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Continuous-state branching processes, extremal processes and super-individuals

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Abstract. The long-term behavior of flows of continuous-state branching processes are characterized through subordinators and extremal processes. The extremal processes arise in the case of supercritical processes with infinite mean and of subcritical processes with infinite variation. The jumps of these extremal processes are interpreted as specific initial individuals whose progenies overwhelm the population. These individuals, which correspond to the records of a certain Poisson point process embedded in the flow, are called super-individuals. They radically increase the growth rate to $+\infty$ in the supercritical case, and slow down the rate of extinction in the subcritical one.

Résumé. Les comportements en temps long des flots de processus de branchement en temps et espace continus sont caractérisés par des subordinateurs et des processus extrémaux. Les processus extrémaux apparaissent dans le cas des processus sur-critiques de moyenne infinie et des processus sous-critiques à variation infinie. Les sauts de ces processus extrémaux sont interprétés comme des individus initiaux spécifiques dont les descendances envahissent la population. Ces individus, qui correspondent aux instants de records d'un certain processus ponctuel de Poisson, sont appelés super-individus. Ils augmentent de façon radicale la vitesse de divergence dans le cas sur-critique et diminuent celle d'extinction dans le cas sous-critique.

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1. Introduction and main result

We consider branching processes in continuous-time with continuous-state space (CSBPs) as defined by Jiřina [17] and Lamperti [22,23]. These processes are the continuous analogues of Galton–Watson Markov chains. A feature of the continuous-state space is that the population can become extinguished asymptotically while maintaining a positive size at any time. The process, in this case, is said to be *persistent*. Grey in [14] and Bingham [5] have studied the long-term behavior of CSBPs. It is shown in [14] that any CSBP with finite mean or finite variation can be linearly renormalized to converge almost-surely. Duquesne and Labbé in [9] have generalized this result by showing that any *flow* of CSBPs with finite mean or finite variation can be renormalized to converge towards a subordinator. This convergence corresponds to the natural idea that all individuals have progenies that grow or decline at the same scale. We show in this article that in the case of CSBPs with infinite mean or infinite variation, the flow can be renormalized (in a non-linear way) to converge towards the partial records of a Poisson point process. The limit process has therefore a very different nature than that in the case of finite mean or finite variation. An intuitive explanation is that the initial individuals have progenies that grow or decline at different rates. The limit process will take into account only the progeny of the individual whose growth rate is *maximal*. An important example of persistent CSBP with infinite mean and infinite variation is the CSBP of Neveu, whose branching mechanism is

$$\Psi(u) = u \log u = \int_0^{+\infty} \big(e^{-ux} - 1 + ux 1_{\{x \le 1\}}\big) x^{-2}\, \mathrm{d}x.$$

Let $(X_t(x), t \ge 0)$ be a Neveu's CSBP with initial value x. A well-known result of Neveu [26], states that for any fixed x

$$e^{-t} \log X_t(x) \underset{t\to+\infty}{\longrightarrow} Z(x) \quad \text{a.s.,} \tag{1}$$

where $Z(x)$ has a Gumbel law: for all $z \in \mathbb{R}$, $\mathbb{P}(Z(x) \le z) = e^{-xe^{-z}}$. We show that for any CSBP starting from a fixed initial value x, which is non-explosive and has infinite mean or is persistent and has infinite variation, one can find a *non-linear* renormalisation, in the same vein as (1), that converges towards a certain random variable $Z(x)$. This non-linear renormalisation reflects that the population grows or declines at a super-exponential rate. The problem of finding a renormalisation in the case of Galton–Watson processes with infinite mean has been considered by many authors, we refer to Grey [15], Barbour and Schuh [1] and the references therein. We will adapt Grey's method to the continuous-state space setting. Following the seminal idea of Bertoin and Le Gall in [4], a *continuous population model* can be defined by considering a flow of subordinators $(X_t(x), t \ge 0, x \ge 0)$. Formally, the progeny at time t of the individual x is given by $\Delta X_t(x) = X_t(x) - X_t(x-)$ the jump at x of the subordinator X_t. The main purpose of this paper is to investigate the limit process $(Z(x), x \ge 0)$ and to interpret it in terms of the population model. We will see that $(Z(x), x \ge 0)$ is an *extremal process* whose law is explicit in terms of the branching mechanism. For instance, in the Neveu case, the process in (1) is an extremal-Λ process with $\Lambda(x) = \exp(-e^{-x})$. In order to give some insights for the interpretation, we borrow some ideas of Bertoin, Fontbona and Martinez in [3], Labbé [21] and Duquesne and Labbé [9]. In [3], the authors show that in a supercritical CSBP with infinite variation, some initial individuals have progenies that tend to $+\infty$. These individuals are called prolific and are responsible for the infinite growth of the process.

Definition 1. The individual x is said to be prolific if $\lim_{t\to+\infty} \Delta X_t(x) = +\infty$. Denote by $\mathcal{P}$ the set of prolific individuals

$$\mathcal{P} := \Big\{x > 0; \lim_{t\to+\infty} \Delta X_t(x) = +\infty\Big\}.$$

We shall see that in a non-explosive CSBP with infinite mean (with or without infinite variation) and in a persistent CSBP with infinite variation, some individuals have a progeny that overwhelms the total progeny of all individuals *below* them (see Definition 2). These individuals will be called *super-individuals*.

Definition 2. The individual x is said to be a super-individual if $\lim_{t\to+\infty} \frac{\Delta X_t(x)}{X_t(x-)} = +\infty$ a.s. Denote by $\mathcal{S}$ the set of super-individuals

$$\mathcal{S} := \Big\{x > 0; \lim_{t\to+\infty} \frac{\Delta X_t(x)}{X_t(x-)} = +\infty\Big\}. \tag{2}$$

We stress that there is an order between the super-individuals: if x_1 and x_2 are in $\mathcal{S}$ and $x_1 < x_2$ then the progeny of x_2 overwhelms that of x_1, since $0 \le \frac{\Delta X_t(x_1)}{\Delta X_t(x_2)} \le \frac{X_t(x_2-)}{\Delta X_t(x_2)} \underset{t\to+\infty}{\longrightarrow} 0$. In the supercritical case, we say that an individual is *super-prolific*, if it is a prolific super-individual. We will see that only certain prolific individuals are super-prolific. In the subcritical case, since all initial individuals have progenies that get extinct (in finite time or not), no prolific individual may exist. However, when the process is persistent with infinite variation, super-individuals do exist. They are individuals whose progeny decays at a much slower rate than all individuals below them. The super-individuals in the subcritical case and the super-prolific individuals in the supercritical case will correspond to the jumps times of the extremal process $(Z(x), x \ge 0)$. In the finite variation case and finite mean case, $\mathcal{S}$ is degenerate (empty or reduced to a single point) and there are basically no super-individuals. A Poisson construction of the flow of subordinators

$(X_t(x), t \geq 0)$ is given in [9] for all branching mechanisms Ψ. In the infinite mean or infinite variation case, this Poisson construction, recalled in Section 2.2, allows us to determine a Poisson point process $\mathcal{M}$, as shown in Lemma 7 and Lemma 16, whose partial records correspond to $(Z(x), x \geq 0)$. We state now our main result.

Theorem 1. *Consider* $(X_t(x), t \geq 0, x \geq 0)$ *a flow of* CSBPs(Ψ) *as defined in* (16) *and* (17).

(i) *Assume* $\Psi'(0+) = -\infty$ *and* $\int_0 \frac{\mathrm{d}u}{\Psi(u)} = -\infty$. *Call* ρ *the largest root of* Ψ, *fix* $\lambda_0 \in (0, \rho)$ *and define* $G(y) := \exp(-\int_y^{\lambda_0} \frac{\mathrm{d}u}{\Psi(u)})$ *on* $(0, \rho)$. *Then, almost-surely for all* $x \geq 0$

$$e^{-t} G\left(\frac{1}{X_t(x)} \wedge \rho\right) \underset{t \to +\infty}{\longrightarrow} Z(x) = \sup_{x_i \leq x} Z_i,$$

where $\mathcal{M} := \sum_{i \in I} \delta_{(x_i, Z_i)}$ *is a Poisson point process over* $\mathbb{R}_+ \times (0, \infty)$ *with intensity* $\mathrm{d}x \otimes \mu(\mathrm{d}z)$ *whose tail is* $\bar{\mu}(z) = G^{-1}(z)$ (*the inverse function of* G). *Moreover,*

$$\mathcal{S} \cap \mathcal{P} = \{x > 0; \Delta Z(x) > 0\} \quad a.s.$$

(ii) *Assume* $\Psi'(0+) \geq 0$, $\Psi(u)/u \underset{u \to +\infty}{\longrightarrow} +\infty$ *and* $\int^{+\infty} \frac{\mathrm{d}u}{\Psi(u)} = +\infty$. *Fix* $\lambda_0 \in (0, +\infty)$ *and define* $G(y) := \exp(-\int_{\lambda_0}^{y} \frac{\mathrm{d}u}{\Psi(u)})$ *on* $(0, +\infty)$. *Then, almost-surely for all* $x \geq 0$

$$e^{t} G\left(\frac{1}{X_t(x)}\right) \underset{t \to +\infty}{\longrightarrow} Z(x) = \sup_{x_i \leq x} Z_i,$$

where $\mathcal{M} := \sum_{i \in I} \delta_{(x_i, Z_i)}$ *is a Poisson point process over* $\mathbb{R}_+ \times (0, \infty)$ *with intensity* $\mathrm{d}x \otimes \mu(\mathrm{d}z)$ *whose tail is* $\bar{\mu}(z) = G^{-1}(z)$. *Moreover,*

$$\mathcal{S} = \{x > 0; \Delta Z(x) > 0\} \quad a.s.$$

The Poisson point process $\mathcal{M}$ represents the initial individuals with their asymptotic growth rates. In the supercritical case, we will see that non-prolific individuals have growth rates Z_i equal to 0. In the subcritical case, we shall see that there are infinitely many super-individuals near zero. We highlight that the function G^{-1} can be written explicitly in terms of the cumulant of the CSBP(Ψ).

In [9], the authors establish a classification of branching mechanims for which the population concentrates on the progeny of a single individual. This phenomenon is called *Eve property*. More precisely, the population started from a *fixed* size x has the Eve property if there exists a random variable $\mathsf{e} \in [0, x]$, such that

$$\frac{\Delta X_t(\mathsf{e})}{X_t(x)} \underset{t \to \zeta_x}{\longrightarrow} 1 \quad \text{a.s.}, \tag{3}$$

where $\zeta_x := \inf\{t \geq 0; X_t(x) \in \{0, +\infty\}\} \in \mathbb{R}_+ \cup \{+\infty\}$. The individual e is called the Eve. We refer to Definition 1.1 in [21] and Definition 0.1 in [9] for equivalent definitions. Unsurprisingly, the Eve property holds precisely when extremal processes arise (i.e. in the infinite variation or infinite mean case). From (2) and (3), we see that the Eve e must be a super-individual whose progeny overwhelms those of individuals in $(\mathsf{e}, x]$. In our setting, the Eve of the population started with size x is therefore characterized as the *last* super-individual in $[0, x]$. In other words, the Eve is the individual whose growth is the fastest in the non-explosive supercritical case with infinite mean or whose decay is the slowest in the persistent subcritical one with infinite variation. This corresponds to the record of the Poisson point process $\mathcal{M}$ in $[0, x]$. In particular, this *forward-in-time* argument allows us to follow the Eve of the population started from x when x evolves in the half-line. Labbé, in [21], has shown that when the population has an Eve, one can define a recursive sequence of Eves on which the population concentrates. We stress that super-individuals and successive Eves are not the same individuals. Indeed, the successive Eves are i.i.d. uniform in $[0, x]$ (see Proposition 4.13 in [21]), whereas the super-individuals are ordered and therefore not independent.

We wish to mention that Bertoin et al. in [3] have shown that the number of prolific individuals, when time evolves, is an immortal branching process. This discrete process is related to the backbone decomposition which has been deeply studied by Berestycki et al [2], Kyprianou et al [19,20] and by Duquesne and Winkel [10] in the framework of random trees. We will not address here the study of the number of super-individuals in time. Moreover, no spatial motion is taken into account in this work. We refer the reader to Fleischmann and Sturm [12] and Fleischmann and Wachtel [13] where superprocesses with Neveu's branching mechanism are studied. Lastly, the CSBP of Neveu and its limit (1) have been used in the study of Derrida's random energy model by Neveu [26], Bovier and Kourkova [6] and Huillet [16].

The paper is organized as follows. In Section 2, we recall the definition of a continuous-state branching process and some of its important properties. We describe the Poisson construction of the continuous population model (as in [9]). Then we gather some results about extremal processes. In Section 3, we start by a brief recall of the convergence of supercritical CSBPs with finite mean towards subordinators, established in [9]. We deduce that in this case except the first prolific individual, there is no super-individual. Then, we focus on CSBPs with infinite mean and we establish the finite-dimensional convergence towards an extremal process (Theorem 3). Afterwards we prove Theorem 1(i) through three Lemmas: Lemma 7, Lemma 9 and Lemma 13. In Section 4, we study the subcritical processes. The organisation is similar. Theorem 4 yields the convergence towards an extremal process characterized by its finite-dimensional marginal laws. Lemma 15, Lemma 16 and Lemma 17 are obtained similarly as in the supercritical case. By combining them with Theorem 4, we obtain Theorem 1(ii). In Section 4.2, we treat the special case of the Neveu's CSBP.

2. Preliminaries

Notation. If x and y are two real numbers. We denote their maximum by $x \vee y$, and their minimum by $x \wedge y$. If X and Y are two random variables, $X \overset{d}{=} Y$ means that X and Y have the same law.

2.1. *Continuous-state branching processes*

Our main references are Chapter 12 of Kyprianou's book [18] and Chapter 3 of Li's book [24]. A positive Markov process $(X_t(x), t \geq 0)$ with $X_0(x) = x \geq 0$ is a continuous-state branching process in continuous time (CSBP for short) if for any $y \in \mathbb{R}_+$

$$\bigl(X_t(x+y), t \geq 0\bigr) \overset{d}{=} \bigl(X_t(x), t \geq 0\bigr) + \bigl(\tilde{X}_t(y), t \geq 0\bigr), \tag{4}$$

where $(\tilde{X}_t(y), t \geq 0)$ is an independent copy of $(X_t(y), t \geq 0)$. The branching property (4) ensures the existence of a map $\lambda \mapsto v_t(\lambda)$, called cumulant, such that for all $\lambda \geq 0$ and all $t, s \geq 0$

$$\mathbb{E}\bigl[e^{-\lambda X_t(x)}\bigr] = \exp\bigl(-x v_t(\lambda)\bigr) \quad \text{and} \quad v_{s+t}(\lambda) = v_s \circ v_t(\lambda). \tag{5}$$

Moreover, there exists a unique function Ψ of the form

$$\Psi(q) = \frac{\sigma^2}{2} q^2 + \gamma q + \int_0^{+\infty} \bigl(e^{-qx} - 1 + qx 1_{\{x \leq 1\}}\bigr) \pi(\mathrm{d}x) \tag{6}$$

with $\gamma \in \mathbb{R}$, $\sigma \geq 0$, and π a σ-finite measure carried on $\mathbb{R}_+$ satisfying

$$\int_0^{+\infty} \bigl(1 \wedge x^2\bigr) \pi(\mathrm{d}x) < +\infty$$

such that the map $t \mapsto v_t(\lambda)$ is the unique solution to the integral equation

$$\forall t \in [0, +\infty), \forall \lambda \in (0, +\infty)/\{\rho\}, \qquad \int_{v_t(\lambda)}^{\lambda} \frac{\mathrm{d}z}{\Psi(z)} = t, \tag{7}$$

where $\rho = \inf\{z > 0, \Psi(z) \geq 0\} \in [0, +\infty]$. The process is said to be supercritical if $\Psi'(0+) \in [-\infty, 0)$ (in which case $\rho \in (0, +\infty]$), subcritical if $\Psi'(0+) \in (0, +\infty)$ and critical if $\Psi'(0+) = 0$ (in these last two cases $\rho = 0$).

Theorem 2 (Grey [14]). *Consider* $(X_t(x), t \geq 0)$ *a* CSBP(Ψ) *started from* x.

(i) *For any* $x \geq 0$,

$$\mathbb{P}\Big(\lim_{t\to+\infty} X_t(x) = 0\Big) = 1 - \mathbb{P}\Big(\lim_{t\to+\infty} X_t(x) = +\infty\Big) = e^{-x\rho}.$$

One has $\rho = +\infty$ *if and only if* $-\Psi$ *is the Laplace exponent of a subordinator. In this case, the process is non-decreasing and tends to* $+\infty$ *almost-surely.*

(ii) *The process is almost-surely not absorbed at* 0 *if and only if*

$$\int^{+\infty} \frac{\mathrm{d}u}{|\Psi(u)|} = +\infty \quad (\textit{persistence}). \tag{8}$$

If $\int^{+\infty} \frac{\mathrm{d}u}{\Psi(u)} < +\infty$, *the limits* $\bar{v}_t := \lim_{\lambda\to+\infty} v_t(\lambda) \in (0, +\infty)$ *for any* $t \geq 0$ *and* $\bar{v} := \lim_{t\to+\infty} \downarrow \bar{v}_t$ *exist. Moreover for all* $t \geq 0$ $\frac{\mathrm{d}}{\mathrm{d}t}\bar{v}_t = -\Psi(\bar{v}_t)$ *with* $\bar{v}_0 = +\infty$ *and*

$$\mathbb{P}\big(X_t(x) = 0\big) = \exp(-x\bar{v}_t) \quad \textit{and} \quad \mathbb{P}\big(\exists t \geq 0 : X_t(x) = 0\big) = \exp(-x\bar{v}).$$

(iii) *The process is almost-surely not absorbed in* $+\infty$ *if and only if*

$$\int_0 \frac{\mathrm{d}u}{|\Psi(u)|} = +\infty \quad (\textit{non-explosion}). \tag{9}$$

If $\int_0 \frac{\mathrm{d}u}{|\Psi(u)|} < +\infty$, *the limits* $\underline{v}_t := \lim_{\lambda\to 0} v_t(\lambda) \in (0, +\infty)$ *for any* $t \geq 0$ *and* $\underline{v} := \lim_{t\to+\infty} \uparrow \underline{v}_t$ *exist. Moreover for all* $t \geq 0$, $\frac{\mathrm{d}}{\mathrm{d}t}\underline{v}_t = -\Psi(\underline{v}_t)$ *with* $\underline{v}_0 = 0$ *and*

$$\mathbb{P}\big(X_t(x) = +\infty\big) = 1 - \exp(-x\underline{v}_t) \quad \textit{and} \quad \mathbb{P}\big(\exists t \geq 0 : X_t(x) = +\infty\big) = 1 - \exp(-x\underline{v}).$$

We classify now the mechanisms Ψ according to their behaviour near 0 and $+\infty$. For any Ψ as in (6),

$$\Psi'(0+) = \lim_{u\to 0} \frac{\Psi(u)}{u} = \gamma - \int_1^{+\infty} x\pi(\mathrm{d}x) \in [-\infty, +\infty) \quad \text{(mean)}.$$

One can show from (5) that $\mathbb{E}(X_t(x)) = xe^{-\Psi'(0+)t}$ for all $t \geq 0$, this leads to the following classification.

- If $\Psi'(0+) \in (-\infty, 0)$, the process has a finite mean and $\int_0 \frac{\mathrm{d}u}{\Psi(u)} = -\infty$. Therefore the process does not explode almost-surely and goes to $+\infty$ with probability $1 - e^{-x\rho}$.
- If $\Psi'(0+) = -\infty$ and $\int_0 \frac{\mathrm{d}u}{\Psi(u)} = -\infty$ then the process has an infinite mean, does not explode almost-surely and goes to $+\infty$ with probability $1 - e^{-x\rho}$. The Neveu's CSBP provides an example of non-explosive CSBP with infinite mean.
- If $\int_0 \frac{\mathrm{d}u}{\Psi(u)} \in (-\infty, 0)$, the process explodes continuously to $+\infty$ with probability $1 - e^{-x\rho}$ if $\underline{v} < +\infty$, with probability 1 if $\underline{v} = +\infty$.

For any Ψ as in (6),

$$\mathbf{d} := \lim_{u\to+\infty} \frac{\Psi(u)}{u} = +\infty 1_{\{\sigma>0\}} + \gamma + \int_0^1 x\pi(\mathrm{d}x) \in (-\infty, +\infty] \quad \text{(variation)}.$$

- If $\mathbf{d} \in \mathbb{R}$, then the process has finite variation sample paths (we will say that Ψ is of finite variation) and $\int^{\infty} \frac{\mathrm{d}u}{\Psi(u)} = +\infty$. Therefore the process is persistent (not absorbed at 0 almost-surely) and goes to 0 with probability $e^{-x\rho}$ ($\rho = +\infty$ if $\mathbf{d} \leq 0$).
- If $\mathbf{d} = +\infty$ and $\int^{+\infty} \frac{\mathrm{d}u}{\Psi(u)} = +\infty$, then the process has infinite variation sample paths, is persistent and goes to 0 with probability $e^{-x\rho}$.

– If $\mathbf{d} = +\infty$ and $\int^{+\infty} \frac{du}{\Psi(u)} < +\infty$, then the process has infinite variation sample paths and is absorbed at 0 with probability $e^{-x\rho}$.

Note that $\mathbf{d} \in \mathbb{R}$ if and only if $\sigma = 0$ and $\int_0^1 u\pi(du) < +\infty$. In this case (6) can be rewritten as

$$\Psi(u) = \mathbf{d}u - \int_0^{+\infty} \pi(dr)\left(1 - e^{-ur}\right). \tag{10}$$

Unless explicitly specified, we shall always work under the non-explosion condition (9).

Lemma 1. *Denote by $\lambda \mapsto v_{-t}(\lambda)$ the inverse of $\lambda \mapsto v_t(\lambda)$. This is a strictly increasing function, well-defined on $[0, \bar{v}_t)$. For all $s, t \in \mathbb{R}^+$, if $0 \le \lambda < \bar{v}_{s+t}$, then*

$$v_{-(s+t)}(\lambda) = v_{-s} \circ v_{-t}(\lambda).$$

Moreover by (7), for all $t \ge 0$ and $\lambda < \bar{v}_t$

$$\int_{v_{-t}(\lambda)}^{\lambda} \frac{du}{\Psi(u)} = \int_{v_{-t}(\lambda)}^{v_t(v_{-t}(\lambda))} \frac{du}{\Psi(u)} = -t. \tag{11}$$

In the supercritical case, $\Psi'(0+) < 0$, $v_{-t}(\lambda) \underset{t\to+\infty}{\longrightarrow} 0$ and for any $\lambda', \lambda \in (0, \rho)$, we have

$$\frac{v_{-t}(\lambda)}{v_{-t}(\lambda')} \underset{t\to+\infty}{\longrightarrow} \exp\left(\Psi'(0+) \int_{\lambda'}^{\lambda} \frac{du}{\Psi(u)}\right). \tag{12}$$

In the subcritical case, $\Psi'(0+) \ge 0$, $v_{-t}(\lambda) \underset{t\to+\infty}{\longrightarrow} +\infty$. One has $\mathbf{d} \in (0, +\infty]$ and for any $\lambda, \lambda' > 0$, we have

$$\frac{v_{-t}(\lambda)}{v_{-t}(\lambda')} \underset{t\to+\infty}{\longrightarrow} \exp\left(\mathbf{d} \int_{\lambda'}^{\lambda} \frac{du}{\Psi(u)}\right). \tag{13}$$

Proof. We refer the reader to [14]. We show (12) and (13). By (5), (7) and (11), one can show that for all $t \in \mathbb{R}$, $\frac{d}{d\lambda} v_t(\lambda) = \frac{\Psi(v_t(\lambda))}{\Psi(\lambda)}$, and therefore for any λ, λ'

$$\frac{v_{-t}(\lambda)}{v_{-t}(\lambda')} = \exp\left(\int_{\lambda'}^{\lambda} \frac{d}{du} \log v_{-t}(u)\, du\right) = \exp\left(\int_{\lambda'}^{\lambda} \frac{\Psi(v_{-t}(u))}{v_{-t}(u)} \frac{du}{\Psi(u)}\right).$$

In the supercritical case, $v_{-t}(u) \underset{t\to+\infty}{\longrightarrow} 0$ and by convexity of Ψ, $\frac{\Psi(v_{-t}(u))}{v_{-t}(u)}$ decreases towards $\Psi'(0+) \in [-\infty, 0)$ as t goes to $+\infty$. In the subcritical case, $v_{-t}(u) \underset{t\to+\infty}{\longrightarrow} +\infty$ and by convexity of Ψ, $\frac{\Psi(v_{-t}(u))}{v_{-t}(u)}$ increases towards $\mathbf{d} \in (-\infty, +\infty]$ as t goes to $+\infty$. The limits (12) and (13) are obtained by monotone convergence. □

The following lemma holds for all non-explosive supercritical CSBPs and persistent subcritical ones. Note that in the case of a subcritical non-persistent CSBP, $\bar{v} = 0$ and the following lemma is degenerate. We refer to Theorem 3.2.1 in [24] for a proof.

Lemma 2 (Grey's martingale [14]). *For all $x \ge 0$, and $\lambda \in (0, \bar{v})$, the process*

$$\left(M_t^{\lambda}(x), t \ge 0\right) := \left(\exp\left(-v_{-t}(\lambda) X_t(x)\right), t \ge 0\right)$$

is a positive martingale.

2.2. *Continuous population model*

As noticed in [4], the branching property (4) allows one to apply the Kolmogorov extension theorem and to define on some probability space a flow of subordinators $(X_t(x), x \geq 0, t \geq 0)$ such that

(i) for all $t \geq 0$ $(X_t(x), x \geq 0)$ is a subordinator with Laplace exponent $\lambda \mapsto v_t(\lambda)$,
(ii) for any $y \geq x$, $(X_t(y) - X_t(x), t \geq 0)$ is a CSBP(Ψ) started from $y - x$, independent of $(X_t(x), t \geq 0)$.

This provides a genuine continuous population model: the individual a living at time 0 has for descendant b at time t, if

$$X_t(a-) < b < X_t(a).$$

Duquesne and Labbé in [9] (see Theorem 0.2 and Theorem 1.8 in [9]), provide a construction for any mechanism Ψ of the flow $(X_t(x), x \geq 0, t \geq 0)$ via a Poisson point process on the space of càdlàg trajectories (see also Dawson and Li in [8] for an approach with Poisson driven stochastic differential equations).

In the case of infinite variation, the Laplace exponent $\lambda \mapsto v_t(\lambda)$ is driftless and thus takes the form $v_t(\lambda) = \int_{(0,+\infty]} \ell_t(\mathrm{d}x)(1 - e^{-\lambda x})$ for a certain Lévy measure ℓ_t on $\mathbb{R}^+ \cup \{+\infty\}$ such that $\int_{(0,+\infty]}(1 \wedge x)\ell_t(\mathrm{d}x) < +\infty$. As noticed in Chapter 3 of [24], $(\ell_t, t > 0)$ is an entrance law for the semi-group of the CSBP(Ψ). This yields the existence of a measure N_Ψ (called cluster measure in [9], and canonical measure in [24]) on the space $\mathcal{D}$ of càdlàg paths from $\mathbb{R}_+^\star$ to $\mathbb{R}_+ \cup \{+\infty\}$ such that for any non-negative function F

$$N_\Psi\big(F(X_{t+\cdot}); X_t > 0\big) = \int_{(0,+\infty]} \ell_t(\mathrm{d}x)\mathbb{E}_x^\Psi(F) \quad \text{and} \quad N_\Psi(X_0 > 0) = 0 \tag{14}$$

and for any non-negative function F, G, the Markov property holds:

$$N_\Psi\big[F(X_{\cdot\wedge t})G(X_{t+\cdot}); X_t > 0\big] = N_\Psi\big[F(X_{\cdot\wedge t})\mathbb{E}_{X_t}[G]; X_t > 0\big]. \tag{15}$$

Consider a Poisson point process $\mathcal{N} = \sum_{i \in I} \delta_{(x_i, X^i)}$ over $\mathbb{R}_+ \times \mathcal{D}$ with intensity $\mathrm{d}x \otimes N_\Psi(\mathrm{d}X)$ and set for all $x \geq 0$ and $t > 0$,

$$X_t(x) = \sum_{x_i \leq x} X_t^i, \tag{16}$$

with $X_0(x) = x$. The flow $(X_t(x), x \geq 0, t \geq 0)$ defined by (16) satisfies the properties (i) and (ii).

In the case of finite variation, one can construct a flow $(X_t(x), x \geq 0, t \geq 0)$ in a Poisson manner as in [9] (see Equation 1.25). Recall $\mathbf{d} := \lim_{u \to +\infty} \frac{\Psi(u)}{u}$. Consider a Poisson point process $\mathcal{N} = \sum_{i \in I} \delta_{(x_i, t_i, X^i)}$ over $\mathbb{R}_+ \times \mathbb{R}_+ \times \mathcal{D}$ with intensity $\mathrm{d}x \otimes e^{-\mathbf{d}t}\,\mathrm{d}t \otimes \int_0^{+\infty} \pi(\mathrm{d}r)\mathbb{P}_r^\Psi(\mathrm{d}X)$. Set

$$X_t(x) = e^{-\mathbf{d}t}x + \sum_{x_i \leq x} 1_{\{t_i \leq t\}} X_{t-t_i}^i. \tag{17}$$

The flow $(X_t(x), x \geq 0, t \geq 0)$ defined by (17) satisfies the properties (i) and (ii).

In both finite or infinite variation case, for any $x \in \mathbb{R}_+$, we define $\Delta X_t(x) := X_t(x) - X_t(x-)$, this represents the progeny at time t of the individual x at time 0. By the Poisson construction, any individual with a non zero progeny at a certain time belongs to $\{x_i, i \in I\}$.

Remark 1. One can understand N_Ψ as the Lévy measure of the path-valued subordinator $(X_t(x), x \geq 0)_{t \geq 0}$. Roughly speaking, we add CSBPs with the same mechanism Ψ started at individuals with zero mass. We refer to Li [25] for a recent work on path-valued processes.

In the rest of the paper, if not otherwise specified, the flows that we consider are all constructed from Poisson point processes as in (16) and (17).

2.3. *Extremal processes*

We gather here some of the fundamental properties of extremal processes that can be found in Chapter 4, Section 3 in Resnick [27]. Let F be a probability distribution function with a given support $(s_l, s_o) \subset \overline{\mathbb{R}}$. A real-valued process $(M_x, x \geq 0)$ is an extremal-F process if for any $0 \leq x_1 \leq \cdots \leq x_n$ and $(z_1, \ldots, z_n) \in \mathbb{R}^n$,

$$\mathbb{P}(M_{x_1} \leq z_1, M_{x_2} \leq z_2, \ldots, M_{x_n} \leq z_n) = F^{x_1}\left(z_1'\right) F^{x_2 - x_1}\left(z_2'\right) \cdots F^{x_n - x_{n-1}}\left(z_n'\right), \tag{18}$$

where $z_i' = \bigwedge_{k=i}^n z_k$ for all $i \geq 1$. Any extremal-F process $(M_x, x \geq 0)$ has the following properties:

(i) $(M_x, x \geq 0)$ is stochastically continuous.
(ii) $(M_x, x \geq 0)$ has a càdlàg version.
(iii) $(M_x, x \geq 0)$ has a version with non-decreasing paths such that $\lim_{x \to +\infty} M_x = s_o$ and $\lim_{x \to 0} M_x = s_l$ almost-surely.
(iv) $(M_x, x \geq 0)$ is a Markov process with for $x > 0$, $y > 0$,

$$\mathbb{P}(M_{x+y} \leq z \mid M_x = v) = \begin{cases} F^y(z) & \text{if } z \geq v, \\ 0 & \text{if } z < v. \end{cases} \tag{19}$$

For all $x \geq 0$, set $Q(x) = -\log F(x)$. The parameter of the exponential holding time in state x is $Q(x)$, and the process jumps from x to $(x, y]$ with probability $1 - \frac{Q(y)}{Q(x)}$. The only possible instantaneous state is s_l and it is instantaneous if and only if $F(s_l) = 0$.

Any process $(M_x, x \geq 0)$ verifying

$$\begin{cases} \mathbb{P}(M_x \leq z) = F(z)^x & \text{for all } z \in \mathbb{R} \text{ and } x \geq 0, \\ M_{x+y} = M_x \vee M_y' & \text{a.s. for all } x, y \in \mathbb{R}_+, \end{cases} \tag{20}$$

where M_y' is independent of $(M_u, 0 \leq u \leq x)$ and $M_y' \overset{d}{=} M_y$, satisfies (18) and is therefore an extremal-F process. A constructive approach of extremal processes is given by the records of a Poisson point process. Let μ be a σ-finite measure on $(0, +\infty)$, and consider a Poisson point process $\mathcal{P} = \sum_{i \in I} \delta_{(x_i, Z_i)}$ with intensity $\mathrm{d}x \otimes \mu$. The process $(M_x, x \geq 0)$ defined by $M_x = \sup_{x_i \leq x} Z_i$ is a càdlàg extremal-F process with for all $z \in \mathbb{R}$, $F(z) = \exp(-\bar{\mu}(z))$ where $\bar{\mu}(z) = \mu(z, +\infty)$. We highlight that the state 0 is instantaneous if the intensity measure μ is infinite. The positive extremal processes will play an important role in the sequel. They correspond to the records of Poisson point processes over $\mathbb{R}_+ \times \mathbb{R}_+$.

An interesting feature of extremal processes lies in their *max-infinite divisibility*. Namely, for any integer m,

$$(M_x, x \geq 0) \overset{d}{=} \Big(\max_{i \in [|1,m|]} M_x^i, x \geq 0\Big), \tag{21}$$

where $(M_x^i, x \geq 0)_{i \in [|1,m|]}$ are i.i.d. extremal-$F^{1/m}$ processes. Indeed,

$$\begin{aligned} \mathbb{P}\Big(\max_{i \in [|1,m|]} M_{x_1}^i \leq z_1, \ldots, \max_{i \in [|1,m|]} M_{x_n}^i \leq z_n\Big) &= \left(F^{\frac{x_1}{m}}\left(z_1'\right) F^{\frac{x_2 - x_1}{m}}\left(z_2'\right) \cdots F^{\frac{x_n - x_{n-1}}{m}}\left(z_n'\right)\right)^m \\ &= \mathbb{P}(M_{x_1} \leq z_1, M_{x_2} \leq z_2, \ldots, M_{x_n} \leq z_n). \end{aligned}$$

We refer the reader to Dwass [11] and Resnick and Rubinovitch [28] for more details about extremal processes.

3. Supercritical processes

3.1. *Prolific individuals*

Consider Ψ a supercritical mechanism, namely such that $\Psi'(0+) \in [-\infty, 0)$. Consider a flow of CSBPs $(X_t(x), x \geq 0, t \geq 0)$ with mechanism Ψ and recall the notion of prolific individual defined in Definition 1. The following lemma generalizes Lemma 2 in [3] to the finite variation case.

Lemma 3. *The set of prolific individuals*

$$\mathcal{P} := \left\{x \in \mathbb{R}_+;\ \lim_{t\to+\infty} \Delta X_t(x) = +\infty\right\}$$

is almost-surely non-empty. If $\rho < +\infty$, $(\#\mathcal{P} \cap [0,x], x \geq 0)$ is a Poisson process with intensity ρ. If $\rho = +\infty$ then $\mathcal{P} = \{x_i, i \in I\}$.

Proof. For any $a_n > 0$, the set $\mathcal{P} \cap [0,a_n]$ is non-empty if and only if $(X_t(a_n), t \geq 0)$ is not extinguishing. Consider a non-decreasing sequence $(a_n)_{n\geq 1}$ such that $a_n \underset{n\to+\infty}{\longrightarrow} +\infty$: $\mathbb{P}(\mathcal{P} \cap [0,a_n] \neq \varnothing) = 1 - e^{-\rho a_n} \underset{n\to+\infty}{\longrightarrow} 1$. Therefore, $\mathbb{P}(\mathcal{P} \neq \varnothing) = 1$. Assume $\rho < +\infty$, in the infinite variation case, one has $N_\Psi(X_t \underset{t\to+\infty}{\longrightarrow} +\infty, X_s > 0) = \int_{(0,+\infty]} \ell_s(\mathrm{d}x)\mathbb{P}_x^\Psi(X_t \underset{t\to+\infty}{\longrightarrow} +\infty) = \int_{(0,+\infty]} \ell_s(\mathrm{d}x)(1 - e^{-x\rho}) = v_s(\rho) = \rho$. By letting s to 0, we see that the restriction of $\mathcal{P}$ to the atoms x_i such that $X_t^i \underset{t\to+\infty}{\longrightarrow} +\infty$ is a Poisson point process with intensity $\rho\,\mathrm{d}x$. In the finite variation case, if $\rho < +\infty$, then $\mathbf{d} > 0$ and

$$\int_0^{+\infty} e^{-\mathbf{d}t}\,\mathrm{d}t \int_0^{+\infty} \pi(\mathrm{d}r)\mathbb{P}_r^\Psi(X_{u-t} \underset{u\to+\infty}{\longrightarrow} +\infty) = \frac{1}{\mathbf{d}} \int_0^{+\infty} \pi(\mathrm{d}r)\big(1 - e^{-r\rho}\big) = \frac{-\Psi(\rho) + \mathbf{d}\rho}{\mathbf{d}} = \rho.$$

The restriction of $\mathcal{P}$ to the atoms x_i such that $X_{t-t_i}^i \underset{t\to+\infty}{\longrightarrow} +\infty$ is therefore also a Poisson point process over $\mathbb{R}_+$ with intensity $\rho\,\mathrm{d}x$. Assume $\rho = +\infty$, then by Theorem 2(i), $\mathbb{P}_r(X_t \underset{t\to+\infty}{\not\longrightarrow} +\infty) = 0$, and thus $\int_0^{+\infty} \pi(\mathrm{d}r)\mathbb{P}_r^\Psi(X_t \underset{t\to+\infty}{\not\longrightarrow} +\infty) = 0$. Therefore $\mathcal{P} = \{x_i; i \in I\}$. □

3.2. *CSBPs with finite mean and subordinators*

We briefly study the case of a branching mechanism with finite mean to show that the notion of super-prolific individual is degenerate. The following proposition is essentially a rewriting of Proposition 2.1 and Lemma 2.2 in [9], but is important before treating the infinite mean case.

Proposition 1 (Proposition 2.1 and Lemma 2.2 in [9]). *Suppose $\Psi'(0+) \in (-\infty, 0)$ and fix $\lambda \in (0,\rho)$. Consider a flow of* CSBPs(Ψ) $(X_t(x), x \geq 0, t \geq 0)$ *defined as in* (16) *or* (17)*. There exists a càdlàg driftless subordinator $(W^\lambda(x), x \geq 0)$ with Laplace exponent $\theta \mapsto v_{\frac{\log(\theta)}{-\Psi'(0+)}}(\lambda)$ such that almost-surely for any $x > 0$,*

$$v_{-t}(\lambda)X_t(x) \underset{t\to+\infty}{\longrightarrow} W^\lambda(x) \quad \textit{and} \quad v_{-t}(\lambda)X_t(x-) \underset{t\to+\infty}{\longrightarrow} W^\lambda(x-).$$

The Lévy measure of $(W^\lambda(x), x \geq 0)$ has total mass $\rho \in (0,+\infty]$. Moreover

$$\mathcal{P} = \big\{x > 0;\ W^\lambda(x) > W^\lambda(x-)\big\}.$$

If $\rho = +\infty$, $\mathcal{S} \cap \mathcal{P} = \varnothing$ a.s.; if $\rho < +\infty$, $\mathcal{S} \cap \mathcal{P} = \{x^\star\}$ a.s. with $x^\star$ the time of the first jump of $(W^\lambda(x), x \geq 0)$.

Remark 2. The infinite divisibility of $(W^\lambda(x), x \geq 0)$ ensures that the random variable $W^\lambda(x)$ has the same law as the sum of n copies of $W^\lambda(x/n)$:

$$W^\lambda(x) \overset{d}{=} W^1(x/n) + \cdots + W^n(x/n).$$

In a loose sense, the progeny of the individuals $[0,x]$ grows as the sum of prolific individuals progenies in $[0,x]$.

Proof. We only show that $\mathcal{P} = \{x > 0; W^\lambda(x) > W^\lambda(x-)\}$ and $\mathcal{S} \cap \mathcal{P} = \{x^\star\}$ a.s. if $\rho < \infty$, $\mathcal{S} \cap \mathcal{P} = \varnothing$ if $\rho = \infty$. Lemma 2.2 in [9] states that $(W^\lambda(x), x \geq 0)$ is a subordinator such that $\lim_{t\to\infty} v_{-t}(\lambda)\Delta X_t(x) = \Delta W^\lambda(x)$. By Lemma 1, $v_{-t}(\lambda) \underset{t\to+\infty}{\longrightarrow} 0$. This implies

$$\big\{x > 0, \Delta W^\lambda(x) > 0\big\} \subset \left\{x > 0, \lim_{t\to\infty} \Delta X_t(x) = \infty\right\}.$$

Moreover, when $\rho < \infty$, $(W^\lambda(x), x \geq 0)$ is a compound Poisson process with intensity ρ. By Lemma 3, for any $n \in \mathbb{N}$, $\#\{0 < x < n; \Delta W^\lambda(x) > 0\} \leq \#\{0 < x < n; \lim_{t\to\infty} \Delta X_t(x) = \infty\}$. For any n, these two random variables have the same Poisson law with parameter ρn and therefore for all n, almost-surely

$$\left\{0 < x < n, \Delta W^\lambda(x) > 0\right\} = \left\{0 < x < n, \lim_{t\to\infty} \Delta X_t(x) = \infty\right\}.$$

By letting n to infinity, we get that $\mathcal{P} = \{x > 0; \Delta W^\lambda(x) > 0\}$ almost-surely. If $\rho = \infty$, one has $\mathbb{P}_r(v_{-t}(\lambda)X_t \underset{t\to\infty}{\longrightarrow} 0) = 0$ and thus $\{x > 0, \Delta W^\lambda(x) > 0\} = \{x_i, i \in I\} = \mathcal{P}$. Moreover, if $\rho < \infty$, then $\mathcal{S} \cap \mathcal{P} = \{x > 0; \Delta W^\lambda(x) > 0 \text{ and } \frac{\Delta W^\lambda(x)}{W^\lambda(x-)} = +\infty\} = \{x^\star\}$, since $W^\lambda_{x^\star-} = 0$. If $\rho = \infty$, then for all $x > 0$, $\lim_{t\to\infty} \frac{\Delta X_t(x)}{X_t(x-)} = \frac{\Delta W^\lambda(x)}{W^\lambda(x-)} < \infty$, therefore $\mathcal{S} \cap \mathcal{P} = \varnothing$. □

3.3. *CSBPs with infinite mean and extremal processes*

Theorem 3. *Suppose* $\Psi'(0+) = -\infty$ *and* $\int_0 \frac{du}{\Psi(u)} = -\infty$. *Fix* $\lambda_0 \in (0, \rho)$, *and define* $G(y) := \exp(-\int_y^{\lambda_0} \frac{du}{\Psi(u)})$ *for* $y \in (0, \rho)$. *Then, for all* $x \geq 0$, *almost-surely*

$$e^{-t} G\left(\frac{1}{X_t(x)} \wedge \rho\right) \underset{t\to+\infty}{\longrightarrow} \tilde{Z}(x),$$

where $(\tilde{Z}(x), x \geq 0)$ *is a positive extremal-F process* (*in the sense of* (18)) *with* $F(z) = \exp(-G^{-1}(z))$ *for* $z \in [0, +\infty]$, *and* $G^{-1}(z) = v_{\log(\frac{1}{z})}(\lambda_0)$ *for all* $z \in [0, +\infty]$.

Example 1. Consider the Neveu's mechanism $\Psi(u) = u \log u$ for which $\rho = 1$. For all $t \in \mathbb{R}$, $v_t(\lambda) = \lambda^{e^{-t}} = e^{-\log(\frac{1}{\lambda})e^{-t}}$. Fix $\lambda_0 = \frac{1}{e}$, one has $G(z) = \log(1/z)$ and the process $(\tilde{Z}(x), x \geq 0)$ is a positive extremal-F process with for $z \geq 0$, $F(z) = e^{-e^{-z}}$ and for $z < 0$, $F(z) = 0$. As $F(0) > 0$, the state 0 is not instantaneous.

Remark 3. By applying the branching property at time $t \geq 0$ in Theorem 3, one can see that for all $x \geq 0$, $\tilde{Z}(x) \overset{d}{=} e^{-t} \tilde{Z}'(X_t(x))$, where $(\tilde{Z}'(x), x \geq 0)$ has the same distribution as $(\tilde{Z}(x), x \geq 0)$ and is independent of $(X_s(x) : 0 \leq s \leq t, x \geq 0)$. This result was observed by Cohn and Pakes in [7] for Galton–Watson processes with infinite mean. The max-infinite divisibility of the process $(\tilde{Z}(x), x \geq 0)$ ensures that the random variable $\tilde{Z}(x)$ has the same law as the maximum of n independent copies of $\tilde{Z}(x/n)$:

$$\tilde{Z}(x) \overset{d}{=} \max\left(\tilde{Z}^1(x/n), \ldots, \tilde{Z}^n(x/n)\right).$$

In a loose sense, the infinite mean of the process transforms the sum into a maximum.

We start the proof of Theorem 3 by two lemmas, the first one shows notably the slow variation of G at 0.

Lemma 4. *If* $\Psi'(0+) = -\infty$, *and* $\int_0 \frac{du}{\Psi(u)} = -\infty$, *the map* $G : y \mapsto \exp(-\int_y^{\lambda_0} \frac{du}{\Psi(u)})$ *is continuous, non-increasing, goes from* $[0, \rho]$ *to* $[0, +\infty]$ *and is slowly varying at* 0. *Moreover, for all* $y \in (0, \rho)$, $G'(y) = \frac{G(y)}{\Psi(y)}$ *with* $G(\lambda_0) = 1$ *and* $G^{-1}(z) = v_{\log(\frac{1}{z})}(\lambda_0)$ *for* $z \in [0, +\infty]$.

Proof. Since Ψ is non-positive on $(0, \rho)$, and $\int_\rho^{\lambda_0} \frac{du}{\Psi(u)} = +\infty$ then $G(y) \underset{y\to\rho}{\longrightarrow} 0$. Moreover since $\int_0 \frac{du}{\Psi(u)} = -\infty$, then $G(y) \underset{y\to 0}{\longrightarrow} +\infty$. By definition $\int_{G^{-1}(z)}^{\lambda_0} \frac{du}{\Psi(u)} = \log(\frac{1}{z})$ and by (7) $G^{-1}(z) = v_{\log(\frac{1}{z})}(\lambda_0)$. By assumption $\Psi'(0+) = -\infty$, then for any fixed $\theta > 0$, there exists $|b|$ arbitrarily large, such that for a small enough y

$$\left|\int_y^{\theta y} \frac{du}{|\Psi(u)|}\right| \leq \left|\int_y^{\theta y} \frac{du}{|b|u}\right| = \frac{|\log(\theta)|}{|b|}$$

this tends to 0 as $b \to +\infty$. Therefore $\lim_{y\to 0} \int_y^{\theta y} \frac{du}{\Psi(u)} = 0$ and G is slowly varying at 0. □

Lemma 5. *Let $x > 0$ and $\lambda \in (0, \bar{v})$. The limit $W^\lambda(x) := \lim_{t\to+\infty} v_{-t}(\lambda)X_t(x)$ exists almost-surely in $\mathbb{R}_+ \cup \{+\infty\}$. Moreover, for all $x \geq 0$*

$$\mathbb{P}\big(W^\lambda(x) = 0\big) = 1 - \mathbb{P}\big(W^\lambda(x) = +\infty\big) = \exp(-x\lambda).$$

Proof. Lemma 2 and the martingale convergence theorem applied to $(M_t^\lambda)_{t\geq 0}$ ensure that $v_{-t}(\lambda)X_t(x)$ converges almost-surely as t goes to infinity towards a random variable $W^\lambda(x)$ with values in $[0, +\infty]$. Let $0 \leq \lambda < \bar{v}$ and $\theta \geq 0$. One has for all $x \geq 0$

$$\mathbb{E}\big[e^{-\theta v_{-t}(\lambda)X_t(x)}\big] = \exp\big(-x v_t\big(\theta v_{-t}(\lambda)\big)\big). \tag{22}$$

By Lemma 1, $v_{-t}(\lambda) \underset{t\to+\infty}{\longrightarrow} 0$. For all $\theta > 0$, and t such that $v_{-t}(\lambda), \theta v_{-t}(\lambda) \in (0, \rho)$; we have by (7) and Lemma 1

$$\begin{aligned}\int_\lambda^{v_t(\theta v_{-t}(\lambda))} \frac{\mathrm{d}z}{\Psi(z)} &= \int_\lambda^{v_{-t}(\lambda)} \frac{\mathrm{d}z}{\Psi(z)} + \int_{v_{-t}(\lambda)}^{\theta v_{-t}(\lambda)} \frac{\mathrm{d}z}{\Psi(z)} + \int_{\theta v_{-t}(\lambda)}^{v_t(\theta v_{-t}(\lambda))} \frac{\mathrm{d}z}{\Psi(z)} \\ &= \int_{v_{-t}(\lambda)}^{\theta v_{-t}(\lambda)} \frac{\mathrm{d}z}{\Psi(z)}.\end{aligned} \tag{23}$$

Fix any positive constant $\theta \neq 1$. For any $b > 0$, there exists a large enough t such that $|\Psi(z)| \geq |b|z$ for all $z \in ((\theta \wedge 1)v_t(\lambda), (\theta \vee 1)v_t(\lambda))$. Therefore by Lemma 4 and (23),

$$\lim_{t\to+\infty}\left|\int_\lambda^{v_t(\theta v_{-t}(\lambda))} \frac{\mathrm{d}z}{\Psi(z)}\right| \leq \lim_{t\to+\infty} \int_{v_{-t}(\lambda)}^{\theta v_{-t}(\lambda)} \frac{\mathrm{d}z}{|\Psi(z)|} = 0.$$

Then $\lim_{t\to+\infty} \int_\lambda^{v_t(\theta v_{-t}(\lambda))} \frac{\mathrm{d}z}{\Psi(z)} = 0$ and thus for all $\theta > 0$

$$\lim_{t\to+\infty} v_t\big(\theta v_{-t}(\lambda)\big) = \lambda. \tag{24}$$

The limit in (22) as t tends to $+\infty$ equals $e^{-x\lambda}$ and does not depend on θ. The random variable $W^\lambda(x)$ is thus equal to 0 or $+\infty$ with probability 1. $\square$

Proof of Theorem 3. The arguments for the almost-sure convergence are adapted from those on pages 711-712 in [15]. Fix $x > 0$. By Lemma 2 and Lemma 5, there exists $\Omega_0 \in \mathcal{F}$ such that $\mathbb{P}(\Omega_0) = 1$ and on Ω_0, for any $q \in (0, \rho) \cap \mathbb{Q}$, $v_{-t}(q)X_t(x) \underset{t\to+\infty}{\longrightarrow} W^q(x)$ with $W^q(x)$ taking values in $\{0, +\infty\}$. Since $q \mapsto v_{-t}(q)$ is increasing then by definition, $W_x^q \leq W_x^{q'}$ if $q \leq q'$. Then $\{W^q(x) : q \in (0, \rho) \cap \mathbb{Q}\}$ steps up from 0 to $+\infty$ at some random threshold Λ_x and is otherwise constant. Define $\Lambda_x := \inf\{q \in (0, \rho) \cap \mathbb{Q} : W^q(x) = +\infty\} \in \mathbb{R}_+$. On Ω_0, for any $\lambda \in [0, \rho)$,

$$\{\Lambda_x \leq \lambda\} = \lim_{q\in\mathbb{Q}, q\downarrow\lambda} \big\{W^q(x) = +\infty\big\} \quad \text{and} \quad \{\Lambda_x = \rho\} = \lim_{q\in\mathbb{Q}, q\uparrow\rho} \big\{W^q(x) = 0\big\}.$$

Then Λ_x is a random variable. By Lemma 5, we have $\mathbb{P}(\Lambda_x \leq \lambda) = 1 - e^{-x\lambda}$ for $\lambda \in [0, \rho)$ and $\mathbb{P}(\Lambda_x = \rho) = e^{-x\rho}$, which implies that $\{\Lambda_x = \rho\} = \{X_t(x) \underset{t\to+\infty}{\longrightarrow} 0\}$ a.s. We then work deterministically on Ω_0 to show that for any $\lambda \in (0, \rho)$,

$$v_{-t}(\lambda)X_t(x) \underset{t\to+\infty}{\longrightarrow} W^\lambda(x) = \begin{cases} 0 & \text{if } \Lambda_x > \lambda, \\ +\infty & \text{if } \Lambda_x < \lambda. \end{cases} \tag{25}$$

Let $\lambda \in (0, \rho)$. If $\Lambda_x > \lambda$, there exists some $q' \in \mathbb{Q}$ such that $\Lambda_x > q' > \lambda$. Since $v_{-t}(q')X_t(x) \underset{t\to+\infty}{\longrightarrow} 0$ and $v_{-t}(q')X_t(x) \geq v_{-t}(\lambda)X_t(x)$ therefore $v_{-t}(\lambda)X_t(x) \underset{t\to+\infty}{\longrightarrow} 0$. If $\Lambda_x < \lambda$, there exists some $q'' \in \mathbb{Q}$ such that $\Lambda_x < q'' < \lambda$. Since $v_{-t}(q'')X_t(x) \leq v_{-t}(\lambda)X_t(x)$ and $v_{-t}(q'')X_t(x) \underset{t\to+\infty}{\longrightarrow} +\infty$, therefore $v_{-t}(\lambda)X_t(x) \underset{t\to+\infty}{\longrightarrow} +\infty$.

Assume $\Lambda_x < \rho$. Choose λ' and λ'' such that $\lambda', \lambda'' \in (0,\rho) \cap \mathbb{Q}$ and $\lambda' < \Lambda_x < \lambda''$, by definition if t is large enough,

$$v_{-t}\big(\lambda''\big) X_t(x) \geq 1 \quad \text{and} \quad v_{-t}\big(\lambda'\big) X_t(x) \leq 1,$$

thus

$$G\big(v_{-t}\big(\lambda'\big)\big) \geq G\big(1/X_t(x)\big) \geq G\big(v_{-t}\big(\lambda''\big)\big).$$

Recall (7) and (11), this yields $G(v_{-t}(\lambda')) = e^{-\int_{\lambda'}^{\lambda_0} \frac{\mathrm{d}x}{\Psi(x)}} e^t = G(\lambda')e^t$, and then

$$G\big(\lambda'\big) \geq e^{-t} G\big(1/X_t(x)\big) \geq G\big(\lambda''\big).$$

Since λ' and λ'' are arbitrarily close to Λ_x, and G is continuous, we get

$$e^{-t} G\left(\frac{1}{X_t(x)}\right) \underset{t\to+\infty}{\longrightarrow} G(\Lambda_x) \quad \mathbb{P}\text{-almost surely on } \{\Lambda_x < \rho\}.$$

If $\Lambda_x = \rho$, $1/X_t(x) \underset{t\to+\infty}{\longrightarrow} +\infty$, and

$$G\left(\frac{1}{X_t(x)} \wedge \rho\right) \underset{t\to+\infty}{\longrightarrow} G(\rho) = 0 \quad \mathbb{P}\text{-almost surely on } \{\Lambda_x = \rho\}.$$

Define

$$\tilde{Z}(x) = G(\Lambda_x). \tag{26}$$

The one-dimensional law of $\tilde{Z}(x)$ follows readily. In order to avoid cumbersome notations, we only show that $(\tilde{Z}(x), x \geq 0)$ satisfies (18) for the two-dimensional marginals. Let $z_1, z_2 \in \mathbb{R}_+$. By (25) the events $\{\tilde{Z}(x_1) < z_1, \tilde{Z}(x_2) < z_2\}$ and $\{W^{G^{-1}(z_1)}(x_1) = 0, W^{G^{-1}(z_2)}(x_2) = 0\}$ are identical. Let $\lambda_1 = G^{-1}(z_1)$ and $\lambda_2 = G^{-1}(z_2)$, then

$$\begin{aligned}
&\mathbb{P}\big(W^{\lambda_1}(x_1) = 0, W^{\lambda_2}(x_2) = 0\big) \\
&\quad = \mathbb{E}\big[e^{-W^{\lambda_1}(x_1) - W^{\lambda_2}(x_2)}\big] \\
&\quad = \lim_{t\to+\infty} \mathbb{E}\big[\exp\big(-v_{-t}(\lambda_1)X_t(x_1) - v_{-t}(\lambda_2)X_t(x_2)\big)\big] \\
&\quad = \lim_{t\to+\infty} \mathbb{E}\big[\exp\big(-\big(v_{-t}(\lambda_1) + v_{-t}(\lambda_2)\big)X_t(x_1) - v_{-t}(\lambda_2)\big(X_t(x_2) - X_t(x_1)\big)\big)\big] \\
&\quad = \lim_{t\to+\infty} \mathbb{E}\big[\exp\big(-\big(v_{-t}(\lambda_1) + v_{-t}(\lambda_2)\big)X_t(x_1)\big)\big]\mathbb{E}\big[\exp\big(-v_{-t}(\lambda_2)\big(X_t(x_2) - X_t(x_1)\big)\big)\big] \\
&\quad = \lim_{t\to+\infty} \exp\big(-x_1 v_t\big(v_{-t}(\lambda_1) + v_{-t}(\lambda_2)\big)\big) \exp\big(-(x_2 - x_1)v_t\big(v_{-t}(\lambda_2)\big)\big).
\end{aligned}$$

By definition $v_t(v_{-t}(\lambda_2)) = \lambda_2$. With no loss of generality assume $\lambda_1 < \lambda_2$, by Lemma 1, since $\Psi'(0+) = -\infty$, then $\frac{v_{-t}(\lambda_1)}{v_{-t}(\lambda_2)} \underset{t\to+\infty}{\longrightarrow} 0$. Fix any $\theta > 1$. For t large enough

$$v_t\big(v_{-t}(\lambda_2)\big) \leq v_t\big(v_{-t}(\lambda_1) + v_{-t}(\lambda_2)\big) \leq v_{-t}\big(\theta v_{-t}(\lambda_2)\big).$$

Recall (24), since $\lim_{t\to+\infty} v_t(\theta v_{-t}(\lambda)) = \lambda$, therefore $v_t(v_{-t}(\lambda_1) + v_{-t}(\lambda_2)) \underset{t\to+\infty}{\longrightarrow} \lambda_1 \vee \lambda_2$. Thus

$$\begin{aligned}
\mathbb{P}\big(\tilde{Z}(x_1) < z_1, \tilde{Z}(x_2) < z_2\big) &= e^{-x_1 G^{-1}(z_1) \vee G^{-1}(z_2)} e^{-(x_2 - x_1)G^{-1}(z_2)} \\
&= e^{-x_1 G^{-1}(z_1 \wedge z_2)} e^{-(x_2 - x_1)G^{-1}(z_2)}.
\end{aligned}$$

□

Remark 4. An alternative route to see that $(\tilde{Z}(x), x \geq 0)$ is an extremal-F process is to verify (20) instead of (18). By applying Theorem 3 to the CSBP $(X_t(x+y) - X_t(x), t \geq 0)$, we get that $\lim_{t\to+\infty} e^{-t} G(\frac{1}{X_t(x+y)-X_t(x)}) =: \tilde{Z}(x, x+y)$ exists almost-surely, has the same law as $\tilde{Z}_y$ and is independent of $\tilde{Z}(x)$. It is readily checked that $\tilde{Z}(x+y)$ and $\tilde{Z}(x) \vee \tilde{Z}(x, x+y)$ have the same law. Since G is non-increasing, one has $\tilde{Z}(x+y) \geq \tilde{Z}(x) \vee \tilde{Z}(x, x+y)$ a.s. Therefore $\tilde{Z}(x+y) = \tilde{Z}(x) \vee \tilde{Z}(x, x+y)$ a.s.

Proposition 2. *If $\Psi'(0+) = -\infty$ and there are $\lambda > 0$ and $\alpha > 0$ such that $|\int_0^\lambda (\frac{1}{\Psi(u)} - \frac{1}{\alpha u \log u}) \mathrm{d}u| < +\infty$, then $G(1/y) \underset{y\to+\infty}{\sim} k_\lambda \log(y)^{1/\alpha}$, with k_λ a positive constant. Fix $x > 0$, on the event $\{X_t(x) \underset{t\to+\infty}{\longrightarrow} +\infty\}$,*

$$\log X_t(x) \underset{t\to+\infty}{\sim} e^{\alpha t} k_\lambda^{-\alpha} \tilde{Z}(x)^\alpha \quad a.s.$$

Proof. By definition of G,

$$\frac{G(1/y)}{(\log y)^{1/\alpha}} = \exp\left(-\int_{\frac{1}{y}}^{\lambda} \frac{\mathrm{d}u}{\Psi(u)} - \frac{1}{\alpha} \log\log y\right)$$

$$= \exp\left(-\int_{\frac{1}{y}}^{\lambda} \left(\frac{1}{\Psi(u)} - \frac{1}{\alpha u \log u}\right) \mathrm{d}u - \frac{1}{\alpha} \log\log\frac{1}{\lambda}\right) \underset{y\to+\infty}{\longrightarrow} k_\lambda \in (0, +\infty).$$

By Theorem 3, $e^{-t}(\log X_t(x))^{1/\alpha} \underset{t\to+\infty}{\longrightarrow} k_\lambda^{-1} \tilde{Z}(x)$ a.s. □

The following proposition shows how to associate an extremal process to a flow of explosive CSBPs through the explosion times.

Proposition 3 (Theorem 0.3(i) in [9] for $\zeta = \zeta_\infty$). *Consider a flow of* CSBPs(Ψ), $(X_t(x), x \geq 0, t \geq 0)$, *with Ψ such that $\int_0 \frac{\mathrm{d}u}{|\Psi(u)|} < +\infty$. Define $\xi_0 = +\infty$ and $\xi_x := \inf\{t > 0; X_t(x) = +\infty\}$. The process $(Z(x), x \geq 0) := (1/\xi_x, x \geq 0)$ is an extremal-F process with $F(z) = \exp(-\underline{v}_{\frac{1}{z}})$. For all $i \in I$, set $Z_i := 1/\xi_i$ with $\xi_i := \inf\{t \geq 0; X_t^i = +\infty\}$ in the infinite variation case and $\xi_i := t_i + \inf\{t \geq t_i; X_{t-t_i}^i = +\infty\}$ in the finite variation case. The point process $\mathcal{M} := \sum_{i\in I} \delta_{(x_i, Z_i)}$ is a Poisson point process with intensity* $\mathrm{d}x \otimes \mu(\mathrm{d}z)$ *with $\bar{\mu}(z) = \underline{v}_{\frac{1}{z}}$ and almost-surely for all $x \geq 0$, $Z(x) = \sup_{x_i \leq x} Z_i$. Moreover*

$$\mathcal{S} := \{x > 0, \exists t > 0; \Delta X_t(x) = +\infty \text{ and } X_t(x-) < +\infty\} = \{x > 0; \Delta Z(x) > 0\}.$$

Proof. By Theorem 2(iii), $\mathbb{P}(\xi_x > 1/z) = e^{-x\underline{v}_{\frac{1}{z}}}$ where $t \mapsto \underline{v}_t$ is the unique solution to $\frac{\mathrm{d}\underline{v}_t}{\mathrm{d}t} = -\Psi(\underline{v}_t)$ and $\underline{v}_0 = 0$. Plainly, $\xi_{x+y} = \xi_x \wedge \xi_{x,x+y}$ with $\xi_{x,x+y} := \inf\{t \geq 0; X_t(x+y) - X_t(x) = +\infty\}$. The random variable $\xi_{x,x+y}$ is independent of $(\xi_u, 0 \leq u \leq x)$ and has the same law as ξ_y. Therefore, the process $(Z(x), x \geq 0)$ satisfies (20) and is an extremal-F process with $F(z) = e^{-\underline{v}_{\frac{1}{z}}}$. Assume Ψ of finite variation, the intensity of $\mathcal{M}$ is $\mathrm{d}x \otimes \mu(\mathrm{d}z)$ where

$$\begin{aligned}
\bar{\mu}(z) &= \int_0^{+\infty} e^{-\mathbf{d}t} \,\mathrm{d}t \int_0^{+\infty} \pi(\mathrm{d}r) \mathbb{P}_r^\Psi\left(\frac{1}{t+\xi} > z\right) \\
&= \int_0^{1/z} e^{-\mathbf{d}t} \,\mathrm{d}t \int_0^{+\infty} \pi(\mathrm{d}r) \mathbb{P}_r^\Psi\left(\frac{1}{t+\xi} > z\right) \quad \text{since } \mathbb{P}_r^\Psi\left(\frac{1}{t+\xi} > z\right) = 0 \text{ if } t > 1/z \\
&= \int_0^{1/z} e^{-\mathbf{d}t} \,\mathrm{d}t \int_0^{+\infty} \pi(\mathrm{d}r)\left(1 - e^{-r\underline{v}_{1/z-t}}\right) \quad \text{by Theorem 2(iii)} \\
&= \int_0^{1/z} e^{-\mathbf{d}t} \,\mathrm{d}t\left(-\Psi(\underline{v}_{1/z-t}) + \mathbf{d}\underline{v}_{1/z-t}\right) \quad \text{since } \Psi \text{ has the form (10)} \\
&= \int_0^{1/z} e^{-\mathbf{d}t} \,\mathrm{d}t\left(-\frac{\mathrm{d}\underline{v}_{1/z-t}}{\mathrm{d}t} + \mathbf{d}\underline{v}_{1/z-t}\right) = \left[-e^{-\mathbf{d}t}\underline{v}_{1/z-t}\right]_{t=0}^{t=1/z} = \underline{v}_{1/z}.
\end{aligned}$$

By definition, $\xi_x = \inf_{x_i \le x}\{t \ge 0;\, X^i_{(t-t_i)_+} = +\infty\} = \inf_{x_i \le x} \xi_i$ and then $Z(x) = \sup_{x_i \le x} Z_i$. In the infinite variation case, by Equation (14),

$$
\begin{aligned}
N_\Psi(Z > z; X_s > 0) &= \int_{(0,+\infty]} \ell_s(\mathrm{d}x)\big(1 - \mathbb{P}^\Psi_x(Z < z)\big) = \int_{(0,+\infty]} \ell_s(\mathrm{d}x)\big(1 - \mathbb{P}^\Psi_x(\xi > 1/z)\big) \\
&= \int_{(0,+\infty]} \ell_s(\mathrm{d}x)\big(1 - e^{-x\underline{v}_{1/z}}\big) = v_s(\underline{v}_{1/z}) \underset{s\to 0}{\longrightarrow} \underline{v}_{1/z}.
\end{aligned}
$$

By definition, $\xi_x = \inf_{x_i \le x}\{t \ge 0;\, X^i_t = +\infty\} = \inf_{x_i \le x} \xi_i$ and then $Z(x) = \sup_{x_i \le x} Z_i$. □

Example 2. Let $\alpha \in (0,1)$ and $\Psi(u) = -c_\alpha u^\alpha$ with $c_\alpha = \frac{1}{1-\alpha}$. Consider $(X_t(x), t \ge 0, x \ge 0)$ a flow of CSBPs(Ψ). By Theorem 2(iii), for all x, the process $(X_t(x), t \ge 0)$ is explosive. The explosion time ξ_x of the process $(X_t(x), t \ge 0)$ has a Weibull law with parameter $\frac{1}{1-\alpha}$. By Proposition 3, the process $(Z(x), x \ge 0)$ is an extremal-F process with F the probability distribution function of a Fréchet law with parameter $\frac{1}{1-\alpha} \in (1,+\infty)$, that is to say $F(z) = e^{-z^{-\frac{1}{1-\alpha}}}$ for all $z \ge 0$.

3.4. *Proof of Theorem* 1(i)

The arguments provided in the sequel could be simplified in the case $\rho < +\infty$ merely because only finitely many individuals in $[0,x]$ are prolific. We shall not distinguish the cases $\rho < +\infty$ and $\rho = +\infty$ and the arguments will hold also in the subcritical case with infinite variation.

Lemma 6. *Suppose that $(X_t, t \ge 0)$ and $(Y_t, t \ge 0)$ are two independent* CSBPs(Ψ) *on the same probability space, satisfying the conditions of Theorem* 3, *with initial value X_0 and Y_0 respectively. Then*

$$
\begin{aligned}
&e^{-t} G\left(\frac{1}{X_t} \wedge \rho\right) \underset{t\to+\infty}{\longrightarrow} \Gamma_1, \qquad e^{-t} G\left(\frac{1}{Y_t} \wedge \rho\right) \underset{t\to+\infty}{\longrightarrow} \Gamma_2 \quad \textit{and} \\
&e^{-t} G\left(\frac{1}{X_t + Y_t} \wedge \rho\right) \underset{t\to+\infty}{\longrightarrow} \Gamma_1 \vee \Gamma_2 \quad \textit{a.s.},
\end{aligned}
$$

where Γ_1 and Γ_2 are independent such that $\Gamma_1 = \Gamma_2$ if and only if $\Gamma_1 = \Gamma_2 = 0$ and

$$
\Gamma_1 \overset{d}{=} \bar{Z}(X_0), \qquad \Gamma_2 \overset{d}{=} \bar{Z}(Y_0) \quad \textit{and} \quad \Gamma_1 \vee \Gamma_2 \overset{d}{=} \bar{Z}(X_0 + Y_0),
$$

where $(\bar{Z}(x), x \ge 0)$ is an extremal-F process, independent of X_0 and Y_0.

Proof. Let $(\bar{Z}(x), x \ge 0)$ be an extremal-F process, independent of X_0 and Y_0. Conditionally given X_0 and Y_0, the processes $(X_t, t \ge 0)$ and $(Y_t, t \ge 0)$ are independent CSBPs with same mechanism Ψ started respectively from X_0 and Y_0. The branching property ensures that $(X_t + Y_t, t \ge 0)$ is a CSBP(Ψ) started from $X_0 + Y_0$. By applying Theorem 3, there exists three random variables Γ_1, Γ_2 and Γ_3 such that almost-surely

$$
\begin{aligned}
&e^{-t} G\left(\frac{1}{X_t} \wedge \rho\right) \underset{t\to+\infty}{\longrightarrow} \Gamma_1, \qquad e^{-t} G\left(\frac{1}{Y_t} \wedge \rho\right) \underset{t\to+\infty}{\longrightarrow} \Gamma_2 \quad \text{and} \\
&e^{-t} G\left(\frac{1}{X_t + Y_t} \wedge \rho\right) \underset{t\to+\infty}{\longrightarrow} \Gamma_3 \quad \text{a.s.}
\end{aligned}
$$

with the same law respectively as $\bar{Z}(X_0)$, $\bar{Z}(Y_0)$ and $\bar{Z}(X_0 + Y_0)$. By (18), for any x and y, $\bar{Z}(x+y) \overset{d}{=} \bar{Z}(x) \vee \bar{Z}'(y)$ with $\bar{Z}'(y)$ independent of $\bar{Z}(x)$, therefore $\Gamma_3 \overset{d}{=} \Gamma_1 \vee \Gamma_2$. Moreover for any $t \ge 0$,

$$
e^{-t} G\left(\frac{1}{X_t + Y_t} \wedge \rho\right) \ge e^{-t} G\left(\frac{1}{X_t} \wedge \rho\right) \vee e^{-t} G\left(\frac{1}{Y_t} \wedge \rho\right).
$$

Thus $\Gamma_3 \geq \Gamma_1 \vee \Gamma_2$ a.s. which, with the equality in law, entails $\Gamma_3 = \Gamma_1 \vee \Gamma_2$ a.s. Since, conditionally on X_0 and Y_0, the laws of Γ_1 and Γ_2 have no atoms in $(0, +\infty)$, one has $\Gamma_1 = \Gamma_2$ a.s. if and only if $\Gamma_1 = \Gamma_2 = 0$ a.s. $\square$

Lemma 7. *Assume $\Psi'(0) = -\infty$, $\int_0 \frac{\mathrm{d}u}{\Psi(u)} = -\infty$ and Ψ is of infinite variation. Consider a flow of CSBPs $(X_t(x), t \geq 0, x \geq 0)$ as in (16). Then almost-surely for all $i \in I$,*

$$Z_i := \lim_{t \to +\infty} e^{-t} G\left(\frac{1}{X_t^i} \wedge \rho\right). \tag{27}$$

The point process $\mathcal{M} := \sum_{i \in I} \delta_{(x_i, Z_i)}$ is a Poisson point process with intensity $\mathrm{d}x \otimes \mu(\mathrm{d}z)$ where $\bar{\mu}(z) = G^{-1}(z)$.

Proof. Consider the flow of CSBP $(X_t(x), t \geq 0, x \geq 0)$ defined by (16). First we will prove that the following event

$$\Omega_0 := \left\{ \text{for all } i \in I, \lim_{t \to +\infty} e^{-t} G\left(\frac{1}{X_t^i} \wedge \rho\right) \text{ exist in } [0, +\infty) \right\}$$

has probability 1. Let $(s_l, l \geq 1)$, $(\epsilon_k, k \geq 1)$ two sequences of positive real numbers decreasing towards 0. For any fixed l and k, define $I_{l,k} = \{i \in I, X_{s_l}^i \geq \epsilon_k\}$. Set

$$\Omega_0^{l,k} := \left\{ \text{for all } i \in I_{l,k}, \lim_{t \to +\infty} e^{-t} G\left(\frac{1}{X_t^i} \wedge \rho\right) \text{ exist in } [0, +\infty) \right\}.$$

For any fixed l, set $\Omega_0^l := \bigcap_{k \geq 1} \Omega_0^{l,k}$. One has $\Omega_0^{l+1} \subset \Omega_0^l$ and $\Omega_0 = \bigcap_{l,k \geq 1} \Omega_0^{l,k}$. Observe that

$$\left\{ \left(x_i, \left(X_{s_l + t}^i\right)_{t \geq 0}\right), i \in I_{l,k} \right\} = \left\{ \left(U_n^{l,k}, \left(V_t^{(n),l,k}\right)_{t \geq 0}\right), n = 1, 2, \ldots \right\},$$

where $(U_n^{l,k}, n \geq 1)$ are the arrival times of the Poisson process

$$\left(N_{l,k}(x), x \geq 0\right) := \left(\#\left\{x_i \leq x, X_{s_l}^i > \epsilon_k\right\}, x \geq 0\right) \tag{28}$$

whose parameter is $\ell_{s_l}((\epsilon_k, +\infty))$, and $(V_{\cdot}^{(n),l,k})_{n \geq 1}$ is a sequence of i.i.d. CSBPs(Ψ) with initial value $V_0^{(n),l,k}$ whose law is $\ell_{s_l}(\mathrm{d}x; x > \epsilon_k)/\ell_{s_l}((\epsilon_k, +\infty))$. It follows from Lemma 6 that for all $i \in I_{l,k}$, almost-surely,

$$e^{-t} G\left(\frac{1}{X_t^i} \wedge \rho\right) = e^{-s_l} e^{-(t - s_l)} G\left(\frac{1}{X_{s_l + (t - s_l)}^i} \wedge \rho\right) \underset{t \to +\infty}{\longrightarrow} Z_i, \tag{29}$$

where $\{(x_i, Z_i), i \in I_{l,k}\} = \{(U_n^{l,k}, Z_n^{l,k}), n = 1, 2, \ldots\}$ and $(Z_n^{l,k})_{n \geq 1}$ is a sequence of i.i.d. non-negative random variables independent of the sequence $(U_n^{l,k}, n \geq 1)$. For each fixed n, $Z_n^{l,k}$ has the same distribution as $e^{-s_l} \bar{Z}(V_0^{(n),l,k})$. Here $(\bar{Z}(x), x \geq 0)$ is an extremal process, independent of $\sigma\{X_u^i(x), 0 \leq u \leq s_l, x \geq 0, i \in I\}$ and thus independent of the sequences $(V_0^{(n),l,k})_{n \geq 1}$. By (29), we have that $\mathbb{P}(\Omega_0^{l,k}) = 1$. Therefore one has $\mathbb{P}(\Omega_0) = \lim_{k \to +\infty, l \to +\infty} \mathbb{P}(\Omega_0^{l,k}) = 1$. Then on Ω_0 define the point process $\mathcal{M} := \sum_{i \in I} \delta_{(x_i, Z_i)}$. It is easy to see that for $\lambda > 0$ and $z > 0$,

$$\mathbb{E}\left[e^{-\lambda \mathcal{M}((0,u] \times (z, +\infty))}\right] = \lim_{\substack{k \to +\infty \\ l \to +\infty}} \mathbb{E}\left[e^{-\lambda \sum_{i \in I_{l,k}} 1_{\{x_i \leq u, Z_i > z\}}}\right] = \lim_{\substack{k \to +\infty \\ l \to +\infty}} \mathbb{E}\left[e^{-\lambda \sum_{n \geq 1} 1_{\{U_n^{l,k} \leq u, Z_n^{l,k} > z\}}}\right],$$

and by (29),

$$\begin{aligned}
N_\Psi(X_{s_l} > \epsilon_k; Z > z) &= \ell_{s_l}\left((\epsilon_k, +\infty)\right) \mathbb{P}\left(Z_n^{l,k} > z\right) \\
&= \ell_{s_l}\left((\epsilon_k, +\infty)\right) \mathbb{P}\left(\bar{Z}_{V_0^{(n),l,k}} > e^{s_l} z\right) \\
&= \int_{(\epsilon_k, \infty]} \ell_s(\mathrm{d}x)\left(1 - \mathbb{P}\left(\bar{Z}_x < e^{s_l} z\right)\right)
\end{aligned}$$

$$= \int_{(\epsilon_k,+\infty]} \ell_{s_l}(\mathrm{d}x)\left(1-e^{-xG^{-1}(ze^{s_l})}\right)$$

$$\underset{k\to+\infty}{\longrightarrow} \int_{(0,+\infty]} \ell_{s_l}(\mathrm{d}x)\left(1-e^{-xG^{-1}(ze^{s_l})}\right) = v_{s_l}\left(G^{-1}\left(ze^{s_l}\right)\right).$$

For any λ, $v_{s_l}(\lambda) \underset{l\to+\infty}{\longrightarrow} \lambda$ and $G^{-1}(e^{s_l}z) \underset{l\to+\infty}{\longrightarrow} G^{-1}(z)$. Thus, $N_\Psi(Z>z) = G^{-1}(z)$ and $\mathcal{M} = \sum_{i\in I} \delta_{(x_i,Z_i)}$ is a Poisson point process with intensity $\mathrm{d}x \otimes \mu(\mathrm{d}z)$. $\square$

Lemma 8. *Assume $\Psi'(0) = -\infty$, $\int_0 \frac{\mathrm{d}u}{\Psi(u)} = -\infty$ and Ψ is of finite variation. Consider a flow of CSBPs $(X_t(x), t \geq 0, x \geq 0)$ as in (17), then almost-surely for all $i \in I$,*

$$Z_i := \lim_{t\to+\infty} e^{-t} G\left(\frac{1}{X^i_{t-t_i}} \wedge \rho\right). \tag{30}$$

The point process $\mathcal{M} := \sum_{i\in I} \delta_{(x_i,Z_i)}$ is a Poisson point process with intensity $\mathrm{d}x \otimes \mu(\mathrm{d}z)$ where $\bar{\mu}(z) = G^{-1}(z)$.

Proof. Consider the flow of CSBP $(X_t(x), t \geq 0, x \geq 0)$ defined by (17). Let $(s_l, l \geq 1)$, $(\epsilon_k, k \geq 1)$ be two sequences of positive real numbers such that $s_l \uparrow \infty$ as $l \to \infty$ and $\epsilon_k \downarrow 0$ as $k \to \infty$. For any fixed l and k, define $I_{l,k} = \{i \in I, t_i \leq s_l, X^i_{t_i} > \epsilon_k\}$. Note that

$$\sum_{i\in I_{l,k}} \delta_{(x_i,t_i,X^i_\cdot)} = \sum_{n=1}^{\infty} \delta_{(U^{l,k}_n, T^{l,k}_n, V^{(n),l,k}_\cdot)},$$

where $(U^{l,k}_n, n \geq 1)$ are the arrival times of the Poisson process

$$\left(N_{l,k}(x), x \geq 0\right) := \left(\#\left\{x_i \leq x, t_i \leq s_l, X^i_{t_i} > \epsilon_k\right\}, x \geq 0\right)$$

whose parameter $C_{l,k} := \mathbf{d}^{-1}(1-e^{-\mathbf{d}s_l})\pi((\epsilon_k,+\infty))$ if $\mathbf{d} \neq 0$, and $C_{l,k} := s_l \pi((\epsilon_k,+\infty))$ if $\mathbf{d} = 0$. The random variables $(T^{l,k}_n)_{n\geq 1}$ form a sequence of i.i.d. random variables with law given by $(1-e^{-\mathbf{d}s_l})^{-1}\,\mathbf{d}e^{-\mathbf{d}t} 1_{[0,s_l]}(t)\,\mathrm{d}t$ if $\mathbf{d} \neq 0$, and by $s_l^{-1} 1_{[0,s_l]}(t)\,\mathrm{d}t$ if $\mathbf{d} = 0$. The processes $(V^{(n),l,k}_\cdot)_{n\geq 1}$ form a sequence of i.i.d. CSBPs(Ψ) with initial value $V^{(n),l,k}_0$ whose law is $\pi((\epsilon_k,+\infty))^{-1} 1_{(\epsilon_k,\infty)}(r)\pi(\mathrm{d}r)$. Moreover $(U^{l,k}_n)_{n\geq 1}$, $(T^{l,k}_n)_{n\geq 1}$ and $(V^{(n),l,k}_\cdot)_{n\geq 1}$ are independent of each other. It follows from Lemma 6 that for all $i \in I_{l,k}$, almost-surely,

$$e^{-t} G\left(\frac{1}{X^i_t} \wedge \rho\right) = e^{-t_i} e^{-(t-t_i)} G\left(\frac{1}{X^i_{t_i+(t-t_i)}} \wedge \rho\right) \underset{t\to+\infty}{\longrightarrow} Z_i, \tag{31}$$

where $\{(x_i, Z_i), i \in I_{l,k}\} = \{(U^{l,k}_n, Z^{l,k}_n), n = 1, 2, \ldots\}$ and $(Z^{l,k}_n)_{n\geq 1}$ is a sequence of i.i.d. non-negative random variables independent of the sequence $(U^{l,k}_n, n \geq 1)$. For each fixed n, $Z^{l,k}_n$ has the same distribution as $e^{-T^{l,k}_n} \bar{Z}(V^{(n),l,k}_0)$ where $(\bar{Z}(x), x \geq 0)$ is an extremal process independent of $T^{l,k}_n$ and $V^{(n),l,k}_0$. As in the infinite variation case, we deduce the existence of an almost-sure event Ω_0 on which the limits in (31) exist for all $i \in I$ almost-surely. Then on Ω_0 set $\mathcal{M} := \sum_{i\in I} \delta_{(x_i,Z_i)}$. It is a Poisson point process whose intensity $\mathrm{d}x \otimes \mu(\mathrm{d}z)$ verifies

$$\bar{\mu}(z) = \lim_{\substack{l\to\infty \\ k\to\infty}} C_{l,k} \mathbb{P}\left(e^{-T^{l,k}_n} Z_{V^{(n),l,k}_0} > z\right)$$

$$= \int_0^{+\infty} e^{-\mathbf{d}t}\,\mathrm{d}t \int_0^{+\infty} \pi(\mathrm{d}r)\mathbb{P}\left(e^{-t} Z_r > z\right)$$

$$= \int_0^{+\infty} e^{-\mathbf{d}t}\,\mathrm{d}t \int_0^{+\infty} \pi(\mathrm{d}r)\left(1-e^{-rG^{-1}(ze^t)}\right) \quad \text{by Theorem 3}$$

$$= \int_0^{+\infty} e^{-\mathbf{d}t}\,\mathrm{d}t \int_0^{+\infty} \pi(\mathrm{d}r)\big(1 - e^{-r v_{-t-\log(z)}(\lambda_0)}\big) \quad \text{by Lemma 4: } G^{-1}\big(ze^t\big) = v_{\log(e^{-t}/z)}(\lambda_0)$$

$$= \int_0^{+\infty} e^{-\mathbf{d}t}\,\mathrm{d}t \big(-\Psi\big(v_{-t-\log(z)}(\lambda_0)\big) + \mathbf{d}v_{-t-\log(z)}(\lambda_0)\big) \quad \text{since } \Psi \text{ has the form (10).}$$

From the last equality above, we see that

$$\begin{aligned}
\bar{\mu}(z) &= \int_0^{\log(1/z)} e^{-\mathbf{d}t}\,\mathrm{d}t \bigg(-\frac{\mathrm{d}v_{-t+\log(1/z)}(\lambda_0))}{\mathrm{d}t} + \mathbf{d}v_{-t+\log(1/z)}(\lambda_0)\bigg) \quad \text{since } v_t(\lambda) \text{ satisfies (7)} \\
&\quad + \int_{\log(1/z)}^{+\infty} e^{-\mathbf{d}t}\,\mathrm{d}t \bigg(-\frac{\mathrm{d}v_{-t}(v_{\log(1/z)}(\lambda_0))}{\mathrm{d}t} + \mathbf{d}v_{-t}\big(v_{\log(1/z)}(\lambda_0)\big)\bigg) \quad \text{since } v_{-t}(\lambda) \text{ satisfies (11)} \\
&= \big[-e^{-\mathbf{d}t} v_{-t+\log(1/z)}(\lambda_0))\big]_{t=0}^{t=\log(1/z)} + \big[-e^{-\mathbf{d}t} v_{-t}\big(v_{\log(1/z)}(\lambda_0)\big)\big]_{t=\log(1/z)}^{t=+\infty} \quad \text{by integration by parts.}
\end{aligned}$$

We show that $\Psi'(0+) = -\infty$ entails $v_{-t}(\lambda)e^{-\mathbf{d}t} \underset{t\to+\infty}{\longrightarrow} 0$ for any λ. Since $\Psi'(0+) = -\infty$, for any $b > 0$, there exists λ_1 such that for all $u \le \lambda_1$, $|\Psi(u)| \ge bu$. Therefore by applying (11), one has

$$t \le \int_\lambda^{\lambda_1} \frac{\mathrm{d}u}{|\Psi(u)|} + \int_{v_{-t}(\lambda)}^{\lambda_1} \frac{\mathrm{d}u}{|\Psi(u)|}$$

and then for any $b > 0$

$$v_{-t}(\lambda) \le c_{\lambda_1} e^{-bt}$$

for a certain constant c_{λ_1}. Therefore $v_{-t}(\lambda)e^{-\mathbf{d}t} \underset{t\to+\infty}{\longrightarrow} 0$ and it comes

$$\bar{\mu}(z) = -e^{-\mathbf{d}\log(1/z)}\lambda_0 + v_{\log(1/z)}(\lambda_0) - 0 + e^{-\mathbf{d}\log(1/z)}\lambda_0 = G^{-1}(z). \qquad \square$$

Lemma 9. *For any fixed $x \ge 0$, almost-surely*

$$\forall y \ge x, \quad \lim_{t\to+\infty} e^{-t} G\bigg(\frac{1}{X_t(y) - X_t(x)} \wedge \rho\bigg) = Z(x, y) := \sup_{x < x_i \le y} Z_i.$$

Proof. Fix $x \in \mathbb{R}_+$, $(X_t(y) - X_t(x) : t \ge 0, y \ge x)$ is a flow of CSBPs(Ψ) independent of $(X_t(y), t \ge 0, 0 \le y \le x)$. Then without loss of generality we only consider the case $x = 0$. It follows from Theorem 3 that for any $y \ge 0$, almost surely $\lim_{t\to+\infty} e^{-t} G(\frac{1}{X_t(y)} \wedge \rho) = \tilde{Z}(y)$. Furthermore, by Lemma 7 and Lemma 8, we have some Ω_0 with $\mathbb{P}(\Omega_0) = 1$ such that on Ω_0 for all $i \in I$ $\lim_{t\to+\infty} e^{-t} G(\frac{1}{X_t^i} \wedge \rho) = Z_i$. Note that G is non-increasing and for all $i \in I$ with $x_i \le y$, $X_t(y) = \sum_{0<x_j\le y} X_t^j \ge X_t^i$. Therefore for fixed y, almost surely $\tilde{Z}(y) \ge Z_i$ for all i such that $0 < x_i \le y$. This entails that for any fixed y,

$$\tilde{Z}(y) \ge \sup_{0<x_i\le y} Z_i =: Z(0, y) \quad \text{a.s.}$$

Since $\tilde{Z}(y)$ and $Z(0, y)$ have the same law, we can conclude that for any fixed y, $\tilde{Z}(y) = Z(0, y)$ a.s. Then we have some Ω_1 with $P(\Omega_1) = 1$ such that on Ω_1 for all $q \in \mathbb{Q}_+$,

$$\lim_{t\to+\infty} e^{-t} G\bigg(\frac{1}{X_t(q)} \wedge \rho\bigg) = \tilde{Z}(q) = Z(0, q).$$

We now work deterministically on $\Omega_0 \cap \Omega_1$. For all y and $q \in \mathbb{Q}$, such that $y < q$, and any $x_i \le y$,

$$e^{-t} G\bigg(\frac{1}{X_t^i} \wedge \rho\bigg) \le e^{-t} G\bigg(\frac{1}{X_t(y)} \wedge \rho\bigg) \le e^{-t} G\bigg(\frac{1}{X_t(q)} \wedge \rho\bigg).$$

Then for all i with $x_i \le y$

$$Z_i \le \liminf_{t\to+\infty} e^{-t} G\left(\frac{1}{X_t(y)} \wedge \rho\right) \le \limsup_{t\to+\infty} e^{-t} G\left(\frac{1}{X_t(y)} \wedge \rho\right) \le Z(0,q).$$

Therefore

$$Z(0,y) = \sup_{0<x_i\le y} Z_i \le \liminf_{t\to+\infty} e^{-t} G\left(\frac{1}{X_t(y)} \wedge \rho\right) \le \limsup_{t\to+\infty} e^{-t} G\left(\frac{1}{X_t(y)} \wedge \rho\right) \le Z(0,q).$$

Proposition 4.7(ii) in [27] ensures that the process $(Z(0,y), y \ge 0)$ is càdlàg, therefore by letting q to y, we obtain $\lim_{t\to+\infty} e^{-t} G(\frac{1}{X_t(y)} \wedge \rho) = Z(0,y)$ for all $y \ge 0$ almost-surely. □

We write $Z(x)$ for $Z(0,x)$. The first statement of Theorem 1(i) is now established. It remains to prove that the super-prolific individuals correspond to the jumps of $(Z(x), x \ge 0)$.

Lemma 10. $\mathcal{P} = \{x_i; Z_i > 0, \forall i \in I\}$ *a.s.*

Proof. If $Z_i > 0$ then $X_t^i \underset{t\to+\infty}{\longrightarrow} +\infty$ and x_i is prolific. One can check that in both infinite variation and finite variation cases, the Poisson point process $\sum_{i\in I} 1_{\{Z_i>0\}}\delta_{x_i}$ has intensity $\rho\, \mathrm{d}x$, therefore $\mathcal{P} = \{x_i, i \in I; Z_i > 0\}$. □

Lemma 11. *Under the conditions of Lemma 7, we have that almost-surely for all $i \in I$,*

$$\lim_{t\to+\infty} e^{-t} G\left(\frac{1}{X_t(x_i-)} \wedge \rho\right) = Z(x_i-) = \sup_{0<x_j<x_i} Z_j.$$

Proof. Recall $I_{l,k} = \{i \in I, X_{s_l}^i \ge \epsilon_k\}$. As in Lemma 7, it suffices to show that for any fixed $k, l > 0$,

$$\mathbb{P}\left(\text{for all } i \in I_{l,k}, \lim_{t\to+\infty} e^{-t} G\left(\frac{1}{X_t(x_i-)} \wedge \rho\right) = Z(x_i-)\right) = 1. \tag{32}$$

Observe that

$$\left\{\left(x_i, \left(X_{s_l+t}(x_i-)\right)_{t\ge 0}\right), i \in I_{l,k}\right\} = \left\{\left(U_n^{l,k}, \left(X_{s_l+t}\left(U_n^{l,k}-\right)\right)_{t\ge 0}\right), n = 1, 2, \ldots\right\}, \tag{33}$$

where $(U_n^{l,k}, n \ge 1)$ are the arrival times of the Poisson process given by (28) and for fixed n, $(X_{s_l+t}(U_n^{l,k}-))_{t\ge 0} = (\sum_{0<x_i<U_n^{l,k}} X_{s_l+t}^i)_{t\ge 0}$ is a CSBP(Ψ) with initial value $X_{s_l}(U_n^{l,k}-)$. For simplicity we write U_n for $U_n^{l,k}$. It follows from Lemma 6 that for each n,

$$e^{-t} G\left(\frac{1}{X_t(U_n-)} \wedge \rho\right) = e^{-s_l} e^{-(t-s_l)} G\left(\frac{1}{X_{s_l+(t-s_l)}(U_n-)} \wedge \rho\right) \underset{t\to+\infty}{\longrightarrow} \hat{Z}_n \quad \text{a.s.} \tag{34}$$

By (33)–(34), to establish (32) we only need to prove that for any fixed n, $\hat{Z}_n = Z(U_n-)$ a.s.

Step 1. We claim that for any fixed n, $\hat{Z}_n \ge Z(U_n-)$ a.s. In fact, note that $X_t(U_n-) = \sum_{x_j<U_n} X_t^j$. Since G is a non-increasing function, we have for all $t \ge 0$ and all $i \in I$ such that $x_i < U_n$, $e^{-t} G(\frac{1}{X_t(U_n-)}) \ge e^{-t} G(\frac{1}{X_t^i})$. Then the claim follows from (34) and Lemma 7.

Step 2. We use the coupling method to prove that for any fixed n, $\hat{Z}_n$ and $Z(U_n-)$ have the same distributions. On an extended probability space, let $(Y_t(y), t \ge 0, y \ge 0)$ be an independent copy of $(X_t(x), t \ge 0, x \ge 0)$ given by

$$Y_t(y) = \sum_{y_j\le y} Y_t^j, \quad t > 0, y \ge 0,$$

with $Y_0(y)=y$, where $\mathcal{N}^Y=\sum_{i\in J}\delta_{(y_j,Y^j)}$ is a Poisson point process over $\mathbb{R}_+\times\mathcal{D}$ with intensity $\mathrm{d}y\otimes N_\Psi(\mathrm{d}Y)$. For $s_l>0$ and for all $i\in I$, define

$$\bar{X}^i_t=\sum_{X_{s_l}(x_i-)<y_j\le X_{s_l}(x_i)} Y^j_t,\quad t>0\quad\text{and}\quad \bar{X}^i_0=\Delta X_{s_l}(x_i). \tag{35}$$

Note that $X^i_{s_l}=\Delta X_{s_l}(x_i)$ and

$$\sum_{i\in I}1_{\{X^i_{s_l}>0\}}\delta_{(x_i,X^i_{s_l+\cdot})}\overset{d}{=}\sum_{i\in I}1_{\{\bar{X}^i_0>0\}}\delta_{(x_i,\bar{X}^i_\cdot)}. \tag{36}$$

Set $\bar{X}_t(x):=\sum_{x_i\le x}\bar{X}^i_t$, one has

$$\big(\bar{X}_t(x),t\ge 0,x\ge 0\big)\overset{d}{=}\big(X_{s_l+t}(x),t\ge 0,x\ge 0\big). \tag{37}$$

Applying Lemma 7 and Lemma 9 to $(Y_t(y),\,y\ge 0,t\ge 0)$, we have that almost surely, for all $j\in J$ and for all $y\ge 0$,

$$Z^Y_j:=\lim_{t\to+\infty}e^{-t}G\bigg(\frac{1}{Y^j_t}\wedge\rho\bigg)\text{ exists}\quad\text{and}\quad\sup_{y_j\le y}Z^Y_j=\lim_{t\to+\infty}e^{-t}G\bigg(\frac{1}{Y_t(y)}\wedge\rho\bigg). \tag{38}$$

By (35), $\bar{X}_t(U_n-)=\sum_{x_i<U_n}\bar{X}^i_t=Y_t(X_{s_l}(U_n-))$ and then

$$\lim_{t\to+\infty}e^{-(s_l+t)}G\bigg(\frac{1}{\bar{X}_t(U_n-)}\wedge\rho\bigg)=e^{-s_l}\sup_{y_j\le X_{s_l}(U_n-)}Z^Y_j\quad\text{a.s.} \tag{39}$$

By (34), (37) $\lim_{t\to+\infty}e^{-(s_l+t)}G(\frac{1}{\bar{X}_t(U_n-)}\wedge\rho)\overset{d}{=}\hat{Z}_n$. Therefore by (39),

$$\hat{Z}_n\overset{d}{=}e^{-s_l}\sup_{y_j\le X_{s_l}(U_n-)}Z^Y_j. \tag{40}$$

By definition of $(\bar{X}^i_t)_{t\ge0}$, for $i\in I$ such that $\Delta X_{s_l}(x_i)>0$, $\bar{X}^i_t=Y_t(X_{s_l}(x_i))-Y_t(X_{s_l}(x_i-))$. If $\Delta X_{s_l}(x_i)=0$, then we set $\bar{X}^i_t=0$ for all $t\ge 0$. By Lemma 6 for any $i\in I$,

$$\bar{Z}_i:=\lim_{t\to+\infty}e^{-(s_l+t)}G\bigg(\frac{1}{\bar{X}^i_t}\wedge\rho\bigg)\text{ exists a.s.}\quad\text{and}\quad\bar{Z}_i\overset{d}{=}e^{-s_l}\sup_{y_j\in(X_{s_l}(x_i-),X_{s_l}(x_i)]}Z^Y_j. \tag{41}$$

Fix i. By (35), $\bar{X}^i_t\ge Y^j_t$ for any $j\in J$ such that $y_j\in(X_{s_l}(x_i-),X_{s_l}(x_i)]$. It follows from the above limit and (38) that $\bar{Z}_i\ge e^{-s_l}\sup_{y_j\in(X_{s_l}(x_i-),X_{s_l}(x_i)]}Z^Y_j$ a.s. Thus for each $i\in I$, $\bar{Z}_i=e^{-s_l}\sup_{y_j\in(X_{s_l}(x_i-),X_{s_l}(x_i)]}Z^Y_j$ a.s. and

$$e^{-s_l}\sup_{y_j\le X_{s_l}(U_n-)}Z^Y_j=e^{-s_l}\sup_{x_i<U_n}\sup_{y_j\in(X_{s_l}(x_i-),X_{s_l}(x_i)]}Z^Y_j=\sup_{x_i<U_n}\bar{Z}_i\quad\text{a.s.}$$

By (36), $\sup_{x_i<U_n}\bar{Z}_i\overset{d}{=}\sup_{x_i<U_n}Z_i=Z(U_n-)$. Therefore, (40) entails that for any n, $\hat{Z}_n\overset{d}{=}Z(U_n-)$. □

Lemma 12. *Under the conditions of Lemma 8, we have that almost-surely for all $i\in I$,*

$$\lim_{t\to+\infty}e^{-t}G\bigg(\frac{1}{X_t(x_i-)}\wedge\rho\bigg)=Z(x_i-)=\sup_{0<x_j<x_i}Z_j.$$

Proof. Recall $I_{l,k} := \{i \in I, t_i \le s_l, X^i_{t_i} > \epsilon_k\}$, $U^{l,k}_n$ and $T^{l,k}_n$ defined in the proof of Lemma 8. For simplicity we write U_n and T_n for $U^{l,k}_n$ and $T^{l,k}_n$. Observe that

$$\big\{\big(x_i, t_i, \big(X_{t_i+t}(x_i-)\big)_{t\ge 0}\big), i \in I_{l,k}\big\} = \big\{\big(U_n, T_n, \big(X_{T_n+t}(U_n-)\big)_{t\ge 0}\big), n = 1, 2, \dots\big\},$$

where for each n,

$$X_{T_n+t}(U_n-) = e^{-\mathbf{d}(t+T_n)} U_n + \sum_{0<x_j<U_n} X^j_{t+T_n-t_j} 1_{\{t_j<t+T_n\}} \tag{42}$$

is a CSBP(Ψ) with initial value $X_{T_n}(U_n-)$. It follows from Lemma 6 that for each n, $e^{-t}G(\frac{1}{X_t(U_n-)} \wedge \rho) \underset{t\to+\infty}{\longrightarrow} \hat{Z}_n$ a.s. As proved in Step 1 of Lemma 11, it follows from (42) and (30) that $\hat{Z}_n \ge \sup_{x_i<U_n} Z_i = Z(U_n-)$ a.s. Then we use the coupling method to prove that $\hat{Z}_n$ and $Z(U_n-)$ have the same distributions. On an extended probability space, let $(Y_t(y), t \ge 0, y \ge 0)$ be an independent copy of $(X_t(x), t \ge 0, x \ge 0)$ given by

$$Y_t(y) = e^{-\mathbf{d}t} y + \sum_{y_j \le y} 1_{\{s_j \le t\}} Y^j_{t-s_j}, \quad t \ge 0, y \ge 0,$$

where $\mathcal{N}^Y = \sum_{j\in J} \delta_{(y_j,s_j,Y^j)}$ is a Poisson point process over $\mathbb{R}_+ \times \mathbb{R}_+ \times \mathcal{D}$ with intensity $\mathrm{d}y \otimes e^{-\mathbf{d}s}\,\mathrm{d}s \otimes \int_0^{+\infty} \pi(\mathrm{d}r)\mathbb{P}^\Psi_r(\mathrm{d}Y)$. For $i \in I$ with $t_i \le T_n$, set

$$\bar{X}^i_t = X^i_t \quad \text{if } 0 \le t \le T_n - t_i;$$
$$\bar{X}^i_t = e^{-\mathbf{d}(t-T_n+t_i)} \Delta X_{T_n}(x_i) + \sum_{j\in J_i} 1_{\{s_j \le t-T_n+t_i\}} Y^j_{t-T_n+t_i-s_j} \quad \text{if } t > T_n - t_i,$$

where $J_i = \{j \in J : X_{T_n}(x_i-) < y_j \le X_{T_n}(x_i)\}$. For $i \in I$ with $t_i > T_n$, set $\bar{X}^i_t = X^i_t$ for $t \ge 0$. One has

$$\sum_{i\in I} \delta_{(x_i,t_i,X^i_\cdot)} \overset{d}{=} \sum_{i\in I} \delta_{(x_i,t_i,\bar{X}^i_\cdot)}.$$

The flow $(\bar{X}_t(x), t \ge 0, x \ge 0) = (e^{-\mathbf{d}t}x + \sum_{x_i\le x} 1_{\{t_i\le t\}} \bar{X}^i_{t-t_i}, t \ge 0, x \ge 0)$ is a flow of CSBPs(Ψ) and one can show that $\hat{Z}_n \overset{d}{=} Z(U_n-)$ as in Step 2 of Lemma 11. □

Lemma 13. $\mathcal{S} \cap \mathcal{P} = \{x > 0; \Delta Z(x) > 0\}$ *a.s.*

Proof. We focus on the infinite variation case, the proof in the finite variation case is similar by replacing X^i_t by $X^i_{t-t_i}$. We first establish that if x_j and x_m are two individuals such that $Z_j > Z_m$, then $\frac{X^j_t}{X^m_t} \underset{t\to+\infty}{\longrightarrow} +\infty$ a.s. If $Z_m = 0$, then clearly $\frac{X^j_t}{X^m_t} \underset{t\to+\infty}{\longrightarrow} +\infty$, as $X^m_t \underset{t\to+\infty}{\longrightarrow} 0$ and $X^j_t \underset{t\to+\infty}{\longrightarrow} +\infty$. Assume now $Z_m > 0$, Recall $G(z) = \exp(-\int_z^{\lambda_0} \frac{\mathrm{d}u}{\Psi(u)})$, by Lemma 7,

$$\frac{e^{-t}G(1/X^m_t)}{e^{-t}G(1/X^j_t)} = \exp\left(-\int_{\frac{1}{X^m_t}}^{\frac{1}{X^j_t}} \frac{\mathrm{d}u}{\Psi(u)}\right) \underset{t\to+\infty}{\longrightarrow} \frac{Z_m}{Z_j} < 1 \quad \text{a.s.}$$

Then, $\lim_{t\to+\infty} \int_{\frac{1}{X^m_t}}^{\frac{1}{X^j_t}} \frac{\mathrm{d}u}{\Psi(u)} \in (0, +\infty)$. Since Ψ is non-positive in a neighbourhood of 0, then for t large enough $\frac{1}{X^j_t} \le \frac{1}{X^m_t}$. This ensures that $\liminf_{t\to+\infty} \frac{X^j_t}{X^m_t} > 0$. Assume $\liminf_{t\to+\infty} \frac{X^j_t}{X^m_t} =: \theta \in (0, +\infty)$, then there exists some sequence $(t_n)_{n\ge1}$ such that $t_n \underset{n\to+\infty}{\longrightarrow} +\infty$ and $\lim_{n\to\infty} \frac{X^j_{t_n}}{X^m_{t_n}} = \theta$. Since G is slowly varying at 0, by locally uniform convergence (see Proposition 0.5 in [27]) we have that

$$\frac{G(\frac{1}{X^m_{t_n}})}{G(\frac{1}{X^j_{t_n}})} = \frac{G(\frac{1}{X^j_{t_n}}\frac{X^j_{t_n}}{X^m_{t_n}})}{G(\frac{1}{X^j_{t_n}})} \underset{n\to+\infty}{\longrightarrow} 1 \quad \text{a.s.}$$

This entails a contradiction. In conclusion, $\theta = +\infty$ and $\frac{X^j_t}{X^m_t} \underset{t\to+\infty}{\longrightarrow} +\infty$ a.s. We show now $\mathcal{S}\cap\mathcal{P} = \{x > 0; \Delta Z(x) > 0\}$.

- If $x = x_m$ is not a jump of $(Z(x), x \geq 0)$, then $Z_m < \sup_{x_i \leq x_m} Z_i$. Therefore there exists j such that $x_j < x_m$ and $Z_j > Z_m$. Since $\frac{X_t(x_m-)}{X^m_t} \geq \frac{X^j_t}{X^m_t}$, we get $\lim_{t\to+\infty} \frac{X^m_t}{X_t(x_m-)} = 0$, that is to say x_m is not a super-individual and $\mathcal{S}\cap\mathcal{P} \subset \{x > 0 : \Delta Z(x) > 0\}$.
- If $x = x_m$ is a jump of $(Z(x), x \geq 0)$, then by Lemma 9, $Z_m = \sup_{x_i\leq x_m} Z_i$ and $Z_m > 0$. By Lemma 11, $e^{-t}G(1/X_t(x_m-)\wedge\rho) \underset{t\to+\infty}{\longrightarrow} Z(x_m-) < Z_m$. Therefore,

$$\frac{G(1/X_t(x_m-)\wedge\rho)}{G(1/X^m_t)} \underset{t\to+\infty}{\longrightarrow} \frac{Z(x_m-)}{Z_m} < 1.$$

  Using the slow variation property of G as before, we show that

$$\lim_{t\to+\infty} \frac{X^m_t}{X_t(x_m-)} = +\infty,$$

  thus x_m is a super-individual. □

By combining Lemma 9 and Lemma 13, one obtains Theorem 1(i).

4. Subcritical processes

Consider a subcritical or critical CSBP $(X_t(x), t \geq 0)$. Since it goes to 0 almost-surely, there is no prolific individuals and $\Delta X_t(x)$ goes to 0 with probability 1 for all $x \geq 0$. If not otherwise specified, we consider Ψ such that $\Psi'(0+) \geq 0$ and $\int^{+\infty} \frac{\mathrm{d}u}{\Psi(u)} = +\infty$. As recalled in Theorem 2, the CSBP(Ψ) goes to zero without being absorbed. We mention that there is no notion of persistence in the framework of discrete state-space branching processes, so that most results in this section have no discrete counterparts. Recall the set of super-individuals

$$\mathcal{S} := \left\{x > 0; \lim_{t\to+\infty} \frac{\Delta X_t(x)}{X_t(x-)} = +\infty\right\}.$$

We start with the case of a branching mechanism with finite variation in which no super-individuals exist.

Proposition 4 (Proposition 2.3 and Lemma 2.4 in [9]). *Suppose* $\mathbf{d} \in \mathbb{R}$ *and fix* $\lambda \in (0, +\infty)$. *Consider a flow of CSBPs*(Ψ) $(X_t(x), x \geq 0, t \geq 0)$ *defined as in* (17). *There exists a càdlàg subordinator* $(V^\lambda(x), x \geq 0)$ *with Laplace exponent* $\theta \mapsto v_{-\frac{\log\theta}{\mathbf{d}}}(\lambda)$ *such that almost-surely for any* $x > 0$,

$$v_{-t}(\lambda)X_t(x) \underset{t\to+\infty}{\longrightarrow} V^\lambda(x) \quad \text{and} \quad v_{-t}(\lambda)X_t(x-) \underset{t\to+\infty}{\longrightarrow} V^\lambda(x-).$$

Moreover, $(V^\lambda(x), x \geq 0)$ *has an infinite Lévy measure and* $\mathcal{S} = \varnothing$ *almost-surely.*

Proof. We only verify that $\mathcal{S} = \varnothing$. Lemma 2.4 in [9] entails that for all $x > 0$, $\lim_{t\to+\infty} \frac{\Delta X_t(x)}{X_t(x-)} = \frac{\Delta V^\lambda(x)}{V^\lambda(x-)} < +\infty$. Therefore $\mathcal{S} = \varnothing$. □

We study now the subcritical persistent processes with infinite variation.

Theorem 4. *Suppose that Ψ is (sub)critical, persistent and of infinite variation. Fix some $\lambda_0 > 0$ and set $G(y) = \exp(-\int_{\lambda_0}^{y} \frac{\mathrm{d}u}{\Psi(u)})$ for $y \in (0, +\infty)$. Then, for all $x \geq 0$, almost-surely*

$$e^t G\left(\frac{1}{X_t(x)}\right) \underset{t\to+\infty}{\longrightarrow} \tilde{Z}(x) \quad a.s.,$$

where $(\tilde{Z}(x), x \geq 0)$ is a positive extremal-F process (in the sense of (18)) with $F(z) = \exp(-G^{-1}(z))$ for $z \geq 0$ and $G^{-1}(z) = v_{\log z}(\lambda_0)$ for all $z \geq 0$.

Example 3. Consider the mechanism $\Psi(u) = (u+1)\log(u+1)$. Fix $\lambda_0 = e - 1$, then for all $z \geq 0$, $G(z) = \frac{1}{\log(1+z)}$ and $G^{-1}(z) = e^{1/z} - 1$. The process $(\tilde{Z}(x), x \geq 0)$ is an extremal-F process with $F(z) = e^{1-e^{1/z}}$ for all $z \geq 0$.

Lemma 14. *If $\mathbf{d} = +\infty$ and $\int^{+\infty} \frac{\mathrm{d}u}{\Psi(u)} = +\infty$, then the map $G : y \mapsto \exp(-\int_{\lambda_0}^{y} \frac{\mathrm{d}u}{\Psi(u)})$ is continuous, non-increasing, goes from $[0, +\infty]$ to $[0, +\infty]$ and is slowly varying at $+\infty$. Moreover $G^{-1}(z) = v_{\log z}(\lambda_0)$ for $z \in [0, +\infty]$.*

Proof. The map G is continuous, non-increasing and maps $(0, +\infty)$ to $(0, +\infty)$. Indeed Ψ is continuous positive on $(0, +\infty)$ and $\lim_{y\to+\infty} G(y) = 0$, since $\int_{\lambda_0}^{+\infty} \frac{1}{\Psi(z)}\,\mathrm{d}z = +\infty$, $\lim_{y\to 0} G(y) = +\infty$ since $\int_0^{\lambda_0} \frac{\mathrm{d}z}{\Psi(z)} = +\infty$. Let $\delta > 0$, since $\int_0^1 u\pi(\mathrm{d}u) = +\infty$, $\mathbf{d} := \lim_{z\to+\infty} \Psi(z)/z = +\infty$ and for any $b > 0$, $\Psi(z) \geq bz$ for large enough z. Therefore $|\int_x^{\delta x} \frac{\mathrm{d}z}{\Psi(z)}| \leq \frac{|\log(\delta)|}{b}$. Since b is arbitrarily large,

$$\lim_{x\to+\infty} \int_x^{\delta x} \frac{\mathrm{d}z}{\Psi(z)} = 0 \tag{43}$$

and the map G is slowly varying at $+\infty$. □

The proof is similar to that of Theorem 3. We provide some details for sake of completeness.

Proof of Theorem 4. Under the assumption $\int^{+\infty} \frac{\mathrm{d}u}{\Psi(u)} = +\infty$, $\bar{v} = +\infty$ and by applying Lemma 2, we have that for a fixed $\lambda > 0$ and any $x \geq 0$, $(v_{-t}(\lambda)X_t(x), t \geq 0)$ converges $\mathbb{P}_x$-almost-surely, as $t \to +\infty$, to some random variable $V^\lambda(x)$ taking value in $\bar{\mathbb{R}}_+$. Let $\delta > 0$, (23) yields

$$\lim_{t\to+\infty}\left|\int_\lambda^{v_t(\delta v_{-t}(\lambda))} \frac{\mathrm{d}z}{\Psi(z)}\right| = \lim_{t\to+\infty}\left|\int_{v_{-t}(\lambda)}^{\delta v_{-t}(\lambda)} \frac{\mathrm{d}z}{\Psi(z)}\right|.$$

Recall that $\lim_{t\to+\infty} v_{-t}(\lambda) = +\infty$, (43) implies that $\lim_{t\to+\infty} v_t(\delta v_{-t}(\lambda)) = \lambda$, for any $\delta > 0$. We have

$$\lim_{t\to+\infty} \mathbb{E}\big[e^{-\delta v_{-t}(\lambda)X_t(x)}\big] = \lim_{t\to+\infty} \exp\big(-x v_t\big(\delta v_{-t}(\lambda)\big)\big) = e^{-x\lambda}.$$

The limit does not depend on δ, therefore $(v_{-t}(\lambda)X_t(x), t \geq 0)$ converges either to 0 or $+\infty$ and $\mathbb{P}(V^\lambda(x) = 0) = e^{-x\lambda}$. For any $x \geq 0$, by considering the collection of random variables $V^\lambda(x)$, one can define the random variable

$$\Lambda_x := \inf\big\{\lambda \in (0, +\infty) \cap \mathbb{Q} : V^\lambda(x) = +\infty\big\}.$$

Note that $\mathbb{P}(\exists\lambda > 0 \text{ such that } V_x^\lambda = 0) = 1$, which implies that $\Lambda_x > 0$ a.s. Similarly as in (25), one can show that almost-surely

$$v_{-t}(\lambda)X_t(x) \underset{t\to+\infty}{\longrightarrow} \begin{cases} 0 & \text{if } \Lambda_x > \lambda, \\ +\infty & \text{if } \Lambda_x < \lambda. \end{cases} \tag{44}$$

Choose λ' and λ'' such that $\lambda' < \Lambda_x < \lambda''$. By (44), for large enough t,

$$v_{-t}\big(\lambda''\big)X_t(x) \geq 1 \quad \text{and} \quad 0 < v_{-t}\big(\lambda'\big)X_t(x) \leq 1.$$

Since $G(v_{-t}(\lambda')) = e^{-\int_{\lambda_0}^{\lambda'} \frac{dx}{\Psi(x)}} e^{-t} = G(\lambda')e^{-t}$ then

$$G(\lambda') \geq e^t G(1/X_t(x)) \geq G(\lambda'').$$

Since λ' and λ'' are arbitrarily close to Λ_x, and G is continuous, we get

$$e^t G(1/X_t(x)) \underset{t\to+\infty}{\longrightarrow} G(\Lambda_x) \quad \mathbb{P}\text{-almost surely.}$$

Let $\lambda_1 \neq \lambda_2$ be two positive real numbers, By Lemma 1, $\lim_{t\to+\infty} v_{-t}(\lambda_2)/v_{-t}(\lambda_1) = 0$ if $\lambda_1 > \lambda_2$ and $+\infty$ otherwise. Thus $\lim_{t\to+\infty} v_t(v_{-t}(\lambda_1) + v_{-t}(\lambda_2)) = \lambda_1 \vee \lambda_2$. We conclude by applying the same arguments as at the end of the proof of Theorem 4. □

The following proposition is obtained similarly as Proposition 2

Proposition 5. *Assume* $\mathbf{d} = +\infty$ *and* $\int^{+\infty} \frac{\mathrm{d}u}{\Psi(u)} = +\infty$. *If there are* $\lambda > 0$ *and* $\alpha > 0$ *such that* $|\int_\lambda^{+\infty} (\frac{1}{\Psi(u)} - \frac{1}{\alpha u \log u})\,\mathrm{d}u| < +\infty$, *then* $G(1/y) \underset{y\to 0}{\sim} k_\lambda (\log 1/y)^{-1/\alpha}$ *for a constant* $k_\lambda > 0$. *Fix* $x > 0$, *then*

$$\log X_t(x) \underset{t\to+\infty}{\sim} -e^{\alpha t} k_\lambda^\alpha \tilde{Z}(x)^{-\alpha} \quad a.s.$$

A natural extremal process associated to a flow of non-persistent subcritical CSBPs is given by the extinction times of each initial individuals. The proof of the following proposition is straightforward.

Proposition 6 (Theorem 0.3(i) in [9] for $\zeta = \zeta_0$). *Assume* $\Psi'(0) \geq 0$ *and* $\int^{+\infty} \frac{\mathrm{d}u}{\Psi(u)} < +\infty$, *all individuals living at time* 0 *get absorbed in* 0. *For all* $x \geq 0$, *set* $Z(x) := \inf\{t \geq 0;\, X_t(x) = 0\}$. *The process* $(Z(x), x \geq 0)$ *is an extremal-F process, with* $F(z) = \exp(-\bar{v}_z)$. *Let* Z_i *be the life-length of* X^i. *The point process* $\mathcal{M} := \sum_{i\in I} \delta_{(x_i, Z_i)}$ *is a Poisson point process with intensity* $\mathrm{d}x \otimes \mu(\mathrm{d}z)$ *with* $\bar{\mu}(z) = \overline{v}_z$. *Moreover, almost-surely for all* $x \geq 0$, $Z(x) = \sup_{x_i \leq x} Z_i$ *and*

$$\mathcal{S} := \{x > 0;\, \exists t > 0;\, \Delta X_t(x) > 0 \text{ and } X_t(x-) = 0\} = \{x > 0;\, \Delta Z(x) > 0\}.$$

Example 4. Let $\alpha \in (0, 1]$ and $\Psi(u) = \alpha u^{\alpha+1}$. Consider a flow of CSBPs(Ψ) $(X_t(x), x \geq 0, t \geq 0)$. By Proposition 6 and Theorem 2(ii), the process $(Z(x), x \geq 0)$ is an extremal-F process with F the probability distribution function of a Fréchet law with parameter $\frac{1}{\alpha} \in [1, +\infty)$, that is to say $F(z) = e^{-z^{-\frac{1}{\alpha}}}$ for all $z \geq 0$.

4.1. *Proof of Theorem* 1(ii)

In the subcritical case with infinite variation, one always has $F(0) = 0$. The state 0 is therefore instantaneous for the process $(Z(x), x \geq 0)$. We work now with Ψ subcritical persistent with infinite variation and consider a flow of CSBPs(Ψ) as in (16). The following lemmas can be established by applying *verbatim* the proofs of the supercritical case.

Lemma 15. *Consider on the same probability space* $(X_t, t \geq 0)$ *and* $(Y_t, t \geq 0)$ *two independent* CSBPs(Ψ), *satisfying the conditions of Theorem* 4 *starting from* X_0 *and* Y_0. *Then, almost-surely*

$$e^t G\left(\frac{1}{X_t}\right) \underset{t\to+\infty}{\longrightarrow} \Gamma_1, \qquad e^t G\left(\frac{1}{Y_t}\right) \underset{t\to+\infty}{\longrightarrow} \Gamma_2 \quad \text{and} \quad e^t G\left(\frac{1}{X_t + Y_t}\right) \underset{t\to+\infty}{\longrightarrow} \Gamma_1 \vee \Gamma_2,$$

with Γ_1 *and* Γ_2 *independent such that* $\Gamma_1 \neq \Gamma_2$ *a.s. and*

$$\Gamma_1 \overset{d}{=} \bar{Z}(X_0), \qquad \Gamma_2 \overset{d}{=} \bar{Z}(Y_0) \quad \text{and} \quad \Gamma_1 \vee \Gamma_2 \overset{d}{=} \bar{Z}(X_0 + Y_0),$$

where $(\bar{Z}(x), x \geq 0)$ *is an extremal-F process, independent of* X_0 *and* Y_0.

Proof. The proof is similar as that of Lemma 6. □

We gather here in a single lemma the analogues of Lemma 7, Lemma 9 and Lemma 11 for the subcritical case.

Lemma 16. *Almost-surely for all $i \in I$, the following limit exists*

$$Z_i := \lim_{t\to+\infty} e^t G\left(\frac{1}{X_t^i}\right). \tag{45}$$

The point process $\mathcal{M} := \sum_{i\in I} \delta_{(x_i,Z_i)}$ is a Poisson point process with intensity $\mathrm{d}x \otimes \mu(\mathrm{d}z)$ where $\bar{\mu}(z) = G^{-1}(z)$. For any $x \geq 0$, almost-surely

$$\forall y \geq x, \quad \lim_{t\to+\infty} e^t G\left(\frac{1}{X_t(y) - X_t(x)}\right) = Z(x,y) := \sup_{x<x_i\leq y} Z_i.$$

Moreover, almost-surely for all $i \in I$,

$$\lim_{t\to+\infty} e^t G\left(\frac{1}{X_t(x_i-)}\right) = Z(x_i-).$$

Lemma 17. $\mathcal{S} = \{x > 0; \Delta Z(x) > 0\}$ *a.s.*

Proof. We only show that if $Z_j > Z_m$ then $\frac{X_t^j}{X_t^m} \underset{t\to+\infty}{\longrightarrow} +\infty$ a.s. The identity for $\mathcal{S}$ can be proven as in Lemma 13 by using the slow variation at $+\infty$. Recall $G(z) = \exp(-\int_{\lambda_0}^z \frac{\mathrm{d}u}{\Psi(u)})$, by Lemma 16,

$$\frac{e^t G(1/X_t^m)}{e^t G(1/X_t^j)} = \exp\left(-\int_{\frac{1}{X_t^j}}^{\frac{1}{X_t^m}} \frac{\mathrm{d}u}{\Psi(u)}\right) \underset{t\to+\infty}{\longrightarrow} \frac{Z_m}{Z_j} < 1.$$

Then, $\lim_{t\to+\infty} \int_{\frac{1}{X_t^j}}^{\frac{1}{X_t^m}} \frac{\mathrm{d}u}{\Psi(u)} \in (0,+\infty)$ a.s. Since Ψ is non-negative then for t large enough $\frac{1}{X_t^m} \geq \frac{1}{X_t^j}$. This ensures that $\liminf_{t\to+\infty} \frac{X_t^j}{X_t^m} > 0$ a.s. Assume $\liminf_{t\to+\infty} \frac{X_t^j}{X_t^m} =: \theta \in (0,+\infty)$ a.s., then there exists (t_n) such that $t_n \underset{n\to+\infty}{\longrightarrow} +\infty$ and $\frac{X_{t_n}^j}{X_{t_n}^m} \underset{n\to+\infty}{\longrightarrow} \theta$ a.s. By locally uniform convergence of slowly varying function (see Proposition 0.5 in [27]) we have that

$$\frac{G(\frac{1}{X_{t_n}^m})}{G(\frac{1}{X_{t_n}^j})} = \frac{G(\frac{1}{X_{t_n}^j}\frac{X_{t_n}^j}{X_{t_n}^m})}{G(\frac{1}{X_{t_n}^j})} \underset{n\to+\infty}{\longrightarrow} 1 \quad \text{a.s.}$$

This entails a contradiction. In conclusion, if $Z_j > Z_m$, then $\frac{X_t^j}{X_t^m} \underset{t\to+\infty}{\longrightarrow} +\infty$ a.s. □

4.2. *Flow of Neveu's continuous-state branching processes*

It is well-known that a supercritical CSBP(Ψ) with $\rho < \infty$ conditioned to be extinct is a subcritical CSBP with mechanism $u \mapsto \Psi(u+\rho)$. When the CSBP is supercritical non-explosive persistent with infinite mean and infinite variation, one can combine Theorems 3 and 4 to obtain both the rates of growth and of decay. In the Neveu case, we can renormalize the population size on the events of extinction and non-extinction by the same function. As claimed in the introduction, the renormalized flow of Neveu CSBPs converges towards an extremal-Λ process with $\Lambda(z) = e^{-e^{-z}}$ for $z \in \mathbb{R}$.

Lemma 18. *Let* $(X_t(x), t \geq 0)$ *be a* CSBP(Ψ) *with* $\Psi(u) = u \log u$. *Then* $e^{-t} \log X_t(x) \underset{t\to+\infty}{\longrightarrow} Z(x)$ *a.s. where the process* $(Z(x), x \geq 0)$ *is an extremal-*Λ *process.*

Proof. We refer the reader to Proposition 10 and its proof in Fleischmann and Sturm [12] for the almost-sure convergence for fixed x, towards a random variable with a Gumbel law. We verify now that the process $(Z(x), x \geq 0)$ is extremal. Clearly $Z(x+y)$ has the same law as $Z(x) \vee Z'(y)$ for a random variable $Z'(y)$ distributed as $Z(y)$ and independent of $Z(x)$. Moreover, since

$$e^{-t} \log\bigl(X_t(x+y)\bigr) \geq e^{-t} \log\bigl(X_t(x)\bigr) \vee e^{-t} \log\bigl(X_t(x+y) - X_t(x)\bigr)$$

then $Z(x+y) \geq Z(x) \vee Z(x, x+y)$ a.s. with $Z(x, x+y) := \lim_{t\to\infty} e^{-t} \log(X_t(x+y) - X_t(x))$. We deduce that $Z(x+y) = Z(x) \vee Z(x, x+y)$ a.s. and conclude by recalling (20). □

If x is such that $Z(x) < 0$, then the population started from x is extinguishing. If x is such that $Z(x) > 0$, the population is not extinguishing. The process $(Z(x), x \geq 0)$ enters in $(0, +\infty)$ with the first prolific individual.

Proposition 7. *Consider* $(X_t(x), t \geq 0, x \geq 0)$ *a flow of Neveu CSBPs* (*constructed as in* (16)). *Then almost-surely for all* $i \in I$, *the limit* $Z_i := \lim_{t\to+\infty} e^{-t} \log X^i_t$ *exists. The point process* $\mathcal{M} := \sum_{i\in I} \delta_{(x_i, Z_i)}$ *is a Poisson point process over* $\mathbb{R}_+ \times \mathbb{R}$ *with intensity* $\mathrm{d}x \otimes e^{-z}\,\mathrm{d}z$ *and almost-surely, for all* $x \geq 0$,

$$e^{-t} \log X_t(x) \underset{t\to+\infty}{\longrightarrow} Z(x) := \sup_{x_i \leq x} Z_i.$$

Moreover $\mathcal{S} = \{x > 0; \Delta Z(x) > 0\}$ *a.s.*

Proof. We stress that Proposition 7 is not a direct combination of Theorem 1(i) and Theorem 1(ii) for $\Psi(u) = u \log u$. Indeed the point process obtained in Theorem 1(i) does not take into account the decay of non-prolific individuals (they are all with $Z_i = 0$). With the same notation as in Lemma 7, for all $i \in I_{l,k}$, by Lemma 18, Z_i exists a.s. and has the same law as $e^{-s_l} \bar{Z}_{V_0^{l,k}}$, with $\bar{Z}_x$ a random variable with a Gumbel Law and $V_0^{l,k}$ a random variable with law $\frac{\ell_{s_l}(\mathrm{d}x; x\geq\epsilon_k)}{\ell_{s_l}([\epsilon_k,+\infty))}$. Since the number of individuals in $I_{l,k}$ is a Poisson random variable with parameter $\ell_{s_l}([\epsilon_k, +\infty))$, one has

$$\begin{aligned} N_\Psi(X_{s_l} > \epsilon_k; Z > z) &= \ell_{s_l}\bigl((\epsilon_k, +\infty)\bigr)\mathbb{P}\bigl(\bar{Z}_{V_0^{l,k}} > e^{s_l} z\bigr) \\ &= \int_{(\epsilon_k,\infty]} \ell_s(\mathrm{d}x)\bigl(1 - \mathbb{P}\bigl(\bar{Z}_x < e^{s_l} z\bigr)\bigr) = \int_{(\epsilon_k,+\infty]} \ell_{s_l}(\mathrm{d}x)\bigl(1 - e^{-xe^{-ze^{s_l}}}\bigr) \\ &\underset{k\to+\infty}{\longrightarrow} \int_{(0,+\infty]} \ell_{s_l}(\mathrm{d}x)\bigl(1 - e^{-xe^{-ze^{s_l}}}\bigr) = v_{s_l}\bigl(e^{-ze^{s_l}}\bigr) = e^{-ze^{2s_l}} \underset{l\to+\infty}{\longrightarrow} e^{-z}. \end{aligned}$$

Thus, the intensity of the Poisson point process $\mathcal{M}$ is $\mu(\mathrm{d}z) = e^{-z}\,\mathrm{d}z$. The rest of the proof follows exactly the same lines as Lemma 7 and Lemma 9. □

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