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Supremum estimates for degenerate, quasilinear stochastic partial differential equations

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Abstract. We prove a priori estimates in L_∞ for a class of quasilinear stochastic partial differential equations. The estimates are obtained independently of the ellipticity constant ε and thus imply analogous estimates for degenerate quasilinear stochastic partial differential equations, such as the stochastic porous medium equation.

Résumé. Nous montrons une estimée a priori dans L_∞ pour une classe d'équations différentielles partielles stochastiques quasi-linéaires. Les estimées sont obtenues indépendamment de la constante d'ellipticité ε et impliquent par conséquent une estimée analogue pour les équations différentielles partielles stochastiques quasi-linéaires dégénérées, telles que l'équation stochastique des milieux poreux.

MSC: 60H15; 60G46

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1. Introduction

We consider quasilinear stochastic partial differential equations (SPDEs) of the form[1]

$$\begin{aligned} du &= \left[\partial_i\left(a_t^{ij}(x,u)\partial_j u + F_t^i(x,u)\right) + F_t(x,u)\right]dt \\ &\quad + \left[\partial_i\left(g_t^{ik}(x,u)\right) + G_t^k(x,u)\right]d\beta_t^k, \qquad (1.1) \\ u_0 &= \xi, \end{aligned}$$

for $(t,x) \in [0,T] \times Q =: Q_T$, with zero Dirichlet conditions on ∂Q, for some bounded open set $Q \subset \mathbb{R}^d$ and β^k being independent Wiener processes.

In this work, using Moser's iteration techniques (see e.g. [21]), we prove the following: First, roughly speaking, we show that if the initial condition ξ is in L_∞ then the solution is in L_∞ for all times $t \geq 0$. Second, we show a regularizing effect, that is, if the initial condition is in L_2, then for all $t > 0$ the solution $u(t)$ is in L_∞ and the corresponding norm blows up at a rate of $t^{-\tilde{\theta}}$, for some constant $\tilde{\theta} > 0$, as $t \searrow 0$.

A key point in these results is that the obtained estimates are uniform with respect to the ellipticity constant of the diffusion coefficients a^{ij} and thus can be applied to the case of degenerate, quasilinear SPDE, such as the porous medium equation.

[1]Throughout the article we use the summation convention with respect to integer valued repeated indices.

More precisely, under certain conditions on the coefficients a^{ij}, F^i, F, g^{ik}, G^k (see Assumptions 2.1–2.2 below for details), we prove the following L^∞ bounds:

Theorem (see Theorems 2.7 and 2.10). *Let $\alpha > 0$, $\mu \in [2,\infty] \cap ((d+2)/2, \infty]$. There exists constants $N, \tilde{\theta} > 0$ such that if u is a solution of (1.1), then*

$$\mathbb{E}\|u\|^\alpha_{L_\infty(Q_T)} \le N\mathbb{E}\big(1 + \|\xi\|^\alpha_{L_\infty(Q)} + \|V^1\|^\alpha_{L_\mu(Q_T)} + \|V^2\|^\alpha_{L_{2\mu}(Q_T)}\big),$$

and

$$\mathbb{E}\|u\|^\alpha_{L_\infty((\rho,T)\times Q)} \le \rho^{-\tilde{\theta}} N\mathbb{E}\big(1 + \|\xi\|^\alpha_{L_2(Q)} + \|V^1\|^\alpha_{L_\mu(Q_T)} + \|V^2\|^\alpha_{L_{2\mu}(Q_T)}\big),$$

for all $\rho \in (0,T)$.

In the above theorem V^1 and V^2 are functions that can be regarded as dominating any existing "free terms" coming from the drift part and the noise part of the equation, respectively (cf. Assumption 2.1 below).

A key point in the two estimates above is that the constants N and $\tilde{\theta}$ are independent of the ellipticity constant of the diffusion coefficients a^{ij}. Hence, the established estimates carry over without change to degenerate SPDE such as stochastic porous media equations

$$\begin{aligned} du &= \big[\Delta\big(|u|^{m-1}u\big) + f_t(x)\big]\,dt + \sum_{i=1}^d \sigma_t \partial_i u \circ d\tilde{\beta}^i_t \\ &\quad + \sum_{k=1}^\infty \big[\nu^k_t(x)u + g^k_t(x)\big],\, dw^k_t \\ u_0 &= \xi, \end{aligned} \tag{1.2}$$

with zero Dirichlet conditions on ∂Q and $m \in (1,\infty)$, where $\tilde{\beta}^1_t, \dots, \tilde{\beta}^d_t$, $w^1_t, w^2_t, \dots$ are independent $\mathbb{R}$-valued standard Wiener processes. The corresponding theorem reads as follows:

Theorem (see Theorems 3.7 and 3.8). *Let $\mu \in [2,\infty] \cap ((d+2)/2, \infty]$. There exists constants $N, \tilde{\theta} > 0$ such that if u is a solution of (1.2), then*

$$\mathbb{E}\|u\|^2_{L_\infty(Q_T)} \le N\mathbb{E}\big(1 + \|\xi\|^2_{L_\infty(Q)} + \|f\|^2_{L_\mu(Q_T)} + \||g|_{l_2}\|^2_{L_{2\mu}(Q_T)}\big),$$

and

$$\mathbb{E}\|u\|^2_{L_\infty((\rho,T)\times Q)} \le \rho^{-\tilde{\theta}} N\mathbb{E}\big(1 + \|\xi\|^2_{H^{-1}} + \|f\|^2_{L_\mu(Q_T)} + \||g|_{l_2}\|^2_{L_{2\mu}(Q_T)}\big),$$

for all $\rho \in (0,T)$.

We restrict to affine operators in the noise in (1.2) for the sole reason that no complete well-posedness theory in L_p spaces of (1.2) with non-linear noise is yet available. We emphasize that this linear structure is *not* required in the derivation of the a priori bounds established in this work. Concerning the well-posedness for nonlinear noise we also refer the reader to [14] for a well-posedness theory of such equations in a kinetic framework.

In the following we will briefly comment on existing literature on the regularity of solutions to stochastic porous media equations. The existence of strong solutions (i.e. $|u|^{m-1}u \in L^2((0,T); H^1_0))$ has been shown in [12] under the assumption that the operators in the noise are bounded and Lipschitz continuous and under the assumption that $\xi \in L_{m+1}$. In the case of linear multiplicative noise (and $\sigma = 0$) (1.2) can be transformed into a PDE with random coefficients. Based on this, the Hölder-continuity and boundedness of solutions has been shown in [2,13].

Concerning the regularity theory for deterministic singular and degenerate quasilinear equations we refer to [3,8,26] (see also the monographs [9,27] and the references therein). The regularity of solutions to non-degenerate SPDE has been addressed in [4–7,10,17]. For general background on SPDE and stochastic evolution equations we refer to [1,19,22,23].

1.1. *Notation*

Let us introduce some notation that will be used throughout this paper. Let T be a positive real number. Let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be a filtered probability space, where the filtration $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$ is right continuous and $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets. We assume that on Ω we are given a sequence of independent one-dimensional $\mathbb{F}$ -Wiener processes $(\beta_t^k)_{k=1}^\infty$. The predictable σ-field on $\Omega_T := \Omega \times [0, T]$ will be denoted by $\mathcal{P}$. Let $Q \subset \mathbb{R}^d$ be a bounded open domain. For $t \in [0, T]$ we set $Q_t = [0, t] \times Q$. The norm in $L_p(Q)$ will be denoted by $\|\cdot\|_{L_p}$. We denote by H_0^1 the completion of $C_c^\infty(Q)$ under the norm

$$\|u\|_{H_0^1}^2 := \int_Q |\nabla u|^2\,dx$$

and by H^{-1} the dual of $H_0^1(Q)$. For $q \in [1, \infty)$, we denote by $\mathbb{H}_q^{-1}$ the set of all H^{-1}-valued, $\mathbb{F}$-adapted, continuous processes u, such that $u \in L_q(\Omega_T, \mathcal{P}; L_q(Q))$. Similarly, we denote by $\mathbb{L}_2$ the set of all $L_2(Q)$-valued, $\mathbb{F}$-adapted, continuous processes u, such that $u \in L_2(\Omega_T, \mathcal{P}; H_0^1(Q))$. We will write $(\cdot,\cdot)_H$ for the inner product in a Hilbert space H. For $m \geq 1$, we will consider the Gel'fand triple

$$L_{m+1}(Q) \hookrightarrow H^{-1} \hookrightarrow \big(L_{m+1}(Q)\big)^*.$$

The duality pairing between $L_{m+1}(Q)$ and $(L_{m+1}(Q))^*$ will be denoted by ${}_{L_{m+1}^*}\langle\cdot,\cdot\rangle_{L_{m+1}}$. Notice that this duality is defined by means of the inner product in H^{-1}. Consequently, for $u, v \in C_c^\infty(Q)$

$${}_{L_{m+1}^*}\langle u, v\rangle_{L_{m+1}} = (u, v)_{H^{-1}} = \big(u, (-\Delta)^{-1} v\big)_{L_2(Q)} \neq \int_Q uv\,dx.$$

For more details we refer to [23, pp. 68–70]. We will use the summation convention with respect to integer valued repeated indices. Moreover, when no confusion arises, we suppress the (t, x)-dependence of the functions for notational convenience.

The article is organized in two sections. In Section 2 we prove our results for the non-degenerate equation. In Section 3, we verify the well-posedness of the degenerate equation, and we approximate the solution by the method of the vanishing viscosity, and by using the estimates of the previous section we pass to the limit.

2. Non-degenerate quasilinear SPDE

As already mentioned in the introduction, in order to obtain the desired estimates for equation (1.2) we first study a class of non-degenerate SPDEs. More precisely, we consider SPDEs of the form

$$\begin{aligned} du = {} & \big[\partial_i\big(a_t^{ij}(x,u)\partial_j u + F_t^i(x,u)\big) + F_t(x,u)\big]\,dt \\ & + \big[\partial_i\big(g_t^{ik}(x,u)\big) + G_t^k(x,u)\big]\,d\beta_t^k, \end{aligned} \tag{2.1}$$

$$u_0 = \xi, \tag{2.2}$$

for $(t, x) \in [0, T] \times Q$, with zero Dirichlet conditions on ∂Q.

Assumption 2.1.

(i) The functions $a^{ij}, F^i, F : \Omega_T \times Q \times \mathbb{R} \to \mathbb{R}$ are $\mathcal{P} \otimes \mathcal{B}(Q) \otimes \mathcal{B}(\mathbb{R})$-measurable.
(ii) The functions $g^i, G : \Omega_T \times Q \times \mathbb{R} \to l_2$ are $\mathcal{P} \otimes \mathcal{B}(Q) \otimes \mathcal{B}(\mathbb{R})$-measurable.

(iii) There exists constants $c > 0$, $\theta > 0$ and $\tilde{m} > 0$ such that for all $(\omega, t, x, r) \in \Omega_T \times Q \times \mathbb{R}$

$$\left(a_t^{ij}(x,r) - \frac{1}{2}\partial_r g_t^{ik}(x,r)\partial_r g_t^{jk}(x,r)\right)\xi^i\xi^j \geq \left(c|r|^{\tilde{m}} + \theta\right)|\xi|^2. \tag{2.3}$$

(iv) For all $(\omega, t) \in \Omega_T$ we have $F_t^i \in C^1(\bar{Q} \times \mathbb{R})$, $g_t^i \in C^2(\bar{Q} \times \mathbb{R}; l_2)$. Moreover, there exist predictable processes $V^1, V^2 : \Omega_T \to L_2(Q)$, such that $V^1, V^2 \in L_2((0,T); L_2(Q))$ almost surely, and a constant K such that for all $(\omega, t, x, r) \in \Omega_T \times Q \times \mathbb{R}$

$$\left|F_t(x,r)\right| + \left|F_t^i(x,r)\right| + \left|\partial_i F_t^i(x,r)\right| \leq V_t^1(x) + K|r|, \tag{2.4}$$

$$\left|G_t(x,r)\right|_{l_2} + \left|g_t^i(x,r)\right|_{l_2} + \left|\partial_i g_t^i(x,r)\right|_{l_2} + \left|\partial_i^2 g_t^i(x,r)\right|_{l_2} \leq V_t^2(x) + K|r|, \tag{2.5}$$

$$\left|\partial_r g_t^i(x,r)\right|_{l_2} + \left|\partial_r\partial_i g_t^i(x,r)\right|_{l_2} \leq K. \tag{2.6}$$

(v) Let $\mathbb{N}_g := \{k \in \mathbb{N} : \exists i \in \{1, \dots, d\}, g^{ik} \not\equiv 0\}$. We assume in addition that for all (ω, t) we have $(G_t^k)_{k\in\mathbb{N}_g} \in C^1(\bar{Q} \times \mathbb{R}; l_2(\mathbb{N}_g))$, and for all $(\omega, t, x, r) \in \Omega_T \times Q \times \mathbb{R}$

$$\sum_{k\in\mathbb{N}_g} \left|\partial_i G_t^k(x,r)\right|^2 \leq V_t^2(x) + K|r|. \tag{2.7}$$

(vi) The initial condition ξ is an $\mathcal{F}_0$-measurable $L_2(Q)$-valued random variable.

Assumption 2.2. There exists a constant N such that for all $(\omega, t, x) \in \Omega_T \times Q$ we have

$$\left|V_t^1(x)\right| + \left|V_t^2(x)\right| \leq N. \tag{2.8}$$

Remark 2.1. Assumption 2.2 is purely technical in the sense that our estimates do not depend on the bound of V^1 and V^2. As it will be seen in the next section, Assumption 2.2 can be removed provided that one has solvability of the equation and some stability properties with respect to the "free-terms".

Definition 2.1. A function $u \in \mathbb{L}_2$ will be called a solution of (2.1)–(2.2) if

(i) For all $i, j \in \{1, \dots, d\}$, almost surely

$$\int_0^T \left\|a^{ij}(u)\partial_i u\right\|_{L_2}^2 dt < \infty.$$

(ii) For all $\phi \in H_0^1$, almost surely, for all $t \in [0,T]$

$$(u_t, \phi)_{L_2} = (\xi, \phi)_{L_2} + \int_0^t \left[\left(F(u), \phi\right)_{L_2} - \left(a^{ij}(u)\partial_j u + F^i(u), \partial_i\phi\right)_{L_2}\right] dt$$
$$+ \int_0^t \left[\left(G^k(u), \phi\right)_{L_2} - \left(g^{ik}(u), \partial_i\phi\right)_{L_2}\right] d\beta_t^k.$$

We first present a collection of lemmas that will be used in the proofs of the main theorems. The following can be found in [24] (see Proposition IV.4.7 and Exercise IV.4.31/1).

Lemma 2.1. *Let X be a non-negative, adapted, right-continuous process, and let Y be a non-decreasing, adapted, continuous process such that*

$$\mathbb{E}(X_\tau|\mathcal{F}_0) \leq \mathbb{E}(Y_\tau|\mathcal{F}_0)$$

for any bounded stopping time $\tau \le T$. Then for any $\sigma \in (0,1)$

$$\mathbb{E}\sup_{t\le T} X_t^\sigma \le \sigma^{-\sigma}(1-\sigma)^{-1}\mathbb{E}Y_T^\sigma.$$

The following lemma is well known (see, e.g., [9, p. 8, Proposition 3.1]). We provide the proof in order to emphasize that the constant C can be chosen independent of λ for $\lambda \in [1,2]$ (see below).

Lemma 2.2. *There exists a constant N such that for all $\lambda \in [1,2]$, $s \in [0,T]$ and all $v \in L_\infty((s,T); L_\lambda(Q)) \cap L_2((s,T); H_0^1(Q))$, we have*

$$\int_s^T \int_Q |v|^q \,dx\,dt \le N^q \left(\int_s^T \int_Q |\nabla v|^2 \,dx\,dt\right)\left(\operatorname*{ess\,sup}_{s\le t\le T} \int_Q |v|^\lambda \,dx\right)^{2/d}, \tag{2.9}$$

where $q = q(\lambda) = 2(d+\lambda)/d$.

Proof. By the Gagliardo-Nirenberg inequality (see [20, p. 62, Theorem 2.2]) we have (notice that $d(2-\lambda)+2\lambda > 0$) for a.e. $t \in (0,T)$

$$\|v_t\|_{L_q} \le N(\lambda)\|\nabla v_t\|_{L_2}^{2/q}\|v_t\|_{L_\lambda}^{(q-2)/q},$$

where

$$N(\lambda) := \left(I_{1=d}\frac{1+\lambda}{\lambda} + I_{d=2}\max\left\{\frac{q(d-1)}{d}, \frac{\lambda+2}{2}\right\} + I_{d>2}\frac{2(d-1)}{d-2}\right)^{2/q}.$$

Since $C := \sup_{\lambda\in[1,2]} N(\lambda) < \infty$, the result follows by taking the q-th power in the inequality above and integrating over $(0,T)$. □

Next is Itô's formula for the p-th power of the L_p norm. It can be proved as [4, Lemma 2] with the help of a localization argument.

Lemma 2.3. *Let Assumption 2.1 hold and let u be a solution of (2.1). Moreover, suppose that for some $p \ge 2$ and some $s \in [0,T)$, almost surely*

$$\|u_s\|_{L_p}^p + \int_s^T \left(\|V^1\|_{L_p}^p + \|V^2\|_{L_p}^p\right) dt < \infty.$$

Then, almost surely

$$\sup_{s\le t\le T} \|u_t\|_{L_p}^p + \int_s^T \int_Q \left(|u|^{p-2} + |u|^{\tilde m+p-2}\right)|\nabla u|^2 \,dx\,dt < \infty. \tag{2.10}$$

Moreover, almost surely

$$\begin{aligned}\|u_t\|_{L_p}^p = \|u_s\|_{L_p}^p &+ \int_s^t \int_Q \left(A_{ij}(x,u)\partial_j u + F^i(x,u)\right)p(1-p)|u|^{p-2}\partial_i u \,dx\,dz \\ &+ \int_s^t \int_Q \left(\frac{1}{2}p(p-1)\left|\partial_i\left(g^i(x,u)\right) + G(x,u)\right|_{l_2}^2 |u|^{p-2} + pF(x,u)u|u|^{p-2}\right) dx\,dz \\ &+ \int_s^t \int_Q \left(p(1-p)g^{ik}(x,u)|u|^{p-2}\partial_i u + pG^k(x,u)u|u|^{p-2}\right) dx\,d\beta_z^k,\end{aligned} \tag{2.11}$$

for all $t \in [s,T]$.

From now on we fix $\mu \in \Gamma_d := [2, \infty] \cap ((d+2)/2, \infty]$, we denote by μ' its conjugate exponent, that is, $\frac{1}{\mu} + \frac{1}{\mu'} = 1$, and we set

$$\gamma := 1 + (2/d),$$
$$\bar{\gamma} := \gamma/\mu',$$
$$\mathfrak{N} := \{l \in [2, \infty) : l = \tilde{m}(1 + \bar{\gamma} + \cdots + \bar{\gamma}^n)/\mu', n \in \mathbb{N}\},$$
$$\kappa := \sup_{p \in \mathfrak{N}} \max\{2p/(p-1), I_{p \neq 2} 4p/(p-2)\} < \infty.$$

Notice that $\bar{\gamma} > 1$.

Lemma 2.4. *Let Assumptions 2.1–2.2 hold and let u be a solution of (2.1). Then, for all $p \in \mathfrak{N}$, $q \geq p$, and $\eta \in (0, 1)$ we have*

$$\mathbb{E}\left(A_q \vee \left(\sup_{t \leq T} \|u_t\|^p_{L_p(Q)} + \int_0^T \int_Q \left|\nabla |u|^{(\tilde{m}+p)/2}\right|^2 dx\, dt\right)\right)^\eta$$
$$\leq \frac{\eta^{-\eta}}{1-\eta} (N p^\kappa)^\eta \mathbb{E}\left(\left(A_q \vee \|u\|^p_{L_{\mu' p}(Q_T)}\right) + p^{-p}\left(\|V^1\|^p_{L_\mu(Q_T)} + \|V^2\|^p_{L_{2\mu}(Q_T)}\right)\right)^\eta, \tag{2.12}$$

where $A_q = (1 + \|\xi\|_{L_\infty})^q$, and N is a constant depending only on $\tilde{m}$, T, c, K, d, μ, and $|Q|$.

Proof. We assume that the right hand side in (2.12) is finite, or else there is nothing to prove. Under this assumption, for each $p \in \mathfrak{N}$ we have the formula (2.11) with $s = 0$. We proceed by estimating the terms that appear at the right hand side of (2.11). We have

$$F^i(x, u) p(1-p)|u|^{p-2} \partial_i u = \partial_i \left(\mathcal{R}_p(F^i)(x, u)\right) - \mathcal{R}_p(\partial_i F^i)(x, u),$$

where for a function f we have used the notation

$$\mathcal{R}_p(f)(x, r) = \int_0^r p(p-1) f(x, s)|s|^{p-2}\, ds.$$

Moreover, from (2.10), (2.4), the fact that V^1 is bounded and the definition of $\mathcal{R}_p(F^i)(x, u)$, it follows (see Lemma A.1) that $\mathcal{R}_p(F^i)(\cdot, u) \in W^{1,1}_0(Q)$ for a.e. $(\omega, t) \in \Omega_T$, which in particular implies that

$$\int_Q \partial_i \left(\mathcal{R}_p(F^i)(\cdot, u)\right) dx = 0.$$

Moreover, one can see from (2.4) that

$$\left|\mathcal{R}_p(\partial_i F^i)(x, r)\right| \leq p V^1(x)|r|^{p-1} + K(p-1)|r|^p.$$

By Hölder's inequality and Young's inequality, we obtain

$$\int_0^t \int_Q \left|\mathcal{R}_p(\partial_i F^i)(x, u)\right| dx\, ds$$
$$\leq p\|V^1\|_{L_\mu(Q_t)} \|u\|^{p-1}_{L_{\mu'(p-1)}(Q_t)} + K(p-1)\|u\|^p_{L_p(Q_t)}$$
$$\leq Np\|V^1\|_{L_\mu(Q_t)} \|u\|^{p-1}_{L_{\mu' p}(Q_t)} + K(p-1)\|u\|^p_{L_{\mu' p}(Q_t)}$$
$$\leq Np^{-p}\|V^1\|^p_{L_\mu(Q_t)} + N\left(Kp + p^{2p/(p-1)}\right)\|u\|^p_{L_{\mu' p}(Q_t)}.$$

Consequently, almost surely, for each $t \in [0, T]$

$$\int_0^t \int_Q F^i(x,u)p(1-p)|u|^{p-2}\partial_i u\,dx\,ds \le Np^{-p}\|V^1\|^p_{L_\mu(Q_t)} + Np^\kappa \|u\|^p_{L_{\mu' p}(Q_t)}, \tag{2.13}$$

where N depends only on K and $|Q|$. We continue with the estimate of the term

$$\frac{1}{2}p(p-1)\int_0^t \int_Q \left|\partial_i\left(g^i(x,u)\right) + G(x,u)\right|^2_{l_2}|u|^{p-2}\,dx\,ds.$$

Obviously,

$$\begin{aligned}
&\int_Q \left|\partial_i\left(g^i(x,u)\right) + G(x,u)\right|^2_{l_2}|u|^{p-2}\,dx \\
&\quad = \sum_{k\in\mathbb{N}_g^c}\int_Q \left|G^k(x,u)\right|^2|u|^{p-2}\,dx + \sum_{k\in\mathbb{N}_g}\int_Q \left|\partial_i\left(g^{ik}(x,u)\right) + G^k(x,u)\right|^2|u|^{p-2}\,dx.
\end{aligned}$$

By the growth condition (2.5), Hölder's inequality and Young's inequality we have

$$\begin{aligned}
&\frac{1}{2}p(p-1)\sum_{k\in\mathbb{N}_g^c}\int_0^t \int_Q \left|G^k(x,u)\right|^2|u|^{p-2}\,dx\,ds \\
&\quad \le Np^{-p}\|V^2\|^p_{L_{2\mu}(Q_t)} + Np^\kappa\|u\|^p_{L_{\mu' p}(Q_t)}.
\end{aligned}$$

Moreover, by Assumption 2.1(v) we have

$$\begin{aligned}
&\sum_{k\in\mathbb{N}_g}\left|\partial_i\left(g^{ik}(x,u)\right) + G^k(x,u)\right|^2 \\
&\quad = \sum_{k\in\mathbb{N}_g}\left|\partial_r g^{ik}(x,u)\partial_i u\right|^2 + \sum_{k\in\mathbb{N}_g}\left|\partial_i g^{ik}(x,u) + G^k(x,u)\right|^2 \\
&\qquad + \sum_{k\in\mathbb{N}_g} 2\partial_r g^{ik}(x,u)\partial_i u\left(\partial_i g^{ik}(x,u) + G^k(x,u)\right).
\end{aligned}$$

By the growth condition (2.5), Hölder's inequality and Young's inequality we have

$$\begin{aligned}
&\frac{1}{2}p(p-1)\sum_{k\in\mathbb{N}_g}\int_0^t \int_Q \left|\partial_i g^{ik}(x,u) + G^k(x,u)\right|^2|u|^{p-2}\,dx\,ds \\
&\quad \le Np^{-p}\|V^2\|^p_{L_{2\mu}(Q_t)} + Np^\kappa\|u\|^p_{L_{\mu' p}(Q_t)}.
\end{aligned}$$

Moreover, we have

$$\begin{aligned}
&p(p-1)\sum_{k\in\mathbb{N}_g}\partial_r g^{ik}(x,u)\partial_i u\left(\partial_i g^{ik}(x,u) + G^k(x,u)\right)|u|^{p-2} \\
&\quad = \partial_i\left(\mathcal{R}_p(\mathfrak{g})(x,u)\right) - \mathcal{R}_p(\partial_i\mathfrak{g})(x,u),
\end{aligned}$$

where

$$\mathfrak{g} := \sum_{k\in\mathbb{N}_g}\partial_r g^{ik}\left(\partial_i g^{ik} + G^k\right).$$

As before, it follows that $\mathcal{R}_p(\mathfrak{g})(\cdot, u) \in W_0^{1,1}(Q)$, which in turn implies that

$$\int_Q \partial_i \big(\mathcal{R}_p(\mathfrak{g})(x,u)\big)\, dx = 0.$$

By (2.5), (2.6), and (2.7), we have

$$\int_0^t \int_Q \big|\mathcal{R}_p(\partial_i \mathfrak{g})(x,u)\big|\, dx\, ds \le N p^{-p} \big\| V^2 \big\|^p_{L_{2\mu}(Q_t)} + N p^{\kappa} \|u\|^p_{L_{\mu' p}(Q_t)}.$$

Consequently, almost surely, for all $t \in [0,T]$ we have

$$\begin{aligned} &\frac{1}{2} p(p-1) \int_0^t \int_Q \big|\partial_i\big(g^i(x,u)\big) + G(x,u)\big|^2_{l_2} |u|^{p-2}\, dx\, ds \\ &\quad \le \frac{1}{2} p(p-1) \int_0^t \int_Q \big|\partial_r g^i(x,u) \partial_i u\big|^2_{l_2} |u|^{p-2}\, dx\, ds \\ &\qquad + N p^{-p} \big\| V^2 \big\|^p_{L_{2\mu}(Q_t)} + N p^{\kappa} \|u\|^p_{L_{\mu' p}(Q_t)}, \end{aligned} \tag{2.14}$$

where N depends only on K and $|Q|$. In a similar manner one gets

$$p \int_Q F(x,u) u |u|^{p-2}\, dx \le N p^{-p} \big\| V^1 \big\|^p_{L_{\mu}(Q_t)} + N p^{\kappa} \|u\|^p_{L_{\mu' p}(Q_t)}.$$

Using the above inequality combined with (2.13), (2.14), and (2.3) we obtain from (2.11)

$$\begin{aligned} &\|u_t\|^p_{L_p} + c p(p-1) \int_0^t \int_Q |u|^{\tilde{m}+p-2} |\nabla u|^2\, dx\, ds \\ &\quad \le \|\xi\|^p_{L_p} + N\big(p^{-p} \big\| V^1 \big\|^p_{L_{\mu}(Q_t)} + p^{-p} \big\| V^2 \big\|^p_{L_{2\mu}(Q_t)} + N p^{\kappa} \|u\|^p_{L_{\mu' p}(Q_t)}\big) + M_t, \end{aligned} \tag{2.15}$$

where M_t is the local martingale from (2.11). For any stopping time $\tau \le T$ and any $B \in \mathcal{F}_0$ we have by the Burkholder-Davis-Gundy inequality

$$\begin{aligned} \mathbb{E} \sup_{t \le \tau} I_B |M_t| \le{}& N \mathbb{E} I_B \bigg(\int_0^\tau \sum_k \bigg(p(1-p) \int_Q g^{ik}(x,u) |u|^{p-2} \partial_i u\, dx \bigg)^2 ds \bigg)^{1/2} \\ &+ N \mathbb{E} I_B \bigg(\int_0^\tau \sum_k \bigg(p \int_Q G^k(x,u) u |u|^{p-2}\, dx \bigg)^2 ds \bigg)^{1/2}. \end{aligned}$$

We have

$$p(1-p) g^{ik}(x,u) |u|^{p-2} \partial_i u = \partial_i \big(\mathcal{R}_p\big(g^{ik}\big)(x,u)\big) - \mathcal{R}_p\big(\partial_i g^{ik}\big)(x,u).$$

As before, we have $\mathcal{R}_p(g^{ik})(\cdot, u) \in W_0^{1,1}(Q)$, which implies that

$$\int_Q \partial_i \big(\mathcal{R}_p\big(g^{ik}\big)(x,u)\big)\, dx = 0.$$

Next notice that by Minkowski's integral inequality, Hölder's inequality, and Young's inequality, we have

$$\sum_k \bigg(\int_Q \mathcal{R}_p\big(\partial_i g^{ik}\big)(x,u)\, dx \bigg)^2 \le \bigg(\int_Q \big|\mathcal{R}_p\big(\partial_i g^{ik}\big)(x,u)\big|_{l_2}\, dx \bigg)^2$$

$$
\begin{aligned}
&\le \Big(\int_Q \int_{-|u|}^{|u|} p(p-1)\big|\partial_i g^i(x,s)\big|_{l_2} |s|^{p-2}\,ds\,dx\Big)^2 \\
&\le N\Big(2p\int_Q \big|V^2(x)\big||u|^{p-1} + |u|^p\,dx\Big)^2 \\
&\le N\|u\|_{L_p}^p \Big(p^2 \int_Q \big|V^2(x)\big|^2 |u|^{p-2}\,dx + p^2\|u\|_{L_p}^p\Big),
\end{aligned}
\tag{2.16}
$$

which implies

$$
\begin{aligned}
&\int_0^t \sum_k \Big(\int_Q \mathcal{R}_p\big(\partial_i g^{ik}\big)(x,u)\,dx\Big)^2 ds \\
&\quad\le N \sup_{s\le t} \|u_s\|_{L_p}^p \big(p^{-p}\big\|V^2\big\|_{L_{2\mu}(Q_t)}^p + p^\kappa \|u\|_{L_{\mu' p}(Q_t)}^p\big).
\end{aligned}
\tag{2.17}
$$

Consequently, by Young's inequality we have for any $\varepsilon > 0$

$$
\begin{aligned}
&N\mathbb{E}I_B\Big(\int_0^\tau \sum_k \Big(p(1-p)\int_Q g^{ik}(x,u)|u|^{p-2}\partial_i u\,dx\Big)^2 ds\Big)^{1/2} \\
&\quad\le \varepsilon \mathbb{E}I_B \sup_{t\le\tau} \|u_t\|_p^p + \frac{1}{\varepsilon} N\mathbb{E}I_B\big(p^{-p}\big\|V^2 I_{[0,\tau]}\big\|_{L_{2\mu}(Q_T)}^p + p^\kappa \|uI_{[0,\tau]}\|_{L_{\mu' p}(Q_T)}^p\big).
\end{aligned}
$$

In a similar manner, for any $\varepsilon > 0$ we get

$$
\begin{aligned}
&N\mathbb{E}I_B\Big(\int_0^\tau \sum_k \Big(p\int_Q G^k(x,u)u|u|^{p-2}\,dx\Big)^2 ds\Big)^{1/2} \\
&\quad\le \varepsilon \mathbb{E}I_B \sup_{t\le\tau} \|u_t\|_p^p + \frac{1}{\varepsilon} N\mathbb{E}I_B\big(p^{-p}\big\|V^2 I_{[0,\tau]}\big\|_{L_{2\mu}(Q_T)}^p + p^\kappa \|uI_{[0,\tau]}\|_{L_{\mu' p}(Q_T)}^p\big).
\end{aligned}
$$

Hence, we obtain for any $\varepsilon > 0$

$$
\begin{aligned}
\mathbb{E}\sup_{t\le\tau} I_B|M_t| &\le \varepsilon \mathbb{E}\sup_{t\le\tau} I_B\|u_t\|_p^p \\
&\quad + \frac{1}{\varepsilon} N\mathbb{E}I_B\big(p^{-p}\big\|V^2 I_{[0,\tau]}\big\|_{L_{2\mu}(Q_T)}^p + p^\kappa \|uI_{[0,\tau]}\|_{L_{\mu' p}(Q_T)}^p\big).
\end{aligned}
\tag{2.18}
$$

By (2.15) we have

$$
\begin{aligned}
\mathbb{E}I_B \sup_{t\le\tau}\|u_t\|_{L_p}^p &\le \mathbb{E}I_B\|\xi\|_{L_p}^p + \mathbb{E}I_B \sup_{t\le\tau}|M_t| \\
&\quad + N\mathbb{E}I_B\big(p^{-p}\big\|V^1 I_{[0,\tau]}\big\|_{L_\mu(Q_T)}^p + p^{-p}\big\|V^2 I_{[0,\tau]}\big\|_{L_{2\mu}(Q_T)}^p + p^\kappa \|uI_{[0,\tau]}\|_{L_{\mu' p}(Q_T)}^p\big).
\end{aligned}
\tag{2.19}
$$

By (2.15) again, after a localization argument we obtain

$$
\begin{aligned}
&\frac{4cp(p-1)}{(p+\tilde m)^2}\mathbb{E}I_B\int_0^\tau\int_Q \big|\nabla|u|^{(p+\tilde m)/2}\big|^2\,dx\,ds \\
&\quad\le N\mathbb{E}I_B\|\xi\|_{L_p}^p \\
&\qquad + N\mathbb{E}I_B\big(p^{-p}\big\|V^1 I_{[0,\tau]}\big\|_{L_\mu(Q_T)}^p + p^{-p}\big\|V^2 I_{[0,\tau]}\big\|_{L_{2\mu}(Q_T)}^p + p^\kappa \|uI_{[0,\tau]}\|_{L_{\mu' p}(Q_T)}^p\big)
\end{aligned}
\tag{2.20}
$$

and notice that for all $p \geq 2$

$$\frac{4cp(p-1)}{(p+\tilde{m})^2} \geq N(\tilde{m}, c),$$

and therefore it can be dropped from the right hand side of (2.20). Let us denote by τ_n the first exit time of $\|u_t\|^p_{L_p} + \|V^1\|^p_{L_\mu(Q_t)} + \|V^2\|^p_{L_{2\mu}(Q_t)}$ from $(-n, n)$, and by $C_n := \{\|\xi\|_{L_p} \leq n\}$. For an arbitrary $C \in \mathcal{F}_0$ and an arbitrary stopping time $\rho \leq T$, we apply (2.18) with $\tau = \tau^n \wedge \rho =: \rho_n$ and $B = C \cap C_n =: H_n$, which combined with (2.19) gives after rearrangement

$$\mathbb{E} \sup_{t \leq \rho_n} I_{H_n} \|u_t\|^p_{L_p} \leq N \mathbb{E} I_{H_n} \|\xi\|^p_{L_p}$$
$$+ N \mathbb{E} I_{H_n} \big(p^{-p} \big\| V^1 I_{[0,\rho_n]} \big\|^p_{L_\mu(Q_T)} + p^{-p} \big\| V^2 I_{[0,\rho_n]} \big\|^p_{L_{2\mu}(Q_T)} + p^\kappa \|u I_{[0,\rho_n]}\|^p_{L_{\mu' p}(Q_T)} \big).$$

By the above inequality and (2.20) (applied with $\tau = \rho_n$, $B = H_n$) one can easily see that for all $q \geq p$ we have

$$\mathbb{E} I_C X^{n,q}_\rho \leq \mathbb{E} I_C Y^{n,q}_\rho < \infty,$$

where

$$X^{n,q}_t := I_{C_n} \bigg(A_q \vee \bigg(\sup_{s \leq \tau_n \wedge t} \|u_s\|^p_{L_p} + \int_0^{t \wedge \tau_n} \int_Q \big| \nabla |u|^{(\tilde{m}+p)/2} \big|^2 \, dx \, ds \bigg) \bigg),$$
$$Y^{n,q}_t := N p^\kappa I_{C_n} \big(\big(A_q \vee \|u I_{[0,t\wedge\tau_n]}\|^p_{L_{\mu' p}(Q_T)} \big)$$
$$+ p^{-p} \big(\big\| V^1 I_{[0,t\wedge\tau_n]} \big\|^p_{L_\mu(Q_T)} + \big\| V^2 I_{[0,t\wedge\tau_n]} \big\|^p_{L_{2\mu}(Q_T)} \big) \big)$$

and $A_q = (1 + \|\xi\|_{L_\infty})^q$. By Lemma 2.1 we have

$$\mathbb{E} \big(X^{n,q}_T \big)^\eta \leq \frac{\eta^{-\eta}}{1-\eta} \mathbb{E} \big(Y^{n,q}_T \big)^\eta.$$

The assertion now follows by letting $n \to \infty$. □

Lemma 2.5. *Let Assumptions 2.1–2.2 hold and let u be a solution of (2.1). Let $\rho \in (0,1)$ and set $r_n = \rho(1 - 2^{-n})$. Then for all $p \in \mathfrak{N}$, $\eta \in (0,1)$, and $n \in \mathbb{N}$ we have*

$$\mathbb{E} \bigg(\sup_{t \in [r_{n+1}, T]} \|u\|^p_{L_p} + \int_{r_{n+1}}^T \int_Q \big| \nabla |u|^{(\tilde{m}+p)/2} \big|^2 \, dx \, dt \bigg)^\eta$$
$$\leq \bigg(N p^\kappa \frac{2^n}{\rho} \bigg)^\eta \frac{\eta^{-\eta}}{1-\eta}$$
$$\times \mathbb{E} \big(\|u I_{[r_n,T]}\|^p_{L_{\mu' p}(Q_T)} + p^{-p} \big(\big\| V^1 \big\|^p_{L_\mu(Q_T)} + \big\| V^2 \big\|^p_{L_{2\mu}(Q_T)} \big) \big)^\eta, \tag{2.21}$$

where N is a constant depending only on $\tilde{m}$, T, c, K, d, μ and $|Q|$.

Proof. We assume that the right hand side of (2.21) is finite and we set

$$c_n = \rho \bigg(1 - \frac{3}{4} 2^{-n} \bigg).$$

There exists a $t' \in (r_n, c_n)$ such that almost surely

$$\|u_{t'}\|^p_{L_p} + \int_{t'}^T \big(\big\| V^1_s \big\|^p_{L_p} + \big\| V^2_s \big\|^p_{L_p} \big) \, ds < \infty.$$

Let $\psi \in C^1([0,T])$ with $0 \le \psi \le 1$, $\psi_t = 0$ for $0 \le t \le c_n$, $\psi_t = 1$ for $r_{n+1} \le t \le T$, and $|\psi'_t| \le 2^{n+2}\rho^{-1}$. By Lemma 2.3 we obtain

$$\begin{aligned}\psi_t \|u_t\|_{L_p}^p &= p(1-p)\int_0^t \int_Q \psi\big(A_{ij}(u)\partial_j u + F^i(u)\big)|u|^{p-2}\partial_i u\, ds \\ &+ \int_0^t \psi\left[\frac{1}{2}p(p-1)\int_Q \big|\partial_i\big(g^i(u)\big) + G(u)\big|_{l_2}^2 |u|^{p-2}\, dx + p\int_Q F(u)u|u|^{p-2}\, dx\right] ds \\ &+ \int_0^t \psi\left[p(1-p)\int_Q g^{ik}(u)|u|^{p-2}\partial_i u\, dx + p\int_Q G^k(u)u|u|^{p-2}\, dx\right] d\beta_s^k \\ &+ \int_0^t \|u_s\|_{L_p}^p \psi'\, ds. \end{aligned} \tag{2.22}$$

By using the estimates obtained in the proof of Lemma 2.4, we obtain

$$\begin{aligned}&\psi_t \|u_t\|_{L_p}^p + cp(p-1)\int_0^t \int_Q \psi |u|^{\tilde{m}+p-2}|\nabla u|^2\, dx\, ds \\ &\le Np^{-p}\big(\|V^1\|_{L_\mu(Q_t)}^p + \|V^2\|_{L_{2\mu}(Q_t)}^p\big) + Np^\kappa \|\psi^{1/p}u\|_{L_{\mu' p}(Q_t)}^p \\ &+ \int_0^t \psi'\|u\|_{L_p}^p\, ds + M_t, \end{aligned} \tag{2.23}$$

where M_t is the local martingale from (2.22). From this, by using arguments almost identical to the ones of the proof of Lemma 2.4 one gets

$$\begin{aligned}&\mathbb{E}\left(\sup_{t\in[0,T]} \psi_t\|u_t\|_{L_p}^p + \int_0^T \int_Q \psi\big|\nabla |u|^{(\tilde{m}+p)/2}\big|^2\, dx\, dt\right)^\eta \\ &\le N^\eta \frac{\eta^{-\eta}}{1-\eta} \times \mathbb{E}\bigg(p^{-p}\big(\|V^1\|_{L_\mu(Q_T)}^p + \|V^2\|_{L_{2\mu}(Q_T)}^p\big) \\ &+ p^\kappa \|\psi^{1/p}u\|_{L_{\mu' p}(Q_T)}^p + \int_0^T \psi'\|u\|_{L_p}^p\, ds\bigg)^\eta. \end{aligned}$$

Having in mind that $\mu' p > p$ and that $p^\kappa + |\psi'| \le 8p^\kappa 2^n \rho^{-1}$, the result follows from the properties of ψ. □

Lemma 2.6. *Let Assumption 2.1 hold and let u be a solution of (2.1). Then for all $p \ge 2$ and all $\alpha > 0$, we have*

$$\mathbb{E}\sup_{t\le T}\|u_t\|_{L_p}^\alpha \le N\mathbb{E}\|\xi\|_{L_p}^\alpha + N\mathbb{E}\left(\int_0^T \|V^1\|_{L_p}^p + \|V^2\|_{L_p}^p\, ds\right)^{\alpha/p},$$

where N depends on α, p, K, T and d.

Proof. We assume that the right hand side is finite. Similarly to (2.15), one can show that

$$\|u_t\|_{L_p}^p \le \|\xi\|_{L_p}^p + N\int_0^t \big(\|V^1\|_{L_p}^p + \|V^2\|_{L_p}^p + \|u\|_{L_p}^p\big)\, ds + M_t,$$

where M_t is the local martingale from (2.11). Moreover, as in the derivation of (2.16), one can check that

$$\langle M\rangle_t \le N\int_0^t \big(\|u\|_{L_p}^p \|V^2\|_{L_p}^p + \|u\|_{L_p}^{2p}\big)\, ds.$$

The result then follows from Lemma 5.2 in [11]. □

We are now ready to present our first main result.

Theorem 2.7. *Let Assumptions 2.1–2.2 hold and let $u \in \mathbb{L}_2$ be a solution of (2.1)–(2.2). Then, for all $\mu \in \Gamma_d$, $\alpha > 0$, we have*

$$\mathbb{E}\|u\|^\alpha_{L_\infty(Q_T)} \le N\mathbb{E}\big(1 + \|\xi\|^\alpha_{L_\infty(Q)} + \|V^1\|^\alpha_{L_\mu(Q_T)} + \|V^2\|^\alpha_{L_{2\mu}(Q_T)}\big), \tag{2.24}$$

where N is a constant depending only on α, $\tilde{m}$, T, c, K, d, μ and $|Q|$.

Proof. We fix $\alpha > 0$, $\mu \in \Gamma_d$, and let μ' be the conjugate exponent of μ. Without loss of generality we assume that the right hand side of (2.24) is finite. Recall also the notations $\gamma = 1 + (2/d)$, $\bar{\gamma} = \gamma/\mu'(> 1)$ and let $\delta := \tilde{m}\bar{\gamma}/(\bar{\gamma} - 1)$. By Lemma 2.2, after raising (2.9) to the power γ^{-1}, we obtain by Young's inequality (with exponents $p = \gamma$, $p^* = \gamma/(\gamma - 1)$, and note that $2/d(\gamma - 1) = 1$)

$$\left(\int_s^T \int_Q |v|^q \,dx\,dt\right)^{1/\gamma} \le C^{q/\gamma}\left(\int_s^T \int_Q |\nabla v|^2 \,dx\,dt + \operatorname*{ess\,sup}_{s\le t\le T} \int_Q |v|^\lambda \,dx\right). \tag{2.25}$$

For $p \ge \tilde{m}$, we apply this inequality with $\lambda = 2p/(\tilde{m} + p) \in [1, 2]$, $q = 2(1 + (\lambda/d))$, $v = |u|^{(\tilde{m}+p)/2}$ (notice that $q(\tilde{m} + p)/2 = \tilde{m} + \gamma p = \tilde{m} + \mu'\bar{\gamma} p$), and we raise to the power $\alpha(\mu')^{n+1}/(\delta\gamma^n)$ to conclude that

$$\begin{aligned}
&\mathbb{E}\left(A_\alpha \vee \left(\int_0^T \int_Q |u|^{\tilde{m}+\mu' p\bar{\gamma}} \,dx\,dt\right)^{\alpha/(\delta\bar{\gamma}^{n+1})}\right)\\
&\quad= \mathbb{E}\left(A_\alpha \vee \left(\int_0^T \int_Q |u|^{\tilde{m}+p\gamma} \,dx\,dt\right)^{\alpha(\mu')^{n+1}/(\delta\gamma^{n+1})}\right)\\
&\quad\le N^{1/\bar{\gamma}^n}\mathbb{E}\left(A_\alpha \vee \left(\sup_{t\le T}\|u_t\|^p_{L_p} + \int_0^T \int_Q \big|\nabla|u|^{(\tilde{m}+p)/2}\big|^2 \,dx\,dt\right)^{\alpha(\mu')^{n+1}/(\delta\gamma^n)}\right)\\
&\quad\le N^{1/\bar{\gamma}^n}\mathbb{E}\left(A_{\delta\bar{\gamma}^n/\mu'} \vee \left(\sup_{t\le T}\|u_t\|^p_{L_p} + \int_0^T \int_Q \big|\nabla|u|^{(\tilde{m}+p)/2}\big|^2 \,dx\,dt\right)\right)^{\alpha\mu'/(\delta\bar{\gamma}^n)}
\end{aligned} \tag{2.26}$$

where, recall that for $q \ge 0$, $A_q := (1 + \|\xi\|_{L_\infty})^q$. Let

$$p_n := \tilde{m}\big(1 + \cdots + \bar{\gamma}^n\big) = \frac{\tilde{m}(\bar{\gamma}^{n+1} - 1)}{\bar{\gamma} - 1},$$

and let n_0 be the minimal positive integer such that

$$p_{n_0} \ge 2\mu' \quad \text{and} \quad \alpha\mu'/\big(\delta\bar{\gamma}^{n_0}\big) < 1.$$

By combining inequality (2.26) (with $p = p_n/\mu'$) with (2.12) (with $\eta = \alpha\mu'\delta^{-1}\bar{\gamma}^{-n}$, $q = \delta\bar{\gamma}^n/\mu' \ge p_n/\mu'$) we obtain for all $n \ge n_0$

$$\begin{aligned}
&\mathbb{E}\left(A_\alpha \vee \left(\int_0^T \int_Q |u|^{p_{n+1}} \,dx\,dt\right)^{\alpha/(\delta\bar{\gamma}^{n+1})}\right)\\
&\quad= \mathbb{E}\left(A_\alpha \vee \left(\int_0^T \int_Q |u|^{\tilde{m}+\bar{\gamma} p_n} \,dx\,dt\right)^{\alpha/(\delta\bar{\gamma}^{n+1})}\right)\\
&\quad\le c_n\mathbb{E}\bigg(\big(A_{\delta\bar{\gamma}^n/\mu'} \vee \|u\|^{p_n/\mu'}_{L_{p_n}(Q_T)}\big)
\end{aligned}$$

$$
\begin{aligned}
&+\left(\frac{p_n}{\mu'}\right)^{-p_n/\mu'}\left(\|V^1\|_{L_\mu(Q_T)}^{p_n/\mu'}+\|V^2\|_{L_{2\mu}(Q_T)}^{p_n/\mu'}\right)\Big)^{\alpha\mu'/(\delta\bar{\gamma}^n)}\\
&\leq c_n\mathbb{E}\left(A_\alpha\vee\left(\int_0^T\int_Q|u|^{p_n}\,dx\,ds\right)^{\alpha/(\delta\bar{\gamma}^n)}\right)\\
&+c_n\left(\frac{p_n}{\mu'}\right)^{-\alpha p_n/(\delta\bar{\gamma}^n)}\mathbb{E}\big(2+\|V^1\|_{L_\mu(Q_T)}^\alpha+\|V^2\|_{L_{2\mu}(Q_T)}^\alpha\big),
\end{aligned}
\tag{2.27}
$$

where

$$
c_n:=N^{1/\bar{\gamma}^n}\big(\delta\bar{\gamma}^n/(\mu'\alpha)\big)^{\alpha\mu'/(\delta\bar{\gamma}^n)}\frac{1}{1-(\alpha\mu'/(\delta\bar{\gamma}^n))}\left(N\frac{p_n^\kappa}{(\mu')^k}\right)^{\alpha\mu'/(\delta\bar{\gamma}^n)},
\tag{2.28}
$$

N does not depend on n, and we have used that $p_n/\bar{\gamma}^n\uparrow\delta$. Notice that the right hand side of (2.27) is finite (by the assumption that the right hand side of (2.24) is finite, Lemma 2.6 and (2.8)). One can easily see that

$$
\prod_{n=1}^\infty N^{1/\bar{\gamma}^n}<\infty,\qquad \prod_{n=1}^\infty\big(\delta\bar{\gamma}^n/(\mu'\alpha)\big)^{\alpha\mu'/(\delta\bar{\gamma}^n)}<\infty.
$$

Also, $p_n\leq\tilde{m}n\bar{\gamma}^n$ which implies that

$$
\prod_{n=1}^\infty\left(N\frac{p_n^\kappa}{(\mu')^k}\right)^{\alpha\mu'/(\delta\bar{\gamma}^n)}<\infty.
$$

Moreover, since $e^{-2x}\leq 1-x$ for all x sufficiently small, we have for some constant N that and all $M\in\mathbb{N}$

$$
\prod_{n=1}^M\frac{1}{1-(\alpha\mu'/(\delta\bar{\gamma}^n))}\leq Ne^{\sum_{n=1}^M 2\alpha\mu'/\delta\bar{\gamma}^n},
$$

which implies that $\prod_{n=1}^\infty\frac{1}{1-(\alpha\mu'/(\delta\bar{\gamma}^n))}<\infty$. Consequently, there exists an $N\in\mathbb{R}$ such that for any $M\in\mathbb{N}$

$$
\prod_{n=1}^M c_n\leq N.
\tag{2.29}
$$

Since $p_n/\bar{\gamma}^n\uparrow\delta$, there exists an N such that for all $n\in\mathbb{N}$ large enough, we have

$$
p_n^{-\alpha p_n/(\delta\bar{\gamma}^n)}\leq N\big(\bar{\gamma}^n\big)^{-\alpha p_n/(\delta\bar{\gamma}^n)}\leq N\big(\bar{\gamma}^{\alpha/2}\big)^{-n}.
$$

Since, $\alpha>0$, $\bar{\gamma}>1$ we have

$$
\sum_{n=1}^\infty\left(\frac{p_n}{\mu'}\right)^{-\alpha p_n/(\delta\bar{\gamma}^n)}<\infty.
\tag{2.30}
$$

Consequently, by iterating (2.27) and using (2.29) we obtain

$$
\Theta_m\leq\left(\prod_{n=n_0}^m c_n\right)\Theta_{n_0}+N\left(\sum_{n=n_0}^m\lambda_n\right)\mathbb{E}(1+\mathcal{V}_\mu)^\alpha,
\tag{2.31}
$$

where

$$
\Theta_n:=\mathbb{E}\left(A_\alpha\vee\left(\int_0^T\int_Q|u|^{p_n}\,dx\,dt\right)^{\alpha/(\delta\bar{\gamma}^n)}\right),\qquad \lambda_n:=\left(\frac{p_n}{\mu'}\right)^{-\alpha p_n/(\delta\bar{\gamma}^n)},
$$

and

$$\mathcal{V}_\mu := \|V^1\|_{L_\mu(Q_T)} + \|V^2\|_{L_{2\mu}(Q_T)}.$$

By virtue of (2.29) and (2.30), we can let $m \to \infty$ in (2.31) and use that $p_m/(\delta\bar{\gamma}^m) \to 1$ to obtain by Fatou's lemma

$$\begin{aligned}\mathbb{E}\|u\|^\alpha_{L_\infty(Q_T)} &\le \liminf_{n\to\infty}\mathbb{E}\left(\int_0^T\int_Q |u|^{p_n}\,dx\,dt\right)^{\alpha/(\delta\bar{\gamma}^n)} \\ &\le N\mathbb{E}\|u\|^{\alpha p_{n_0}/(\delta\bar{\gamma}^{n_0})}_{L_{p_{n_0}}(Q_T)} + N\mathbb{E}\big(1+\|\xi\|^\alpha_{L_\infty(Q)} + |\mathcal{V}_\mu|^\alpha\big).\end{aligned}$$

Since $p_n/\bar{\gamma}^n$ is increasing in n, we have $p_{n_0}/(\delta\bar{\gamma}^{n_0}) \le 1$ and thus

$$\mathbb{E}\|u\|^\alpha_{L_\infty(Q_T)} \le N\mathbb{E}\big(1+\|u\|^\alpha_{L_{p_{n_0}}(Q_T)} + \|\xi\|^\alpha_{L_\infty(Q)} + |\mathcal{V}_\mu|^\alpha\big). \tag{2.32}$$

Notice that by the assumption that the right hand side of (2.24) is finite, combined with (2.8) and Lemma 2.6, we get that the right hand side of (2.32) is finite. By the interpolation inequality

$$\|u\|_{L_{p_{n_0}}(Q_T)} \le \varepsilon\|u\|_{L_\infty(Q_T)} + N_\varepsilon\|u\|_{L_2(Q_T)},$$

we obtain after rearrangement in (2.32)

$$\mathbb{E}\|u\|^\alpha_{L_\infty(Q_T)} \le N\mathbb{E}\big(1+\|u\|^\alpha_{L_2(Q_T)} + \|\xi\|^\alpha_{L_\infty(Q)} + |\mathcal{V}_\mu|^\alpha\big),$$

which again by virtue of Lemma 2.6 gives (since $\mu \ge 2$)

$$\mathbb{E}\|u\|^\alpha_{L_\infty(Q_T)} \le N\mathbb{E}\big(1+\|\xi\|^\alpha_{L_\infty(Q)} + |\mathcal{V}_\mu|^\alpha\big).$$

This finishes the proof. □

Remark 2.2. In [4], in the non-degenerate case, mixed $L^t_\nu L^x_\mu$-norms of the free terms appear at the right hand side of the estimates. This is also achievable in our setting provided that one has a mixed-norm version of the embedding Lemma 2.2 (see also [4, Lemma 1]).

Next we present the "regularizing" effect. Recall that $\gamma = 1 + (2/d)$, $\bar{\gamma} = \gamma/\mu'$, $\delta = \tilde{m}\bar{\gamma}/(\bar{\gamma}-1)$, and $p_n = \tilde{m}(1+\bar{\gamma}+\cdots+\bar{\gamma}^n)$. We will need the following two lemmata.

Lemma 2.8. *Let $\alpha > 0$, and let $q := p_{n_0}$, where n_0 is the minimal positive integer such that $p_{n_0} \ge 2$ and $\alpha/(\delta\bar{\gamma}^{n_0}) < 1$. Suppose that Assumptions 2.1–2.2 are satisfied and let u be a solution of (2.1). Then, for all $\rho \in (0,1)$ we have*

$$\mathbb{E}\|u\|^\alpha_{L_\infty((\rho,T)\times Q)} \le \rho^{-\tilde{\theta}}N\mathbb{E}\big(1+\|u\|^\alpha_{L_q((r_{n_0},T)\times Q)} + |\mathcal{V}_\mu|^\alpha\big), \tag{2.33}$$

where

$$r_{n_0} = \rho\big(1-2^{-n_0}\big), \qquad |\mathcal{V}_\mu| = \|V^1\|_{L_\mu(Q_T)} + \|V^2\|_{L_{2\mu}(Q_T)},$$

N is a constant depending only on α, $\tilde{m}$, T, c, K, d, μ, and $|Q|$, and $\tilde{\theta} > 0$ is a constant depending only on α, d, μ and $\tilde{m}$.

Proof. Similarly to the proof of Theorem 2.7, by Lemma 2.2 and Lemma 2.5, we have for all $n \ge n_0$

$$\mathbb{E}\left(\int_{r_{n+1}}^T\int_Q |u|^{p_{n+1}}\,dx\,dt\right)^{\alpha/(\delta\bar{\gamma}^{n+1})}$$

$$\leq \big(\rho^{-1}2^n\big)^{\mu'\alpha/(\delta\bar{\gamma}^n)} c_n \mathbb{E}\bigg(\int_{r_n}^T \int_Q |u|^{p_n}\,ds\bigg)^{\alpha/(\delta\bar{\gamma}^n)}$$
$$+ \big(\rho^{-1}2^n\big)^{\alpha\mu'/(\delta\bar{\gamma}^n)} c_n \bigg(\frac{p_n}{\mu'}\bigg)^{-\alpha p_n/(\delta\bar{\gamma}^n)} \mathbb{E}\big(2 + \|V^1\|^\alpha_{L_\mu(Q_T)} + \|V^2\|^\alpha_{L_{2\mu}(Q_T)}\big), \tag{2.34}$$

where c_n is given in (2.28). Under the assumption that the right hand side of (2.33) is finite, it follows that the right hand side of the above inequality it is also finite for $n = n_0$, and by the same inequality and induction it follows that is finite for all $n \geq n_0$. Also notice that for all $M \in \mathbb{N}$

$$\prod_{n=n_0}^M \big(\rho^{-1}2^n\big)^{\alpha\mu'/(\delta\bar{\gamma}^n)} \leq N\rho^{-\tilde{\theta}},$$

with $\tilde{\theta} = (\alpha\mu'/\delta)\sum_n \bar{\gamma}^{-n}$. Consequently, by iterating (2.34) and passing to the limit as $n \to \infty$ we obtain

$$\begin{aligned}\mathbb{E}\|u\|^\alpha_{L_\infty((\rho,T)\times Q)} &\leq \rho^{-\tilde{\theta}} N \mathbb{E}\bigg(\int_{r_{n_0}}^T \int_Q |u|^{p_{n_0}}\,ds\bigg)^{\alpha/\delta\bar{\gamma}^{n_0}} \\ &\quad + \rho^{-\tilde{\theta}} N\big(1 + |\mathcal{V}_\mu|^\alpha\big) \\ &\leq \rho^{-\tilde{\theta}} N \mathbb{E}\big(1 + \|u\|^\alpha_{L_{p_{n_0}}((r_{n_0},T)\times Q)} + |\mathcal{V}_\mu|^\alpha\big),\end{aligned}$$

where we have used that $p_{n_0} \leq \delta\bar{\gamma}^{n_0}$. □

Lemma 2.9. *Suppose that Assumptions 2.1–2.2 are satisfied, let $\alpha > 0$, and let $u \in \mathbb{L}_2$ be a solution of (2.1)–(2.2). Then, for all $\rho \in (0, 1)$ we have*

$$\mathbb{E}\|u\|^\alpha_{L_\infty((\rho,T)\times Q)} \leq \rho^{-\tilde{\theta}} N \mathbb{E}\big(1 + \|u\|^\alpha_{L_2(Q_T)} + |\mathcal{V}_\mu|^\alpha\big), \tag{2.35}$$

where N is a constant depending only on α, $\tilde{m}$, T, c, K, d, μ, and $|Q|$, and $\tilde{\theta} > 0$ is a constant depending only on α, d, μ and $\tilde{m}$.

Proof. Due to Lemma 2.8, we only need to estimate $\mathbb{E}\|u\|^\alpha_{L_{p_{n_0}}((r_{n_0},T)\times Q)}$ by the right hand side of (2.35). For this, it suffices to show that for all $\beta > 0$, $p > 2$, and $\varrho \in (0, 1)$ we have

$$\mathbb{E}\|u\|^\beta_{L_p((\varrho,T)\times Q)} \leq N\varrho^{-\tilde{\theta}} \mathbb{E}\big(1 + \|u\|^\beta_{L_2(Q_T)} + |\mathcal{V}_\mu|^\beta\big), \tag{2.36}$$

where N is a constant depending only on β, p, ϱ, $\tilde{m}$, T, c, K, d, r and $|Q|$, and $\tilde{\theta} > 0$ depends only on β, d and $\tilde{m}$. We assume that the right hand side of (2.36) is finite. Let us set $p_0 = 2$, $p_{n+1} = \tilde{m} + p_n\bar{\gamma}$, $n' = \min\{n \in \mathbb{N}: p_n \geq p\}$, and $\varrho_k = k\varrho/n'$, for $k = 0, \ldots, n'$. Clearly, it suffices to prove that for all $k = 0, \ldots, n'$ we have

$$\mathbb{E}\|u\|^\beta_{L_{p_{k+1}}((\varrho_{k+1},T)\times Q)} \leq \varrho^{-\tilde{\theta}} N \mathbb{E}\big(1 + \|u\|^\beta_{L_{p_k}((\varrho_k,T)\times Q)} + |\mathcal{V}_\mu|^\beta\big), \tag{2.37}$$

since (2.36) follows by iterating (2.37) finitely many times. We assume that the right hand side of (2.37) is finite and we first prove it for $k \geq 1$. Let $\varrho'_k = (\varrho_k + \varrho_{k+1})/2$. Let $\psi \in C^1([0, T])$ with $0 \leq \psi \leq 1$, $\psi_t = 0$ for $0 \leq t \leq \varrho'_k$, $\psi_t = 1$ for $\varrho_{k+1} \leq t \leq T$, and $|\psi'_t| \leq 2n'\varrho^{-1}$. Then, similarly to (2.15) we have for $p \geq 2$

$$\begin{aligned}&\psi_t\|u_t\|^p_{L_p} + cp(p-1)\int_0^t \int_Q \psi|u|^{\tilde{m}+p-2}|\nabla u|^2\,dx\,ds \\ &\quad \leq N\big(\|V^1\|^p_{L_\mu(Q_t)} + \|V^2\|^p_{L_{2\mu}(Q_t)}\big) + N\|\psi^{1/p}u\|^p_{L_{\mu'p}(Q_t)} \\ &\qquad + \int_0^t \psi'\|u\|^p_{L_p}\,ds + M_t,\end{aligned} \tag{2.38}$$

with M_t is the martingale from (2.22). If $\beta\gamma/p_{k+1} < 1$, then by virtue of Lemma 2.1 and the familiar techniques of Lemma 2.4 we obtain

$$\begin{aligned}
&\mathbb{E}\Big(\sup_{t\in[0,T]}\psi_t\|u_t\|_{L_p}^p+\int_0^T\int_Q\psi\big|\nabla|u|^{(\tilde{m}+p)/2}\big|^2\,dx\,dt\Big)^{\frac{\beta\gamma}{p_{k+1}}}\\
&\quad\le N\mathbb{E}\Big(\big(\|V^1\|^p_{L_\mu(Q_T)}+\|V^2\|^p_{L_{2\mu}(Q_T)}\big)+\|\psi^{1/p}u\|^p_{L_{\mu'p}(Q_T)}+\int_0^T\psi'\|u\|^p_{L_p}\,ds\Big)^{\frac{\beta\gamma}{p_{k+1}}},
\end{aligned}\tag{2.39}$$

where N depends on β, p, ϱ, $\tilde{m}$, T, c, K, d, r and $|Q|$. If $\beta\gamma/p_{k+1}\ge 1$ we have by the Burkholder-Davis-Gundy inequality

$$\mathbb{E}\sup_{t\le T}|M_t|^{\beta\gamma/p_{k+1}}\le N\mathbb{E}\langle M\rangle_T^{\beta\gamma/2p_{k+1}}.\tag{2.40}$$

Again, as in the derivation of (2.17) we have

$$\langle M\rangle_T\le\sup_{t\le T}\psi_t\|u_t\|^p_{L_p}\big(\|V^2\|^p_{L_{2\mu}(Q_T)}+\|\psi^{1/p}u\|^p_{L_{\mu'p}(Q_T)}\big)$$

which combined with (2.40) gives by virtue of Young's inequality, for any $\varepsilon>0$

$$\begin{aligned}
\mathbb{E}\sup_{t\le T}|M_t|^{\beta\gamma/p_{k+1}}&\le\varepsilon\mathbb{E}\sup_{t\le T}\big(\psi_t\|u_t\|^p_{L_p}\big)^{\beta\gamma/p_{k+1}}\\
&\quad+N\mathbb{E}\big(\|V^2\|^p_{L_{2\mu}(Q_T)}+\|\psi^{1/p}u\|^p_{L_{\mu'p}(Q_T)}\big)^{\beta\gamma/p_{k+1}}.
\end{aligned}$$

Using this and (2.38) we get (2.39) (for $\beta\gamma/p_{k+1}\ge 1$), provided that the quantity $\mathbb{E}\sup_{t\le T}(\psi_t\|u_t\|^p_{L_p})^{\beta\gamma/p_{k+1}}$ is finite, which can be achieved by a localization argument. Now we use (2.39) with $p=p_k/\mu'$ (notice that $p_k/\mu'\ge 2$ for $k\ge 1$) and using the properties of ψ and the fact that $\bar{\gamma}/p_{k+1}\le 1/p_k$, (2.39) yields

$$\begin{aligned}
&\mathbb{E}\Big(\sup_{t\in[\varrho_{k+1},T]}\|u_t\|^{p_k/\mu'}_{L_{p_k/\mu'}}+\int_{\varrho_{k+1}}^T\int_Q\big|\nabla|u|^{(\tilde{m}+p_k/\mu')/2}\big|^2\,dx\,dt\Big)^{\beta\gamma/p_{k+1}}\\
&\quad\le\varrho^{-\tilde{\theta}}N\mathbb{E}\big(1+\|u\|^\beta_{L_{p_k}((\varrho_k,T)\times Q)}+|\mathcal{V}_\mu|^\beta\big).
\end{aligned}$$

An application of Lemma 2.2 (see also (2.25) and (2.26)) gives (2.37). Recall that we have assumed that $k\ge 1$. For $k=0$, instead of (2.38), we use the estimate

$$\begin{aligned}
&\psi_t\|u_t\|^p_{L_p}+cp(p-1)\int_0^t\int_Q\psi|u|^{\tilde{m}+p-2}|\nabla u|^2\,dx\,ds\\
&\quad\le N\big(\|V^1\|^p_{L_p(Q_t)}+\|V^2\|^p_{L_p(Q_t)}\big)+N\|\psi^{1/p}u\|^p_{L_p(Q_t)}\\
&\qquad+\int_0^t\psi'\|u\|^p_{L_p}\,ds+M_t.
\end{aligned}$$

We apply it with $p=2$ and we proceed as above, this time raising to the power $\gamma\beta/(\tilde{m}+2\gamma)$. Following the same steps, one arrives at the estimate

$$\begin{aligned}
&\mathbb{E}\Big(\int_{\varrho_1}^T\int_Q|u|^{\tilde{m}+2\gamma}\,dx\,dt\Big)^{\beta/(\tilde{m}+2\gamma)}\\
&\quad\le\varrho^{-\tilde{\theta}}N\mathbb{E}\big(1+\|u\|^\beta_{L_2(Q_T)}+\|V^1\|^\beta_{L_2(Q_T)}+\|V^2\|^\beta_{L_2(Q_T)}\big).
\end{aligned}$$

This finishes the proof since $\tilde{m}+2\gamma\ge\tilde{m}+2\bar{\gamma}=p_1$. □

Theorem 2.10. *Suppose that Assumptions 2.1–2.2 are satisfied. Let $u \in \mathbb{L}_2$ be a solution of (2.1)–(2.2) and let $\alpha > 0$ and $\mu \in \Gamma_d$. Then, for all $\rho \in (0,1)$ we have*

$$\mathbb{E}\|u\|^\alpha_{L_\infty((\rho,T)\times Q)} \le \rho^{-\tilde\theta} N\mathbb{E}\big(1 + \|\xi\|^\alpha_{L_2(Q)} + \|V^1\|^\alpha_{L_\mu(Q_T)} + \|V^2\|^\alpha_{L_{2\mu}(Q_T)}\big), \tag{2.41}$$

where N is a constant depending only on α, $\tilde m$, T, c, K, d, μ, and $|Q|$, and $\tilde\theta > 0$ is a constant depending only on α, d, μ and $\tilde m$.

Proof. The conclusion of the theorem follows immediately from Lemma 2.6 and Lemma 2.9. □

Remark 2.3. In Theorems 2.7 and 2.10, the expressions $\|u\|_{L_\infty(Q_T)}$ and $\|u\|_{L_\infty((\rho,T)\times Q)}$ can be replaced by $\sup_{t\in[0,T]}\|u_t\|_{L_\infty(Q)}$ and $\sup_{t\in[\rho,T]}\|u_t\|_{L_\infty(Q)}$, respectively. This follows from the fact that u is a continuous $L_2(Q)$-valued process.

Remark 2.4. A cut-off argument in space, similar to the cut-off in time as it was used in the proof of Theorem 2.10, can be used in order to derive local in space-time estimates that are applicable not only to solutions of the Dirichlet problem but to any u satisfying (2.1) (see, e.g., [4]).

3. Degenerate quasilinear SPDE

In this section, we proceed with the degenerate equation (1.2). Notice that the constant N in Theorem 2.7 and Theorem 2.9 of the previous section does not depend on the non-degeneracy constant θ. Using this fact we can deduce similar estimates for the stochastic porous medium equation (1.2).

Suppose that on $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ we are given independent $\mathbb{R}$-valued Wiener processes $\tilde\beta^1_t, \dots, \tilde\beta^d_t, w^1_t, w^2_t, \dots$. Moreover, in this section we will assume that the domain Q is convex and that the boundary ∂Q is of class C^2. The Stratonovich integral

$$\sum_{i=1}^d \sigma_t \partial_i u_t \circ d\tilde\beta^i_t$$

in (1.2) is a short notation for

$$\frac{1}{2}\sigma_t^2 \Delta u_t\, dt + \sum_{i=1}^d \sigma_t \partial_i u_t\, d\tilde\beta^i_t.$$

In the following, we consider a slightly more general class of equations. Namely, on $(0,T)\times Q$ we consider the stochastic porous medium equation (SPME) of the form

$$\begin{aligned} du &= \big[\Delta\big(\Phi(u)\big) + H_t u + f_t(x)\big]\,dt + \sum_{i=1}^d \sigma_t \partial_i u\, d\tilde\beta^i_t \\ &\quad + \sum_{k=1}^\infty \big[\nu^k_t(x)u + g^k_t(x)\big],\, dw^k_t \\ u_0 &= \xi, \end{aligned} \tag{3.1}$$

with zero Dirichlet boundary condition on ∂Q, where

$$H_t u := \frac{\sigma_t^2}{2}\Delta u + b^i_t(x)\partial_i u + c_t(x) u.$$

If we set

$$\beta_t^k := \begin{cases} \tilde\beta_t^k, & \text{for } k \in \{1,\dots,d\}, \\ w_t^{k-d}, & \text{for } k \in \{d+1, d+2, \dots\} \end{cases}$$

and

$$M_t^k(u) := \begin{cases} \sigma_t \partial_k u, & \text{for } k \in \{1,\dots,d\}, \\ \nu_t^{k-d}(x)u + g_t^{k-d}(x), & \text{for } k \in \{d+1, d+2, \dots\}, \end{cases}$$

we can rewrite (3.1) in the more compact form

$$\begin{aligned} du &= \big[\Delta\big(\Phi(u)\big) + H_t u + f_t(x)\big]\,dt + \sum_{k=1}^{\infty} M_t^k(u)\,d\beta_t^k \\ u_0 &= \xi. \end{aligned} \tag{3.2}$$

Assumption 3.1.

(1) The function $\Phi : \mathbb{R} \to \mathbb{R}$ is continuously differentiable, non-decreasing, such that $\Phi(0) = 0$. There exists constants $\lambda > 0$, $C \geq 0$, $m \in [1, \infty)$ such that for all $r \in \mathbb{R}$ we have

$$r\Phi(r) \geq \lambda |r|^{m+1} - C, \qquad \big|\Phi'(r)\big| \leq C|r|^{m-1} + C.$$

(2) For each $i = 1, \dots, d$, the functions $b^i, c : \Omega_T \times \overline{Q} \to \mathbb{R}$ are $\mathcal{P} \otimes \mathcal{B}(\overline{Q})$-measurable, and for all $(\omega, t) \in \Omega_T$ we have $b^i \in C^2(\overline{Q}; \mathbb{R})$, $c \in C^1(\overline{Q}; \mathbb{R})$, and $b^i = 0$ on ∂Q. The functions $\sigma : \Omega_T \to \mathbb{R}$ and $\nu : \Omega_T \times \overline{Q} \to l_2$ are $\mathcal{P}$- and $\mathcal{P} \times \mathcal{B}(\overline{Q})$-measurable, respectively, and for all $(\omega, t) \in \Omega_T$ we have $\nu \in C^1(\overline{Q}; l_2)$. Moreover, there exists a constant K such that for all $(\omega, t, x) \in \Omega_T \times Q$ we have

$$\begin{aligned} &|\sigma_t| + \big|b_t^i(x)\big| + \big|c_t(x)\big| + \big|\nu_t(x)\big|_{l_2} \\ &\quad + \big|\nabla b_t^i(x)\big| + \big|\nabla c_t(x)\big| + \big|\nabla \nu_t(x)\big|_{l_2} + \big|\nabla^2 b_t^i(x)\big| \leq K \end{aligned}$$

(3) The functions $f : \Omega_T \to H^{-1}$ and $g^k : \Omega_T \to H^{-1}$, for $k \in \mathbb{N}$, are $\mathcal{P}$-measurable and it holds that

$$\mathbb{E}\int_0^T \left(\|f\|_{H^{-1}}^2 + \sum_{k=1}^{\infty} \|g^k\|_{H^{-1}}^2 \right) dt < \infty.$$

(4) The initial condition ξ is an $\mathcal{F}_0$-measurable H^{-1}-valued random variable such that $\mathbb{E}\|\xi\|_{H^{-1}}^2 < \infty$.

Assumption 3.2. There exists a constant $\bar c > 0$ such that $\bar c|r|^{m-1} \leq \Phi'(r)$ for all $r \in \mathbb{R}$.

We note that Assumption 3.2 implies the first part of Assumption 3.1(1), that is $r\Phi(r) \geq \bar c|r|^{m+1}$.

Let us set $A_t(u) := \Delta(\Phi(u)) + H_t u_t + f_t$. The operators are understood in the following sense: For $u \in L_{m+1}(Q)$, we have $A_t(u) \in (L_{m+1})^*$, $M_t^k(u) \in H^{-1}$, given by

$$\begin{aligned} {}_{(L_{m+1})^*}\big\langle A_t(u), \phi\big\rangle_{L_{m+1}} := &-\int_Q \Phi(u)\phi\,dx - \int_Q \frac{1}{2}\sigma_t^2 u\phi\,dx \\ &- \int_Q u\partial_i\big(b_t^i(-\Delta)^{-1}\phi\big) + \int_Q (c_t u + f_t)(-\Delta)^{-1}\phi\,dx, \end{aligned}$$

$$\big(M_t^k(u)\big)(\psi) := \begin{cases} -\int_Q \sigma_t u \partial_k \psi\,dx, & \text{for } k \in \{1,\dots,d\}, \\ \int_Q (\nu_t^{k-d} u + g_t^{k-d})\psi\,dx, & \text{for } k \in \{d+1, d+2, \dots\}, \end{cases}$$

for $\phi \in L_{m+1}(Q)$, $\psi \in H_0^1(Q)$, where $(-\Delta)^{-1}$ denotes the inverse of the Dirichlet Laplacian on Q. Notice that for $\phi \in H^{-1}$ (in particular for $\phi \in L_{m+1}(Q)$), it holds that (see, e.g., p. 69 in [23])

$$\big(M_t^k(u), \phi\big)_{H^{-1}} = \big(M_t^k(u)\big)\big((-\Delta)^{-1}\phi\big).$$

Definition 3.1. A solution of equation (3.1) is a process $u \in \mathbb{H}_{m+1}^{-1}$, such that for all $\phi \in L_{m+1}(Q)$, with probability one we have

$$(u_t, \phi)_{H^{-1}} = (\xi, \phi)_{H^{-1}} + \int_0^t {}_{(L_{m+1})^*}\big\langle A(u), \phi\big\rangle_{L_{m+1}}\, ds + \int_0^t \big(M^k(u), \phi\big)_{H^{-1}}\, d\beta_s^k,$$

for all $t \in [0, T]$.

3.1. *Well-posedness*

In this subsection we show that the problem (3.1) has a unique solution. This will be a consequence of [19, Theorems 3.6 and 3.8], once the respective assumptions are shown to be fulfilled. This is the purpose of the following lemmata.

Remark 3.1. In Definition 3.1, the set of full probability on which the equality is satisfied can be chosen independently of $\phi \in L_{m+1}$. This follows by the fact that the expression

$$\int_0^{\cdot} M^k(u)\, d\beta_s^k$$

is a continuous H^{-1}-valued martingale, combined with the separability of L_{m+1}.

Lemma 3.1. *Under Assumption 3.1 there is a constant N depending only on K such that for all $(\omega, t) \in \Omega_T$, $u \in L_{m+1}(Q)$ we have*

$$\left|\int_Q u\partial_i\big(b_t^i(-\Delta)^{-1}u\big)\, dx\right| \le N\|u\|_{H^{-1}}^2, \tag{3.3}$$

$$\left|\int_Q uc_t(-\Delta)^{-1}u\, dx\right| \le N\|u\|_{H^{-1}}^2. \tag{3.4}$$

Proof. By continuity it suffices to show the conclusion for $u \in C_c^\infty(Q)$. We have

$$\int_Q u\partial_i\big(b_t^i(-\Delta)^{-1}u\big)\, dx = \int_Q \big(\partial_i b_t^i\big)u(-\Delta)^{-1}u\, dx + \int_Q b_t^i u\partial_i(-\Delta)^{-1}u\, dx. \tag{3.5}$$

Recall that $\partial Q \in C^2$ which implies $(-\Delta)^{-1}u \in H^2(Q)$. Hence, writing $u = (-\Delta)(-\Delta)^{-1}u$, integration by parts gives ($(-\Delta)^{-1}u$ vanishes on ∂Q)

$$\begin{aligned}\left|\int_Q \big(\partial_i b_t^i\big)u(-\Delta)^{-1}u\, dx\right| &\le \left|\int_Q \big(\partial_{ij} b_t^i\big)\big(\partial_j(-\Delta)^{-1}u\big)(-\Delta)^{-1}u\, dx\right| \\ &\quad + \left|\int_Q \big(\partial_i b_t^i\big)\big(\partial_j(-\Delta)^{-1}u\big)^2\, dx\right| \\ &\le N\|u\|_{H^{-1}}^2,\end{aligned} \tag{3.6}$$

where we have used Young's and Poincaré's inequalities. For the other term, since $\partial_j(-\Delta)^{-1}u \in H^1(Q)$ (recall that $\partial Q \in C^2$) and b^i vanishes on ∂Q, we have

$$\begin{aligned}\int_Q b_t^i u\partial_i(-\Delta)^{-1}u\, dx &= \int_Q b_t^i\big(\partial_j(-\Delta)^{-1}u\big)\partial_j\partial_i(-\Delta)^{-1}u\, dx \\ &\quad + \int_Q \big(\partial_j b_t^i\big)\big(\partial_j(-\Delta)^{-1}u\big)\partial_i(-\Delta)^{-1}u\, dx.\end{aligned}$$

For the second term in the above equality we have by Hölder's inequality

$$\left| \int_Q (\partial_j b_t^i)(\partial_j (-\Delta)^{-1} u) \partial_i (-\Delta)^{-1} u \, dx \right| \le N \|u\|_{H^{-1}}^2,$$

while for the first term we have

$$\begin{aligned} \left| \int_Q b_t^i (\partial_j (-\Delta)^{-1} u) \partial_i \partial_j (-\Delta)^{-1} u \, dx \right| &= \frac{1}{2} \left| \int_Q b_t^i \partial_i (\partial_j (-\Delta)^{-1} u)^2 \, dx \right| \\ &= \frac{1}{2} \left| \int (\partial_i b_t^i)(\partial_j (-\Delta)^{-1} u)^2 \, dx \right| \\ &\le N \|u\|_{H^{-1}}^2, \end{aligned}$$

where we have used again that $b^i = 0$ on ∂Q. Hence,

$$\left| \int_Q b_t^i u \partial_i (-\Delta)^{-1} u \, dx \right| \le N \|u\|_{H^{-1}}^2. \tag{3.7}$$

Combining (3.6) with (3.7) we obtain (3.3). Inequality (3.4) follows similarly from the fact that $|c_t(x)| + |\nabla c_t(x)| \le K$. □

Lemma 3.2. *Under Assumption 3.1, there exists a constant N depending only on K and d such that for all $(\omega, t) \in \Omega_T$ and all $\varphi, \psi \in L_{m+1}(Q)$ we have*

$$\begin{aligned} &2_{(L_{m+1})^*} \langle H_t \phi + f_t, \phi \rangle_{L_{m+1}} + \sum_{k=1}^{\infty} \| M_t^k(\phi) \|_{H^{-1}}^2 \\ &\quad \le N \left(\|\phi\|_{H^{-1}}^2 + \|f_t\|_{H^{-1}}^2 + \sum_{k=1}^{\infty} \|g_t^k\|_{H^{-1}}^2 \right), \end{aligned} \tag{3.8}$$

and

$$2_{(L_{m+1})^*} \langle A_t(\phi) - A_t(\psi), \phi - \psi \rangle_{L_{m+1}} + \sum_{k=1}^{\infty} \| M_t^k(\phi) - M_t^k(\psi) \|_{H^{-1}}^2 \le N \|\phi - \psi\|_{H^{-1}}^2. \tag{3.9}$$

Proof. We start by proving (3.8). By virtue of the previous lemma, it suffices to show that

$$\begin{aligned} &-\sigma_t^2 \|\phi\|_{L_2}^2 + (f_t, \phi)_{H^{-1}} + \sum_{k=1}^{\infty} \| M_t^k(\phi) \|_{H^{-1}}^2 \\ &\quad \le N \left(\|\phi\|_{H^{-1}}^2 + \|f_t\|_{H^{-1}}^2 + \sum_{k=1}^{\infty} \|g_t^k\|_{H^{-1}}^2 \right). \end{aligned}$$

Clearly it suffices to show the last inequality for $\phi \in C_c^\infty(Q)$. To this end, we have

$$\begin{aligned} &-\sigma_t^2 \|\phi\|_{L_2}^2 + (f_t, \phi)_{H^{-1}} + \sum_{k=1}^{\infty} \| M_t^k(\phi) \|_{H^{-1}}^2 \\ &\quad = -\sigma_t^2 \|\phi\|_{L_2}^2 + (f_t, \phi)_{H^{-1}} + \sigma_t^2 \sum_{i=1}^{d} \|\partial_i \phi\|_{H^{-1}}^2 + \sum_{k=1}^{\infty} \| \nu_t^k \phi + g_t^k \|_{H^{-1}}^2 \\ &\quad \le \sigma_t^2 \left(\sum_{i=1}^{d} \|\partial_i \phi\|_{H^{-1}}^2 - \|\phi\|_{L_2}^2 \right) + N \left(\|\phi\|_{H^{-1}}^2 + \|f_t\|_{H^{-1}}^2 + \sum_{k=1}^{\infty} \|g_t^k\|_{H^{-1}}^2 \right). \end{aligned}$$

Hence, we only have to show that

$$\sum_{i=1}^{d}\|\partial_i\phi\|_{H^{-1}}^2 \le \|\phi\|_{L_2}^2.$$

Let $\zeta \in C_c^\infty(Q)$. The action of $\partial_i\phi$ on ζ is given by $-(\phi, \partial_i\zeta)_{L_2}$. We have

$$\begin{aligned}\left|(\phi, \partial_i\zeta)_{L_2}\right| &= \left|\left((-\Delta)(-\Delta)^{-1}\phi, \partial_i\zeta\right)_{L_2}\right| = \left|\sum_{l=1}^{d}\left(\partial_i\partial_l(-\Delta)^{-1}\phi, \partial_l\zeta\right)_{L_2}\right| \\ &\le \|\zeta\|_{H_0^1}^2 \sum_{l=1}^{d}\left\|\partial_i\partial_l(-\Delta)^{-1}\phi\right\|_{L_2}^2.\end{aligned}$$

Consequently,

$$\|\partial_i\phi\|_{H^{-1}}^2 \le \sum_{l=1}^{d}\left\|\partial_i\partial_l(-\Delta)^{-1}\phi\right\|_{L_2}^2.$$

It now suffices to show that

$$\sum_{l,i=1}^{d}\left\|\partial_i\partial_l(-\Delta)^{-1}\phi\right\|_{L_2}^2 \le \|\phi\|_{L_2}^2.$$

This follows from the convexity of Q. Namely, if Q is a convex, open, bounded subset of $\mathbb{R}^d$ with boundary of class C^2, then it holds that

$$\sum_{l,i=1}^{d}\|\partial_i\partial_l v\|_{L_2(Q)}^2 \le \|\Delta v\|_{L_2(Q)}^2$$

for all $v \in H^2(Q) \cap H_0^1(Q)$ (see [16, p. 139, Theorem 3.1.2.1, inequality (3,1,2,2)]). Applying this to $v := (-\Delta)^{-1}\phi$ finishes the proof of (3.8).

For (3.9), by considering (3.8) (with $f = 0$, $g = 0$), it is clear that we only have to show that

$${}_{(L_{m+1})^*}\left\langle\Delta\big(\Phi(\phi)\big) - \Delta\big(\Phi(\psi)\big), \phi - \psi\right\rangle_{L_{m+1}} \le 0.$$

This follows from the well-known fact (see, e.g., p. 71 in [23]) that

$${}_{(L_{m+1})^*}\left\langle\Delta\big(\Phi(\phi)\big) - \Delta\big(\Phi(\psi)\big), \phi - \psi\right\rangle_{L_{m+1}} = -\big(\Phi(\phi) - \Phi(\psi), \phi - \psi\big)_{L_2} \le 0,$$

since Φ is non-decreasing. This completes the proof. □

Theorem 3.3. *Under Assumption* 3.1 *there exists a unique solution of equation* (3.1).

Proof. It is straightforward to check that the operator A satisfies (A_1) (hemi-continuity) from [19]. The fact that A and M satisfy (A_2) (monotonicity) was proved in (3.9). Coercivity or (A_3), follows from (3.8) combined with (1) of Assumption 3.1, which implies that

$${}_{(L_{m+1})^*}\left\langle\Delta\big(\Phi(\phi)\big), \phi\right\rangle_{L_{m+1}} = -\big(\Phi(\phi), \phi\big)_{L_2} \le -\lambda\|\phi\|_{L_{m+1}}^{m+1} + C.$$

For the growth condition (A_4) we have, for $v \in L_{m+1}$,

$$\begin{aligned}\big\|\Delta\big(\Phi(v)\big) + H_t v + f_t\big\|_{(L_{m+1})^*} &\leq \big\|\Phi(v)\big\|_{L_{(m+1)/m}} + \|H_t v + f_t\|_{(L_{m+1})^*} \\ &\leq N + N\|v\|^m_{L_{m+1}} + \|H_t v + f_t\|_{(L_{m+1})^*},\end{aligned}$$

where we have used Assumption 3.1(1). Then, notice that for $\phi \in C_c^\infty(Q)$

$$\begin{aligned}&{}_{(L_{m+1})^*}\langle H_t v + f_t, \phi\rangle_{L_{m+1}} \\ &\quad = -\int_Q \frac{1}{2}\sigma_t^2 u\phi\, dx - \int_Q u\partial_i\big(b_t^i(-\Delta)^{-1}\phi\big)\, dx + \int_Q (c_t u + f_t)(-\Delta)^{-1}\phi\, dx \\ &\quad \leq N\|u\|_{L_{(m+1)/m}}\|\phi\|_{L_{m+1}} + N\|u\|_{L_2}\|\phi\|_{H^{-1}} + N\|f\|_{H^{-1}}\|\phi\|_{H^{-1}} \\ &\quad \leq N\big(1 + \|f_t\|^{2m/(m+1)}_{H^{-1}} + \|v\|^m_{L_{m+1}}\big)\|\phi\|_{L_{m+1}}.\end{aligned}$$

Hence,

$$\big\|\Delta\big(\Phi(v)\big) + H_t v + f_t\big\|_{(L_{m+1})^*} \leq N\big(1 + \|f_t\|^{2m/m+1}_{H^{-1}} + \|v\|^m_{L_{m+1}}\big),$$

where N depends only on m, K, C, d and $|Q|$. This finishes the verification of the assumptions of [19, Theorems 3.6 and 3.8], an application of which concludes the proof. □

3.2. *Regularity*

In this section we add a viscosity term of magnitude ε to equation (3.1) and show that the corresponding equation and its solution u^ε satisfy the assumptions of Theorem 2.7, which yields supremum estimates for u^ε uniformly in ε. Then, we show that u^ε converges to u and pass to the limit to obtain the desired estimates for the u.

First, we consider an approximating equation where the non-linear term is Lipschitz continuous, that is, for $\varepsilon > 0$, on Q_T

$$\begin{aligned}du_t &= \big[\Delta\big(\bar\Phi(u_t)\big) + \varepsilon\Delta u_t + Hu_t + f_t\big]\, dt + M_t^k(u_t)\, d\beta_t^k, \\ u_0 &= \xi,\end{aligned} \tag{3.10}$$

with zero Dirichlet boundary conditions on ∂Q. Let us set

$$\mathcal{K}_p := \mathbb{E}\|\xi\|^p_{L_p} + \mathbb{E}\int_0^T \big(\|f\|^p_{L_p} + \big\||g|_{l_2}\big\|^p_{L_p}\big)\, dt.$$

As in [15, Lemma B.1] we have the following.

Lemma 3.4. *Assume that Assumption 3.1(2), (3), (4) is satisfied and that $\bar\Phi : \mathbb{R} \to \mathbb{R}$ is a Lipschitz continuous, non-decreasing function with $\bar\Phi(0) = 0$. Then, equation (3.10) has a unique solution u in $\mathbb{H}_2^{-1}$. Moreover, if $K_2 < \infty$, then $u \in \mathbb{L}_2$ and for any $p \geq 2$ the following estimate holds*

$$\mathbb{E}\sup_{t\leq T}\|u_t\|^p_{L_p} + \mathbb{E}\int_0^T \big\||u|^{(p-2)/2}|\nabla u|\big\|^2_{L_2}\, dt \leq N\mathcal{K}_p, \tag{3.11}$$

where N is a constant depending only on ε, K, d, T and p.

Proof. The existence and uniqueness of solutions in $\mathbb{H}_2^{-1}$ follows from Theorem 3.3. Therefore, we only have to show that $u \in \mathbb{L}_2$ and (3.11) under the assumption that $\mathcal{K}_2 < \infty$. Let $(e_i)_{i=1}^\infty$ be an orthonormal basis of H^{-1} consisting of eigenvectors of $-\Delta$ and let $\Pi_n : H^{-1} \to \mathrm{span}\{e_1, \dots, e_n\}$ be the orthogonal projection onto the span of the first n eigenvectors. Consider the Galerkin approximation

$$\begin{aligned} du_t^n &= \Pi_n\big[\Delta\big(\bar\Phi\big(u_t^n\big)\big) + \varepsilon\Delta u_t^n + Hu_t^n + f_t\big]\,dt + \Pi_n M_t^k\big(u_t^n\big)\,d\beta_t^k, \\ u_0^n &= \Pi_n\xi. \end{aligned} \tag{3.12}$$

Under the assumptions of the lemma, it is very well known from the theory of stochastic evolution equations (see [19]) that the Galerkin scheme above has a unique solution u^n which converges weakly in $L_2(\Omega_T, \mathcal{P}; L_2(Q))$ to u (in fact, this is how the solution u is constructed). Notice that the restriction of Π_n to L_2 is again the orthogonal projection (in L_2) onto $\mathrm{span}\{e_1, \dots, e_n\}$. Consequently, for any $\phi, \psi \in C_c^\infty$ we have $-(\phi, \Pi_n\partial_i\psi)_{L_2} = (\partial_i\Pi_n\phi, \psi)_{L_2}$ which remains true for $\phi, \psi \in L_2$. Hence, by Itô's formula, we have

$$\begin{aligned} \big\|u_t^n\big\|_{L_2}^2 \le \|\xi\|_{L_2}^2 &- \int_0^t 2\left(\partial_j u^n, \partial_j\big(\bar\Phi\big(u^n\big)\big) + \varepsilon\partial_j u^n + \frac{\sigma^2}{2}\partial_j u^n\right)_{L_2} ds \\ &+ \int_0^t 2\big(b^i\partial_i u^n + cu^n + f, u^n\big)_{L_2}\,ds \\ &+ \int_0^t \left(\sum_{i=1}^d \sigma^2\big\|\partial_i u^n\big\|_{L_2}^2 + \sum_{k=1}^\infty \big\|\nu^k u^n + g^k\big\|_{L_2}^2\right) ds \\ &+ \int_0^t 2\big(M^k u^n + g^k, u^n\big)\,d\beta_s^k. \end{aligned}$$

Since $\bar\Phi$ is a non-decreasing Lipschitz continuous function, we have

$$\big(\partial_j u^n, \partial_j\big(\bar\Phi\big(u^n\big)\big)\big)_{L_2} \ge 0.$$

It then follows by standard arguments (see, e.g., the proof of Theorem 4 in [25]) that

$$\mathbb{E}\int_0^T \big\|u^n\big\|_{H_0^1}^2\,dt \le N\mathcal{K}_2,$$

where N depends only on ε, K, d, and T. Since the Galerkin approximation u^n converges weakly in $L_2(\Omega_T, \mathcal{P}; L_2(Q))$ to u, taking $\liminf$ as $n \to \infty$ in the above inequality gives

$$\mathbb{E}\int_0^T \|u\|_{H_0^1}^2\,dt \le N\mathcal{K}_2.$$

Moreover, since $u \in L_2(\Omega_T, \mathcal{P}; H_0^1(Q))$ and satisfies (3.10), it follows (see [19, Theorem 2.16]) that is has a continuous L_2-valued version which implies that $u \in \mathbb{L}_2$. From here, one can deduce (3.11) by following step by step the proof of Lemma 2 from [4], keeping in mind that $\bar\Phi'(u) \ge 0$. □

We use the previous result to obtain the required regularity for the solution of the SPME in the presence of a non-degenerate viscosity term,

$$\begin{aligned} du_t &= \big[\Delta\big(\Phi(u)\big) + \varepsilon\Delta u_t + H_t u_t + f_t\big]\,dt + M_t^k(u_t)\,d\beta_t^k, \\ u_0 &= \xi. \end{aligned} \tag{3.13}$$

Lemma 3.5. *Suppose that Assumption 3.1 holds. Then, there exists a unique $\mathbb{H}_{m+1}^{-1}$-solution of equation (3.13). If $\xi \in L_{m+1}(\Omega; L_{m+1}(Q))$, $f \in L_{m+1}(\Omega_T; L_{m+1}(Q))$, and $g \in L_{m+1}(\Omega_T; L_{m+1}(Q; l_2))$, then we have that $u, \Phi(u) \in L_2(\Omega_T; H_0^1)$ and $\nabla\Phi(u) = \Phi'(u)\nabla u$. In particular, $u \in \mathbb{L}_2$.*

Proof. The fact that (3.13) has a unique $\mathbb{H}^{-1}_{m+1}$-solution follows from Theorem 3.3. For the remaining properties, let us consider the approximation

$$\begin{aligned} du^n_t &= \big[\Delta\big(\Phi_n\big(u^n_t\big)\big) + \varepsilon \Delta u^n_t + H_t u^n_t + f_t\big]\,dt + M^k_t\big(u^n_t\big)\,d\beta^k_t, \\ u_0 &= \xi, \end{aligned} \tag{3.14}$$

where for $n \in \mathbb{N}$, $\Phi_n : \mathbb{R} \to \mathbb{R}$ is defined by

$$\Phi_n(r) = \int_0^r \min\big\{\Phi'(s), n\big\}\,ds.$$

By Lemma 3.4, equation (3.14) has a unique solution u^n in $\mathbb{H}^{-1}_2$ which moreover belongs to $\mathbb{L}_2$, and for all $q \in [2, m+1]$ we have

$$\begin{aligned} &\mathbb{E}\sup_{t\le T}\big\|u^n\big\|^q_{L_q} + \mathbb{E}\int_0^T \int_Q \big|u^n\big|^{q-2}\big|\nabla u^n\big|^2\,dx\,dt \\ &\quad \le N\bigg(\mathbb{E}\|\xi\|^q_{L_q} + \mathbb{E}\int_0^T \|f\|^q_{L_q} + \big\||g|_{l_2}\big\|^q_{L_q}\,dt\bigg) < \infty, \end{aligned} \tag{3.15}$$

where N depends only on K, T, d, q and ε.

Let $\Psi_n(r) = \int_0^r \Phi_n(s)\,ds$. By Theorem 3.1 in [18] we have

$$\begin{aligned} \int_Q \Psi_n\big(u^n_t\big)\,dx = \int_Q \Psi_n(\xi)\,dx &- \int_0^t \int_Q \big|\nabla\Phi_n\big(u^n\big)\big|^2 + \varepsilon\Phi'\big(u^n\big)\big|\nabla u^n\big|^2\,dx\,ds \\ &+ \int_0^t \big(b^i\partial_i u^n + cu^n + f, \Phi_n\big(u^n\big)\big)_{L_2}\,ds \\ &+ \int_0^t \frac{1}{2}\big(\Phi'_n\big(u^n\big), \big|\nu u^n + g\big|^2_{l_2}\big)_{L_2}\,ds \\ &+ \int_0^t \big(M^k\big(u^n\big), \Phi_n\big(u^n\big)\big)_{L_2}\,d\beta^k_s. \end{aligned} \tag{3.16}$$

Notice that by Assumption 3.1 we have

$$\begin{aligned} &-\Phi'_n\big(u^n\big)\big|\nabla u^n\big|^2 \le 0, \\ &\big(cu^n + f, \Phi_n\big(u^n\big)\big)_{L_2} \le N\big(\big\|u^n\big\|^{m+1}_{L_{m+1}} + \|f\|^{m+1}_{L_{m+1}}\big), \\ &\big(b^i\partial_i u^n, \Phi_n\big(u^n\big)\big)_{L_2} = -\big(\big(\partial_i b^i\big)u^n, \Phi_n\big(u^n\big)\big)_{L_2} - \big(b^i u^n, \partial_i\Phi_n\big(u^n\big)\big)_{L_2} \\ &\qquad\qquad \le N\big(1 + \big\|u^n\big\|^{m+1}_{L_{m+1}}\big) + \frac{1}{2}\big\|\nabla\Phi_n\big(u^n\big)\big\|^2_{L_2} \end{aligned}$$

and

$$\big(\Phi'_n\big(u^n\big), \big|\nu u^n + g\big|^2_{l_2}\big)_{L_2} \le N\big(1 + \big\|u^n\big\|^{m+1}_{L_{m+1}} + \big\||g|_{l_2}\big\|^{m+1}_{L_{m+1}}\big),$$

for a constant N depending only on C, K, d, and $|Q|$. Hence, after a localization argument we obtain for all $t \in [0, T]$

$$\begin{aligned} &\mathbb{E}\int_Q \Psi_n\big(u^n_t\big)\,dx + \mathbb{E}\int_0^t \big\|\nabla\Phi_n\big(u^n\big)\big\|^2_{L_2}\,dt \\ &\quad \le N\mathbb{E}\bigg(1 + \|\xi\|^{m+1}_{L_{m+1}} + \int_0^t \|f\|^{m+1}_{L_{m+1}} + \big\||g|_{l_2}\big\|^{m+1}_{L_{m+1}} + \big\|u^n\big\|^{m+1}_{L_{m+1}}\,dt\bigg), \end{aligned} \tag{3.17}$$

for a constant N depending only on C, K, d, and $|Q|$, which in particular gives by (3.15)

$$\mathbb{E}\int_0^T \|\nabla\Phi_n(u^n)\|_{L_2}^2\,dt \le N\mathbb{E}\bigg(1+\|\xi\|_{L_{m+1}}^{m+1}+\int_0^T \|f\|_{L_{m+1}}^{m+1}+\||g|_{l_2}\|_{L_{m+1}}^{m+1}\,dt\bigg)<\infty, \tag{3.18}$$

where N depends only on K, T, d, m, C, $|Q|$ and ε. By (3.15) and (3.18) we have for a (non-relabeled) subsequence

$$\begin{aligned} &u^n \rightharpoonup v \quad \text{in } L_2(\Omega_T; H_0^1(Q)),\\ &u^n \rightharpoonup v \quad \text{in } L_{m+1}(\Omega_T; L_{m+1}(Q)),\\ &\Phi_n(u_n) \rightharpoonup \eta \quad \text{in } L_2(\Omega_T; H_0^1(Q)),\\ &u_T^n \rightharpoonup u^\infty \quad \text{in } L_{m+1}(\Omega; L_{m+1}(Q)), \end{aligned} \tag{3.19}$$

for some v, η and u^∞. Recall that we want to show that $u, \Phi(u) \in L_2(\Omega_T; H_0^1)$. For this, we will show that $u = v$ and $\Phi(v) = \eta$ by using standard techniques from the theory of monotone operators (see, e.g., [19]). Notice that by (3.19) we also have

$$\begin{aligned} &u^n \rightharpoonup v \quad \text{in } L_2(\Omega_T; H^{-1}),\\ &\Delta\Phi_n(u_n) \rightharpoonup \Delta\eta \quad \text{in } L_2(\Omega_T; H^{-1}),\\ &u_T^n \rightharpoonup u^\infty \quad \text{in } L_2(\Omega; H^{-1}). \end{aligned} \tag{3.20}$$

As in Section 3.5 in [19], by passing to the weak limit in (3.14) we have in H^{-1} for almost all (ω, t)

$$v_t = \xi + \int_0^t (\Delta\eta + \varepsilon\Delta v + Hv + f)\,ds + \int_0^t M^k(v)\,d\beta_s^k, \tag{3.21}$$

and, almost surely,

$$u^\infty = \xi + \int_0^T (\Delta\eta + \varepsilon\Delta v + Hv + f)\,ds + \int_0^T M^k(v)\,d\beta_s^k. \tag{3.22}$$

Hence, we can choose a version of v that is a continuous, adapted, H^{-1}-valued process. It follows that (3.21) holds for all $t \in [0, T]$ on a set of probability one and that almost surely $v_T = u^\infty$. To ease the notation let us set

$$\begin{aligned} &A_t^n(\varphi) := \Delta(\Phi_n(\varphi)) + \varepsilon\Delta\varphi + H_t\varphi + f_t,\\ &A_t(\varphi) := \Delta(\Phi(\varphi)) + \varepsilon\Delta\varphi + H_t\varphi + f_t \end{aligned}$$

for $\varphi \in L_{m+1}(Q)$. Let y be a predictable $L_{m+1}(Q)$-valued process, such that

$$\mathbb{E}\int_0^T \|y\|_{L_{m+1}}^{m+1}\,dt < \infty.$$

For $c > 0$ we set

$$\begin{aligned} \mathcal{O}_n := {}& \mathbb{E}\int_0^T e^{-ct}2\,{}_{(L_{m+1})^*}\langle A^n(u^n) - A(y), u^n - y\rangle_{L_{m+1}}\,dt\\ &+ \mathbb{E}\int_0^T e^{-ct}\sum_{k=1}^\infty \|M^k(u^n) - M^k(y)\|_{H^{-1}}^2\,dt - \mathbb{E}\int_0^T ce^{-ct}\|u^n - y\|_{H^{-1}}^2\,dt. \end{aligned}$$

Notice that due to Lemma 3.2 we have for $c > 0$ large enough (independent of n)

$$\begin{aligned}\mathcal{O}_n &= \mathbb{E}\int_0^T 2e^{-ct}{}_{(L_{m+1})^*}\langle A^n(u^n) - A^n(y), u^n - y\rangle_{L_{m+1}}\,dt \\ &\quad + \mathbb{E}\int_0^T e^{-ct}\sum_{k=1}^\infty \|M^k(u^n) - M^k(y)\|_{H^{-1}}^2\,dt \\ &\quad - \mathbb{E}\int_0^T ce^{-ct}\|u^n - y\|_{H^{-1}}^2\,dt \\ &\quad + \mathbb{E}\int_0^T e^{-ct}2_{(L_{m+1})^*}\langle A^n(y) - A(y), u^n - y\rangle_{L_{m+1}}\,dt \\ &\le \mathbb{E}\int_0^T e^{-ct}2_{(L_{m+1})^*}\langle A^n(y) - A(y), u^n - y\rangle_{L_{m+1}}\,dt.\end{aligned}$$

Moreover, one can easily see that by the properties of Φ_n we have that $\Phi_n(y) \to \Phi(y)$ strongly in $L_{m+1}(\Omega_T; L_{m+1}(Q))$, which combined with (3.19) gives

$$\lim_{n\to\infty}\mathbb{E}\int_0^T e^{-ct}2_{(L_{m+1})^*}\langle A^n(y) - A(y), u^n - y\rangle_{L_{m+1}}\,dt = 0.$$

Consequently, we have

$$\limsup_{n\to\infty}\mathcal{O}_n \le 0. \tag{3.23}$$

We also set

$$\mathcal{O}_n^1 := \mathbb{E}\int_0^T e^{-ct}\left(2_{(L_{m+1})^*}\langle A^n(u^n), u^n\rangle_{L_{m+1}} + \sum_{k=1}^\infty \|M^k(u^n)\|_{H^{-1}}^2 - c\|u^n\|_{H^{-1}}^2\right)dt$$

and $\mathcal{O}_n^2 = \mathcal{O}_n - \mathcal{O}_n^1$. By Itô's formula (see [19, Theorem 2.17]) we have

$$\begin{aligned}&e^{-cT}\|u_T^n\|_{H^{-1}}^2 - \|\xi\|_{H^{-1}}^2 \\ &\quad = \int_0^T e^{-ct}\left(2_{(L_{m+1})^*}\langle A^n(u^n), u^n\rangle_{L_{m+1}} + \sum_{k=1}^\infty \|M^k(u^n)\|_{H^{-1}}^2\right)dt \\ &\qquad - \int_0^T e^{-ct}c\|u^n\|_{H^{-1}}^2\,dt + \int_0^T e^{-ct}\big(M^k(u^n), u^n\big)_{H^{-1}}\,d\beta_t^k.\end{aligned}$$

By the estimates in (3.15) one can easily see that

$$\mathbb{E}\left(\int_0^T\sum_{k=1}^\infty \big(M^k(u^n), u^n\big)_{H^{-1}}^2\,dt\right)^{1/2} < \infty,$$

which implies that the expectation of the last term at the right hand side of the above equality vanishes. Hence,

$$\mathcal{O}_n^1 = \mathbb{E}e^{-cT}\|u_T^n\|_{H^{-1}}^2 - \mathbb{E}\|\xi\|_{H^{-1}}^2,$$

from which we get that

$$\limsup_{n\to\infty}\mathcal{O}_n^1 = \mathbb{E}e^{-cT}\|v_T\|_{H^{-1}}^2 - \mathbb{E}\|\xi\|_{H^{-1}}^2 + \delta e^{-cT} \tag{3.24}$$

with $\delta := \limsup_{n\to\infty} \mathbb{E}\|u_T^n\|_{H^{-1}}^2 - \mathbb{E}\|v_T\|_{H^{-1}}^2 \geq 0$, due to (3.20). On the other hand, by (3.19) and (3.15) it follows that the quantity

$$\mathbb{E} \operatorname*{ess\,sup}_{t\in[0,T]} \|v_t\|_{L_2}^2 + \mathbb{E}\int_0^T \|\nabla v\|_{L_2}^2\,dt$$

can be estimated by the right hand side of (3.15) with $q = 2$. In particular, this implies that

$$\mathbb{E}\left(\int_0^T \sum_{k=1}^\infty \big(M^k(v), v\big)_{H^{-1}}^2\,dt\right)^{1/2} < \infty.$$

Hence, by (3.21) and Itô's formula we obtain

$$\begin{aligned}\mathbb{E}e^{-cT}\|v_T\|_{H^{-1}}^2 = \mathbb{E}\|\xi\|_{H^{-1}}^2 &\\ &+ \mathbb{E}\int_0^T e^{-ct} 2_{(L_{m+1})^*}\langle \Delta\eta + \varepsilon\Delta v + Hv + f, v\rangle_{L_{m+1}}\,dt \\ &+ \mathbb{E}\int_0^T e^{-ct}\sum_{k=1}^\infty \big\|M^k(v)\big\|_{H^{-1}}^2\,dt - \mathbb{E}\int_0^T e^{-ct}c\|v\|_{H^{-1}}^2\,dt.\end{aligned}$$

Hence,

$$\begin{aligned}\limsup_{n\to\infty} \mathcal{O}_n^1 = {}&\mathbb{E}\int_0^T e^{-ct} 2_{(L_{m+1})^*}\langle \Delta\eta + \varepsilon\Delta v + Hv + f, v\rangle_{L_{m+1}}\,dt \\ &+ \mathbb{E}\int_0^T e^{-ct}\sum_{k=1}^\infty \big\|M^k(v)\big\|_{H^{-1}}^2\,dt - \mathbb{E}\int_0^T e^{-ct}c\|v\|_{H^{-1}}^2\,dt + \delta e^{-cT}.\end{aligned} \tag{3.25}$$

Moreover, by (3.19) we have

$$\begin{aligned}\lim_{n\to\infty} \mathcal{O}_n^2 = {}&\mathbb{E}\int_0^T e^{-ct} 2\big({}_{(L_{m+1})^*}\big\langle A(y), y\big\rangle_{L_{m+1}} - {}_{(L_{m+1})^*}\big\langle A(y), v\big\rangle_{L_{m+1}}\big)\,dt \\ &- \mathbb{E}\int_0^T e^{-ct} 2_{(L_{m+1})^*}\langle \Delta\eta + \varepsilon\Delta v + Hv + f, y\rangle_{L_{m+1}}\,dt \\ &+ \mathbb{E}\int_0^T e^{-ct}\left(\sum_{k=1}^\infty \big\|M^k(y)\big\|_{H^{-1}}^2 - 2\big(M^k(y), M^k(v)\big)_{H^{-1}}\right)dt \\ &+ \mathbb{E}\int_0^T e^{-ct}c\big(2(v,y)_{H^{-1}} - \|y\|_{H^{-1}}^2\big)\,dt.\end{aligned} \tag{3.26}$$

Consequently,

$$\begin{aligned}&\mathbb{E}\int_0^T e^{-ct} 2_{(L_{m+1})^*}\big\langle \Delta\eta + \varepsilon\Delta v + Hv + f - A(y), v - y\big\rangle_{L_{m+1}}\,dt \\ &\quad + \mathbb{E}\int_0^T e^{-ct} 2\sum_{k=1}^\infty \big\|M^k(v) - M^k(y)\big\|_{H^{-1}}^2\,dt \\ &\quad - \mathbb{E}\int_0^T ce^{-ct}\|v - y\|_{H^{-1}}^2\,dt + \delta e^{-cT} = \limsup_{n\to\infty} \mathcal{O}_n^1 + \lim_{n\to\infty} \mathcal{O}_n^2 \\ &= \limsup_{n\to\infty} \mathcal{O}_n \leq 0,\end{aligned} \tag{3.27}$$

by (3.23). By choosing $y = v$ in (3.27) we obtain that $\delta = 0$. Moreover, it follows that

$$\mathbb{E}\int_0^T e^{-ct} 2_{(L_{m+1})^*}\langle \Delta\eta + \varepsilon\Delta v + Hv + f - A(y), v - y\rangle_{L_{m+1}}\, dt$$
$$- \mathbb{E}\int_0^T ce^{-ct}\|v - y\|^2_{H^{-1}}\, dt \le 0.$$

Let z be a predictable process with values in $L_{m+1}(Q)$ with $\mathbb{E}\int_0^T \|z\|^{m+1}_{L_{m+1}}\, dt < \infty$ and choose in the above inequality $y = v - \lambda z$ for $\lambda > 0$. Then, we have

$$\mathbb{E}\int_0^T e^{-ct} 2\lambda_{(L_{m+1})^*}\langle \Delta\eta + \varepsilon\Delta v + Hv + f - A(v - \lambda z), z\rangle_{L_{m+1}}\, dt$$
$$- \mathbb{E}\int_0^T c\lambda^2 e^{-ct}\|z\|^2_{H^{-1}}\, dt \le 0.$$

Dividing by λ, letting $\lambda \to 0$ and using the hemi-continuity property we obtain

$$\mathbb{E}\int_0^T {}_{(L_{m+1})^*}\langle \Delta\eta - \Delta(\Phi(v)), z\rangle_{L_{m+1}}\, dt \le 0.$$

Since z was arbitrary, we have $\Delta\eta = \Delta(\Phi(v))$. This shows that v is a solution of (3.13), and by uniqueness, we have $u = v$, $\Phi(u) = \Phi(v) = \eta$. This finishes the proof. □

Remark 3.2. Suppose that Assumption 3.2 also holds and let u^ε be the solution of (3.13). By writing Itô's formula for $\|\Psi(u_t^\varepsilon)\|_{L_1(Q)}$, where $\Psi(r) = \int_0^r \Phi(s)\, ds$, similarly to (3.17) and after applying Gronwal's lemma one has

$$\mathbb{E}\int_0^T \|\nabla\Phi(u^\varepsilon)\|^2_{L_2}\, dt \le N\mathbb{E}\left(1 + \|\xi\|^{m+1}_{L_{m+1}} + \int_0^T \|f\|^{m+1}_{L_{m+1}} + \||g|_{l_2}\|^{m+1}_{L_{m+1}}\, dt\right),$$

where N is independent of ε.

We will also need the following.

Lemma 3.6. *Let u^n be real-valued functions on Q such that $u^n \rightharpoonup u$ in H^{-1} for some $u \in H^{-1}$. Then for any $p \in [1, \infty]$*

$$\|u\|_{L_p} \le \liminf_{n\to\infty}\|u^n\|_{L_p}.$$

Proof. Suppose first that $p \in [1, \infty)$. We assume that $\liminf_{n\to\infty}\|u^n\|_{L_p} < \infty$ or else there is nothing to prove. Under this assumption there exists a subsequence u^{n_k} with $\lim_k \|u^{n_k}\|^p_{L_p} = \liminf_n \|u^n\|^p_{L_p}$ and $v \in L_p(Q)$ such that $u^{n_k} \rightharpoonup v$ in $L_p(Q)$. It follows that $u = v \in L_p(Q)$, which finishes the proof since $\|v\|_{L_p} \le \liminf \|u^{n_k}\|^p_{L_p}$. For $p = \infty$ we have the following. We know that the conclusion holds for all $p \in [1, \infty)$. Hence,

$$\|u\|_{L_p} \le |Q|^{1/p} \liminf_{n\to\infty}\|u^n\|_{L_\infty}.$$

The assertion follows by letting $p \to \infty$. □

We can now present our main theorems.

Theorem 3.7. *Suppose that Assumptions 3.1 and 3.2 are satisfied with $m > 1$. Let $\mu \in \Gamma_d$ and let $u \in \mathbb{H}^{-1}_{m+1}$ be the unique solution of (3.1). Then, we have*

$$\mathbb{E}\|u\|^2_{L_\infty(Q_T)} \le N\mathbb{E}\left(\|\xi\|^2_{L_\infty(Q)} + |S_\mu(f, g)|^2\right), \tag{3.28}$$

where

$$\mathcal{S}_r(f,g) = 1 + \|f\|_{L_\mu(Q_T)} + \big\||g|_{l_2}\big\|_{L_{2\mu}(Q_T)},$$

and N is a constant depending only on m, T, $\bar{c}$, K, d, μ and $|Q|$.

Proof. *Step 1:* In a first step we assume that $|\xi(x)|$, $|f_t(x)|$, $|g_t(x)|_{l_2}$ are bounded uniformly in (ω,t,x). Let u^ε denote the unique solution of the problem (3.13). By Assumptions 3.1 and 3.2 we have that equation (3.13) satisfies Assumptions 2.1–2.2 with $\theta=\varepsilon$, $c=\bar{c}$, $\tilde{m}=m-1$, and

$$\begin{aligned}
&a_t^{ij}(x,r) = \Phi'(r)I_{i=j} + \varepsilon I_{i=j}r + \sigma_t^2 r/2,\\
&F_t^i(x,r) = b_t^i(x)r, \qquad F_t(x,r) = \big(c_t(x) - \partial_i b_t^i(x)\big)r + f_t(x),\\
&g_t^{ik}(x,r) = I_{i=k,k\le d}\sigma_t r, \qquad G_t^k(x,r) = I_{k>d}\big(\nu_t^{k-d}(x)r + g_t^{k-d}(x)\big),\\
&V_t^1(x) = \big|f_t(x)\big|, \qquad V_t^2(x) = \big|g_t(x)\big|_{l_2}.
\end{aligned}$$

By Lemma 3.5 we have that u^ε satisfies equation (3.13) also in the sense of Definition 2.1. Therefore, by Theorem 2.7 we have

$$\mathbb{E}\big\|u^\varepsilon\big\|^2_{L_\infty(Q_T)} \le N\mathbb{E}\big(1 + \|\xi\|^2_{L_\infty(Q)} + \big|S_\mu(f,g)\big|^2\big), \tag{3.29}$$

where N is a constant depending only on m, T, $\bar{c}$, K, d, r, and $|Q|$. By Itô's formula, the monotonicity of Φ, and (3.8), one can easily see that for a constant N independent of ε we have

$$\big\|u_t - u_t^\varepsilon\big\|^2_{H^{-1}} \le N\int_0^t \big\|u - u^\varepsilon\big\|^2_{H^{-1}} + \varepsilon 2_{(L_{m+1})^*}\big\langle \Delta u^\varepsilon, u^\varepsilon - u\big\rangle_{L_{m+1}}\,ds + M_t^\varepsilon,$$

for a local martingale M_t^ε. Hence,

$$\mathbb{E}\big\|u_t^\varepsilon - u_t\big\|^2_{H^{-1}} \le N\mathbb{E}\int_0^T \varepsilon\big|_{(L_{m+1})^*}\big\langle\Delta u^\varepsilon, u^\varepsilon - u\big\rangle_{L_{m+1}}\big|\,dt. \tag{3.30}$$

Moreover, by Itô's formula, Assumption 3.1(1), and (3.8) we have for a constant N independent of ε

$$\mathbb{E}\int_0^T \big\|u^\varepsilon\big\|^{m+1}_{L_{m+1}}\,dt \le N\mathbb{E}\left(1 + \|\xi\|^2_{H^{-1}} + \int_0^T \|f\|^2_{H^{-1}} + \sum_{k=1}^\infty \big\|g^k\big\|^2_{H^{-1}}\,dt\right) < \infty.$$

The same estimate holds for u. Hence, by virtue of (3.30), we have

$$\lim_{\varepsilon\to 0}\mathbb{E}\int_0^T \big\|u_t - u_t^\varepsilon\big\|^2_{H^{-1}}\,dt = 0.$$

In particular, for a sequence $\varepsilon_k\to 0$ we have $\|u_t^{\varepsilon_k} - u_t\|_{H^{-1}}\to 0$ for almost all (ω,t). By Lemma 3.6, Fatou's lemma, and (3.29), we have for any $p\in[1,\infty)$

$$\begin{aligned}
\mathbb{E}\|u\|^2_{L_p(Q_T)} &\le \liminf_k \mathbb{E}\big\|u^{\varepsilon_k}\big\|^2_{L_p(Q_T)} \le \liminf_k N\mathbb{E}\big\|u^{\varepsilon_k}\big\|^2_{L_\infty(Q_T)}\\
&\le N\mathbb{E}\big(1 + \|\xi\|^2_{L_\infty(Q)} + \big|S_\mu(f,g)\big|^2\big),
\end{aligned}$$

with N independent of p, and the result follows by letting $p\to\infty$.

Step 2: For general ξ, f, g we set

$$\xi^n = \big((-n) \vee \xi\big) \wedge n, \qquad f^n = \big((-n) \vee f\big) \wedge n,$$
$$g^n = \sum_{k=1}^{n} \big(\big((-C_n) \vee g^k\big) \wedge C_n\big) e_k,$$

where $(e_k)_{k=1}^\infty$ is the usual orthonormal basis of l_2 and $C_n \geq 0$ are chosen such that $\mathbb{E}\int_0^T \||g^n - g|_{l_2}\|_{L_2}^2\, dt \to 0$. Let u^n be the solution of the equation corresponding to the truncated data. Then by Itô's formula one can easily check that

$$\mathbb{E}\int_0^T \|u_t^n - u_t\|_{H^{-1}}^2\, dt \leq N\mathbb{E}\|\xi - \xi^n\|_{H^{-1}}^2 + N\mathbb{E}\left(\int_0^T \|f - f^n\|_{H^{-1}}^2 + \sum_{k=1}^\infty \|g^k - g^{n,k}\|_{L_2}^2\, dt\right) \to 0,$$

as $n \to \infty$. Since for each $n \in \mathbb{N}$ we have

$$\mathbb{E}\|u^n\|_{L_\infty(Q_T)}^2 \leq N\mathbb{E}\big(1 + \|\xi\|_{L_\infty(Q)}^2 + \big|S_\mu(f, g)\big|^2\big),$$

the result follows by virtue of Lemma 3.6. □

Theorem 3.8. *Suppose that Assumptions 3.1 and 3.2 are satisfied with $m > 1$. Let $\mu \in \Gamma_d$ and let u be the solution of (3.1). Then, for all $\rho \in (0, 1)$ we have*

$$\mathbb{E}\|u\|_{L_\infty((\rho,T)\times Q)}^2 \leq \rho^{-\tilde{\theta}} N\mathbb{E}\big(\|\xi\|_{H^{-1}}^2 + \big|S_\mu(f, g)\big|^2\big), \tag{3.31}$$

where $S_\mu(f, g)$ is as in Theorem 3.7, N is a constant depending only on C, m, T, c, K, d, μ and $|Q|$, and $\tilde{\theta} > 0$ is a constant depending only on d, μ and m.

Proof. *Step 1:* In a first step we assume that $\xi \in L_{m+1}(\Omega, L_{m+1}(Q))$ and that $|f_t(x)|$, $|g_t(x)|_{l_2}$ are bounded uniformly in (ω, t, x). Let u^ε denote the unique solution of the problem (3.13). By Lemma 3.5 u^ε is a solution also in the sense of Definition 2.1. Hence, by Lemma 2.9 we have

$$\mathbb{E}\|u^\varepsilon\|_{L_\infty((\rho,T)\times Q)}^2 \leq \rho^{-\tilde{\theta}} N\mathbb{E}\big(1 + \|u^\varepsilon\|_{L_2(Q_T)}^2 + \big|\mathcal{S}_r(f, g)\big|^2\big), \tag{3.32}$$

with N independent of ε. By Itô's formula, Assumption 3.1(1) and Lemma 3.2 we have

$$\|u_t^\varepsilon\|_{H^{-1}}^2 + \int_0^t \|u^\varepsilon\|_{L_{m+1}}^{m+1}\, ds \leq N\left(1 + \|\xi\|_{H^{-1}}^2 + \int_0^t \|u^\varepsilon\|_{H^{-1}}^2 + \|f\|_{H^{-1}}^2 + \sum_{k=1}^\infty \|g^k\|_{H^{-1}}^2\, ds\right) + M_t, \tag{3.33}$$

for a local martingale, with N independent of ε. After a localization argument and Gronwal's lemma one gets

$$\mathbb{E}\int_0^T \|u^\varepsilon\|_{L_{m+1}}^{m+1}\, ds \leq N\mathbb{E}\left(1 + \|\xi\|_{H^{-1}}^2 + \int_0^T \left(\|f\|_{H^{-1}}^2 + \sum_{k=1}^\infty \|g^k\|_{H^{-1}}^2\right) ds\right).$$

Plugging this in (3.32) ($m + 1 > 2$) gives the desired inequality.

Step 2: For general ξ, f and g, one can proceed as in the proof of Theorem 3.7, this time choosing $\xi^n \in L_{m+1}(\Omega; L_{m+1}(Q))$ such that $\lim_n \mathbb{E}\|\xi^n - \xi\|_{H^{-1}}^2 = 0$ and $\|\xi_n\|_{H^{-1}} \leq \|\xi\|_{H^{-1}}$ almost surely. This finishes the proof. □

Remark 3.3. As already seen, for any $\xi \in L_2(\Omega; H^{-1})$, the corresponding solution u of (3.1) belongs to the space $L_{m+1}(\Omega_T; L_{m+1}(Q))$. Consequently, there exists arbitrarily small $s > 0$ such that $\mathbb{E}\|u_s\|_{L_{m+1}}^{m+1} < \infty$. By Remark 3.2 the quantity $\mathbb{E}\|\Phi(u^\varepsilon)\|^2_{L_2(s,T;H_0^1)}$ (where u^ε is the solution of (3.13) starting at time s from $u_s^\varepsilon = u_s$) can be controlled by $\mathbb{E}\|u_s\|_{L_{m+1}}^{m+1}$. Using this, one can use again the theory of monotone operators to show that the weak limit of $\Phi(u^\varepsilon)$ in $L_2(\Omega \times (s,T); H_0^1)$ coincides with $\Phi(u)$. In particular, the solution u is strong on the time interval (s,T), that is, $\Phi(u_t) \in H_0^1(Q)$ for a.e. $(\omega, t) \in \Omega \times (s,T)$.

Appendix

Lemma A.1. *Let $Q \subset \mathbb{R}^d$ be an open bounded set and let $R \in C^1(\overline{Q} \times \mathbb{R})$ be such that there exist $N \in \mathbb{R}$, $p \in [2,\infty)$ and $g \in L_p(Q)$ such that for all $(x,r) \in \bar{Q} \times \mathbb{R}$*

$$\left|R(x,r)\right| + \left|\nabla_x R(x,r)\right| \le N + \left|g(x)\right||r|^{p-2} + N|r|^{p-1}. \tag{A.1}$$

Set

$$G(x,r) := \int_0^r R(x,s)\,ds,$$

and let $u \in H_0^1(Q)$ be such that

$$\int_Q |u|^p\,dx + \int_Q |\nabla u|^2 |u|^{p-2}\,dx < \infty. \tag{A.2}$$

Then $G(\cdot, u) \in W_0^{1,1}(Q)$.

Proof. Let us set

$$R_n(x,r) := \begin{cases} R(x,r), & \text{for } |r| \le n, \\ R(x,n), & \text{for } r > n, \\ R(x,-n), & \text{for } r < -n \end{cases}$$

and

$$G_n(x,r) := \int_0^r R_n(x,s)\,ds.$$

It follows that $\nabla_x G_n(x,r)$ and $\partial_r G_n(x,r)$ are continuous in $(x,r) \in \overline{Q} \times \mathbb{R}$, and they satisfy with some constant $N(n)$, for all $(x,r) \in \overline{Q} \times \mathbb{R}$

$$\left|\nabla_x G_n(x,r)\right| \le N(n)|r|, \qquad \left|\partial_r G_n(x,r)\right| \le N(n).$$

Moreover, we have $G_n(x,0) = 0$. Hence, by approximating u in $H_0^1(Q)$ with $u^m \in C_c^\infty(Q)$, one concludes easily that $G_n(\cdot, u) \in W_0^{1,1}(Q)$. Notice that there exists a constant N, such that for all $n \in \mathbb{N}$, $(x,r) \in \overline{Q} \times \mathbb{R}$ we have

(i) $|G_n(x,r)| \le N(1 + |g(x)|^p + |r|^p)$,
(ii) $|\nabla_x G_n(x,r)| \le N(1 + |g(x)|^p + |r|^p)$,
(iii) $|\partial_r G_n(x,r)| \le N(1 + |g(x)||r|^{p-2} + |r|^{p-1})$.

This implies by Young's inequality

$$\left|G_n(x,u)\right| \le N\left(1 + \left|g(x)\right|^p + |u|^p\right),$$

$$\left|\nabla_x G_n(x,u)\right| + \left|\partial_r G_n(x,u)\right||\nabla u| \le N\left(1 + \left|g(x)\right|^p + |u|^p + |u|^{p-2}|\nabla u|^2\right).$$

By Lebesgue's theorem on dominated convergence we have $G_n(\cdot, u) \to G(\cdot, u)$ and $\nabla_x(G_n(\cdot, u)) \to \nabla_x G(\cdot, u) + \partial_r G(\cdot, u)\nabla_x u$ in $L_1(Q)$, and the claim follows since $G_n(\cdot, u) \in W_0^{1,1}(Q)$ for all $n \in \mathbb{N}$. □

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