Annales de l'Institut Henri Poincaré - Probabilités et Statistiques
2019, Vol. 55, No. 3, 1751–1764
https://doi.org/10.1214/18-AIHP933

ANNALES DE L'INSTITUT HENRI POINCARÉ
PROBABILITÉS ET STATISTIQUES
www.imstat.org/aihp

Active phase for activated random walks on $\mathbb{Z}^d$, $d \geq 3$, with density less than one and arbitrary sleeping rate

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Received 3 January 2018; revised 5 June 2018; accepted 17 September 2018

Abstract. It has been conjectured that the critical density of the Activated Random Walk model is strictly less than one for any value of the sleeping rate. We prove this conjecture on $\mathbb{Z}^d$ when $d \geq 3$ and, more generally, on graphs where the random walk is transient. Moreover, we establish the occurrence of a phase transition on non-amenable graphs, extending previous results which require that the graph is amenable or a regular tree.

Résumé. Il a été conjecturé que la densité critique pour le modèle de marches aléatoires activées était strictement inférieur à 1 pour toute valeur du taux d'endormissement. Nous démontrons cette conjecture pour $\mathbb{Z}^d$ quand $d \geq 3$ et, plus généralement, pour les graphes sur lesquels la marche aléatoire est transitoire. De plus, nous montrons l'existence d'une transition de phase pour les graphes non moyennables, généralisant ainsi des résultats antérieurs qui demandaient que le graphe soit moyennable ou un arbre régulier.

MSC: Primary 82C22; secondary 60K35; 82C26

Keywords: Interacting particle systems; Abelian networks; Absorbing-state phase transition; Self-organized criticality

1. Introduction

The activated random walk model (ARW) is a system of interacting particles on a graph $G = (V, E)$. Together with Abelian and Stochastic Sandpiles, it belongs to a class of systems which have been introduced in order to study a physical phenomenon known as self-organized criticality. Moreover, it can be interpreted as a toy model for an epidemic spreading, with infected individuals moving diffusively on a graph. The model is defined as follows. Every particle is in one of two states, A (active) or S (inactive, sleeping). Initially, the number of particles at each vertex of G is an independent Poisson random variable with mean $\mu \in (0, \infty)$, usually called the *particle density*, and all particles are of type A. Active particles perform an independent, continuous time random walk on G with jump rate 1, and with each jump being to a uniformly random neighbour. Moreover, every A-particle has a Poisson clock of rate $\lambda > 0$ (called *sleeping rate*). When the clock of a particle rings, if the particle does not share the vertex with other particles, the particle becomes of type S; otherwise nothing happens. Each S-particle does not move and remains sleeping until another particle jumps into its location. At such an instant, the S-particle is activated and turns into type A.

For any value of λ, a phase transition as a μ varies is expected to occur. When μ is small, there is a lot of free space between the particles. This allows every particle to turn into type S eventually and never become active again. When this happens, we say that ARW *fixates*. This is not expected to occur when μ is large, since the active particles

[1] Supported by German Research Foundation (DFG).

will repetitively jump on top of other particles, activating the ones that had turned into type S. In this case, we say that ARW is *active*.

In a seminal paper [4], Rolla and Sidoravicius prove a 0–1 law (i.e., the process is either active or fixates with probability 1) and a monotonicity property with respect to μ. This leads to the existence of a critical curve $\mu_c = \mu_c(\lambda)$,

$$\mu_c = \mu_c(\lambda) := \inf\{\mu \geq 0 : \mathbb{P}(\text{ARW is active}) > 0\}, \tag{1}$$

which is such that, for any $\mu > \mu_c$ the system is almost surely active, and for any $\mu < \mu_c$ the system fixates almost surely. Though [4] is restricted to the case of G being $\mathbb{Z}^d$, the above properties hold for any vertex-transitive graph. Throughout this paper we always consider that G is an infinite simple graph that is locally-finite and vertex transitive, which ensures the existence of μ_c.

In recent years considerable effort has been made to prove basic properties of the critical curve $\mu_c = \mu_c(\lambda)$ [1,2,4–6,8–10]. A quite natural bound for this curve is $\mu_c \leq 1$ for any value of $\lambda \in (0,\infty)$, which was proved in [1,4,7]. Indeed, one does not expect fixation when the average number of particles per vertex is more than one, since a particle can be in the S-state only if it is alone on a given vertex and, for this reason, there is not enough space for all the A-particles to turn to the S-state. A more challenging question is whether μ_c is strictly less than one for any value of $\lambda \in (0,\infty)$, which is expected to hold true under wide generality. In other words, one expects that, for any value of $\lambda \in (0,\infty)$, there exists a value of μ which is strictly less than one such that, even though there is enough space for all the particles to turn into the S-state, particle motion prevents this from happening, so the system does not fixate. This question was asked by Rolla and Sidoravicius in their seminal paper [4] and appears also in [2,3]. Such a question received much attention in the last few years [2,4,6,9,10] but, despite much effort, a complete answer was provided only in two cases: on vertex-transitive graphs where the random walk has a positive speed [9] and for a simplified model on $\mathbb{Z}^d$ where the jump distribution of active particles is biased in a fixed direction [10]. A partial answer which requires the assumption that λ is smaller than a finite constant $\lambda_0 < \infty$ was also provided in [9] when G is vertex-transitive and transient and in [2] when $G = \mathbb{Z}$.

The first main result of this paper is the next theorem, which provides a positive answer to this question for any $\lambda \in (0,\infty)$ on $\mathbb{Z}^d$, when $d \geq 3$, for the original model where active particles jump uniformly to nearest-neighbours. More generally, our result holds for any vertex-transitive amenable graph where the random walk is transient.

Theorem 1.1. *If G is vertex-transitive, amenable and transient, then*

$$\mu_c(\lambda) < 1 \quad \forall \lambda \in (0,\infty).$$

Moreover, $\limsup_{\lambda \to 0} \frac{\mu_c(\lambda)}{\lambda^{\frac{1}{2}}} < \infty$.

A second basic question concerning the behaviour of the critical curve $\mu_c = \mu_c(\lambda)$ is whether its value is positive. A positive answer has been proved by Sidoravicius and Teixeira in [8] when $G = \mathbb{Z}^d$ by means of renormalization techniques. A shorter proof was also provided by Stauffer and Taggi in [9] when G is amenable and vertex-transitive and when G is a regular tree. The proofs of [8,9] crucially rely on the amenability property of the graph or on the assumption that G is a regular tree.

Our second main theorem provides a positive answer to this question on vertex-transitive graphs that are non-amenable, establishing the occurrence of a phase transition for this class of graphs and extending the previous results [8,9]. Moreover, we also obtain that $\lim_{\lambda \to \infty} \mu_c(\lambda) = 1$.

Theorem 1.2. *If G is vertex-transitive and non-amenable, then $\mu_c(\lambda) > 0$ for any value of $\lambda \in (0,\infty)$. More specifically,*

$$\mu_c(\lambda) \geq \frac{\lambda}{1+\lambda} \quad \forall \lambda \in (0,\infty).$$

Theorem 1.2 and the results of [8,9] imply that $\mu_c(\lambda) > 0$ for any $\lambda \in (0,\infty)$ and that $\lim_{\lambda \to \infty} \mu_c(\lambda) = 1$ on any vertex transitive graph. Moreover, Theorem 1.1 and the results of [9] imply that $\mu_c(\lambda) < 1$ for any $\lambda \in (0,\infty)$ and that $\lim_{\lambda \to 0} \mu_c(\lambda) = 0$ on any vertex-transitive graph where the random walk is transient. Furthermore, it has been proved in [5] that if the initial particle configuration is a spatially ergodic distribution with density μ, then ARW is a.s. active whenever $\mu > \mu_c(\lambda)$ and fixates a.s. whenever $\mu < \mu_c(\lambda)$.

Description of the proofs. Our proofs are simple and rely on a graphical representation, which is called *Diaconis–Fulton* and has been introduced in [4], and on *weak stabilization*, a procedure that has been introduced in [9] which consists of using the random instructions of such a representation by following a certain strategy.

A fundamental quantity for the mathematical analysis of the activated random walks is the number of times m_{B_L} the origin is visited by a particle when the dynamics take place in a finite ball of radius L, B_L, with particles being absorbed whenever they leave B_L. As it was proved in [4], activity for ARW is equivalent to the limit $L \to \infty$ of this quantity being infinite almost surely. A quantity that plays a central role in this paper is the probability $Q(x, B_L)$ that an S-particle is at a vertex $x \in B_L$ when B_L becomes stable. This quantity is important since the values $\{Q(x, B_L)\}_{x \in B_L}$ are related to the expectation of m_{B_L} by mass-conservation arguments. Thus, one can deduce whether the system is active by estimating these values.

The proof of Theorem 1.1 consists of bounding away from one the probabilities $\{Q(x, B_L)\}_{x \in B_L}$ for any $\lambda \in (0, \infty)$ uniformly in L and in $x \in B_L$. This improves the upper bound that was provided in [9], where the probabilities $\{Q(x, B_L)\}_{x \in B_L}$ were bounded away from one only for λ small enough. Such an enhancement is obtained by introducing a stabilization procedure that allows to recover independence from sleep instructions at one vertex. This gained independence and the fact that we do not count the total number of instructions but only jump instructions, allows to obtain an additional factor in the upper bound for $Q(x, B_L)$ which prevents this bound from exploding when λ is infinitely large. Our upper bound on $Q(x, B_L)$ implies that for any $\lambda \in (0, \infty)$ one can find $\varepsilon > 0$ and set the value of μ such that $1 > \mu \geq Q(x, B_L) + \varepsilon$ for all L and $x \in B_L$. This implies that a positive density ε of particles eventually leaves B_L and, as it was proved in [6], that the system is active, proving Theorem 1.1.

Theorem 1.2 extends to non-amenable graphs the analogous result that was proved in [9] for amenable graphs. The idea of the proof that is presented in [9] is that one assumes activity and uses this assumption and the weak stabilization procedure to show that for any $\varepsilon > 0$, there exists a large enough constant $r_0 = r_0(\varepsilon)$ such that, for any large enough L and for any vertex $x \in B_L$ which has a distance at least r_0 from the boundary of B_L,

$$Q(x, B_L) \geq \frac{\lambda}{1+\lambda} - \varepsilon. \tag{2}$$

This leads to the conclusion that the particle density after the stabilization of B_L is at least $\frac{\lambda}{1+\lambda}$. The amenability assumption is crucial here, since the number of particles which start "close" to the boundary, for which (2) does not hold, can be neglected only if the graph is amenable (i.e. their number is of order $o(|B_L|)$). Since the initial particle density is μ and since the particle density cannot increase, we conclude that $\mu \geq \frac{\lambda}{1+\lambda}$. Since this is a consequence of activity, we obtain that $\mu_c \geq \frac{\lambda}{1+\lambda}$.

In this paper, we use a different strategy that allows us to extend this result to non-amenable graphs. By assuming that the system is active and by using (2), one obtains that the particle density in a small ball $B_{(1-\delta)L} \subset B_L$ after the stabilization of the larger ball B_L is at least $\frac{\lambda}{1+\lambda}$, for some $\delta > 0$ and all L large enough. Thus, if we set $\mu < \frac{\lambda}{1+\lambda}$, this means that the particle density inside the smaller ball must have increased during the stabilization of the larger ball. Due to the conservation law, the only way this might have happened is if a large number of particles which started from $B_L \setminus B_{(1-\delta)L}$ turns into the S-state for the last time in $B_{(1-\delta)L}$. We show that, if the graph is non-amenable, this cannot happen simply because, even though the number of the boundary particles is not negligible if compared to $|B_{(1-\delta)L}|$, the bias towards the outside of the ball allows only a few of them to penetrate inside the ball. So, the particle density in the smaller ball cannot increase and this leads to the conclusion that $\mu \geq \frac{\lambda}{1+\lambda}$. Since this is a consequence of activity, we obtain that $\mu_c \geq \frac{\lambda}{1+\lambda}$.

The remaining part of the paper is organized as follows. In Section 2 we introduce the Diaconis–Fulton representation following [4], we recall the notion of weak stabilization following [9] and we fix the notation. In Section 3 we provide an explicit upper bound for $Q(x, B_L)$, which is presented in Theorem 3.1, and we prove Theorem 1.1. Finally, Section 4 is dedicated to the proof of Theorem 1.2.

2. Diaconis–Fulton representation and weak stabilization

In this section we describe the Diaconis–Fulton graphical representation for the dynamics of ARW, following [4], and we recall the notion of *weak stabilization*, following [9]. Before starting, we fix the notation.

Notation. A graph is denoted by $G = (V, E)$ and is always assumed to be simple, infinite and locally-finite. The simple random walk measure is denoted by P_x, where x is the starting vertex of the random walk. The expectation with respect to P_x is denoted by E_x. For any set $Z \subset V$ and any pair of vertices $x, y \in V$, we let

$$G_Z(x, y) = E_x\left(\sum_{t=0}^{\tau_Z - 1} \mathbb{1}\{X(t) = y\}\right)$$

be the expected number of times a discrete time random walk $X(t)$ starting from x hits y before reaching Z (Green's function), where τ_Z is the hitting time of the set Z. If $Z = \varnothing$, then we set $\tau_Z = \infty$ and we simply write $G(x, y)$. We also denote by τ_Z^+ the return time to Z. The origin of the graph will be denoted by $0 \in V$. We let $B_r(x) = \{y \in V : d(x, y) < r\}$ be the ball of radius $r > 0$ centred at x, where $d(\cdot, \cdot)$ is the graph distance, and write B_r for $B_r(0)$.

2.1. *Stabilization*

Diaconis–Fulton representation. For a graph $G = (V, E)$, the state of configurations is $\Omega = \{0, \rho, 1, 2, 3, \ldots\}^V$, where a vertex being in state ρ denotes that the vertex has one S-particle, while being in state $i \in \{0, 1, 2, \ldots\}$ denotes that the vertex contains i A-particles. We employ the following order on the states of a vertex: $0 < \rho < 1 < 2 < \cdots$. In a configuration $\eta \in \Omega$, a vertex $x \in V$ is called *stable* if $\eta(x) \in \{0, \rho\}$, and it is called *unstable* if $\eta(x) \geq 1$. We fix an array of *instructions* $\tau = (\tau^{x,j} : x \in V, j \in \mathbb{N})$, where $\tau^{x,j}$ can either be of the form τ_{xy} or $\tau_{x\rho}$. We let τ_{xy} with $x, y \in V$ denote the instruction that a particle from x jumps to vertex y, and $\tau_{x\rho}$ denote the instruction that a particle from x falls asleep. Henceforth we call τ_{xy} a *jump instruction* and $\tau_{x\rho}$ a *sleep instruction*. Therefore, given any configuration η, performing the instruction τ_{xy} in η yields another configuration η' such that $\eta'(z) = \eta(z)$ for all $z \in V \setminus \{x, y\}$, $\eta'(x) = \eta(x) - \mathbb{1}(\eta(x) \geq 1)$, and $\eta'(y) = \eta(y) + \mathbb{1}(\eta(x) \geq 1)$. We use the convention that $1 + \rho = 2$. Similarly, performing the instruction $\tau_{x\rho}$ to η yields a configuration η' such that $\eta'(z) = \eta(z)$ for all $z \in V \setminus \{x\}$, and if $\eta(x) = 1$ we have $\eta'(x) = \rho$, otherwise $\eta'(x) = \eta(x)$.

Let $h = (h(x) : x \in V)$ count the number of instructions used at each vertex. We say that we *use* an instruction at x (or that we *topple* x) when we act on the current particle configuration η through the operator Φ_x, which is defined as,

$$\Phi_x(\eta, h) = \left(\tau^{x,h(x)+1}\eta, h + \delta_x\right), \tag{3}$$

where $\delta_x(y) = 1$ if $y = x$ and $\delta_x(y) = 0$ otherwise. The operation Φ_x is *legal* for η if x is unstable in η, otherwise it is *illegal*.

Properties. We now describe the properties of this representation. Later we discuss how they are related to the stochastic dynamics of ARW. For a sequence of vertices $\alpha = (x_1, x_2, \ldots, x_k)$, we write $\Phi_\alpha = \Phi_{x_k}\Phi_{x_{k-1}}, \ldots, \Phi_{x_1}$ and we say that Φ_α is *legal* for η if Φ_{x_ℓ} is legal for $\Phi_{(x_{\ell-1},\ldots,x_1)}(\eta, h)$ for all $\ell \in \{1, 2, \ldots, k\}$. Let $m_\alpha = (m_\alpha(x) : x \in V)$ be given by, $m_\alpha(x) = \sum_\ell \mathbb{1}(x_\ell = x)$, the number of times the vertex x appears in α. We write $m_\alpha \geq m_\beta$ if $m_\alpha(x) \geq m_\beta(x)$ $\forall x \in V$. Analogously we write $\eta' \geq \eta$ if $\eta'(x) \geq \eta(x)$ for all $x \in V$. We also write $(\eta', h') \geq (\eta, h)$ if $\eta' \geq \eta$ and $h' = h$.

Let η, η' be two configurations, x be a vertex in V and τ be a realization of the array of instructions. Let V' be a finite subset of V. A configuration η is said to be *stable* in V' if all the vertices $x \in V'$ are stable. We say that α is contained in V' if all its elements are in V', and we say that α *stabilizes* η in V' if every $x \in V'$ is stable in $\Phi_\alpha\eta$. The following lemmas give fundamental properties of the Diaconis–Fulton representation. For the proof, we refer to [4].

Lemma 2.1 (Abelian property). *Given any $V' \subset V$, if α and β are both legal sequences for η that are contained in V' and stabilize η in V', then $m_\alpha = m_\beta$. In particular, $\Phi_\alpha\eta = \Phi_\beta\eta$.*

For any subset $V' \subset V$, any $x \in V$, any particle configuration η, and any array of instructions τ, we denote by $m_{V',\eta,\tau}(x)$ the number of times that x is toppled in the stabilization of V' starting from configuration η and using the instructions in τ. Note that by Lemma 2.1, we have that $m_{V',\eta,\tau}$ is well defined.

Lemma 2.2 (Monotonicity). *If $V' \subset V'' \subset V$ and $\eta \leq \eta'$, then $m_{V',\eta,\tau} \leq m_{V'',\eta',\tau}$.*

By monotonicity, given any growing sequence of subsets $V_1 \subseteq V_2 \subseteq V_3 \subseteq \cdots \subseteq V$ such that $\lim_{m\to\infty} V_m = V$, the limit

$$m_{\eta,\tau} = \lim_{m\to\infty} m_{V_m,\eta,\tau},$$

exists and does not depend on the particular sequence $\{V_m\}_m$.

We now introduce a probability measure on the space of instructions and of particle configurations. We denote by $\mathcal{P}$ the probability measure according to which, for any $x \in V$ and any $j \in \mathbb{N}$, $\mathcal{P}(\tau^{x,j} = \tau_{x\rho}) = \frac{\lambda}{1+\lambda}$ and $\mathcal{P}(\tau^{x,j} = \tau_{xy}) = \frac{1}{d(1+\lambda)}$ for any $y \in V$ neighboring x, where d is the degree of each vertex of G and the $\tau^{x,j}$ are independent across diffent values of x or j. Finally, we denote by $\mathcal{P}^\nu = \mathcal{P} \otimes \nu$ the joint law of η and τ, where ν is a distribution on Ω giving the law of η. Let $\mathbb{P}^\nu$ denotes the probability measure induced by the ARW process when the initial distribution of particles is given by ν. We shall often omit the dependence on ν by writing $\mathcal{P}$ and $\mathbb{P}$ instead of $\mathcal{P}^\nu$ and $\mathbb{P}^\nu$. The following lemma relates the dynamics of ARW to the stability property of the representation.

Lemma 2.3 (0–1 law). *Let ν be a translation-invariant, ergodic distribution with finite density. Let $x \in V$ be any given vertex of G. Then $\mathbb{P}^\nu(\text{ARW fixates}) = \mathcal{P}^\nu(m_{\eta,\tau}(x) < \infty) \in \{0, 1\}$.*

Roughly speaking, the next lemma gives that removing an instruction sleep, cannot decrease the number of instructions used at a given vertex for stabilization. In order to state the lemma, consider an additional instruction ι besides τ_{xy} and $\tau_{x\rho}$. The effect of ι is to leave the configuration unchanged; i.e., $\iota\eta = \eta$, so we will call this instruction *neutral*. Then given two arrays $\tau = (\tau^{x,j})_{x,j}$ and $\tilde{\tau} = (\tilde{\tau}^{x,j})_{x,j}$, we write $\tau \leq \tilde{\tau}$ if for every $x \in V$ and $j \in \mathbb{N}$, we either have $\tilde{\tau}^{x,j} = \tau^{x,j}$ or we have $\tilde{\tau}^{x,j} = \iota$ and $\tau^{x,j} = \tau_{x\rho}$.

Lemma 2.4 (Monotonicity with enforced activation). *Let τ and $\tilde{\tau}$ be two arrays of instructions such that $\tau \leq \tilde{\tau}$. Then, for any finite subset $V' \subset V$ and configuration $\eta \in \Omega$, we have $m_{V',\eta,\tau} \leq m_{V',\eta,\tilde{\tau}}$.*

When we average over η and τ using the measure $\mathcal{P}$, we will simply write $m_{V'}$ instead of $m_{V',\eta,\tau}$ and we will do the same for the other quantities that will be introduced later.

2.2. *Weak stabilization*

We now recall the notion of weak stabilization following [9].

Definition 2.5 (Weakly stable configurations). We say that a configuration η is *weakly stable* in a subset $K \subset V$ with respect to a vertex $x \in K$ if $\eta(x) \leq 1$ and $\eta(y) \leq \rho$ for all $y \in K \setminus \{x\}$. For conciseness, we just write that η is weakly stable for (x, K).

Definition 2.6 (Weak stabilization). Given a subset $K \subset V$ and a vertex $x \in K$, the *weak stabilization* of (x, K) is a sequence of topplings of unstable vertices of $K \setminus \{x\}$ and of topplings of x whenever x has at least two active particles, until a weakly stable configuration for (x, K) is obtained. The order of the topplings of a weak stabilization can be arbitrary.

The Abelian property (Lemma 2.1), the monotonicity property (Lemma 2.2) and monotonicity with enforced activation (Lemma 2.4) hold for weak stabilization as well. Since the proof of these lemmas is the same as for stabilization, for the proofs we refer to [4]. For any given particle configuration η and instruction array τ, we let $m^1_{(x,K),\eta,\tau}(y)$ be the number of instructions that are used at y for the weak stabilization of (x, K). By the Abelian property, this quantity is well defined.

We now formulate the Least Action Principle for weak stabilization of (x, K). In order to state the lemma, we need to extend the notion of unstable vertex and of legal operations to weak stabilization of (x, K). We call a vertex y *WS-unstable* (that is, unstable for weak stabilization) in $\eta \in \Omega$ if $\eta(y) \geq 1 + \delta_x(y)$, where $\delta_x(y) = 1$ if $x = y$ and $\delta_x(y) = 0$ otherwise. We call a vertex y *WS-stable* in $\eta \in \Omega$ if it is not WS-unstable. We call the operation Φ_y defined in (3) *WS-legal* for η if y is WS-unstable in η. Note that a WS-legal operation is always legal but a legal operation is not necessarily WS-legal. For a sequence of vertices $\alpha = (x_1, x_2, \dots, x_k)$, we say that Φ_α is WS-legal for η if Φ_{x_ℓ} is WS-legal for $\Phi_{(x_{\ell-1},\dots,x_1)}(\eta, h)$ for all $\ell \in \{1, 2, \dots, k\}$. We say that that α *stabilizes* η weakly in (x, K) if every $x \in V$ is WS-stable in $\Phi_\alpha \eta$.

Lemma 2.7 (Least action principle for weak stabilization of (x, K)). *If α and β are sequences of topplings for η such that α is legal and stabilizes η weakly in (x, K) and β is WS-legal and is contained in K, then $m_\beta \le m_\alpha$.*

For the proof of the lemma, we refer to [9].

We now introduce a stabilization procedure of K consisting of a sequence of weak stabilizations of (x, K). This stabilization procedure is called *stabilization via weak stabilization* and was used also in [9]. From now on, we will omit the dependence of the quantities on η and τ, unless necessary, in order to lighten the notation.

Stabilization via weak stabilization. Let η be the initial particle configuration. *At the first step*, we perform the weak stabilization of (x, K). Recall that $m^1_{(x,K)}(y)$ is the total number of instructions that are used at y for the weak stabilization of (x, K) and let η_1 be the resulting particle configuration. If η_1 has no particle at x, then η_1 is stable and we complete the stabilization procedure. If η_1 has an active particle at x, then we move to the next step. *At the ith step*, $i \ge 2$, we use an instruction at x and, if such an instruction is not sleep, then we perform a weak stabilization of (x, K). We let $m^i_{(x,K)}(y)$ be the number of instructions that have been used at $y \in K$ up to this time, and denote by η_i the configuration we then obtained. We iterate the procedure until we obtain a stable configuration and we let $T_{(x,K)}$ denote the number of iterations. More precisely,

$$T_{(x,K)} := \min\{n \in \mathbb{N}_{>0} : \eta_n \text{ is stable}\}.$$

Note that $T_{(x,K)}$ is strictly positive for any η and τ and that if $T_{(x,K)} = 1$, then the stable configuration $\eta_{T_{(x,K)}}$ hosts no particle at x. For consistency, for any $i > T_{(x,K)}$, let η_i be the stable configuration obtained after stabilizing K and, for any $y \in K$, define $m^i_{(x,K)}(y) = m_K(y)$, which is the total number of instructions used at y for the complete stabilization of K. By the Abelian property, the quantities $T_{(x,K)}$ and $m^i_{(x,K)}$ are all well defined. Note that the quantity $T_{(x,K)}$ is defined slightly differently than in [9].

In Section 3 we will show that the number of weak stabilizations of (x, K) that is necessary to perform to stabilize K is related to the probability that the stabilization of K ends with one particle at x, which is an important quantity for the proof of Theorem 1.1. In Section 3 we will upper bound this probability by introducing a new stabilization procedure.

3. Active phase on transient graphs

In this section we prove Theorem 1.1. We first state Theorem 3.1, where the probability $Q(x, K)$ that the vertex $x \in K$ hosts an S-particle after the stabilization of the finite set $K \subset V$ is bounded away from one for any value of $\lambda \in (0, \infty)$. In order for the next theorem to hold true the graph G does not need to be vertex-transitive.

Theorem 3.1. *Let $G = (V, E)$ be a locally-finite graph and let $K \subset V$ be a finite set. Then, for any set $K \subset V$, for any vertex $x \in K$, for any positive integer H,*

$$Q(x, K) \le 1 - \left(1 - \frac{G_{K^c}(x, x)}{H + 1}\right)\left(\frac{1}{1 + \lambda}\right)^H. \tag{4}$$

Theorem 1.1 is proved in the end of this section and will be a direct consequence of Theorem 3.1. We now introduce a new stabilization procedure that consists of ignoring sleep instructions at one fixed vertex and prove some auxiliary lemmas that are necessary for the proof of Theorem 3.1. After that, we prove Theorem 3.1 and Theorem 1.1.

We introduce the function T^x that associates to any instruction array τ a new instruction array $T^x(\tau)$ that is obtained from τ by ignoring all sleep instruction at x. More precisely, we define for any $y \in V$ and $j \in \mathbb{N}$,

$$\big(T^x(\tau)\big)^{y,j} := \begin{cases} \tau^{y,j} & \text{if } y \neq x, \\ \tau^{y,j} & \text{if } y = x \text{ and } \tau^{y,j} \neq \tau_{y\rho}, \\ \iota & \text{if } y = x \text{ and } \tau^{y,j} = \tau_{y\rho}, \end{cases}$$

recalling that ι denotes a neutral instruction. Moreover, for any $y, x \in V$, we let

$$m^e_{(x,K),\eta,\tau}(y) := m_{K,\eta,T^x(\tau)}(y) \tag{5}$$

be the number of instructions that are used at y when we stabilize the set K by ignoring sleep instructions at x. This function plays an important role in this section. For the proof of Theorem 3.1, we will not count the total number of instructions, but only the number of jump instructions. Thus, for any $y \in K$, we let

$$M^e_{(x,K),\eta,\tau}(y) := \left|\left\{\tau^{y,j} : j \in \left[0, m^e_{(x,K),\eta,\tau}(y)\right], \tau^{y,j} \neq \tau_{y\rho}\right\}\right|$$

be the number of jump instructions that are used at y when we stabilize K by ignoring sleep instructions at x. Similarly, we let $M_{K,\eta,\tau}(y)$ be the number of jump instructions that are used at y for the stabilization of K and $M^1_{(x,K),\eta,\tau}(y)$ be the number of jump instructions that are used at y for the weak stabilization of (x, K). In the next lemma we state some simple but important relations between these quantities.

Lemma 3.2. *Let η be an arbitrary particle configuration, let τ be an arbitrary instruction array, let $\tilde{\tau} = T^x(\tau)$ be obtained from τ by turning all the sleep instructions at x into neutral instructions. Then, we have that, for any vertex $y \in K$,*

$$m^1_{(x,K),\eta,\tau}(y) = m^1_{(x,K),\eta,\tilde{\tau}}(y), \qquad M^1_{(x,K),\eta,\tau}(y) = M^1_{(x,K),\eta,\tilde{\tau}}(y), \tag{6}$$

$$m^e_{(x,K),\eta,\tau}(y) \geq m_{K,\eta,\tau}(y), \qquad M^e_{(x,K),\eta,\tau}(y) \geq M_{K,\eta,\tau}(y). \tag{7}$$

Proof. When we perform the weak stabilization of (x, K), we topple x only if x contains at least two particles, so the sleep instructions at x have no effect. This leads to (6). The relations (7) follow from a direct application of monotonicity with enforced activation for stabilization (Lemma 2.4). □

For the next lemma we need to recall the notion of stabilization via weak stabilization that has been introduced in Section 2. Thus, let

$$A_{(x,K),\eta,\tau} := M^e_{(x,K),\eta,\tau}(x) - M^1_{(x,K),\eta,\tau}(x) \tag{8}$$

be the total number of jump instructions that are used at x when we stabilize K by ignoring sleep instructions at x and that are not used for the weak stabilization of (x, K).

Recall the stabilization-via-weak-stabilization procedure that has been introduced in Section 2.

Lemma 3.3. *Let $G = (V, E)$ be an arbitrary locally-finite graph and let $K \subset V$ be a finite set. Let η' be the particle configuration that is obtained after the stabilization of K. Then, for any $x \in K$, for any integer $\ell \geq 2$,*

$$\mathcal{P}\big(\eta'(x) = \rho, T_{(x,K)} = \ell\big) \leq \frac{\lambda}{1+\lambda}\left(\frac{1}{1+\lambda}\right)^{\ell-2}\mathcal{P}(A_{(x,K)} \geq \ell - 2). \tag{9}$$

Proof. First of all, note that for any integer $\ell \geq 2$,

$$\begin{aligned}\mathcal{P}\big(\eta'(x) = \rho, T_{(x,K)} = \ell\big) &= \mathcal{P}\big(T_{(x,K)} \geq \ell, \tau^{x,m^{\ell-1}_{(x,K)}+1} = \tau_{x\rho}\big) &(10)\\ &= \frac{\lambda}{1+\lambda}\mathcal{P}(T_{(x,K)} \geq \ell). &(11)\end{aligned}$$

This first equality holds true since, if after having completed $\ell - 1$ weak stabilizations the next instruction at x is sleep, then the stabilization is completed with one particle at x. The second equality follows from independence of instructions.

Recall that η_1 is the particle configuration that we obtain when the first weak stabilization of (x, K) is complete. Note that,

$$\begin{aligned}\{T_{(x,K)} \geq \ell\} &= \big\{\eta_1(x) = 1\big\} \cap \big\{\forall i \in [1, \ell - 2], \tau^{x,m^i_{(x,K)}(x)+1} \neq \tau_{x\rho}\big\} \\ &\subset \big\{M_K(x) - M^1_{(x,K)}(x) \geq \ell - 2\big\}. \end{aligned} \tag{12}$$

The equality holds true since, in order for the stabilization-via-weak-stabilization procedure to consist of at least $\ell \geq 2$ weak stabilizations, it is necessary that the first weak stabilization ends with one particle at x (if the weak stabilization ended with no particle at x then the stabilization of K would be completed and $T_{(x,K)}$ would be equal to one) and that the next $\ell - 2$ times the next instruction at x is a jump instruction. From the previous formula we obtain that

$$\begin{aligned}&\mathcal{P}(T_{(x,K)} \geq \ell) \\ &\quad = \mathcal{P}\big(\big\{\eta_1(x) = 1\big\} \cap \big\{\forall i \in [1, \ell - 2], \tau^{x,m^i_{(x,K)}(x)+1} \neq \tau_{x\rho}\big\} \cap \big\{M_K(x) - M^1_{(x,K)}(x) \geq \ell - 2\big\}\big) \\ &\quad = \sum_{\tilde{\eta} \in \Omega : \tilde{\eta}(x) = 1} \mathcal{P}\big(\big\{\forall i \in [1, \ell - 2], \tau^{x,m^i_{(x,K)}(x)+1} \neq \tau_{x\rho}\big\} \\ &\qquad \cap \big\{M_K(x) - M^1_{(x,K)}(x) \geq \ell - 2\big\} | \eta_1 = \tilde{\eta}\big)\mathcal{P}(\eta_1 = \tilde{\eta}). \end{aligned} \tag{13}$$

Now note that, for any particle configuration $\tilde{\eta}$ in the previous sum, we have that,

$$\begin{aligned}&\mathcal{P}\big(\big\{\forall i \in [1, \ell - 2], \tau^{x,m^i_{(x,K)}(x)+1} \neq \tau_{x\rho}\big\} \cap \big\{M_K(x) - M^1_{(x,K)}(x) \geq \ell - 2\big\} | \eta_1 = \tilde{\eta}\big) \\ &\quad \leq \mathcal{P}\big(\big\{\forall i \in [1, \ell - 2], \tau^{x,m^i_{(x,K)}(x)+1} \neq \tau_{x\rho}\big\} \cap \{A_{(x,K)} \geq \ell - 2\} | \eta_1 = \tilde{\eta}\big) \\ &\quad = \mathcal{P}\big(\forall i \in [1, \ell - 2], \tau^{x,m^i_{(x,K)}(x)+1} \neq \tau_{x\rho} | \eta_1 = \tilde{\eta}\big)\mathcal{P}(A_{(x,K)} \geq \ell - 2 | \eta_1 = \tilde{\eta}) \\ &\quad \leq \left(\frac{1}{1+\lambda}\right)^{\ell-2} \mathcal{P}(A_{(x,K)} \geq \ell - 2 | \eta_1 = \tilde{\eta}), \end{aligned} \tag{14}$$

where the first inequality follows from (7) and the equality follows from independence of instructions. Indeed, from Lemma 3.2 we have that, for any realization of η and τ, both $M^e_{(x,K)}(x)$ and $M^1_{(x,K)}(x)$ do not change if we turn sleep instructions at x into a neutral instruction, so the function $A_{(x,K)} = M^e_{(x,K)}(x) - M^1_{(x,K)}(x)$ is independent from the sleep instructions which are located at x. By replacing (14) in (13) we conclude the proof of the lemma. □

Remark 3.4. In [9] the quantity in the left-hand side of (9) is bounded from above by the probability that at least $\ell - 2$ instructions are used at x after the first weak stabilization, without distinguishing between jump and sleep instructions. Our enhancement is obtained by considering only jump instructions and, more importantly, by introducing a stabilization procedure (5) that ignores sleep instructions at one vertex, on which the quantity $A_{(x,K)}$ depends. Indeed, even though replacing $M_K(x) - M^1_{(x,K)}(x)$ by $A_{(x,K)}$ in the first inequality of (14) might make the probability larger, it allows to recover independence from sleep instructions and, thus, to split the second term of (14) into the product of two factors, which are then bounded from above separately.

In the next lemma we will bound from above the expectation of $A_{(x,K)}$. This will lead to an upper bound on the second factor of the last term in (14).

Lemma 3.5. *Let G be a locally-finite graph and let $K \subset V$ be a finite set. Then, for any $x \in K$,*

$$\boldsymbol{E}(A_{(x,K)}) \leq G_{K^c}(x, x), \tag{15}$$

where $\boldsymbol{E}$ is the expectation with respect to $\mathcal{P}$.

Proof. Note that the expectation of $A_{(x,K)}$ can be written as follows,

$$\boldsymbol{E}(A_{(x,K)}) = \sum_{k=0}^{\infty} \mathrm{Poi}_{\mu}(k)\big[\boldsymbol{E}_k\big(M^e_{(x,K)}(x)\big) - \boldsymbol{E}_k\big(M^1_{(x,K)}(x)\big)\big], \tag{16}$$

where $\boldsymbol{E}_k$ is the expectation $\boldsymbol{E}$ conditional on having precisely k particles starting from x at time 0 and $\mathrm{Poi}_{\mu}(k)$ is the probability that a Poisson random variable with mean μ has outcome k. We claim that, for any $k \in \mathbb{N}$,

$$\boldsymbol{E}_{k+1}\big(M^1_{(x,K)}(x)\big) = \boldsymbol{E}_k\big(M^e_{(x,K)}(x)\big), \tag{17}$$

and that

$$\boldsymbol{E}_{k+1}\big(M^1_{(x,K)}(x)\big) \leq \boldsymbol{E}_k\big(M^1_{(x,K)}(x)\big) + G_{K^c}(x,x). \tag{18}$$

By using (17) and (18) we obtain from (16) that,

$$\begin{aligned}\boldsymbol{E}(A_{(x,K)}) &= \sum_{k=0}^{\infty} \mathrm{Poi}_{\mu}(k)\big[\boldsymbol{E}_{k+1}\big(M^1_{(x,K)}(x)\big) - \boldsymbol{E}_k\big(M^1_{(x,K)}(x)\big)\big] \\ &\leq \sum_{k=0}^{\infty} \mathrm{Poi}_{\mu}(k) G_{K^c}(x,x) \\ &= G_{K^c}(x,x),\end{aligned}$$

obtaining the desired inequality (15). So, in order to conclude the proof, it remains to prove (17) and (18).

The equality (17) holds true since adding one particle at a x and never moving that particle is equivalent to stabilizing K by ignoring all the sleep instructions at x. For a formal proof, let η^{k+1} be an arbitrary particle configuration with $k+1$ particles at x and let η^k be obtained from η^{k+1} by removing one of the particles at x. Let τ be an arbitrary array and let $\tilde{\tau}$ be obtained from τ by turning sleep instructions at x into neutral instructions. We use the instructions of τ for η^{k+1} and the instructions of $\tilde{\tau}$ for η^k simultaneously. More specifically, let $\alpha = (x_1, x_2, \dots, x_{|\alpha|})$ be a sequence that stabilizes η^k in K by using the instructions of $\tilde{\tau}$. Since any step of α is legal for η^k when we use $\tilde{\tau}$ (a neutral instruction is always legal), it is also WS-legal for η^{k+1} when we use τ. Moreover, since $\Phi_\alpha \eta^k$ is stable in K and has no particle at x when we use $\tilde{\tau}$, then $\Phi_\alpha \eta^{k+1}$ is weakly stable in (x, K) when we use τ. Thus, the sequence α stabilizes η^k in K when we use the instrutions of $\tilde{\tau}$ and stabilizes η^{k+1} weakly in (x, K) when we use the instructions of τ. From the Abelian property we deduce that for any $y \in K$, $m^1_{(x,K),\eta^{k+1},\tau}(y) = m^e_{(x,K),\eta^k,\tau}(y)$. This implies (17).

We now prove (18), adapting the steps of a similar proof that appears in [9] to our setting. Let η be an arbitrary particle configuration with $k+1$ particles at x. In the first step, we move one of the particles that is at x until it leaves the set K, ignoring any sleep instruction. During this step of the procedure we might use some instruction at x that is WS-illegal (but legal). The expected number of times a jump instruction is used at x during this step is $G_{K^c}(x,x)$. In the second step, we perform the weak stabilization of (x, K) with the remaining particles. Let $M'_{K,\eta,\tau}(x)$ be the total number of jump instructions that are used at x. We have that, by monotonicity with enforced activation and by the least action principle for weak stabilization,

$$M^1_{(x,K),\eta,\tau}(x) \leq M'_{K,\eta,\tau}(x).$$

Moreover, since in the second step we start from a configuration with k particles at x and instructions are independent,

$$\boldsymbol{E}_{k+1}\big(M'_K(x)\big) = \boldsymbol{E}_k\big(M^1_{(x,K)}(x)\big) + G_{K^c}(x,x).$$

By using the two previous relations we obtain (18). $\square$

3.1. *Proof of Theorem* 3.1

The proof of Theorem 3.1 is a direct consequence of Lemma 3.3 and Lemma 3.5.

From Lemma 3.3 we obtain that,

$$
\begin{aligned}
Q(x,K) &= \sum_{\ell=2}^{\infty} \mathcal{P}\big(\eta'(x)=\rho, T_{(x,K)}=\ell\big) \\
&\leq \sum_{\ell=2}^{\infty} \frac{\lambda}{1+\lambda}\left(\frac{1}{1+\lambda}\right)^{\ell-2} \mathcal{P}(A_{(x,K)} \geq \ell-2),
\end{aligned}
$$

having used the fact that $\mathcal{P}(\eta'(x)=\rho, T_{(x,K)}=1)=0$ and that $T_{(x,K)}>0$ almost surely. We now perform simple calculations in order to prove the quantitative upper bound of Theorem 3.1. By using the Markov's inequality and Lemma 3.5, we obtain that for any positive integer H,

$$
\begin{aligned}
\mathcal{P}\big(\eta'(x)=\rho\big) &\leq \sum_{\ell=0}^{H-1} \frac{\lambda}{1+\lambda}\left(\frac{1}{1+\lambda}\right)^{\ell} + \sum_{\ell=H}^{\infty} \frac{\lambda}{1+\lambda}\left(\frac{1}{1+\lambda}\right)^{\ell} \mathcal{P}(A_{(x,K)} \geq \ell) \\
&\leq \frac{\lambda}{1+\lambda}\left[\sum_{\ell=0}^{H-1}\left(\frac{1}{1+\lambda}\right)^{\ell} + \frac{G_{K^c}(x,x)}{H+1}\sum_{\ell=H}^{\infty}\left(\frac{1}{1+\lambda}\right)^{\ell}\right] \\
&= \frac{\lambda}{1+\lambda}\left[\frac{1}{1-\frac{1}{1+\lambda}} - \left(1-\frac{G_{K^c}(x,x)}{H+1}\right)\left(\frac{1}{1+\lambda}\right)^{H}\frac{1}{(1-\frac{1}{1+\lambda})}\right] \\
&= 1-\left(1-\frac{G_{K^c}(x,x)}{H+1}\right)\left(\frac{1}{1+\lambda}\right)^{H}.
\end{aligned}
$$

This concludes the proof of Theorem 3.1.

3.2. *Proof of Theorem* 1.1

Suppose that the graph is vertex-transitive and transient. We have that for any set $K \subset V$ and any vertex $x \in K$, $G_{K^c}(x,x) \leq G(0,0) < \infty$. Define,

$$
g(\lambda) := \inf\left\{1-\left(1-\frac{G(0,0)}{H+1}\right)\left(\frac{1}{1+\lambda}\right)^{H} : H \in \mathbb{N}\right\},
$$

and note that from Theorem 3.1 it follows that,

$$
\forall K \subset V, \forall x \in K, \quad Q(x,K) \leq g(\lambda) < 1. \tag{19}
$$

By choosing $H^* := \lceil\sqrt{\frac{G(0,0)}{\log(1+\lambda)}}\rceil$, we deduce that $\limsup_{\lambda\to 0}\frac{g(\lambda)}{\lambda^{\frac{1}{2}}} < \infty$.

Suppose now that $\mu > g(\lambda)$. Since from (19) we have that the expected number of particles after the stabilization of K is at most $g(\lambda)|K|$, it follows that the expected number of particles leaving K during the stabilization of K is at least $(\mu - g(\lambda))|K|$. Since a positive density of particles leaves the set, since the graph is amenable, and since K is an arbitrary finite set, we deduce from [6], Proposition 2, that the system is active. This implies that $\mu_c(\lambda) \leq g(\lambda)$ for any $\lambda \in (0,\infty)$ and concludes the proof.

4. Fixation on non-amenable graphs

In this section we prove Theorem 1.2. We start stating an auxiliary lemma which provides an upper bound and a lower bound for the expected number of times that particles starting “close” or “far” from the boundary of a ball visit the centre of that ball. Afterwards, we state Proposition 4.2, which connects the values of the probabilities $\{Q(x, B_L)\}_{x\in B_L}$ to the expected number of times a particle visits the origin. Finally, we prove Theorem 1.2 by showing that, if one assumes that ARW is active and that $\mu < \frac{\lambda}{1+\lambda}$, then Proposition 4.2 cannot hold. This leads to the desired contradiction.

Lemma 4.1. *Let G be a vertex-transitive graph where the random walk has a positive speed. There exists $C_1 = C_1(G) \in (0, \infty)$ such that, for any $\delta \in (0, 1)$, there exists an infinite increasing sequence of integers $\{L_n\}_{n\in\mathbb{N}}$ such that*

$$\sum_{x\in B_{L_n}\setminus B_{(1-\delta)L_n}} G_{B^c_{L_n}}(x, 0) \leq C_1\delta L_n. \tag{20}$$

Moreover, there exists $C_2 = C_2(G) \in (0, \infty)$ such that, for any $L \in \mathbb{N}$,

$$\sum_{y\in B_{(1-\delta)L}} G_{B^c_L}(y, 0) \geq C_2(1-\delta)L. \tag{21}$$

Proof. We start with the proof of (20). For any pair of real numbers $r_2 > r_1$, let

$$\Xi(r_1, r_2) := E_0\left(\sum_{t=0}^{\infty} \mathbb{1}\left\{X(t) \in B_{r_2} \setminus B_{r_1}\right\}\right),$$

be the expected number of vertices in the ring $B_{r_2} \setminus B_{r_1}$ which are visited by the random walk, with $\Xi(0, r_2)$ being the expected number of vertices in the ball B_{r_2} which are visited by the random walk. By regularity of the graph, for any integer n and $x \in B_n$, we have that,

$$P_x(\tau_0 < \tau_{B^c_n}) = G_{B^c_n\cup\{0\}}(x, x)P_0\big(\tau_x < \tau^+_{\{0\}\cup B^c_n}\big). \tag{22}$$

Then, for any $\delta' \in (0, 1)$,

$$\begin{aligned}
\sum_{x\in B_n\setminus B_{(1-\delta')n}} G_{B^c_n}(x, 0) &= G_{B^c_n}(0, 0) \sum_{x\in B_n\setminus B_{(1-\delta')n}} P_x(\tau_0 < \tau_{B^c_n}) && (23)\\
&\leq G(0, 0)^2 \sum_{x\in B_n\setminus B_{(1-\delta')n}} P_0(\tau_x < \tau_{B^c_n}) && (24)\\
&\leq G(0, 0)^2\,\Xi\big((1-\delta')n, n\big), && (25)
\end{aligned}$$

where we used (22) and vertex-transitivity. We have that,

$$\forall n \in \mathbb{N}, \quad \Xi(0, n) \geq \sum_{\ell=1}^{\lfloor \frac{1}{\delta'} \rfloor} \Xi\big(\delta' n(\ell-1), \delta' n\ell\big). \tag{26}$$

Since the random walk has a positive speed, we have that there exists $K = K(G)$ such that,

$$\forall n \in \mathbb{N}, \quad \Xi(0, n) \leq Kn \tag{27}$$

(see for example [9], eq. (5.16), for a proof, and note the two typos present there: $P(X(\Delta_{(2k-1)L}) > L)$ should be replaced by $P(\forall \ell \in \{1, \dots, k\}, X(\Delta_{(2\ell-1)L}) \in B_L)$, which is bounded from above by ξ^k). Conditions (26) and (27) imply that,

$$\forall n \in \mathbb{N}, \exists \ell_n \in \left[\frac{1}{2\delta'}, \frac{1}{\delta'}\right] \quad \text{s.t.} \quad \Xi\big(\delta' n(\ell_n - 1), \delta' n\ell_n\big) \leq 4K\delta' n. \tag{28}$$

For any $n \in \mathbb{N}$, define now $L_n := \lfloor \delta' n \ell_n \rfloor$. From (28) we obtain that, for any large enough n,

$$\begin{aligned}\Xi\left(L_n\left(1-\frac{\delta'}{2}\right), L_n\right) &\le \Xi\left(L_n\left(\frac{\delta' n \ell_n}{L_n}-\delta'\right), L_n\right) \le \Xi\left(L_n\left(\frac{\delta' n \ell_n}{L_n}-\frac{1}{\ell_n}\right), L_n\right)\\ &= \Xi\left(\delta' n \ell_n - \frac{L_n}{\ell_n}, L_n\right) \le \Xi\big(\delta' n(\ell_n-1), \delta' n \ell_n\big) \le 4K\frac{\delta' n \ell_n}{\ell_n}\\ &\le 5K\frac{L_n}{\ell_n} \le 10K\delta' L_n. \end{aligned} \tag{29}$$

The proof of (20) follows by defining $\delta = \frac{\delta'}{2}$, $C_1 = 20KG(0,0)^2$ and by selecting an infinite increasing subsequence of $\{L_n\}_{n\in\mathbb{N}}$. We now prove (21). We have that,

$$\sum_{y\in B_L} G_{B^c_{(1-\delta)L}}(y,0) = G_{B^c_L}(0,0)\mathbb{E}_0\left(\sum_{t=0}^{\tau^+_{B^c_L\cup\{0\}}} \mathbb{1}\big\{X(t)\in B_{(1-\delta)L}\big\}\right) \tag{30}$$

$$\ge G_{B^c_L}(0,0)p(1-\delta)L \tag{31}$$

$$\ge C_2(1-\delta)L, \tag{32}$$

where for the second inequality we used conditional expectation and we let $p = P_0\ (X(t)\neq 0\ \forall t>0) > 0$ be the probability that the random walk does not return to its starting vertex, which is positive since the random walk has a positive speed and is then transient. □

The next proposition relates the expected number of particles visiting the origin to the quantities $\{Q(y,B_L)\}_{y\in B_L}$. The proof of Theorem 1.2 will only use the fact that (33) is nonnegative.

Proposition 4.2. *Let G be a graph. For any $L\in\mathbb{N}$,*

$$\mathbb{E}_{B_L}\big(M_{B_L}(0)\big) = \sum_{y\in B_L} G_{B^c_L}(y,0)\big(\mu - Q(y,B_L)\big). \tag{33}$$

Proof. In order to prove (33), we use the ghost explorer technique similarly to [7,9]. First, we let the particles move until the ball B_L is stable. This means that some particles leave B_L being absorbed at the boundary and other particles remain in B_L after having turned into the S-state.

We now let a *ghost particle* start an independent simple random walk from every vertex that is occupied by an S-particle in B_L. Ghost particles are "killed" whenever they visit B^c_L. We let $R_{B_L}(x)$ be the total number of visits at $x\in B_L$ by ghost particles. We let $W_{B_L}(x)$ be the total number of visits at x by normal particles or by ghost particles. Now we have that,

$$M_{B_L}(x) = W_{B_L}(x) - R_{B_L}(x).$$

As both particles and ghost particles stop only when they leave B_L and as random walks are independent, we have that,

$$\tilde{E}_{B_L}\big[W_{B_L}(x)\big] = \mu\sum_{y\in B_L} G_{B^c_L}(y,x),$$

where $\tilde{E}_{B_L}$ is the expectation in the enlarged probability space of activated random walks and ghost particles. Moreover, since precisely one ghost leaves from every vertex where an S-particle is located,

$$\tilde{E}_{B_L}\big[R_{B_L}(x)\big] = \sum_{y\in B_L} G_{B^c_L}(y,x)Q(y,B_L),$$

by linearity of expectation. The proof of (33) is concluded again by using linearity of expectation. □

Proof of Theorem 1.2. The proof is by contradiction. First of all note that from Lemma 3.5 and (33) we obtain that, for any $\delta \in (0,1)$, there exists an infinite increasing sequence of integers $\{L_n\}_{n\in\mathbb{N}}$ such that for any $\mu > 0$ and $\lambda \in (0,\infty)$,

$$\mathbb{E}\big(M_{B_{L_n}}(0)\big) \leq \sum_{y\in B_{(1-\delta)L_n}} \big(\mu - Q(y, B_L)\big) G_{B^c_{L_n}}(y,0) + \mu C_1 \delta L_n.$$

Since the number of visits cannot be negative, we obtain that,

$$\sum_{y\in B_{(1-\delta)L_n}} G_{B^c_{L_n}}(y,0)\big(Q(y, B_{L_n}) - \mu\big) \leq \mu C_1 \delta L_n. \tag{34}$$

We will show that, if G is such that the random walk has a positive speed, then condition (34) cannot be satisfied when $\mu < \frac{\lambda}{1+\lambda}$ and ARW is active, obtaining a contradiction and concluding that ARW fixates when $\mu < \frac{\lambda}{1+\lambda}$. Thus, first note that,

$$Q(x, B_L) \geq \mathcal{P}\big(m_{B_L(x)}(x) \geq 1\big)\frac{\lambda}{1+\lambda}. \tag{35}$$

This inequality was proved in [9] and follows from the next relation,

$$\big\{m^1_{(x,B_L(x))}(x) \geq 1\big\} \cap \big\{\tau^{x, m^1_{(x,K)}(x)+1} = \tau_{x\rho}\big\} \subset \big\{\eta'(x) = \rho\big\}. \tag{36}$$

Indeed, if one concludes weak stabilization of (x,K) with one particle at x and the next instruction at x is sleep, then the stabilization is completed with one particle at x. Moreover, at least one instruction is used at x during the stabilization of K if and only if at least one instruction is used at x during the weak stabilization of (x,K). By independence of instructions, one obtains (35).

Thus, assume that $\mu < \frac{\lambda}{1+\lambda}$ and that ARW is active and let $D := \frac{\frac{\lambda}{1+\lambda}-\mu}{2} > 0$. We have that, for any $\delta > 0$ and for any L large enough depending on δ,

$$\forall x \in B_{(1-\delta)L}, \quad Q(x, B_L) \geq \mathcal{P}\big(m_{B_{\delta L}(x)}(x) \geq 1\big)\frac{\lambda}{1+\lambda} \tag{37}$$

$$\geq \mathcal{P}\big(m_{B_{\delta L}}(0) \geq 1\big)\frac{\lambda}{1+\lambda} \tag{38}$$

$$\geq \mu + D, \tag{39}$$

where the first inequality follows from (35) and from monotonicity (Lemma 2.2), for the second inequality we used vertex-transitivity and for the third inequality we used the definition of activity and Lemma 2.3.

Then, for any $\delta \in (0,1)$ and for any L we deduce from Lemma 3.5 that,

$$\sum_{y\in B_{(1-\delta)L}} G_{B^c_L}(y,0)\big(Q(y, B_L) - \mu\big) \geq DC_2(1-\delta)L. \tag{40}$$

Choose now $\delta \in (0,1)$ small enough such that, for any L large enough,

$$\sum_{y\in B_{(1-\delta)L}} G_{B^c_L}(y,0)\big(Q(y, B_L) - \mu\big) > C_1\delta L.$$

Since the previous condition holds for any L large enough, we deduce that an infinite increasing sequence $\{L_n\}_{n\in\mathbb{N}}$ satisfying (34) cannot exist when $\mu \leq \frac{\lambda}{1+\lambda} \leq 1$. This leads to the desired contradiction. □

Acknowledgements

The author thanks Elisabetta Candellero and Alexandre Stauffer for very interesting discussions, Shirshendu Ganguly and Leonardo Rolla for useful comments, and the anonymous referee for his useful suggestions.

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