p^D(t, x, y)$ (cf. [19], Lemma 5.13).

Again, by [13], Theorem 1.19, we have the interior near-diagonal lower bound of $p^B(t, x, y)$ (and parabolic Harnack inequality).

THEOREM 3.11. *There exist $\varepsilon \in (0, 1)$ and $c_1 > 0$ such that for any $x_0 \in \mathbb{R}^d$, $r > 0$, $0 < t \leq \Phi(\varepsilon r)$ and $B = B(x_0, r)$, $p^B(t, x, y) \geq c_1\Phi^{-1}(t)^{-d}$ for all $x, y \in B(x_0, \varepsilon\Phi^{-1}(t))$.*

The next lemma follows from [10], Theorem 1.15, Theorem 3.8 and the conservativeness of $(\mathcal{E}, \mathcal{F})$.

LEMMA 3.12. *For any $r > 0$ and $x \in \mathbb{R}^d$, $\mathbb{E}^x[\tau_{B(x,r)}] \asymp \Phi(r)$.*

4. Off-diagonal estimates.

4.1. *Off-diagonal upper heat kernel estimates.* Recall from the previous section that for $\rho > 0$, $(\mathcal{E}^\rho, \mathcal{F})$ is ρ-truncated Dirichlet form of $(\mathcal{E}, \mathcal{F})$. Also, the Hunt process associated with $(\mathcal{E}^\rho, \mathcal{F})$ is denoted by X^ρ, and $p^\rho(t, x, y)$ is the transition density function of X^ρ.

For any open set $D \subset \mathbb{R}^d$, let $\{P_t^D\}$ and $\{Q_t^{\rho,D}\}$ be the semigroups of $(\mathcal{E}, \mathcal{F}_D)$ and $(\mathcal{E}^\rho, \mathcal{F}_D)$, respectively. We write $\{Q_t^{\rho,\mathbb{R}^d}\}$ as $\{Q_t^\rho\}$ for simplicity. We also use τ_D^ρ to denote the first exit time of the process $\{X_t^\rho\}$ in D.

LEMMA 4.1 ([10], Lemma 5.2). *There exist constants $c, C_1, C_2 > 0$ such that for any $t, \rho > 0$ and $x, y \in \mathbb{R}^d$,*

$$p^\rho(t, x, y) \le c\Phi^{-1}(t)^{-d} \exp\left(C_1 \frac{t}{\Phi(\rho)} - C_2 \frac{|x-y|}{\rho}\right).$$

PROOF. Note that by Lemma 2.4, Φ satisfies $U(\beta_2 \wedge 2, C_U)$ and $L(\beta_1, C_L)$. By Theorem 3.5, (2.2), and Lemma 3.12, the assumptions of [10], Lemma 5.2, are satisfied. Thus, the lemma follows. □

The next lemma was proved in [10], Lemma 7.11, and [20], Theorem 3.1, under the assumption that $\phi(r, \cdot)$ is nondecreasing for all $r > 0$. We will prove the lemma without such assumption.

LEMMA 4.2. *Let $r, t, \rho > 0$. Assume that $\mathbb{P}^w(\tau^\rho_{B(x,r)} \le t) \le \phi(r, t)$ for all $x \in \mathbb{R}^d$ and $w \in B(x, r/4)$, where ϕ is a nonnegative measurable function on $\mathbb{R}_+ \times \mathbb{R}_+$. Then, for any integer $k \ge 1$, $Q^\rho_t \mathbf{1}_{B(x,k(r+\rho))^c}(z) \le \phi(r, t)^k$ for all $x \in \mathbb{R}^d$ and $z \in B(x, r/4)$.*

PROOF. Fix $x \in \mathbb{R}^d$. Note that $X^\rho_{\tau^\rho_{B(x,r)}} = X^\rho(\tau^\rho_{B(x,r)}) \in B(x, r+\rho)$, and $|w - y| \ge |x - w| - |y - x| \ge k(r + \rho)$ for any $w \notin B(x, (k+1)(r+\rho))^c$ and $y \in B(x, r+\rho)$. Thus by the strong Markov property, for all $s \le t$ and $z \in B(x, r/4)$ we have

$$\begin{aligned}
&Q^\rho_s \mathbf{1}_{B(x,(k+1)(r+\rho))^c}(z) \\
&\quad = \mathbb{E}^z\left[\mathbf{1}_{\{\tau^\rho_{B(x,r)} < s\}} \mathbb{P}^{X^\rho(\tau^\rho_{B(x,r)})}\left(X^\rho\left(s - \tau^\rho_{B(x,r)}\right) \notin B\left(x, (k+1)(r+\rho)\right)\right)\right] \\
&\quad \le \mathbb{P}^z\left(\tau^\rho_{B(x,r)} < s\right) \sup_{y \in B(x, r+\rho), s_1 \le s} Q^\rho_{s_1} \mathbf{1}_{B(y,k(r+\rho))^c}(y) \\
&\quad \le \phi(r, t) \sup_{y \in \mathbb{R}^d, s_1 \le t} Q^\rho_{s_1} \mathbf{1}_{B(y,k(r+\rho))^c}(y).
\end{aligned}$$

By using the above step $k - 1$ times we conclude that for all $z \in B(x, r/4)$

$$\begin{aligned}
Q^\rho_t \mathbf{1}_{B(x,k(r+\rho))^c}(z) &\le \phi(r, t) \sup_{y_1 \in \mathbb{R}^d, s_1 \le t} Q^\rho_{s_1} \mathbf{1}_{B(y_1,(k-1)(r+\rho))^c}(y_1) \\
&\le \cdots \le \phi(r, t)^{k-1} \sup_{y_{k-1} \in \mathbb{R}^d, s_{k-1} \le t} Q^\rho_{s_{k-1}} \mathbf{1}_{B(y_{k-1}, r+\rho)^c}(y_{k-1}) \\
&\le \phi(r, t)^{k-1} \sup_{y_{k-1} \in \mathbb{R}^d, s_{k-1} \le t} \mathbb{P}^{y_{k-1}}\left(\tau^\rho_{B(y_{k-1},r)} \le t\right) \le \phi(r, t)^k.
\end{aligned}$$

□

The following lemma is a key to obtain upper bound of transition density function and will be used in several times.

LEMMA 4.3. *Let $f : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ be a measurable function satisfying that $t \mapsto f(r,t)$ is nonincreasing for all $r > 0$ and that $r \mapsto f(r,t)$ is nondecreasing for all $t > 0$. Fix $T \in (0,\infty]$. Suppose that the following hold*: (i) *For each $b > 0$, $\sup_{t \le T} f(b\Phi^{-1}(t), t) < \infty$ (resp. $\sup_{t \ge T} f(b\Phi^{-1}(t), t) < \infty$)*; (ii) *there exist $\eta \in (0, \beta_1]$, $a_1 > 0$ and $c_1 > 0$ such that*

$$\mathbb{P}^x(|X_t - x| > r) \le c_1(\psi^{-1}(t)/r)^\eta + c_1 \exp(-a_1 f(r,t)) \tag{4.1}$$

for all $t \in (0,T)$ (resp. $t \in [T,\infty)$) and $r > 0$, $x \in \mathbb{R}^d$.

Then, there exist constants $k, c > 0$ such that

$$p(t,x,y) \le \frac{ct}{|x-y|^d \psi(|x-y|)} + c\Phi^{-1}(t)^{-d} \exp(-a_1 k f(|x-y|/(16k), t))$$

for all $t \in (0,T)$ (resp. $t \in [T,\infty)$) and $x, y \in \mathbb{R}^d$.

PROOF. Since the proofs for the case $t \in (0,T)$ and the case $t \in [T,\infty)$ are similar, we only prove for $t \in (0,T)$. For $x_0 \in \mathbb{R}^d$, let $B(r) = B(x_0, r)$. By the strong Markov property, (4.1), and the fact that $t \mapsto f(r,t)$ is nonincreasing, we have that for $x \in B(r/4)$ and $t \in (0, T/2)$,

$$\begin{aligned}
\mathbb{P}^x(\tau_{B(r)} \le t) &\le \mathbb{P}^x(X_{2t} \in B(r/2)^c) + \mathbb{P}^x(\tau_{B(r)} \le t, X_{2t} \in B(r/2)) \\
&\le \mathbb{P}^x(X_{2t} \in B(x, r/4)^c) \\
&\quad + \sup_{z \in B(r)^c, s \le t} \mathbb{P}^z(X_{2t-s} \in B(z, r/4)^c) \\
&\le c_1(4\psi^{-1}(2t)/4)^\eta + c_1 \exp(-a_1 f(r/4, 2t)).
\end{aligned} \tag{4.2}$$

From this and Lemma 2.2, we have that for $x \in B(r/4)$ and $t \in (0, T/2)$,

$$1 - P_t^B \mathbf{1}_B(x) = \mathbb{P}^x(\tau_B \le t) \le c_2\left(\frac{\psi^{-1}(t)}{r}\right)^\eta + c_1 \exp(-a_1 f(r/4, 2t)). \tag{4.3}$$

By [20], Proposition 4.6, and Lemma 2.3, letting $\rho = r$ we have

$$|P_t^{B(r)} \mathbf{1}_{B(r)}(x) - Q_t^{r,B(r)} \mathbf{1}_{B(r)}(x)| \le 2t \operatorname*{ess\,sup}_{z \in \mathbb{R}^d} \int_{B(z,r)^c} J(z,y)\, dy \le \frac{c_3 t}{\psi(r)}.$$

Combining this with (4.3), we see that for all $x \in B(r/4)$ and $t \in (0, T/2)$,

$$\begin{aligned}
\mathbb{P}^x(\tau^r_{B(r)} \le t) &= 1 - Q_t^{r,B(r)} \mathbf{1}_{B(r)}(x) \le 1 - P_t^{B(r)} \mathbf{1}_{B(r)}(x) + \frac{c_3 t}{\psi(r)} \\
&\le c_2(\psi^{-1}(t)/r)^\eta + c_1 \exp(-a_1 f(r/4, 2t)) + c_3(t/\psi(r)) \\
&=: \phi_1(r,t).
\end{aligned} \tag{4.4}$$

Applying Lemma 4.2 with $r = \rho$ to (4.4), we see that for $t \in (0, T/2)$,

$$\int_{B(x,2kr)^c} p^r(t,x,y)\, dy = Q_t^r \mathbf{1}_{B(x,2kr)^c}(x) \le \phi_1(r,t)^k. \tag{4.5}$$

Let $k = \lceil (\beta_2 + d)/\eta \rceil$. For $t \in (0, T)$ and $x, y \in \mathbb{R}^d$ satisfying $4k\Phi^{-1}(t) \ge |x - y|$, by using that $r \mapsto f(r, t)$ is nondecreasing and the assumption (i), we have $f(|x - y|/(16k), t) \le f(\Phi^{-1}(t)/4, t) \le M < \infty$. Thus, by Theorem 3.5,

$$p(t, x, y) \le c_4 e^{a_1 k M} \Phi^{-1}(t)^{-d} \exp\big(-a_1 k f\big(|x - y|/(16k), t\big)\big). \tag{4.6}$$

For the remainder of the proof, assume $t \in (0, T)$ and $4k\Phi^{-1}(t) < |x - y|$, and let $r = |x - y|$ and $\rho = r/(4k)$. By (4.5) and Lemmas 2.2, 2.4 and 4.1, we have

$$\begin{aligned} p^\rho(t, x, y) &= \int_{\mathbb{R}^d} p^\rho(t/2, x, z) p^\rho(t/2, z, y)\, dz \\ &\le \Big(\int_{B(x,r/2)^c} + \int_{B(y,r/2)^c}\Big) p^\rho(t/2, x, z) p^\rho(t/2, y, z)\, dz \\ &\le \Big(\sup_{z\in\mathbb{R}^d} p^\rho(t/2, z, y)\Big) \int_{B(x,2k\rho)^c} p^\rho(t/2, x, z)\, dz \\ &\quad + \Big(\sup_{z\in\mathbb{R}^d} p^\rho(t/2, x, z)\Big) \int_{B(y,2k\rho)^c} p^\rho(t/2, y, z)\, dz \\ &\le c_5 \Phi^{-1}(t)^{-d} \phi_1(\rho, t/2)^k. \end{aligned} \tag{4.7}$$

Note that $k\beta_1 \ge k\eta \ge \beta_2 + d$, and $\rho \ge \Phi^{-1}(t) > \psi^{-1}(t)$. Thus, by $L(\beta_1, C_L)$ on ψ and using Lemmas 2.2 and 2.4,

$$\begin{aligned} &\Phi^{-1}(t)^{-d} \big(\big(\psi^{-1}(t)/\rho\big)^{\eta k} + \big(t/\psi(\rho)\big)^k\big) \\ &\quad \le \frac{c_6}{r^d} \frac{\psi^{-1}(t)^d}{\Phi^{-1}(t)^d} \big(\psi^{-1}(t)/r\big)^{\beta_2} \\ &\quad \le \frac{c_6}{r^d} \big[\psi^{-1}(t)/\psi^{-1}\big(\psi(r)\big)\big]^{\beta_2} \le \frac{c_7 t}{r^d \psi(r)}. \end{aligned}$$

Applying this to (4.7), we have

$$\begin{aligned} p^\rho(t, x, y) &\le c_8 \Phi^{-1}(t)^{-d} \bigg(\Big(\frac{\psi^{-1}(t)}{\rho}\Big)^{\eta k} + \big(-a_1 k f(\rho/4, t)\big) + \Big(\frac{t}{\psi(\rho)}\Big)^k\bigg) \\ &\le \frac{c_9 t}{r^d \psi(r)} + c_8 \Phi^{-1}(t)^{-d} \exp\big(-a_1 k f\big(r/(16k), t\big)\big). \end{aligned}$$

Thus, by Lemma 3.6 and $U(\beta_2, C_U)$ on ψ, we have

$$\begin{aligned} p(t, x, y) &\le p^\rho(t, x, y) + \frac{c_{10} t}{\rho^d \psi(\rho)} \\ &\le \frac{c_{11} t}{|x - y|^d \psi(|x - y|)} + c_{11} \Phi^{-1}(t)^{-d} \exp\big(-a_1 k f\big(r/(16k), t\big)\big). \end{aligned} \tag{4.8}$$

Now the lemma follows immediately from (4.6) and (4.8). □

The following inequality will be used several times in the proofs of this section: For any $c_0 > 0$ and $\alpha \in (0, 1)$, there exists $c_1 = c_1(c_0, \alpha) > 0$ such that $2n \le \frac{c_0}{2d} 2^{n(1-\alpha)} + c_1$ holds for every $n \ge 0$. Thus, for any $n \ge 0$ and $\kappa \ge 1$,

$$\begin{aligned}2^{nd} \exp\bigl(-c_0 2^{n(1-\alpha)}\kappa\bigr) &\le 2^{-nd} \exp\bigl(2nd - c_0 2^{n(1-\alpha)}\kappa\bigr)\\ &\le e^{c_1 d} 2^{-nd} \exp\biggl(\frac{c_0}{2} 2^{n(1-\alpha)} - c_0 2^{n(1-\alpha)}\kappa\biggr)\\ &\le e^{c_1 d} 2^{-nd} \exp\biggl(-\frac{c_0}{2}\kappa\biggr).\end{aligned} \tag{4.9}$$

The next proposition is an intermediate step toward to Theorem 1.2.

PROPOSITION 4.4. *There exist constants $a_1, C > 0$ and $N \in \mathbb{N}$ such that*

$$p(t, x, y) \le \frac{Ct}{|x - y|^d \psi(|x - y|)} + C\Phi^{-1}(t)^{-d} \exp\biggl(-\frac{a_1 |x - y|^{1/N}}{\Phi^{-1}(t)^{1/N}}\biggr) \tag{4.10}$$

for all $t > 0$ and $x, y \in \mathbb{R}^d$.

PROOF. Fix $\alpha \in (d/(d + \beta_1), 1)$ and let $N := \lceil(\beta_1 + d)/\beta_1\rceil + 1$, and $\eta := \beta_1 - (\beta_1 + d)/N > 0$. We first claim that there exist $a_2 > 0$ and $c_1 > 0$ such that

$$\int_{B(x,r)^c} p(t, x, y)\, dy \le c_2\bigl(\psi^{-1}(t)/r\bigr)^\eta + c_1 \exp\bigl(-a_2\bigl(r/\Phi^{-1}(t)\bigr)^{1/N}\bigr), \tag{4.11}$$

for any $t, r > 0$ and $x \in \mathbb{R}^d$.

When $r \le \Phi^{-1}(t)$, we immediately obtain (4.11) by letting $c_1 = \exp(a_2)$. Thus, we will only consider the case $r > \Phi^{-1}(t)$. Define $\rho_n = \rho_n(r, t) = 2^{n\alpha} r^{1-1/N} \Phi^{-1}(t)^{1/N}$ for all $n \in \mathbb{N}$. Since $r > \Phi^{-1}(t)$, we have $\Phi^{-1}(t) < \rho_n \le 2^n r$. In particular, $t < \Phi(\rho_n)$. Thus, by Lemmas 3.6 and 4.1, we have that for every $t > 0$ and $2^n r \le |x - y| < 2^{n+1} r$,

$$\begin{aligned}p(t, x, y) &\le c_2 \Phi^{-1}(t)^{-d} \exp\biggl(C_1 \frac{t}{\Phi(\rho_n)} - C_2 \frac{|x - y|}{\rho_n}\biggr) + \frac{c_2 t}{\rho_n^d \psi(\rho_n)}\\ &\le c_3 \Phi^{-1}(t)^{-d} \exp\biggl(-C_2 \frac{2^{n(1-\alpha)} r^{1/N}}{\Phi^{-1}(t)^{1/N}}\biggr) + \frac{c_2 t}{\rho_n^d \psi(\rho_n)}.\end{aligned}$$

Using the above estimate, we get that

$$\begin{aligned}&\int_{B(x,r)^c} p(t, x, y)\, dy\\ &\quad = \sum_{n=0}^{\infty} \int_{B(x,2^{n+1}r)\setminus B(x,2^n r)} p(t, x, y)\, dy\end{aligned}$$

$$\le c_4 \sum_{n=0}^{\infty} (2^n r)^d \Phi^{-1}(t)^{-d} \exp\biggl(-C_2 \frac{2^{n(1-\alpha)} r^{1/N}}{\Phi^{-1}(t)^{1/N}}\biggr) + c_4 \sum_{n=0}^{\infty} (2^n r)^d \frac{t}{\rho_n^d \psi(\rho_n)}$$
$$=: I_1 + I_2.$$

Using $\Phi^{-1}(t) < r$, (4.9), and the fact that $\sup_{s\ge 1} s^d \exp(-\frac{C_2}{4} s^{1/N}) < \infty$,

$$\begin{aligned} I_1 &= c_4 \sum_{n=0}^{\infty} \biggl(\frac{r}{\Phi^{-1}(t)}\biggr)^d 2^{nd} \exp\biggl(-C_2 \frac{2^{n(1-\alpha)} r^{1/N}}{\Phi^{-1}(t)^{1/N}}\biggr) \\ &\le c_4 e^{c_1 d} \biggl(\frac{r}{\Phi^{-1}(t)}\biggr)^d \exp\biggl(-\frac{C_2}{2} \frac{r^{1/N}}{\Phi^{-1}(t)^{1/N}}\biggr) \sum_{n=0}^{\infty} 2^{-nd} \\ &\le c_5 \exp\biggl(-\frac{C_2 r^{1/N}}{4\Phi^{-1}(t)^{1/N}}\biggr). \end{aligned} \tag{4.12}$$

We next estimate I_2. By (2.1), $t < \Phi(\rho_n) < \psi(\rho_n)$ and $\Phi^{-1} > \psi^{-1}$, $L(\beta_1, C_L)$ on ψ and $\alpha(d+\beta_1) > d$, we have

$$\begin{aligned} I_2 &= c_4 \sum_{n=0}^{\infty} \frac{(2^n r)^d}{\rho_n^d} \frac{\psi(\psi^{-1}(t))}{\psi(\rho_n)} \le c_4 C_L^{-1} \sum_{n=0}^{\infty} \biggl(\frac{2^n r}{\rho_n}\biggr)^d \biggl(\frac{\psi^{-1}(t)}{\rho_n}\biggr)^{\beta_1} \\ &= c_4 C_L^{-1} \biggl(\frac{\Phi^{-1}(t)}{r}\biggr)^{-\frac{d+\beta_1}{N}} \biggl(\frac{\psi^{-1}(t)}{r}\biggr)^{\beta_1} \sum_{n=0}^{\infty} 2^{n(d-\alpha(d+\beta_1))} \\ &= c_6 \biggl(\frac{\Phi^{-1}(t)}{r}\biggr)^{-\frac{d+\beta_1}{N}} \biggl(\frac{\psi^{-1}(t)}{r}\biggr)^{\beta_1} \le c_6 \biggl(\frac{\psi^{-1}(t)}{r}\biggr)^{\beta_1 - \frac{d+\beta_1}{N}} \\ &= c_6 \biggl(\frac{\psi^{-1}(t)}{r}\biggr)^{\eta}. \end{aligned}$$

Thus, by above estimates of I_1 and I_2, we obtain (4.11).

By $\eta < \beta_1$ and (4.11), assumptions in Lemma 4.3 hold with $f(r,t) := (r/\Phi^{-1}(t))^{1/N}$. Now (4.10) follows from Lemma 4.3. $\square$

By using Proposition 4.4, we obtain the upper bound in Theorem 1.2.

PROOF OF (1.11). Similar to the proof of Proposition 4.4, we will show that there exist $a_2 > 0$ and $c_1 > 0$ such that for any $x \in R^d$ and $t, r > 0$,

$$\int_{B(x,r)^c} p(t,x,y)\,dy \le c_1 \bigl(\psi^{-1}(t)/r\bigr)^{\beta_2/2} + c_1 \exp\bigl(-a_2\bigl(r/\Phi^{-1}(t)\bigr)^2\bigr). \tag{4.13}$$

Let $\theta := \frac{\beta_1}{4d+3\beta_1} \in (0,1)$ and $C_0 = \frac{2C_1}{C_2}$, where C_1 and C_2 are the constants in Lemma 4.1. Without loss of generality, we may and do assume that $C_0 \ge 1$. First, when $r \le C_0 \Phi^{-1}(t)$, the claim is clear. Second, we consider the case

$r > C_0\frac{\Phi^{-1}(t)^{1+\theta}}{\psi^{-1}(t)^{\theta}}$. For $|x-y|>r$, there is a $\theta_0 \in (\theta,\infty)$ such that $|x-y| = C_0\Phi^{-1}(t)^{1+\theta_0}/\psi^{-1}(t)^{\theta_0}$. Let a_1 be the constant in Proposition 4.4. Note that there exists a positive constant $c_2 = c_2(\theta)$ such that $s^{-d-\beta_2-\beta_2/\theta} \geq c_2 \exp(-a_1 s^{1/N})$ for $s \geq 1$. Using this and $U(\beta_2, C_U)$ condition on ψ,

$$\begin{aligned}
&\frac{t}{|x-y|^d\psi(|x-y|)}\\
&\quad \geq C_0^{-d-\beta_2} C_U^{-2}\Phi^{-1}(t)^{-d}\big(\big(\Phi^{-1}(t)/\psi^{-1}(t)\big)^{\theta_0}\big)^{-d-\beta_2-\beta_2/\theta_0}\\
&\quad \geq c_2 C_0^{-d-\beta_2} C_U^{-2}\Phi^{-1}(t)^{-d}\exp\big(-a_1\big(|x-y|/\Phi^{-1}(t)\big)^{1/N}\big). \qquad (4.14)
\end{aligned}$$

Thus, by Proposition 4.4, we have that for $|x-y|>r$, $p(t,x,y) \leq \frac{c_3 t}{|x-y|^d\psi(|x-y|)}$. Using this, Lemma 2.3, $L(\beta_1, C_L)$ of ψ, and the fact that $r > C_0\psi^{-1}(t)$ (which follows from (2.1)),

$$\begin{aligned}
&\int_{B(x,r)^c} p(t,x,y)\,dy\\
&\quad \leq c_3 \int_{B(x,r)^c} \frac{t}{|x-y|^d\psi(|x-y|)}\,dy \leq c_4\frac{t}{\psi(r)}\\
&\quad \leq c_4 C_L^{-1}\big(\psi^{-1}(t)/r\big)^{\beta_1} \leq c_4 C_L^{-1}\big(\psi^{-1}(t)/r\big)^{\beta_1/2}. \qquad (4.15)
\end{aligned}$$

Now consider the case $C_0\Phi^{-1}(t) < r \leq C_0\Phi^{-1}(t)^{1+\theta}/\psi^{-1}(t)^{\theta}$. In this case, there exists $\theta_0 \in (0,\theta]$ such that $r = C_0\Phi^{-1}(t)^{1+\theta_0}/\psi^{-1}(t)^{\theta_0}$. Define $\rho_n = C_0 2^{n\alpha}\Phi^{-1}(t)^2/r$, where $\alpha \in (d/(d+\beta_1), 1)$. Since $C_0 = \frac{2C_1}{C_2}$, using (2.3)

$$\begin{aligned}
\frac{C_1 t}{\Phi(\rho_n)} - \frac{C_2 2^n r}{\rho_n} &\leq \frac{C_1\Phi(\Phi^{-1}(t))}{\Phi(\Phi^{-1}(t)C_0\Phi^{-1}(t)/r)} - \frac{C_2 2^{n(1-\alpha)} r^2}{C_0\Phi^{-1}(t)^2}\\
&\leq C_1\left(\frac{r}{C_0\Phi^{-1}(t)}\right)^2 - C_2 2^{n(1-\alpha)}\frac{r^2}{C_0\Phi^{-1}(t)^2}\\
&\leq \left(\frac{C_1}{C_0^2} - \frac{C_2}{C_0}2^{n(1-\alpha)}\right)\frac{r^2}{\Phi^{-1}(t)^2} \leq -\frac{C_2}{2C_0}2^{n(1-\alpha)}\frac{r^2}{\Phi^{-1}(t)^2}.
\end{aligned}$$

Let $a_2 := 2^{-1}C_2/C_0$. By the above inequality, Lemma 3.6, and Lemma 4.1,

$$\begin{aligned}
&\int_{B(x,r)^c} p(t,x,y)\,dy\\
&\quad = \sum_{n=0}^{\infty}\int_{B(x,2^{n+1}r)\setminus B(x,2^n r)} p(t,x,y)\,dy\\
&\quad \leq c_5\sum_{n=0}^{\infty}(2^n r)^d\Phi^{-1}(t)^{-d}\exp\left(-a_2\frac{2^{n(1-\alpha)}r^2}{\Phi^{-1}(t)^2}\right) + c_5\sum_{n=0}^{\infty}(2^n r)^d\left(\frac{2^n r}{\rho_n}\right)^d\frac{t}{\psi(\rho_n)}\\
&\quad =: c_6(I_1 + I_2).
\end{aligned}$$

Using (4.9) and $r > C_0\Phi^{-1}(t)$, the proof of the upper bound of I_1 is the same as the one in (4.12). Thus, we have

$$I_1 \le c_6\left(\frac{r}{\Phi^{-1}(t)}\right)^d \exp\left(-\frac{a_2 r^2}{2\Phi^{-1}(t)^2}\right)\sum_{n=0}^{\infty} 2^{-nd} \le c_7 \exp\left(-\frac{a_2 r^2}{4\Phi^{-1}(t)^2}\right).$$

We next estimate I_2. Note that $\rho_n \ge \rho_0 = C_0\Phi^{-1}(t)^{1+\theta_0}\psi^{-1}(t)^{\theta_0} \ge C_0\psi^{-1}(t)$. Thus, we have

$$\begin{aligned} I_2 &= \sum_{n=0}^{\infty}\left(\frac{2^{n(1-\alpha)}r^2}{C_0\Phi^{-1}(t)^2}\right)^d \frac{\psi(C_0\psi^{-1}(t))}{\psi(\rho_n)}\frac{\psi(\psi^{-1}(t))}{\psi(C_0\psi^{-1}(t))} \\ &\le c_8\left(\frac{\Phi^{-1}(t)}{r}\right)^{-2(d+\beta_1)}\left(\frac{\psi^{-1}(t)}{r}\right)^{\beta_1}. \end{aligned}$$

Since $r = C_0\frac{\Phi^{-1}(t)^{1+\theta_0}}{\psi^{-1}(t)^{\theta_0}}$, $C_0\psi^{-1}(t) < C_0\Phi^{-1}(t) < r$, and $\theta_0 \le \theta$, by using $\theta = \frac{\beta_1}{4d+3\beta_1}$, we have

$$\left(\frac{\Phi^{-1}(t)}{r}\right)^{-2(d+\beta_1)} \le c_9\left(\frac{\psi^{-1}(t)}{r}\right)^{-\frac{2(d+\beta_1)\theta}{1+\theta}} = c_9\left(\frac{\psi^{-1}(t)}{r}\right)^{-\beta_1/2}.$$

Thus, $I_2 \le c_9c_8(\psi^{-1}(t)/r)^{\beta_1/2}$. Using estimates of I_1 and I_2, we arrive

$$\int_{B(x,r)^c} p(t,x,y)\,dy \le c_6(I_1+I_2) \le c_{10}\left(\frac{\psi^{-1}(t)}{r}\right)^{\beta_1/2} + c_{10}\exp\left(-\frac{a_2r^2}{4\Phi^{-1}(t)^2}\right).$$

Combining above inequality with (4.15), we obtain (4.13). Now, applying (4.13) to Lemma 4.3 with $f(r,t) := (r/\Phi^{-1}(t))^2$, the conclusion follows. $\square$

Recall that, without loss of generality, whenever Φ satisfies the weak lower scaling property at infinity with index $\delta > 1$, we have assumed that Φ satisfies $L^1(\delta, \widetilde{C}_L)$ instead of $L^a(\delta, \widetilde{C}_L)$.

We are now ready to prove the sharp upper bound of $p(t,x,y)$, which is the most delicate part of this paper.

THEOREM 4.5. (1) *Assume that Φ satisfies $L_a(\delta, \widetilde{C}_L)$ with $\delta > 1$. Then for any $T > 0$, there exist constants $a_U > 0$ and $c > 0$ such that for every $x, y \in \mathbb{R}^d$ and $t < T$,*

$$p(t,x,y) \le \frac{ct}{|x-y|^d\psi(|x-y|)} + \frac{c}{\Phi^{-1}(t)^d}\exp\left(-\frac{a_U|x-y|}{\mathscr{K}^{-1}(t/|x-y|)}\right). \tag{4.16}$$

Moreover, if Φ satisfies $L(\delta, \widetilde{C}_L)$, then (4.16) holds for all $t < \infty$.

(2) *Assume that Φ satisfies $L^1(\delta, \widetilde{C}_L)$ with $\delta > 1$. Then for any $T > 0$, there exist constants $a'_U > 0$ and $c' > 0$ such that for every $x, y \in \mathbb{R}^d$ and $t \ge T$,*

$$p(t,x,y) \le \frac{c't}{|x-y|^d\psi(|x-y|)} + \frac{c'}{\Phi^{-1}(t)^d}\exp\left(-\frac{a'_U|x-y|}{\mathscr{K}_\infty^{-1}(t/|x-y|)}\right).$$

PROOF. Take $\theta = \frac{\beta_1(\delta-1)}{2\delta d+\delta\beta_1+\beta_1}$ and $\widetilde{C}_0 = (\frac{2C_1}{C_2\widetilde{C}_L^2})^{1/(\delta-1)}$, where C_1 and C_2 are constants in Lemma 4.1. Without loss of generality, we may and do assume that $\widetilde{C}_0 \geq 1$. Note that θ satisfies $\frac{\delta(d+\beta_1)}{\delta-1}\frac{\theta}{1+\theta} = \frac{\beta_1}{2}$ and $\theta < \delta - 1$. Let $\alpha \in (d/(d+\beta_1), 1)$.

(1) Again we will show that there exist $a_1 > 0$ and $c_1 > 0$ such that for any $t \leq T$ and $r > 0$,

$$\int_{B(x,r)^c} p(t,x,y)\,dy \leq c_1\big(\psi^{-1}(t)/r\big)^{\beta_1/2} + c_1 \exp\left(-\frac{a_1 r}{\mathscr{K}^{-1}(t/r)}\right). \tag{4.17}$$

When $r \leq \widetilde{C}_0\Phi^{-1}(t)$ using (2.7), we have for $t \leq T$

$$\int_{B(x,r)^c} p(t,x,y)\,dy \leq 1 \leq e^{c_2}\exp\left(-\frac{r}{\mathscr{K}^{-1}(t/r)}\right). \tag{4.18}$$

The proof of case $r > \widetilde{C}_0\frac{\Phi^{-1}(t)^{1+\theta}}{\psi^{-1}(t)^{\theta}}$ is exactly same as the corresponding part in the proof of (1.11) in Theorem 1.2.

Now consider the case $\widetilde{C}_0\Phi^{-1}(t) < r \leq \widetilde{C}_0\Phi^{-1}(t)^{1+\theta}/\psi^{-1}(t)^{\theta}$. In this case, there exists $\theta_0 \in (0,\theta]$ such that $r = \widetilde{C}_0\Phi^{-1}(t)^{1+\theta_0}/\psi^{-1}(t)^{\theta_0}$. Define $\rho = \mathscr{K}^{-1}(t/r)$ and $\rho_n = \widetilde{C}_0 2^{n\alpha}\rho$ for integer $n \geq 0$. Note that for $t \leq T$ and $\widetilde{C}_0\Phi^{-1}(t) < r$, we have $t \leq T \wedge \Phi(r)$. Thus, by (2.8)

$$\rho \leq \rho_0 = \widetilde{C}_0\rho \leq \widetilde{C}_0\Phi^{-1}(t)^2/r \leq \widetilde{C}_0\Phi^{-1}(T)\Phi^{-1}(t)/r \leq \Phi^{-1}(T). \tag{4.19}$$

By Remark 2.1, we may assume that $\Phi^{-1}(T) < a$. Thus, by (4.19), Lemma 2.5, the condition $L_a(\delta, \widetilde{C}_L)$ on Φ, and the definition of $\widetilde{C}_0$, we have

$$\begin{aligned}
C_1\frac{t}{\Phi(\rho_n)} - C_2\frac{2^n r}{\rho_n} &\leq C_1\frac{\Phi(\rho)}{\Phi(\rho_0)}\frac{t}{\Phi(\rho)} - \frac{C_2}{\widetilde{C}_0}\frac{2^{n(1-\alpha)}r}{\rho} \\
&\leq \frac{C_2 r}{\widetilde{C}_0\rho}\left(\frac{\widetilde{C}_0 C_1}{C_2\widetilde{C}_L}\frac{\Phi(\rho)}{\Phi(\rho_0)} - 2^{n(1-\alpha)}\right) \\
&\leq \frac{C_2 r}{\widetilde{C}_0\rho}\left(\frac{\widetilde{C}_0^{1-\delta}C_1}{C_2\widetilde{C}_L^2} - 2^{n(1-\alpha)}\right) \\
&= \frac{C_2 r}{\widetilde{C}_0\rho}\left(\frac{1}{2} - 2^{n(1-\alpha)}\right) \leq -c_3 2^{n(1-\alpha)}\frac{r}{\rho}.
\end{aligned} \tag{4.20}$$

Combining (4.20), Lemmas 3.6 and 4.1, we have that we get that

$$\begin{aligned}
&\int_{B(x,r)^c} p(t,x,y)\,dy \\
&\quad\leq \sum_{n=0}^{\infty}\int_{B(x,2^{n+1}r)\setminus B(x,2^n r)} p(t,x,y)\,dy
\end{aligned}$$

$$
\begin{aligned}
&\le c_4 \sum_{n=0}^{\infty} \left(\frac{2^n r}{\Phi^{-1}(t)}\right)^d \exp\left(-c_5 \frac{2^{n(1-\alpha)} r}{\rho}\right) + c_4 \sum_{n=0}^{\infty} \left(\frac{2^n r}{\rho_n}\right)^d \frac{t}{\psi(\rho_n)} \\
&:= I_1 + I_2.
\end{aligned}
$$

We first estimate I_1. Note that by (2.8), $r/\rho \ge (r/\Phi^{-1}(t))^2 \ge \widetilde{C}_0^2$. Using this, (2.8), and (4.9) we have

$$
I_1 \le c_6 \sum_{n=0}^{\infty} (r/\rho)^{d/2} 2^{nd} \exp(-c_5 2^{n(1-\alpha)} r/\rho) \le c_7 \exp\left(-2^{-2} c_5 r/\rho\right).
$$

We next estimate I_2. By using (2.8), $r = \widetilde{C}_0 \Phi^{-1}(t)^{1+\theta_0}/\psi^{-1}(t)^{\theta_0}$, $\psi^{-1}(t) \le \Phi^{-1}(t)$, and $\theta_0 \le \theta < \delta - 1$, we have

$$
\frac{\Phi^{-1}(t)}{c_8 \rho} \le \left(\frac{r}{\Phi^{-1}(t)}\right)^{1/(\delta-1)} = \widetilde{C}_0^{1/(\delta-1)} \left(\frac{\Phi^{-1}(t)}{\psi^{-1}(t)}\right)^{\theta_0/(\delta-1)} \le \widetilde{C}_0^{1/(\delta-1)} \frac{\Phi^{-1}(t)}{\psi^{-1}(t)}.
$$

Thus, we have $\rho_n > \rho \ge C_3^{-1} \widetilde{C}_0^{-1/(\delta-1)} \psi^{-1}(t)$. Using this, $L(\beta_1, C_L)$ condition on ψ, and (2.8),

$$
I_2 \le c_9 \left(\frac{r}{\rho}\right)^{d+\beta_1} \left(\frac{\psi^{-1}(t)}{r}\right)^{\beta_1} \le c_{10} \left(\frac{\Phi^{-1}(t)}{r}\right)^{-\frac{\delta}{\delta-1}(d+\beta_1)} \left(\frac{\psi^{-1}(t)}{r}\right)^{\beta_1}.
$$

Since $\psi^{-1}(t) \le \Phi^{-1}(t) < r = \widetilde{C}_0 \Phi^{-1}(t)^{1+\theta_0}/\psi^{-1}(t)^{\theta_0}$ and $\theta_0 \le \theta$, using $\frac{\delta(d+\beta_1)}{\delta-1} \frac{\theta}{1+\theta} = \frac{\beta_1}{2}$, we have

$$
\left(\frac{\Phi^{-1}(t)}{r}\right)^{-\frac{\delta}{\delta-1}(d+\beta_1)} \le \widetilde{C}_0^{\delta(d+\beta_1)/(\delta-1)} \left(\frac{\psi^{-1}(t)}{r}\right)^{-\beta_1/2},
$$

which implies $I_2 \le c_{11} (\psi^{-1}(t)/r)^{\beta_1/2}$. Using estimates of I_1 and I_2 and combining (4.18) and (4.15) we obtain (4.17).

Let $f(r,t) := \frac{r}{\mathscr{K}^{-1}(t/r)}$. Then, by (4.19) and Lemma 2.8, we see that $f(r,t)$ satisfies the condition in Lemma 4.3. Thus, by Lemma 4.3, we obtain

$$
p(t,x,y) \le \frac{c_{12} t}{|x-y|^d \psi(|x-y|)} + c_{12} \Phi^{-1}(t)^{-d} \exp\left(-\frac{c_{13}|x-y|}{\mathscr{K}^{-1}(c_{14} t/|x-y|)}\right).
$$

Since $|x-y| \ge \widetilde{C}_0 \Phi^{-1}(t) \ge c_{15} t^{1/\delta} \ge c_{15} T^{-1+1/\delta} t$, we can apply (2.4) and get $\mathscr{K}^{-1}(c_{18} t/|x-y|) \le c_{21} \mathscr{K}^{-1}(t/|x-y|)$. We have proved the first claim of the theorem.

(2) The proof of the second claim is similar to the proof of the first claim. We skip the proof. $\square$

Combining Theorems 3.5 and 4.5 and Lemma 2.8, we get the desired upper bounds of $p(t,x,y)$.

4.2. *Off diagonal lower bound estimates.* We first prove (1.12).

PROPOSITION 4.6. *There exist constants* $\delta_1 \in (0, 1/2)$ *and* $C_3 > 0$ *such that*

$$p(t,x,y) \geq C_3 \frac{\mathbf{1}_{\{|x-y|\leq \delta_1 \Phi^{-1}(t)\}}}{\Phi^{-1}(t)^d} + \frac{C_3 t}{|x-y|^d \psi(|x-y|)} \mathbf{1}_{\{|x-y|\geq \delta_1 \Phi^{-1}(t)\}}. \tag{4.21}$$

PROOF. Let $\delta_1 = \varepsilon/2 < 1/2$ where ε is the constant in Theorem 3.11. Then by Theorem 3.11, for all $|x-y| \leq \delta_1 \Phi^{-1}(t)$,

$$p(t,x,y) \geq p^{B(x,\Phi^{-1}(t)/\varepsilon)}(t,x,y) \geq c_0 \Phi^{-1}(t)^{-d}. \tag{4.22}$$

Thus, we have (4.21) when $|x-y| \leq \delta_1 \Phi^{-1}(t)$.

By Lemma 3.9, we have $\mathbb{P}^x(\tau_{B(x,r)} \leq t) \leq c_1 t/\Phi(r)$ for any $r > 0$ and $x \in \mathbb{R}^d$. Let $\delta_2 := (C_L/2)^{1/\beta_1}\delta_1 \in (0, \delta_1)$ so that $\delta_1 \Phi^{-1}((1-b)t) \geq \delta_2 \Phi^{-1}(t)$ holds for all $b \in (0, 1/2]$. Then choose $\lambda \leq c_1^{-1} C_U^{-1} (2\delta_2/3)^{\beta_2}/2 < 1/2$ small enough so that $c_1 \lambda t/\Phi(2\delta_2 \Phi^{-1}(t)/3) \leq \lambda c_1 C_U (2\delta_2/3)^{-\beta_2} \leq 1/2$. Thus, we have $\lambda \in (0, 1/2)$ and $\delta_2 \in (0, \delta_1)$ (independent of t) such that

$$\delta_1 \Phi^{-1}\big((1-\lambda)t\big) \geq \delta_2 \Phi^{-1}(t) \quad \text{for all } t > 0 \tag{4.23}$$

and

$$\mathbb{P}^x(\tau_{B(x,2\delta_2\Phi^{-1}(t)/3)} \leq \lambda t) \leq 1/2 \quad \text{for all } t > 0 \text{ and } x \in \mathbb{R}^d. \tag{4.24}$$

For the remainder of the proof, we assume that $|x-y| \geq \delta_1 \Phi^{-1}(t)$. Since, using (4.22) and (4.23),

$$\begin{aligned} p(t,x,y) &\geq \int_{B(y,\delta_1\Phi^{-1}((1-\lambda)t))} p(\lambda t, x, z) p\big((1-\lambda)t, z, y\big)\,dz \\ &\geq \inf_{z\in B(y,\delta_1\Phi^{-1}((1-\lambda)t))} p\big((1-\lambda)t, z, y\big) \int_{B(y,\delta_1\Phi^{-1}((1-\lambda)t))} p(\lambda t, x, z)\,dz \\ &\geq c_0 \Phi^{-1}(t)^{-d} \mathbb{P}^x\big(X_{\lambda t} \in B\big(y, \delta_2\Phi^{-1}(t)\big)\big), \end{aligned}$$

it suffices to prove

$$\mathbb{P}^x\big(X_{\lambda t} \in B\big(y, \delta_2 \Phi^{-1}(t)\big)\big) \geq c_2 \frac{t\Phi^{-1}(t)^d}{|x-y|^d \psi(|x-y|)}. \tag{4.25}$$

Using the strong Markov property, Lévy system, the lower bound of $J(x,y)$, (1.7), and (4.24), the proof of (4.25) is standard. (See [10], Proposition 5.4(ii).) We omit the details. □

We now give the two-sided sharp estimate for the Green function.

PROOF OF COROLLARY 1.3. Let $\widetilde{\delta} := \beta_2 \wedge 2 < d$ and $r = |x-y|$. By Lemma 2.4, Φ satisfies $L(\beta_1, C_L)$ and $U(\widetilde{\delta}, C_U)$.

For the lower bound, we use Proposition 4.6 and Lemma 2.2 and get

$$G(x,y) \ge \int_{\Phi(r/\delta_1)}^{\infty} p(t,x,y)\,dt \ge c_1 \int_{\Phi(r/\delta_1)}^{2\Phi(r/\delta_1)} \Phi^{-1}(t)^{-d}\,dt \ge c_2 r^{-d}\Phi(r).$$

For the upper bound, by using the condition $\widetilde{\delta} < d$ and (2.3), we get that $\int_{\Phi(r)}^{\infty} \Phi^{-1}(t)^{-d}\,dt \le c_3\Phi(r)r^{-d}$. Using this inequality and Theorem 3.8, we conclude that

$$G(x,y) \le \frac{c_3}{\Phi(r)r^d}\int_0^{\Phi(r)} t\,dt + c_3\int_{\Phi(r)}^{\infty} \Phi^{-1}(t)^{-d}\,dt \le c_4 r^{-d}\Phi(r). \qquad \square$$

By using $\mathscr{K}$ and $\mathscr{K}_\infty$, we give the lower bound of $p(t,x,y)$ under $L_a(\delta,\widetilde{C}_L)$ or $L^1(\delta,\widetilde{C}_L)$ on Φ with $\delta > 1$. See [44], Lemmas 3.1–3.2, for similar bound for Lévy processes.

PROPOSITION 4.7. *Suppose Φ satisfies $L_a(\delta,\widetilde{C}_L)$ with $\delta > 1$ and for some $a > 0$. For $T > 0$ there exist $C > 0$ and $a_L > 0$ such that for any $t \le T$ and $x, y \in \mathbb{R}^d$,*

$$p(t,x,y) \ge C\Phi^{-1}(t)^{-d}\exp\biggl(-a_L\frac{|x-y|}{\mathscr{K}^{-1}(t/|x-y|)}\biggr). \tag{4.26}$$

Moreover, if $a = \infty$, then (4.26) holds for all $t < \infty$.

PROOF. Let $r = |x-y|$. By Proposition 4.6 and Remark 2.1, without loss of generality, we assume that $\delta_1\Phi^{-1}(t) \le r$ and $a \ge \delta_1\Phi^{-1}(T)$ where δ_1 is the constants in Proposition 4.6. Let $k = \lceil 3r\delta_1^{-1}/\mathscr{K}^{-1}(3^{-1}\delta_1 t/r)\rceil$. Note that by (2.8), $\mathscr{K}^{-1}(t/r) \le \Phi^{-1}(t)^2/r \le \delta_1\Phi^{-1}(t) \le \delta_1\Phi^{-1}(T) \le a$. Thus by (2.4), we have $\mathscr{K}^{-1}(t/r) \le \widetilde{C}_L^{-1}(3/\delta_1)\mathscr{K}^{-1}(3^{-1}\delta_1 t/r)$. Since $3^{-1}\delta_1 t/r \le 3^{-1}\delta_1\Phi(r/\delta_1)/r \le 3^{-1}\mathscr{K}(r/\delta_1)$, we see that $\mathscr{K}^{-1}(3^{-1}\delta_1 t/r) \le \frac{r}{\delta_1}$, hence

$$3 \le k \le \frac{4r}{\delta_1\mathscr{K}^{-1}(3^{-1}\delta_1 t/r)} \le \frac{12\widetilde{C}_L^{-1}r}{\delta_1^2\mathscr{K}^{-1}(t/r)}. \tag{4.27}$$

On the other hand, by Lemma 2.5 and our choice of k we have

$$\Phi\biggl(\frac{3r}{\delta_1 k}\biggr)\frac{\delta_1 k}{r} \le 3\mathscr{K}\biggl(\frac{3r}{\delta_1 k}\biggr) \le \frac{\delta_1 t}{r}.$$

Thus, we obtain $\frac{r}{k} \le \frac{\delta_1}{3}\Phi^{-1}(t/k)$. Let $z_l = x + \frac{l}{k}(y-x)$, $l = 0,1,\dots,k-1$. For $\xi_l \in B(z_l, \frac{\delta_1}{3}\Phi^{-1}(\frac{t}{k}))$ and $\xi_{l-1} \in B(z_{l-1}, \frac{\delta_1}{3}\Phi^{-1}(\frac{t}{k}))$, $|\xi_l - \xi_{l-1}| \le |\xi_l - z_l| + |z_l - z_{l-1}| + |z_{l-1} - \xi_{l-1}| \le \delta_1\Phi^{-1}(t/k)$. Thus by Proposition 4.6, $p(\frac{t}{k}, \xi_{l-1}, \xi_l) \ge$

$C_3\Phi^{-1}(t/k)^{-d}$. Using the semigroup property and (4.27), we get

$$\begin{aligned}
p(t,x,y) &\geq \int_{B(z_{k-1},\frac{\delta_1}{3}\Phi^{-1}(\frac{t}{k}))} \cdots \int_{B(z_1,\frac{\delta_1}{3}\Phi^{-1}(\frac{t}{k}))} p\left(\frac{t}{k},x,\xi_1\right)\cdots \\
&\quad \times p\left(\frac{t}{k},\xi_{k-1},y\right) d\xi_1 \cdots d\xi_{k-1} \\
&\geq C_5^k \Phi^{-1}\left(\frac{t}{k}\right)^{-dk} \prod_{l=1}^{k-1} \left| B\left(z_l, \frac{\delta_1}{3}\Phi^{-1}\left(\frac{t}{k}\right)\right)\right| \\
&= c_2 c_3^k \Phi^{-1}\left(\frac{t}{k}\right)^{-dk} \left(\frac{\delta_1}{3}\Phi^{-1}\left(\frac{t}{k}\right)\right)^{d(k-1)} \\
&\geq c_2 \left(\frac{c_3\delta_1^d}{3^d}\right)^k \Phi^{-1}(t)^{-d} \geq c_2\Phi^{-1}(t)^{-d} e^{-C_4 k} \\
&\geq c_2 \Phi^{-1}(t)^{-d} e^{-c_4 \frac{r}{\mathscr{K}^{-1}(t/r)}}. \qquad (4.28)
\end{aligned}$$

This finishes the proof. Here we record that the constant C_4 in (4.28) depends only on d and constants δ_1, C_3 in (4.21). $\square$

PROPOSITION 4.8. *Suppose Φ satisfies $L^1(\delta, \widetilde{C}_L)$ with $\delta > 1$. For any $T > 0$ and $\theta > 0$ satisfying $\frac{1}{\delta} + \theta(\frac{1}{\delta} - \frac{1}{\beta_2}) \leq 1$, there exist c_1, $c_2 > 0$ and $a'_L > 0$ such that for $(t,x,y) \in [T,\infty) \times \mathbb{R}^d \times \mathbb{R}^d$ satisfying $\delta_1\Phi^{-1}(t) < |x-y| \leq c_1\Phi^{-1}(t)^{1+\theta}/\psi^{-1}(t)^\theta$,*

$$p(t,x,y) \geq c_2\Phi^{-1}(t)^{-d} \exp\left(-a'_L \frac{|x-y|}{\mathscr{K}_\infty^{-1}(t/|x-y|)}\right),$$

where δ_1 is the constant in Proposition 4.6.

PROOF. Without loss of generality, we assume that $T \geq \Phi(1)$. Take $c_1 > 0$ small so that

$$c_1\big(\Phi^{-1}(T)\widetilde{C}_L^{-1/\delta}T^{-1/\delta}\big)^{1+\theta}\big(\psi^{-1}(T)^{-1}C_U^{1/\beta_2}T^{1/\beta_2}\big)^\theta\big(1 \vee T^{-1}\big) \leq \frac{\delta_1}{3\mathscr{K}_\infty(2)}.$$

Since ψ satisfies $U(\beta_2, C_U)$ and Φ satisfies $L^1(\delta, \widetilde{C}_L)$, we see that for $t \geq T \geq \Phi(1)$, $\psi^{-1}(t) \geq \psi^{-1}(T)C_U^{-1/\beta_2}(t/T)^{1/\beta_2}$ and $\Phi^{-1}(t) \leq \Phi^{-1}(T)\widetilde{C}_L^{-1/\delta}(t/T)^{1/\delta}$ by Lemma 2.2. Thus, we have

$$|x-y| \leq c_1\Phi^{-1}(t)^{1+\theta}/\psi^{-1}(t)^\theta \leq \frac{\delta_1 t}{3\mathscr{K}_\infty(2)}, \qquad (4.29)$$

where the third inequality follows from $\frac{1}{\delta} + \theta(\frac{1}{\delta} - \frac{1}{\beta_2}) \leq 1$. Let $r = |x-y|$ and $k = \lceil 3r\delta_1^{-1}/\mathscr{K}_\infty^{-1}(3^{-1}\delta_1 t/r)\rceil$. Since $r \geq \delta_1\Phi^{-1}(t) \geq \delta_1$, we have by Lemma 2.7

that $\delta_1 t/r \le \delta_1 \Phi(r/\delta_1)/r = \delta_1 \widetilde{\Phi}(r/\delta_1)/r \le \mathscr{K}_\infty(r/\delta_1)$. Thus, $\mathscr{K}_\infty^{-1}(\delta_1 t/3r) \le \mathscr{K}_\infty^{-1}(\frac{1}{3}\mathscr{K}_\infty(r/\delta_1)) \le r/\delta_1$, which implies that

$$3 \le k \le \frac{4r}{\delta_1 \mathscr{K}_\infty^{-1}(3^{-1}\delta_1 t/r)} \le \frac{12\widetilde{C}_L^{-1} r}{\delta_1^2 \mathscr{K}_\infty^{-1}(t/r)}. \tag{4.30}$$

On the other hand, since $\mathscr{K}_\infty^{-1}(3^{-1}\delta_1 t/r) \ge \mathscr{K}_\infty^{-1}(\mathscr{K}_\infty(2)) = 2$ and $3r/\delta_1 \ge 3\Phi^{-1}(T) \ge 3$, we see that

$$\frac{3r}{\delta_1 \mathscr{K}_\infty^{-1}(3^{-1}\delta_1 t/r)} \le k < \frac{3r}{\delta_1}.$$

Thus, by the above inequality and Lemma 2.7, we get

$$\Phi\left(\frac{3r}{\delta_1 k}\right)\frac{\delta_1 k}{3r} = \widetilde{\Phi}\left(\frac{3r}{\delta_1 k}\right)\frac{\delta_1 k}{3r} \le \mathscr{K}_\infty\left(\frac{3r}{\delta_1 k}\right) \le 3^{-1}\delta_1 t/r,$$

which yields $\frac{r}{k} \le \frac{\delta_1}{3}\Phi^{-1}(t/k)$. Using this, Proposition 4.6, the semigroup property and (4.30), the remaining part of the proof is same as the one in the proof of Proposition 4.7. □

PROOF OF THEOREM 1.4. The both upper bounds of $p(t,x,y)$ in Theorem 1.4 follows from Theorems 3.5 and 4.5 and Lemma 2.8. The lower bound in (1.14) is a direct consequence of Propositions 4.6 and 4.7.

By Propositions 4.6 and 4.8, to complete the proof of Theorem 1.4, it is enough to show that for $t \ge T$ and $(c_1\Phi^{-1}(t)^{1+\theta}/\psi^{-1}(t)^\theta) \vee \delta_1\Phi^{-1}(t) < r$,

$$p(t,x,y) \ge C\Phi^{-1}(t)^{-d} \exp\left(-a'_L \frac{r}{\mathscr{K}_\infty^{-1}(t/r)}\right), \tag{4.31}$$

where c_1 and θ are the constants in Proposition 4.8.

By Proposition 4.6 and the same argument as in (4.14), we have that,

$$p(t,x,y) \ge \frac{c_2 t}{r^d \psi(r)} \ge c_3 \Phi^{-1}(t)^{-d} \exp\left(-\frac{a_2 r^2}{\Phi^{-1}(t)^2}\right).$$

By (2.5) and Lemma 2.2, $\mathscr{K}_\infty$ satisfies $L(\delta - 1, \widetilde{C}_L^{-2/(\delta-1)})$. Using this property and (2.10) (note that $r \ge \delta_1 \Phi^{-1}(t)$ and $T \le t$), we get

$$\left(\frac{\delta_1^{-1} r}{\Phi^{-1}(t)}\right)^2 \le c_4 \frac{\delta_1^{-1} r}{\mathscr{K}_\infty^{-1}(\delta_1 t/r)} \le c_4 \frac{\widetilde{C}_L^{-2/(\delta-1)} \delta_1^{-\delta/(\delta-1)} r}{\mathscr{K}_\infty^{-1}(t/r)}.$$

Thus, (4.31) holds. □

PROOF OF COROLLARY 1.5. Since the upper bound is a direct consequence of Theorem 1.2, we show the lower bound in (1.15). Let $r = |x-y|$ and $\phi(s) := \Phi(s^{-1/2})^{-1}$. Since Φ satisfies $L_a(\delta, \widetilde{C}_L)$, ϕ satisfies $L^{1/a^2}(\delta/2, \widetilde{C}_L)$. Let Z be a subordinate Brownian motion whose Laplace exponent is ϕ and $p^Z(t, |z-w|)$ be its transition density. Then, by [35], Proposition 3.5, and Theorem 1.4, for any $T > 0$, there exist positive constants $\widetilde{a}_L$, a_U, c_1 and c_2 such that for all $(t, x, y) \in (0, T) \times \mathbb{R}^d \times \mathbb{R}^d$,

$$c_1 \exp\biggl(-\frac{\widetilde{a}_L r^2}{\Phi^{-1}(t)^2}\biggr) \le \frac{p^Z(t,r)}{\Phi^{-1}(t)^{-d}} \le c_2 \exp\biggl(-\frac{a_U r}{\mathscr{K}^{-1}(t/r)}\biggr) + \frac{c_2 t \Phi^{-1}(t)^d}{r^d \psi(r)}.$$

Let $a_L \ge a_U$ be a constant in Theorem 1.4 and $A := a_L/a_U \ge 1$. Then, for all $t \in (0, T)$, $s > 0$,

$$\begin{aligned} c_1 \exp\biggl(-\frac{\widetilde{a}_L A^2 s^2}{\Phi^{-1}(t)^2}\biggr) &\le c_2 \exp\biggl(-\frac{a_U A s}{\mathscr{K}^{-1}(t/As)}\biggr) + \frac{c_2 t \Phi^{-1}(t)^d}{(As)^d \psi(As)} \\ &\le c_2 \exp\biggl(-\frac{a_L s}{\mathscr{K}^{-1}(t/s)}\biggr) + \frac{c_3 t \Phi^{-1}(t)^d}{s^d \psi(s)}. \end{aligned}$$

Thus, by Theorem 1.4, we obtain the desired results. $\square$

5. Application to the Khintchine-type law of iterated logarithm. In this section, we apply our main results in previous sections and show that, if our symmetric jump process has the finite second moment, the Khintchine-type law of iterated logarithm at the infinity holds. Furthermore, we will also prove the converse.

We first establish the zero-one law for tail events.

THEOREM 5.1. *Let A be a tail event. Then, either $\mathbb{P}^x(A) = 0$ for all x or else $\mathbb{P}^x(A) = 1$ for all $x \in \mathbb{R}^d$.*

PROOF. Fix $t_0, \varepsilon > 0$ and $x_0 \in \mathbb{R}^d$. Note that, by Lemma 3.9, there exists $c_1 > 0$ such that

$$\mathbb{P}^{x_0}\Bigl(\sup_{s \le t_0} |X_s - x_0| > c_1 \Phi^{-1}(t_0)\Bigr) < \varepsilon. \tag{5.1}$$

While, using Theorem 3.10 to $P_t f$, the semigroups of $(\mathcal{E}, \mathcal{F})$, we can choose $t_1 > 0$ large so that for all $f \in L^\infty(\mathbb{R}^d)$ and $x \in \mathbb{R}^d$ with $|x - x_0| \le c_1 \Phi^{-1}(t_0)$,

$$\bigl|P_{t_1} f(x) - P_{t_1} f(x_0)\bigr| \le c_2 \biggl(\frac{|x_0 - x|}{\Phi^{-1}(t_1)}\biggr)^\theta \sup_{t>0} \|P_t f\|_\infty < \varepsilon \|f\|_\infty. \tag{5.2}$$

Note that (5.1) and (5.2) are same as [27], (A.6) and (A.7), and the proof of the theorem is exactly same as that of [27], Theorem 2.10. $\square$

From (1.6) and (1.7), we see that the following three conditions are equivalent:

$$\sup_{x \in \mathbb{R}^d} \Bigl(\text{or } \inf_{x \in \mathbb{R}^d}\Bigr) \int_{\mathbb{R}^d} J(x,y)|x-y|^2 \, dy < \infty; \tag{5.3}$$

$$c^{-1}r^2 \le \Phi(r) \le cr^2, \quad r > 1; \tag{5.4}$$

$$\int_0^\infty \frac{s\,ds}{\psi(s)} < \infty. \tag{5.5}$$

Using Theorem 1.2, we see that under the assumption (1.6), the above conditions (5.3)–(5.5) are also equivalent to the following finite second moment condition:

$$\sup_{x\in\mathbb{R}^d}\Bigl(\text{or } \inf_{x\in\mathbb{R}^d}\Bigr)\mathbb{E}^x\bigl[|X_t - x|^2\bigr] < \infty \quad \forall \text{ (or } \exists\text{) } t > 0.$$

Here is the main result of this section.

THEOREM 5.2. *Suppose X is symmetric pure-jump process whose jumping density J satisfies* (1.7). (1) *If* (5.4) *holds, then there exists a constant $c \in (0,\infty)$ such that for all $x \in \mathbb{R}^d$*,

$$\limsup_{t\to\infty} \frac{|X_t - x|}{(t \log\log t)^{1/2}} = c \quad \text{for } \mathbb{P}^x\text{-a.e.} \tag{5.6}$$

(2) *Suppose that* (1.6) *holds but* (5.4) *does not hold. Then for all $x \in \mathbb{R}^d$*, (5.6) *holds with $c = \infty$.*

PROOF. Without loss of generality, we assume that $\Phi(1) = 1$. Let $h(t) = t^{1/2}(\log\log t)^{1/2}$. We first observe that, by the change of variable $s = h(t)$,

$$\int_{h(4)}^\infty \frac{s\,ds}{\psi(s)} = \frac{1}{2}\int_4^\infty \frac{(\log\log t) + (\log t)^{-1}}{\psi(h(t))}\,dt \asymp \int_4^\infty \frac{\log\log t}{\psi(h(t))}\,dt, \tag{5.7}$$

and

$$\int_{h(4)}^\infty \frac{s\,ds}{\psi(s)\log\log s} = \frac{1}{2}\int_4^\infty \frac{(\log\log t) + (\log t)^{-1}}{\psi(h(t))\log\log h(t)}\,dt \asymp \int_4^\infty \frac{dt}{\psi(h(t))}. \tag{5.8}$$

(1) By (5.5) and (5.7),

$$\sum_{k=3}^\infty \frac{2^k}{\psi(h(2^k))} \le 2\sum_{k=2}^\infty \int_{2^k}^{2^{k+1}} \frac{dt}{\psi(h(t))} = \int_4^\infty \frac{dt}{\psi(h(t))} < \infty. \tag{5.9}$$

Since we have $L^1(2, C_L)$ under (5.4), by Theorem 1.4(2) we have that for all $C > 0$, $t > 4$ and $t \le u \le 4t$,

$$\begin{aligned}
&\mathbb{P}^x\bigl(|X_u - x| > Ch(t)\bigr)\\
&\quad= \int_{|x-y|>Ch(t)} p(u,x,y)\,dy\\
&\quad\le c_1\biggl(t\int_{Ch(t)}^\infty \frac{1}{s\psi(s)}\,ds + t^{-d/2}\int_{Ch(t)}^\infty \exp\Bigl(-c_2\frac{s^2}{t}\Bigr)s^{d-1}\,ds\biggr)\\
&\quad=: c_3(I + II).
\end{aligned}$$

Let $C := 1 \vee 2c_2^{-1/2}$. By the change of variable $s_1 = \frac{s^2}{t}$, we obtain

$$II \le c_3 \int_{C^2 \log\log t}^{\infty} e^{-c_2 s/2}\, ds \le \frac{c_3}{c_2} (\log t)^{-C^2 c_2/2} = c_3 (\log t)^{-2}.$$

While, by Lemma 2.3, $I \le c_4 \frac{t}{\psi(Ch(t))} \le c_5 \frac{t}{\psi(h(t))}$. Thus, for every $t > 4$ and $t \le u \le 4t$, $\mathbb{P}^x(|X_u - x| > Ch(t)) \le c_6((\log t)^{-2} + \psi(h(t))^{-1})$. Using this and the strong Markov property, with $t_k = 2^k$, $k \ge 3$ we get

$$\begin{aligned}
&\mathbb{P}^x\big(|X_s - x| > 2Ch(s) \text{ for some } s \in [t_{k-1}, t_k]\big) \\
&\quad \le \mathbb{P}^x(\tau_{B(x, Ch(t_{k-1}))} \le t_k) \\
&\quad \le 2 \sup_{s \le t_k, z \in \mathbb{R}^d} \mathbb{P}^z\big(|X_{t_{k+1}-s} - z| > Ch(t_{k-1})\big) \le c_7 \bigg(\frac{1}{k^2} + \frac{2^k}{\psi(h(2^k))}\bigg),
\end{aligned}$$

where we followed the calculations in (4.2). Therefore, by (5.9) and the Borel–Cantelli lemma, the above implies that

$$\mathbb{P}^x\big(|X_t - x| \le 2Ch(t) \text{ for all sufficiently large } t\big) = 1.$$

Thus, $\limsup_{t\to\infty} \frac{|X_t - x|}{h(t)} \le 2C$. Since $\psi(r) \ge \Phi(r)$, by (5.4), $J(x, y) \le c_7 |x - y|^{-d-2}$ for $|x - y| > 1$. Thus, by following the proof of [42], Theorem 1.2(2), line by line using our Theorem 1.4(2), we have that there exists $c_8 > 0$ such that $\mathbb{P}^x(|X_t - x| > c_8 h(t) \text{ for infinitely many } t) = 1$. Therefore,

$$\mathbb{P}^x\bigg(c_8 \le \limsup_{t\to\infty} \frac{|X_t - x|}{h(t)} \le 2C\bigg) = 1.$$

Now using the zero-one law in Theorem 5.1, we conclude that there exists $c_9 \in [c_8, 2C]$ such that

$$\mathbb{P}^x\bigg(\limsup_{t\to\infty} \frac{|X_t - x|}{h(t)} = c_9\bigg) = 1 \quad \text{for all } x \in \mathbb{R}^d.$$

(2) Using Theorem 3.8, there is $\lambda \in (0, 1)$ such that

$$\sup_{t\ge 1} \sup_{y \in \mathbb{R}^d} \int_{|z-y| < \lambda\Phi^{-1}(t)} p(t, y, z)\, dz \le c_0 \sup_{t \ge 1} \big|\lambda \Phi^{-1}(t)\big|^d \big(\Phi^{-1}(t)\big)^{-d} = c_0 \lambda^d < \frac{1}{2}.$$

Let $t_k = 2^k$. By the strong Markov property, we have that for all $C > 0$

$$\mathbb{P}^x\big(|X_{t_{k+1}} - X_{t_k}| \ge Ch(t_{k+1}) \mid \mathcal{F}_{t_k}\big) \ge \inf_{y \in \mathbb{R}^d} \int_{|z-y| \ge Ch(t_{k+1})} p(t_k, y, z)\, dz.$$

We claim that for every $C > 1$,

$$\sum_{k=1}^{\infty} \inf_{y \in \mathbb{R}^d} \int_{|z-y| \ge Ch(t_{k+1})} p(t_k, y, z)\, dz = \infty, \tag{5.10}$$

which implies the theorem. In fact, by the second Borel–Cantelli lemma, $\mathbb{P}^x(\limsup\{|X_{t_{k+1}} - X_{t_k}| \geq Ch(t_{k+1})\}) = 1$. Whence, for infinitely many $k \geq 1$, $|X_{t_{k+1}} - x| \geq Ch(t_{k+1})/2$ or $|X_{t_k} - x| \geq Ch(t_{k+1})/2 \geq Ch(t_k)/2$. Therefore, for all $x \in \mathbb{R}^d$,

$$\limsup_{t\to\infty} \frac{|X_t - x|}{h(t)} = \limsup_{k\to\infty} \frac{|X_{t_k} - x|}{h(t_k)} \geq \frac{C}{2}, \quad \mathbb{P}^x\text{-a.e.}$$

Since the above holds for every $C > 1$, the theorem follows.

We now prove the claim (5.10) by considering two cases separately.

Case 1: Suppose $\int_4^\infty \frac{s\,ds}{\psi(s)\log\log s} = \infty$.

If there exist infinitely many $k \geq 1$ such that $Ch(t_{k+1}) \leq a\Phi^{-1}(t_k)$, then, for infinitely many $k \geq 1$,

$$\inf_{y\in\mathbb{R}^d} \int_{|z-y|\geq a\Phi^{-1}(t_k)} p(t_k, y, z)\,dz = 1 - \sup_{y\in\mathbb{R}^d} \int_{|z-y|< a\Phi^{-1}(t_k)} p(t_k, y, z)\,dz > 1/2.$$

Thus, we get (5.10).

If there is $k_0 \geq 3$ such that for all $k \geq k_0$, $Ch(t_{k+1}) > a\Phi^{-1}(t_k)$, then by Lemma 2.3 and Proposition 4.6, for all $k \geq k_0$

$$\inf_{y\in\mathbb{R}^d} \int_{|z-y|\geq a\Phi^{-1}(t_k)} p(t_k, y, z)\,dz \geq c_1 \int_{Ch(t_{k+1})}^\infty \frac{t_k}{r\psi(r)} dr \geq c_2 \frac{t_{k+1}}{\psi(h(t_{k+1}))}.$$

Combining this with (5.8) and the assumption that $\int_0^\infty \frac{s\,ds}{\psi(s)\log\log s} = \infty$, we also get (5.10).

Case 2: We assume that $\int_4^\infty \frac{s\,ds}{\psi(s)\log\log s} < \infty$. Then for any $s > 4$ we have

$$\Phi^{-1}(s) \leq c_3 s^{1/2}(\log\log s)^{1/2} = c_3 h(s). \tag{5.11}$$

Also, using the assumption $\int_0^\infty \frac{s\,ds}{\psi(s)} = \infty$ to (1.10) we obtain

$$\lim_{s\to\infty} \frac{\Phi^{-1}(s)}{s^{1/2}} = \infty. \tag{5.12}$$

Let $r = |x - y|$ and $\delta_1, C_3 > 0$ be the constants in (4.21). Also, let $C_4 = C_4(d, \delta_1, C_3) > 0$ be the constant C_4 in (4.28). Now define $C_0 = (2C_4)^{-1}$ and $N = \lceil C_0 \log k \rceil$. Then, by (5.12) we have $\lim_{k\to\infty} \Phi^{-1}(t_k/N)/(t_k/N)^{1/2} = \infty$. Thus, there exists $k_0 \in \mathbb{N}$ such that for any $k \geq k_0$, we have $N(k) \geq 3$ and $\frac{\Phi^{-1}(t_k/N)}{(t_k/N)^{1/2}} \geq \frac{12C}{\delta_1 C_0^{1/2}}$. Then there exists a constant $c_4 > 0$ such that for any $k \geq k_0$ and $Ch(t_{k+1}) \leq |x - y| \leq 2Ch(t_{k+1})$,

$$p(t_k, x, y) \geq c_4 \Phi^{-1}(t_k)^{-d} k^{-1/2}. \tag{5.13}$$

Indeed, for $k \geq k_0$ we have $\frac{\delta_1}{3}\Phi^{-1}(\frac{t_k}{N}) = \frac{\delta_1}{3}(\frac{t_k}{N})^{1/2} \frac{\Phi^{-1}(t_k/N)}{(t_k/N)^{1/2}} \geq \frac{2Ch(t_{k+1})}{N} \geq \frac{r}{N}$. Since we have (4.21), following the proof of Proposition 4.7 we obtain

$$p(t_k, x, y) \geq c_5 \Phi^{-1}(t_k)^{-d} \exp(-C_4 N) \geq c_5 \Phi^{-1}(t_k)^{-d} k^{-1/2}.$$

By (5.13) and (5.11) we have that for every $k \geq k_0$,

$$\inf_{y\in\mathbb{R}^d} \int_{Ch(t_{k+1})\leq|z-y|\leq 2Ch(t_{k+1})} p(t_k, y, z)\, dz \geq c_5 \frac{h(t_{k+1})^d}{k^{1/2}\Phi^{-1}(t_k)^d} \geq \frac{c_6}{k^{1/2}}.$$

Therefore, we conclude that

$$\sum_{k=k_0}^{\infty} \inf_{y\in\mathbb{R}^d} \int_{|z-y|\geq Ch(t_{k+1})} p(t_k, y, z)\, dz \geq c_6 \sum_{k=k_0}^{\infty} \frac{1}{k^{1/2}} = \infty.$$

We have proved (5.10). $\square$

We end this section by showing that Khintchine-type law of iterated logarithm at zero does not hold.

PROPOSITION 5.3. *Suppose X is symmetric pure-jump process whose jumping density J satisfies* (1.6) *and* (1.7). *Then, for all $x \in \mathbb{R}^d$,*

$$\limsup_{t\to 0} \frac{|X_t - x|}{(t \log\log 1/t)^{1/2}} = 0 \quad \textit{for } \mathbb{P}^x\textit{-a.e.} \tag{5.14}$$

PROOF. Without loss of generality, we assume that $\Phi(1) = 1$. Let $h(t) = (t \log\log 1/t)^{1/2}$. We first observe that, by the change of variable $s = h(t)$,

$$\int_0^{h(1/4)} \frac{s\, ds}{\psi(s) \log\log \frac{1}{s}} \asymp \int_0^{1/4} \frac{dt}{\psi(h(t))}.$$

Thus, using (1.6) we obtain

$$\sum_{k=3}^{\infty} \frac{2^{-k}}{\psi(h(2^{-k}))} \leq 2 \sum_{k=2}^{\infty} \int_{2^{-k-1}}^{2^{-k}} \frac{dt}{\psi(h(t))} = \int_0^{1/4} \frac{dt}{\psi(h(t))} < \infty. \tag{5.15}$$

By (1.11) in Theorem 1.2, for all $C > 0$, $t < 1/4$ and $t/4 \leq u \leq t$,

$$\begin{aligned}
&\mathbb{P}^x\big(|X_u - x| > Ch(t)\big) \\
&\quad = \int_{|x-y|>Ch(t)} p(u, x, y)\, dy \\
&\quad \leq c_1\bigg(t \int_{Ch(t)}^{\infty} \frac{1}{s\psi(s)}\, ds + \Phi^{-1}(t)^{-d} \int_{Ch(t)}^{\infty} \exp\bigg(-c_2 \frac{s^2}{\Phi^{-1}(t)^2}\bigg) s^{d-1}\, ds\bigg) \\
&\quad =: c_1(I + II).
\end{aligned}$$

Using Lemma 2.3, $I \leq \frac{c_3 t}{\psi(h(t))}$. Also, by the change of variable $v = \frac{s^2}{\Phi^{-1}(t)^2}$,

$$II \leq c_4 \int_{C^2h(t)^2/\Phi^{-1}(t)^2}^{\infty} \exp(-c_2 v) v^{(d-2)/2}\, dv \leq c_5 \exp\bigg(-\frac{c_2 C^2 t}{2\Phi^{-1}(t)^2} \log\log \frac{1}{t}\bigg).$$

Note that $\Phi^{-1}(t)^2/t = \int_0^{\Phi^{-1}(t)} \frac{s}{\psi(s)}\,ds \to 0$ as $t \to 0$. Therefore, when t is sufficiently small, $II \le c_6 \exp(-2\log\log 1/t)$. Thus, letting $t = 2^{-k}$ and using (5.15), we have for sufficiently large $N \in \mathbb{N}$,

$$\sum_{k=N}^{\infty} \sup_{s \le t_k, z \in \mathbb{R}^d} \mathbb{P}^z\big(|X_{t_{k+1}-s} - z| > Ch(t_{k-1})\big) \le c_7 \sum_{k=N}^{\infty} \left(\frac{1}{k^2} + \frac{2^{-k}}{\psi(h(2^{-k}))}\right) < \infty.$$

Now, by the same argument as that in the proof of Theorem 5.2(1), we conclude that $\limsup_{t\to 0} \frac{|X_t - x|}{(t\log\log 1/t)^{1/2}} \le C$, $\mathbb{P}^x$-a.e. for all $C > 0$, which implies (5.14). $\square$

6. Examples. In this section, we will use the notation $f(\cdot) \simeq g(\cdot)$ at ∞ (resp. 0) if $\frac{f(t)}{g(t)} \to 1$ as $t \to \infty$ (resp. $t \to 0$). We denote $\mathcal{R}_0^\infty$ (resp. $\mathcal{R}_0^0$) by the class of slowly varying functions at ∞ (resp. 0). For $\ell \in \mathcal{R}_0^\infty$, we denote Π_ℓ^∞ (resp. Π_ℓ^0) by the class of real-valued measurable function f on $[c, \infty)$ (resp. $(0, c)$) such that for all $\lambda > 0$, $f(\lambda\cdot) - f(\cdot) \simeq \log\lambda\,\ell(\cdot)$ at ∞ (resp. 0) Π_ℓ^∞ (resp. Π_ℓ^0) is called de Haan class at ∞ (resp. 0) determined by ℓ.

For $\ell \in \mathcal{R}_0^\infty$ (resp. $\mathcal{R}_0^0$), we say $\ell_\#$ is de Bruijn conjugate of ℓ if both $\ell(t)\ell_\#(t\ell(t)) \simeq 1$ and $\ell_\#(t)\ell(t\ell_\#(t)) \simeq 1$ at ∞ (resp. 0). Note that $|f| \in \mathcal{R}_0^\infty$ if $f \in \Pi_\ell^\infty$ (see [4], Theorem 3.7.4).

In the following corollary and examples $a_i = a_{i,L}$ or $a_i = a_{i,U}$ depending on whether we consider lower or upper bound.

COROLLARY 6.1. *Let $T \in (0, \infty)$ and ψ be a nondecreasing function that satisfies $L(\beta_1, C_L)$ and $U(\beta_2, C_U)$.*

(1) *Let $\ell \in \mathcal{R}_0^0$ be such that $\int_0^1 \frac{\ell(s)}{s}\,ds < \infty$ and $f(s) := \int_0^s \frac{\ell(t)}{t}\,dt \in \Pi_\ell^0$ satisfies $f(sf^\gamma(s)) \simeq f(s)$ at 0 for $\gamma = 1/2, 1$. Suppose that $\psi(s) \asymp \frac{s^2}{\ell(s)}$ for $s < 1$. Then for $t < T$,*

$$p(t,x,y) \asymp \frac{1}{(tf(t^{1/2}))^{d/2}} \wedge \left(\frac{t}{|x-y|^d\psi(|x-y|)} + \frac{1}{(tf(t^{1/2}))^{d/2}} e^{-\frac{a_1|x-y|^2}{tf(t/|x-y|)}}\right).$$

Furthermore, if $f(s^2) \asymp f(s)$ for $s < 1$, then for $t < T$,

$$p(t,x,y) \asymp \frac{1}{(tf(t))^{d/2}} \wedge \left(\frac{t}{|x-y|^d\psi(|x-y|)} + \frac{1}{(tf(t))^{d/2}} e^{-a_2\frac{|x-y|^2}{tf(t)}}\right). \tag{6.1}$$

(2) *Assume that $\ell \in \mathcal{R}_0^\infty$ satisfies $\int_1^\infty \frac{\ell(t)}{t}\,dt = \infty$.*

Suppose that $\psi(s) \asymp \frac{s^2}{\ell(s)}$ for $s > 1$ and $f \in \Pi_\ell^\infty$ satisfies $f(sf^\gamma(s)) \simeq f(s)$ at ∞ for $\gamma = 1/2, 1$. Then for $t > T$,

$$p(t,x,y) \asymp \frac{1}{(tf(t^{1/2}))^{d/2}} \wedge \left(\frac{t\ell(|x-y|)}{|x-y|^{d+2}} + \frac{1}{(tf(t^{1/2}))^{d/2}} e^{-\frac{a_3|x-y|^2}{tf(t/|x-y|)}}\right).$$

Furthermore, if $f(s^2) \asymp f(s)$ *for* $s > 1$, *then for* $t > T$,

$$p(t,x,y) \asymp \frac{1}{(tf(t))^{d/2}} \wedge \left(\frac{t\ell(|x-y|)}{|x-y|^{d+2}} + \frac{1}{(tf(t))^{d/2}} e^{-a_4 \frac{|x-y|^2}{tf(t)}}\right). \tag{6.2}$$

PROOF. Let $r = |x-y|$ and $\delta_1 > 0$ be the constant in Proposition 4.6.

(1) By [4], Corollary 2.3.4, $(f^\gamma)_\# \simeq 1/f^\gamma$ at 0. Thus, using [4], Theorem 3.6.8, we have for $s < T$,

$$\Phi(s) \asymp \frac{s^2}{f(s)}, \qquad \Phi^{-1}(s) \asymp s^{1/2} f^{1/2}(s^{1/2}), \qquad \mathscr{K}_\infty^{-1}(s) \asymp sf(s).$$

Therefore, by Theorem 1.4(1) and Theorem 1.2, we obtain the first claim and the upper bound in the second claim.

For the lower bound in the second claim, choose small $\theta > 0$ such that $\frac{1}{2} + \theta(\frac{1}{2} - \frac{1}{\beta_1}) =: \varepsilon_1 < 1$. Note that $f(s) \asymp f(s^2)$ for $s < 1$ implies $f(s^b) \asymp f(s)$ for all $b > 0$ since f is nondecreasing. Since the last term in the heat kernel estimate dominates other terms only in the case $\delta_1 \Phi^{-1}(t) < r \le \delta_1 \frac{\Phi^{-1}(t)^{1+\theta}}{\psi^{-1}(t)^\theta}$, it suffices to show $f(t/r) \ge cf(t)$ for this case. Using (2.3) and $L(\beta_1, C_L)$ for ψ we have $\Phi^{-1}(t)/\psi^{-1}(t) \le c_1 t^{\frac{1}{2} - \frac{1}{\beta_1}}$ for $t \le T$. Thus we have $f(t/r) \ge f(c_2 t^{1-\varepsilon_1}) \asymp f(t)$ for every $t \le T$ and $\delta_1 \Phi^{-1}(t) < r \le \delta_1 \frac{\Phi^{-1}(t)^{1+\theta}}{\psi^{-1}(t)^\theta}$.

(2) Similarly, $(f^\gamma)_\# \simeq 1/f^\gamma$ at ∞ by [4], Corollary 2.3.4. Thus, using [4], (1.5.8), Theorem 3.7.3, we have that for $s > T$,

$$\Phi(s) \asymp s^2/f(s), \qquad \Phi^{-1}(s) \asymp s^{1/2} f^{1/2}(s^{1/2}), \qquad \mathscr{K}_\infty^{-1}(s) \asymp sf(s).$$

Note that $\psi(r) \asymp \frac{r^2}{\ell(r)}$ when $r > \delta_1 \Phi^{-1}(t)$ since $r > \delta_1 \Phi^{-1}(t) \ge \delta_1 \Phi^{-1}(T)$. Since the second term in the heat kernel estimate dominates only in the case $r > \delta_1 \Phi^{-1}(t)$, the first claim and upper bound in the second one follow from Theorem 1.4(2) and Theorem 1.2.

Now choose small $\theta' > 0$ such that $\frac{1}{\delta} + \theta'(\frac{1}{\delta} - \frac{1}{\beta_2}) =: \varepsilon_2 < 1$. Without loss of generality we can assume that f is nondecreasing since $f(s) \asymp \int_1^s \frac{\ell(t)}{t}\,dt$. Now $f(s) \asymp f(s^2)$ for $s > 1$ implies $f(s^b) \asymp f(s)$ for all $b > 0$. Similarly, using $L^a(\delta, \widetilde{C}_L)$ for Φ and $U(\beta_2, C_U)$ for ψ we have $\frac{\Phi^{-1}(t)}{\psi^{-1}(t)} \le c_3 t^{\frac{1}{\delta} - \frac{1}{\beta_2}}$ so $f(t/r) \ge f(c_4 t^{1-\varepsilon_2}) \asymp f(t)$ for every $t \ge T$ and $r \le \delta_1 \frac{\Phi^{-1}(t)^{1+\theta'}}{\psi^{-1}(t)^{\theta'}}$. This finishes the proof. $\square$

Table 1 provides nontrivial examples where (6.1) holds.

Table 2 provides nontrivial examples where (6.2) holds.

Note that when $\psi(r) = r^2(\log r)^\beta$, $r > 16$ with $\beta > 1$, (5.5) holds which is equivalent to (5.4).

TABLE 1
Examples where (6.1) holds

$\psi(\lambda)$, $\alpha > 1$, $0 < \lambda \le 2^{-4}$	$\ell(s)$, $s \le 2^{-4}$	$f(s)$, $s \le 2^{-4}$
$\lambda^2(\log \frac{1}{\lambda})^{\alpha}$	$(\log \frac{1}{s})^{-\alpha}$	$\frac{1}{\alpha-1}(\log \frac{1}{s})^{1-\alpha}$
$\lambda^2(\log \frac{1}{\lambda})(\log\log \frac{1}{\lambda})^{\alpha}$	$(\log \frac{1}{s})^{-1}(\log\log \frac{1}{s})^{-\alpha}$	$\frac{1}{\alpha-1}(\log\log \frac{1}{s})^{1-\alpha}$

TABLE 2
Examples where (6.2) holds

$\psi(r) = r^2(\log r)^{\beta}$, $r > 16$	$\ell(s)$, $s > 16$	$f(s)$, $s > 16$
When $\beta < 1$	$(\log r)^{-\beta}$	$(1-\beta)^{-1}(\log s)^{1-\beta}$
When $\beta = 1$	$(\log r)^{-\beta}$	$\log\log s$

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