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On the exit time and stochastic homogenization of isotropic diffusions in large domains

Benjamin Fehrman[1]

Max Planck Institute for Mathematics in the Sciences, Inselstrasse 22, 04103 Leipzig, Germany. E-mail: fehrman@mis.mpg.de

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Abstract. Stochastic homogenization is achieved for a class of elliptic and parabolic equations describing the lifetime, in large domains, of stationary diffusion processes in random environment which are small, statistically isotropic perturbations of Brownian motion in dimension at least three. Furthermore, the homogenization is shown to occur with an algebraic rate. Such processes were first considered in the continuous setting by Sznitman and Zeitouni (*Invent. Math.* **164** (2006) 455–567), upon whose results the present work relies strongly.

Résumé. On effectue l'homogénéisation stochastique d'une certaine classe d'équations elliptiques et paraboliques. Ces équations décrivent la durée de vie, dans des domaines grands, de processus de diffusion stationnaire en environnement aléatoire qui sont des petites perturbations statistiquement isotropes du mouvement brownien, en dimension au moins trois. On démontre que l'homogénéisation a lieu à vitesse algébrique. De tels processus ont été étudiés dans un cadre continu en premier lieu par Snitzman et Zeitouni (*Invent. Math.* **164** (2006) 455–567), sur les résultats desquels le présent travail s'appuie fortement.

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1. Introduction

The purpose of this paper is to characterize, in dimensions greater than two, the lifetime of diffusion processes in large domains which are associated to generators of the form

$$\frac{1}{2}\sum_{i,j=1}^{d} a_{ij}(x,\omega)\frac{\partial^2}{\partial x_i\,\partial x_j}+\sum_{i=1}^{d} b_i(x,\omega)\frac{\partial}{\partial x_i}, \tag{1}$$

where the uniformly elliptic diffusion matrix $A=(a_{ij})$ and drift $b=(b_i)$ are bounded and Lipschitz continuous. These are assumed to describe a stationary, strongly mixing random environment, as indexed by an underlying probability space $(\Omega,\mathcal{F},\mathbb{P})$, which corresponds to a small, statistically isotropic perturbation of Brownian motion.

Precisely, the stationarity is quantified by a measure preserving transformation group $\{\tau_x\}_{x\in\mathbb{R}^d}$ of the probability space which satisfies, for each $x, y\in\mathbb{R}^d$ and $\omega\in\Omega$,

$$A(x+y,\omega)=A(x,\tau_y\omega)\quad\text{and}\quad b(x+y,\omega)=b(x,\tau_y\omega). \tag{2}$$

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The coefficients are statistically isotropic in the sense that, for every orthogonal transformation r of $\mathbb{R}^d$ which preserves the coordinate axes, for each $x \in \mathbb{R}^d$, the random variables

$$\big(A(rx,\omega), b(rx,\omega)\big) \quad \text{and} \quad \big(rA(x,\omega)r^t, rb(x,\omega)\big) \quad \text{have the same law.} \tag{3}$$

The environment is strongly mixing in the way of a finite range dependence. Whenever subsets A, B of $\mathbb{R}^d$ are sufficiently separated in space, the sigma algebras

$$\sigma\big(A(x,\cdot), b(x,\cdot)|x \in A\big) \quad \text{and} \quad \sigma\big(A(x,\cdot), b(x,\cdot)|x \in B\big) \quad \text{are independent.} \tag{4}$$

Finally, there exists a constant $\eta > 0$ to be chosen small such that, for every $x \in \mathbb{R}^d$ and $\omega \in \Omega$,

$$\big|A(x,\omega) - I\big| < \eta \quad \text{and} \quad \big|b(x,\omega)\big| < \eta. \tag{5}$$

Condition (5) implies that the stochastic process determined by (1) is a small perturbation of Brownian motion. Such environments were first considered in the continuous setting by Sznitman and Zeitouni [20], and correspond to the analogue of the discrete framework studied by Bricmont and Kupiainen [5].

The lifetime of these processes in large domains will be understood, for small $\epsilon > 0$ and bounded subsets $U \subset \mathbb{R}^d$ satisfying an exterior ball condition, in terms of solutions to the associated elliptic equation

$$\begin{cases} \frac{1}{2}\operatorname{tr}(A(x,\omega)D^2 v^\epsilon) + b(x,\omega)\cdot Dv^\epsilon = \epsilon^2 g(\epsilon x) & \text{on } U/\epsilon, \\ v^\epsilon = f(\epsilon x) & \text{on } \partial U/\epsilon, \end{cases} \tag{6}$$

which admit the representation

$$v^\epsilon(x) = E_{x,\omega}\left(f(\epsilon X_{\tau^\epsilon}) - \epsilon^2 \int_0^{\tau^\epsilon} g(\epsilon X_s)\, ds\right) \quad \text{on } \overline{U}/\epsilon, \tag{7}$$

for the canonical process $(X_s)_{s\geq 0}$, for the expectation $E_{x,\omega}$ associated to the diffusion in environment ω beginning from x, and for τ^ϵ the exit time from U/ϵ. Since the rescaling $u^\epsilon(x) = v^\epsilon(\frac{x}{\epsilon})$ satisfies

$$\begin{cases} \frac{1}{2}\operatorname{tr}(A(\frac{x}{\epsilon},\omega)D^2 u^\epsilon) + \frac{1}{\epsilon}b(\frac{x}{\epsilon},\omega)\cdot Du^\epsilon = g(x) & \text{on } U, \\ u^\epsilon = f(x) & \text{on } \partial U, \end{cases} \tag{8}$$

it follows from a change of variables in the final integral of (7) that

$$u^\epsilon(x) = v^\epsilon\left(\frac{x}{\epsilon}\right) = E_{\frac{x}{\epsilon},\omega}\left(f(\epsilon X_{\frac{\epsilon^2\tau^\epsilon}{\epsilon^2}}) - \int_0^{\epsilon^2\tau^\epsilon} g(\epsilon X_{s/\epsilon^2})\, ds\right) \quad \text{on } \overline{U}, \tag{9}$$

for the stopping time

$\epsilon^2\tau^\epsilon$ quantifying the exit of the rescaled process $\epsilon X_{\frac{\cdot}{\epsilon^2}}$ from U.

The limiting behavior of the rescaling $\epsilon X_{\cdot/\epsilon^2}$ was characterized almost surely in [20], where it was shown that, provided the perturbation η in (5) is sufficiently small, there exists a deterministic $\overline{\alpha} > 0$ for which, on a subset of full probability, as $\epsilon \to 0$,

$$\epsilon X_{\frac{\cdot}{\epsilon^2}} \text{ converges in law on } \mathbb{R}^d \text{ to a Brownian motion with variance } \overline{\alpha}. \tag{10}$$

The purpose of this paper is to obtain the analogous result for the lifetime (i.e. for the exit time and exit distribution) of such processes in large domains, and the result is stated in terms of the stochastic homogenization of (8) for continuous data on the boundary and interior.

Theorem 1. *There exists a subset of full probability on which, for every bounded domain $U \subset \mathbb{R}^d$ satisfying an exterior ball condition, the solutions of* (8) *converge uniformly on $\overline{U}$, as $\epsilon \to 0$, to the solution*

$$\begin{cases} \frac{\overline{\alpha}}{2}\Delta\overline{u} = g(x) & \text{on } U, \\ \overline{u} = f(x) & \text{on } \partial U. \end{cases} \tag{11}$$

Furthermore, the convergence is shown to occur with an algebraic rate. The rate is first established for boundary data which is the restriction of a bounded, uniformly continuous function and interior data which is the restriction of a bounded, Lipschitz function.

$$\text{Assume } f \in \mathrm{BUC}(\mathbb{R}^d) \text{ and } g \in \mathrm{Lip}(\mathbb{R}^d). \tag{12}$$

For the modulus of continuity σ_f and gradient Dg satisfying, for each $x, y \in \mathbb{R}^d$,

$$\left|f(x) - f(y)\right| \leq \sigma_f\left(|x-y|\right) \quad \text{and} \quad \left|g(x) - g(y)\right| \leq \|Dg\|_{L^\infty(\mathbb{R}^d)}|x-y|,$$

the result is the following.

Theorem 2. *Assume* (12). *There exists a subset of full probability and $c_1, c_2, c_3, c_4 > 0$ such that, for all $\epsilon > 0$ sufficiently small depending upon ω, the respective solutions u^ϵ and $\overline{u}$ of* (8) *and* (11) *satisfy, for $C > 0$ independent of ω and ϵ,*

$$\left\|u^\epsilon - \overline{u}\right\|_{L^\infty(\overline{U})} \leq C\left(\|f\|_{L^\infty(\mathbb{R}^d)}\epsilon^{c_1} + \sigma_f\left(\epsilon^{c_2}\right) + \|g\|_{L^\infty(\mathbb{R}^d)}\epsilon^{c_3} + \|Dg\|_{L^\infty(\mathbb{R}^d)}\epsilon^{c_4}\right).$$

Condition (12) can be relaxed in the case that the domain is smooth via a standard extension argument.

$$\text{Assume } f \in \mathrm{C}(\partial U),\ g \in \mathrm{Lip}(\overline{U}),\ \text{and that the domain } U \text{ is smooth.} \tag{13}$$

Then, the rate obtained in Theorem 2 is preserved up to a domain dependent factor.

Theorem 3. *Assume* (13). *There exists a subset of full probability, $c_1, c_2, c_3, c_4 > 0$ and $C_1 = C_1(U) > 0$ such that, for all $\epsilon > 0$ sufficiently small depending upon ω, the respective solutions u^ϵ and $\overline{u}$ of* (8) *and* (11) *satisfy, for $C > 0$ independent of ω and ϵ,*

$$\left\|u^\epsilon - \overline{u}\right\|_{L^\infty(\overline{U})} \leq C\left(\|f\|_{L^\infty(\partial U)}\epsilon^{c_1} + \sigma_f\left(C_1\epsilon^{c_2}\right) + \|g\|_{L^\infty(\overline{U})}\epsilon^{c_3} + \|Dg\|_{L^\infty(\overline{U})}\epsilon^{c_4}\right).$$

The methods of this paper also apply to the analogous parabolic equation

$$\begin{cases} u_t^\epsilon = \frac{1}{2}\operatorname{tr}(A(\frac{x}{\epsilon},\omega)D^2u^\epsilon) + \frac{1}{\epsilon}b(\frac{x}{\epsilon},\omega)\cdot Du^\epsilon + g(x) & \text{on } U \times (0,\infty), \\ u^\epsilon = f(x) & \text{on } \overline{U} \times \{0\} \cup \partial U \times [0,\infty), \end{cases} \tag{14}$$

whose solutions admit the representation

$$u^\epsilon(x,t) = E_{\frac{x}{\epsilon},\omega}\left(f(\epsilon X_{(\epsilon^2\tau^\epsilon \wedge t)/\epsilon^2}) + \int_0^{(\epsilon^2\tau^\epsilon \wedge t)} g\left(\epsilon X_{\frac{s}{\epsilon^2}}\right)ds\right) \quad \text{on } \overline{U} \times [0,\infty).$$

In this case, on a subset of full probability, the solutions of (14) are shown to convergence uniformly, as $\epsilon \to 0$, to the solution

$$\begin{cases} \overline{u}_t = \frac{\overline{\alpha}}{2}\Delta\overline{u} + g(x) & \text{on } U \times (0,\infty), \\ \overline{u} = f(x) & \text{on } \overline{U} \times \{0\} \cup \partial U \times (0,\infty). \end{cases} \tag{15}$$

Since the proof follows by combining the techniques used in this paper and the author's work [8], the details are omitted.

Theorem 4. *There exists a subset of full probability on which, for every bounded domain $U \subset \mathbb{R}^d$ satisfying an exterior ball condition, the respective solutions u^ϵ and $\overline{u}$ of* (14) *and* (15) *satisfy*

$$\lim_{\epsilon\to 0}\|u^\epsilon - \overline{u}\|_{L^\infty(\overline{U}\times[0,\infty))} = 0.$$

Furthermore, the convergence occurs with an algebraic rate in exact analogy with Theorems 2 *and* 3.

The essential novelty of this paper is to handle the case $g \neq 0$, since when $g = 0$ the results of [8] proved, on a subset of full probability, as $\epsilon \to 0$, the solutions of (8) converge uniformly on $\overline{U}$ to the solution

$$\begin{cases} \Delta\overline{u} = 0 & \text{on } U, \\ \overline{u} = f & \text{on } \partial U. \end{cases}$$

The simplification is that, when dealing with merely the exit distribution, events of vanishing probability necessarily pose a vanishing threat. Or, in terms of the analysis, solutions of (8) are uniformly bounded in $\epsilon > 0$, and satisfy the estimate

$$\|u^\epsilon\|_{L^\infty(\overline{U})} \le \|f\|_{L^\infty(\partial U)} \quad \text{whenever } g = 0.$$

In the case $g \neq 0$, it is not a priori obvious that even such L^∞-estimates are obtainable, since the statistical isotropy (3) imposes no symmetry, in general, on the quenched environments. More precisely, in Section 2 the diffusion beginning from x in environment ω will be described in the space of continuous paths by a measure and expectation denoted respectively

$$P_{x,\omega} \quad \text{and} \quad E_{x,\omega}.$$

It is manifestly not the case that these objects are, in any sense, translationally or rotationally invariant in space or that they are in any way symmetric.

The invariance implied by the stationarity (2) and isotropy (3) is seen only after averaging with respect to the entire collection of environments. That is, the annealed measures and expectations, which are defined as the semi-direct products

$$\mathbb{P}_x = \mathbb{P} \ltimes P_{x,\omega} \quad \text{and} \quad \mathbb{E}_x = \mathbb{E} \ltimes E_{x,\omega},$$

do satisfy a translational and rotational invariance in the sense that, for all $x, y \in \mathbb{R}^d$,

$$\mathbb{E}_{x+y}(X_t) = \mathbb{E}_y(x + X_t) = x + \mathbb{E}_y(X_t), \tag{16}$$

and, for all orthogonal transformations r preserving the coordinate axis, for every $x \in \mathbb{R}^d$,

$$\mathbb{E}_x(rX_t) = \mathbb{E}_{rx}(X_t). \tag{17}$$

While this fact plays an important role in [20] to preclude, with probability one, the emergence of ballistic behavior of the rescaled process in the asymptotic limit, it does not yield an immediate control, with respect to the quenched expectations, for the exit time of the process from large domains. And, therefore, it does not readily imply that the solutions of (8) are uniformly bounded as ϵ approaches zero.

The proof of Theorem 1 is founded strongly in the results of [20], which in particular establish, on scales of order $\frac{1}{\epsilon}$ in space and $\frac{1}{\epsilon^2}$ in time and with high probability, a comparison between solutions

$$\begin{cases} v^\epsilon_t = \operatorname{tr}(A(x,\omega)D^2v^\epsilon) + b(x,\omega)\cdot Dv^\epsilon & \text{on } \mathbb{R}^d \times (0,\infty), \\ v^\epsilon = f(\epsilon x) & \text{on } \mathbb{R}^d \times \{0\}, \end{cases} \tag{18}$$

and the solution of the homogenized problem

$$\begin{cases} \overline{v}^{\epsilon}_t = \frac{\overline{\alpha}}{2}\Delta \overline{v}^{\epsilon} & \text{on } \mathbb{R}^d \times (0,\infty), \\ \overline{v}^{\epsilon} = f(\epsilon x) & \text{on } \mathbb{R}^d \times \{0\}, \end{cases} \tag{19}$$

with respect to rescaled Hölder-norms defined in (49). This comparison is used in Section 4.1, similar to its use in [20, Proposition 3.1] and later in [8, Proposition 5.1], to establish a global coupling, on larges scales in space and time and with high probability, between the diffusion in random environment associated to the generator

$$\frac{1}{2}\sum_{i,j=1}^{d} a_{ij}(x,\omega)\frac{\partial^2}{\partial x_i\,\partial x_j} + \sum_{i=1}^{d} b_i(x,\omega)\frac{\partial}{\partial x_i} \tag{20}$$

and a Brownian motion with variance approximately $\overline{\alpha}$. See Proposition 10 and, in particular, Corollary 11.

This coupling will be achieved along a discrete sequence of time steps which, while small with respect to the scale $\frac{1}{\epsilon^2}$, are typically insufficient to characterize the asymptotic behavior of solutions of (6) due to the emergence of the singular in $\frac{1}{\epsilon}$ drift. The difficulties are twofold.

First, the drift can trap the particle in the domain to create, in expectation, an exponentially in $\frac{1}{\epsilon}$ increasing exit time. To counteract this, the probability that the exit time is large is first controlled by Proposition 12 in Section 4.2. Essentially, it is shown that there exists a small $a_1 > 0$ and a constant $a_2 > 0$ such that the exit time τ^{ϵ} from U/ϵ satisfies, for $C > 0$ independent of ϵ,

$$\sup_{x\in\overline{U}/\epsilon} P_{x,\omega}\left(\tau^{\epsilon} > \frac{1}{\epsilon^{2+a_1}}\right) \le C\epsilon^{a_2}. \tag{21}$$

Note that although this estimate is an improvement upon the generic behavior of processes associated to generators like (20), it remains far from implying a uniform in ϵ control for the expectation of the rescaled exit times $\epsilon^2\tau^{\epsilon}$.

Second, the drift can repel the process from the boundary, and thereby make impossible the existence of barriers which are effective except at scales much smaller than ϵ. To overcome this, a proxy for a barrier is essentially obtained by Propositions 15 and 17 of Section 5. The proof relies upon the coupling established in Corollary 11 and estimates for the exit time of Brownian motion from Proposition 13 of Section 4.3. The latter of these follows from the exterior ball condition and an explicit formula for the exit time of Brownian motion in annular domains.

The primary argument of the paper comes in Theorem 19 of Section 5, and a precise outline is presented between lines (75) and (86). The idea is to introduce a discretely stopped version of the process, and to consider the corresponding discrete version of the representation (7). The efficacy of this approximation follows from localization estimates obtained in [20, Proposition 2.2], see Control 7, and the substitute for boundary barriers implied by Propositions 15 and 17. The discrete proxy is then compared with the analogous approximation defined by a Brownian motion of variance $\overline{\alpha}$ using the coupling from Corollay 11. Finally, the results from Section 4.3 together with standard exponential estimates for Brownian motion allow for the recovery of the homogenized solution (11) from its discrete representation and thereby complete the proof. The rate is presented in Section 6, and the proof is a straightforward consequence of the methods used to prove Theorem 19.

Diffusion processes in the stationary ergodic setting were first considered in the case $b(x,\omega) = 0$ by Papanicolaou and Varadhan [17]. Furthermore, in the case that (6) can be rewritten in divergence form or in the case that $b(x,\omega)$ is divergence free or the gradient of a stationary field, such processes and various boundary value problems have been studied by Papanicolaou and Varadhan [16], De Masi, Ferrari, Goldstein and Wick [6], Kozlov [11], Olla [14] and Osada [15]. However, outside of this framework, much less is understood.

In the continuous setting, the results of [20], which apply to the isotropic, perturbative regime described above, are the only such available. These have been more recently extended by the author in [7–9]. In particular, the results of [8] prove that the exit distributions of such processes from large domains converge to that of a Brownian motion, a result which is the continuous analogue of work in the discrete setting by Bolthausen and Zeitouni [4], who characterized the exit distributions from large balls (so, for $U = B_1$) of random walks in random environment which are small, isotropic perturbations of a simple random walk. Their work was later refined by Baur and Bolthausen [3] under a somewhat less stringent isotropy assumption.

The almost-sure characterization of the exit time and the general homogenization statement contained in Theorem 1 remain open in the discrete case. However, under the assumptions of [3], and by using an additional quenched symmetry assumption along a single coordinate direction, Baur [2] has obtained a quenched invariance principle analogous to (10) and a characterization of the exit times from large balls. The symmetry with respect to the quenched measures $P_{x,\omega}$ allows for the exit of the one-dimensional projection $X_t \cdot e_1$ to be estimated by standard martingale methods, and yields an effective a priori control of the rescaled exit times $\epsilon^2\tau^\epsilon$. Therefore, when dealing with the continuous analogue of such environments, many of the arguments in this paper can be simplified.

It should be noted that the techniques presented here differ substantially from [2–4], which employ renormalization schemes to propagate estimates controlling the convergence of the exit law of the diffusion in random environment to the uniform measure on the boundary of the ball. The arguments of this paper begin instead from the parabolic results of [20], and apply immediately to general domains.

The organization of the paper is as follows. Section 2 contains the notation and assumptions. Section 3 reviews those aspects of [20] most relevant to this work and presents the primary probabilistic statement concerning the random environment. In Section 4, necessary results from [8] are recalled and improved for the arguments of this paper. The global coupling is presented in Section 4.1 and a tail estimate for the exit time associated to the process in random environment is obtained in Section 4.2. Section 4.3 controls the expectation of the exit time of Brownian motion near the boundary. The proof of homogenization is presented in Section 5 and the rate of convergence is established in Section 6.

2. Preliminaries

2.1. *Notation*

The elements of $\mathbb{R}^d$ and $[0,\infty)$ are written x or y and t respectively and (x, y) denotes the standard inner product. The spacial gradient and derivative in time of a scalar function v are written Dv and v_t, while D^2v denotes the Hessian matrix. The spaces of $k\times l$ and $k\times k$ symmetric matrices with real entries are written $\mathcal{M}^{k\times l}$ and $\mathcal{S}(k)$ respectively. If $M \in \mathcal{M}^{k\times l}$, then M^t is its transpose and $|M|$ is the norm defined by $|M| = \mathrm{tr}(MM^t)^{1/2}$. The trace of a square matrix M is written $\mathrm{tr}(M)$. The distance between subsets $A, B \subset \mathbb{R}^d$ is

$$d(A,B) = \inf\bigl\{|a-b| \bigm| a\in A, b\in B\bigr\}$$

and, for an index $\mathcal{A}$ and a family of measurable functions

$$\bigl\{f_\alpha : \mathbb{R}^d\times\Omega \to \mathbb{R}^{n_\alpha}\bigr\}_{\alpha\in\mathcal{A}},$$

the sigma algebra generated by the random variables $f_\alpha(x,\omega)$, for $x\in A$ and $\alpha\in\mathcal{A}$, is denoted

$$\sigma\bigl(f_\alpha(x,\omega) \mid x\in A, \alpha\in\mathcal{A}\bigr).$$

For domains $U\subset\mathbb{R}^d$, $\mathrm{BUC}(U;\mathbb{R}^d)$, $\mathrm{C}(U;\mathbb{R}^d)$, $\mathrm{Lip}(U;\mathbb{R}^d)$, $\mathrm{C}^{0,\beta}(U;\mathbb{R}^d)$ and $\mathrm{C}^k(U;\mathbb{R}^d)$ are the spaces of bounded continuous, continuous, Lipschitz continuous, β-Hölder continuous and k-continuously differentiable functions on U with values in $\mathbb{R}^d$. Furthermore, $C_c^\infty(\mathbb{R}^d)$ denotes the space of smooth, compactly supported functions on $\mathbb{R}^d$. The closure and boundary of $U\subset\mathbb{R}^d$ are denoted $\overline{U}$ and ∂U. The support of a function $f:\mathbb{R}^d\to\mathbb{R}$ is written $\mathrm{Supp}(f)$. The open balls of radius R centered at zero and $x\in\mathbb{R}^d$ are respectively written B_R and $B_R(x)$. For a real number $r\in\mathbb{R}$, the notation $[r]$ denotes the largest integer less than or equal to r. Finally, throughout the paper C represents a constant which may change within a line and from line to line but is independent of $\omega\in\Omega$ unless otherwise indicated.

2.2. *The random environment*

A probability space $(\Omega,\mathcal{F},\mathbb{P})$ indexes the random environment, and the elements $\omega\in\Omega$ correspond to realizations described by the coefficients $A(\cdot,\omega)$ and $b(\cdot,\omega)$ on $\mathbb{R}^d$. Their stationarity is quantified by an

$$\text{ergodic group of measure-preserving transformations } \{\tau_x:\Omega\to\Omega\}_{x\in\mathbb{R}^d} \tag{22}$$

such that $A : \mathbb{R}^d \times \Omega \to \mathcal{S}(d)$ and $b : \mathbb{R}^d \times \Omega \to \mathbb{R}^d$ are bi-measurable stationary functions satisfying, for each $x, y \in \mathbb{R}^d$ and $\omega \in \Omega$,

$$A(x+y,\omega) = A(x,\tau_y\omega) \quad \text{and} \quad b(x+y,\omega) = b(x,\tau_y\omega). \tag{23}$$

The diffusion matrix and drift are bounded, Lipschitz functions on $\mathbb{R}^d$ for each $\omega \in \Omega$. There exists $C > 0$ such that, for all $x \in \mathbb{R}^d$ and $\omega \in \Omega$,

$$\left|b(x,\omega)\right| \le C \quad \text{and} \quad \left|A(x,\omega)\right| \le C, \tag{24}$$

and, for all $x, y \in \mathbb{R}^d$ and $\omega \in \Omega$,

$$\left|b(x,\omega) - b(y,\omega)\right| \le C|x-y| \quad \text{and} \quad \left|A(x,\omega) - A(y,\omega)\right| \le C|x-y|. \tag{25}$$

In addition, the diffusion matrix is uniformly elliptic. There exists $\nu > 1$ such that, for all $x \in \mathbb{R}^d$ and $\omega \in \Omega$,

$$\frac{1}{\nu} I \le A(x,\omega) \le \nu I. \tag{26}$$

The environment is strongly mixing in the sense that the coefficients satisfy a finite range dependence. There exists $R > 0$ such that, for every $A, B \subset \mathbb{R}^d$ satisfying $d(A,B) \ge R$, the sigma algebras

$$\sigma\big(A(x,\cdot), b(x,\cdot) | x \in A\big) \quad \text{and} \quad \sigma\big(A(x,\cdot), b(x,\cdot) | x \in B\big) \quad \text{are independent.} \tag{27}$$

The environment is statistically isotropic in the sense that, for every orthogonal transformation $r : \mathbb{R}^d \to \mathbb{R}^d$ which preserves the coordinate axes, for every $x \in \mathbb{R}^d$,

$$\big(A(rx,\omega), b(rx,\omega)\big) \quad \text{and} \quad \big(rA(x,\omega)r^t, rb(x,\omega)\big) \quad \text{have the same law.} \tag{28}$$

Finally, the diffusion is a small perturbation of Brownian motion. There exists $\eta_0 > 0$, to be fixed small in line (52) of Section 3, such that, for all $x \in \mathbb{R}^d$ and $\omega \in \Omega$,

$$\left|b(x,\omega)\right| \le \eta_0 \quad \text{and} \quad \left|A(x,\omega) - I\right| \le \eta_0. \tag{29}$$

The remaining two assumptions concern the domain. First, the domain

$$U \subset \mathbb{R}^d \quad \text{is open and bounded.} \tag{30}$$

Second, U satisfies an exterior ball condition. There exists $r_0 > 0$ so that, for each $x \in \partial U$ there exists $x^* \in \mathbb{R}^d$ satisfying

$$\overline{B}_{r_0}\big(x^*\big) \cap \overline{U} = \{x\}. \tag{31}$$

To avoid lengthy statements, a steady assumption is made.

$$\text{Assume (22), (23), (24), (25), (26), (27), (28), (29), (30) and (31).} \tag{32}$$

Observe that (24), (25) and (26) guarantee, for every environment $\omega \in \Omega$ and initial distribution $x \in \mathbb{R}^d$, the well-posedness of the martingale problem associated to the generator

$$\frac{1}{2}\sum_{i,j=1}^d a_{ij}(x,\omega)\frac{\partial^2}{\partial x_i\,\partial x_j} + \sum_{i=1}^d b_i(x,\omega)\frac{\partial}{\partial x_i},$$

see Strook and Varadhan [19, Chapters 6,7]. The associated probability measure and expectation on the space of continuous paths $C([0,\infty);\mathbb{R}^d)$ will be respectively denoted $P_{x,\omega}$ and $E_{x,\omega}$ where, almost surely with respect to $P_{x,\omega}$, paths $X_t \in C([0,\infty);\mathbb{R}^d)$ satisfy the stochastic differential equation

$$\begin{cases} dX_t = b(X_t,\omega)\,dt + \sigma(X_t,\omega)\,dB_t, \\ X_0 = x, \end{cases} \tag{33}$$

for $A(x,\omega) = \sigma(x,\omega)\sigma(x,\omega)^t$ and for B_t a standard Brownian motion under $P_{x,\omega}$ with respect to the canonical right-continuous filtration on $C([0,\infty);\mathbb{R}^d)$.

Define as well, for each $n \geq 0$ and $x \in \mathbb{R}^d$, the Wiener measure W_x^n and expectation $E^{W_x^n}$ on $C([0,\infty);\mathbb{R}^d)$ corresponding to Brownian motion on $\mathbb{R}^d$ with variance α_n beginning from x. Almost surely with respect to W_x^n, paths $X_t \in C([0,\infty);\mathbb{R}^d)$ satisfy the stochastic differential equation

$$\begin{cases} dX_t = \sqrt{\alpha_n}\,dB_t, \\ X_0 = x, \end{cases} \tag{34}$$

for B_t some standard Brownian motion under W_x^n with respect to the canonical right-continuous filtration on $C([0,\infty);\mathbb{R}^d)$.

2.3. *A remark on existence and uniqueness*

The boundedness, Lipschitz continuity and ellipticity of the coefficients, see (24), (25) and (26), together with the boundedness and regularity of the domain, see (30) and (31), guarantee the well-posedness, for every $\omega \in \Omega$, of equations like

$$\begin{cases} \frac{1}{2}\operatorname{tr}(A(x,\omega)D^2w) + b(x,\omega)\cdot Dw = g(x) & \text{on } U, \\ u = f(x) & \text{on } \partial U, \end{cases}$$

for every $f \in C(\partial U)$ and $g \in C(\overline{U})$ in the class of bounded continuous functions. See, for instance, Friedman [10, Chapter 3]. Furthermore, if τ denotes the exit time from U, then the solution admits the representation

$$u(x) = E_{x,\omega}\left(f(X_\tau) - \int_0^\tau g(X_s)\,ds\right) \quad \text{on } \overline{U},$$

see Øksendal [13, Exercise 9.12].

The same assumptions ensure the well-posedness of parabolic equations like

$$\begin{cases} w_t = \frac{1}{2}\operatorname{tr}(A(x,\omega)D^2w) + b(x,\omega)\cdot Dw & \text{on } \mathbb{R}^d \times (0,\infty), \\ w = f(x) & \text{on } \mathbb{R}^d \times \{0\}, \end{cases}$$

for continuous initial data $f(x)$ satisfying, for instance and to the extent that it will be applied in this paper, $|f(x)| \leq C(1+|x|^2)$ on $\mathbb{R}^d$, in the class of continuous functions satisfying a quadratic estimate of the same form locally in time. See [10, Chapter 1]. Furthermore,

$$w(x,t) = E_{x,\omega}\big(f(X_t)\big) \quad \text{on } \mathbb{R}^d \times (0,\infty),$$

see [13, Exercise 9.12].

Analogous formulas hold for the constant coefficient elliptic and parabolic equations associated to Brownian motion and the measures W_x^n. Since these facts are well-known, and since the solution to every equation encountered in this paper admits an explicit probabilistic description, the presentation will not further emphasize these points.

3. The inductive framework and probabilistic statement

In this section, the aspects of [20] most relevant to this work are briefly explained. A complete description of the inductive framework can be found in [20], and it was later reviewed in the introduction of [9].

Assume the dimension d satisfies

$$d \geq 3, \tag{35}$$

and fix a Hölder exponent

$$\beta \in \left(0, \frac{1}{2}\right] \quad \text{and a scaling constant} \quad a \in \left(0, \frac{\beta}{1000d}\right]. \tag{36}$$

The following constants will come to define the scales in length and time along which the induction scheme is propagated. Let L_0 be integer multiple of five which will later be fixed large in (52). For each $n \geq 0$, define inductively

$$\ell_n = 5\left[\frac{L_n^a}{5}\right] \quad \text{and} \quad L_{n+1} = \ell_n L_n, \tag{37}$$

where it follows that, for every L_0 sufficiently large, $\frac{1}{2}L_n^{1+a} \leq L_{n+1} \leq 2L_n^{1+a}$. For $c_0 > 0$ to be fixed small in (52), for each $n \geq 0$, define

$$\kappa_n = \exp\bigl(c_0\bigl(\log\log(L_n)\bigr)^2\bigr) \quad \text{and} \quad \tilde{\kappa}_n = \exp\bigl(2c_0\bigl(\log\log(L_n)\bigr)^2\bigr), \tag{38}$$

and observe that, as $n \to \infty$, the constants κ_n are eventually dominated by every positive power of L_n. Furthermore, for each $n \geq 0$, define

$$D_n = L_n\kappa_n \quad \text{and} \quad \tilde{D}_n = L_n\tilde{\kappa}_n, \tag{39}$$

where, using the preceding remark, the scales D_n and $\tilde{D}_n$ are larger but grow comparably with the previously defined scales L_n.

The remaining constants enter into the primary probabilistic statement, see Theorem 8, and the Hölder estimates governing the convergence of solutions to the parabolic equation (43), see Theorem 5 and Control 6. Fix $m_0 \geq 2$ satisfying

$$(1+a)^{m_0-2} \leq 100 < (1+a)^{m_0-1}, \tag{40}$$

and $\delta > 0$ and $M_0 > 0$ satisfying

$$\delta = \frac{5}{32}\beta \quad \text{and} \quad M_0 \geq 100d(1+a)^{m_0+2}. \tag{41}$$

In what follows, it is essential that δ and M_0 are sufficiently larger than a.

In order to exploit the environment's mixing properties, it will be frequently necessary to introduce a stopped version of the process. Define for every element $X_t \in \mathrm{C}([0,\infty);\mathbb{R}^d)$ the path

$$X_t^* = \sup_{0\leq s\leq t} |X_s - X_0|, \tag{42}$$

and, for each $n \geq 0$, the stopping time

$$T_n = \inf\bigl\{s \geq 0 | X_s^* \geq \tilde{D}_n\bigr\}.$$

The effective diffusivity of the ensemble at scale L_n is defined by

$$\alpha_n = \frac{1}{2dL_n^2}\mathbb{E}_0\bigl(|X_{L_n^2\wedge T_n}|^2\bigr),$$

where the localization ensures that the α_n are local quantities on scale $\tilde{D}_n$. The convergence of the α_n to a limiting diffusivity $\overline{\alpha}$ is proven in [20, Proposition 5.7].

Theorem 5. *Assume* (32). *There exists L_0 and c_0 sufficiently large and $\eta_0 > 0$ sufficiently small such that, for all $n \geq 0$,*

$$\frac{1}{2\nu} \leq \alpha_n \leq 2\nu \quad \textit{and} \quad |\alpha_{n+1} - \alpha_n| \leq L_n^{-(1+\frac{9}{10})\delta},$$

which implies the existence of $\overline{\alpha} > 0$ satisfying

$$\frac{1}{2\nu} \leq \overline{\alpha} \leq 2\nu \quad \textit{and} \quad \lim_{n\to\infty} \alpha_n = \overline{\alpha}.$$

The results of [20] obtain an effective comparison on the parabolic scale (L_n, L_n^2) in space and time, with improving probability as $n \to \infty$, between solutions

$$\begin{cases} u_t = \frac{1}{2}\operatorname{tr}(A(x,\omega)D^2u) + b(x,\omega)\cdot Du & \text{on } \mathbb{R}^d \times (0,\infty), \\ u = f(x) & \text{on } \mathbb{R}^d \times \{0\}, \end{cases} \tag{43}$$

and solutions of the approximate limiting equation

$$\begin{cases} u_{n,t} = \frac{\alpha_n}{2}\Delta u_n & \text{on } \mathbb{R}^d \times (0,\infty), \\ u_n = f(x) & \text{on } \mathbb{R}^d \times \{0\}. \end{cases} \tag{44}$$

To simplify the notation, for each $n \geq 0$, define the operators

$$R_n f(x) = u\big(x, L_n^2\big) \quad \text{and} \quad \overline{R}_n f(x) = u_n\big(x, L_n^2\big), \tag{45}$$

and the difference

$$S_n f(x) = R_n f(x) - \overline{R}_n f(x). \tag{46}$$

Since solutions of (43) are not, in general, effectively comparable with solutions of (44) globally in space, it is necessary to localize using a cutoff function. For each $\upsilon > 0$, define

$$\chi(y) = 1 \wedge \big(2 - |y|\big)_+ \quad \text{and} \quad \chi_\upsilon(y) = \chi\left(\frac{y}{\upsilon}\right), \tag{47}$$

and, for each $x \in \mathbb{R}^d$ and $n \geq 0$,

$$\chi_{n,x}(y) = \chi_{30\sqrt{d}L_n}(y - x). \tag{48}$$

Furthermore, since the comparison of the solutions must necessarily respect the scaling associated to (6) and (8), it is obtained with respect to the rescaled global Hölder-norms defined, for each $n \geq 0$, by

$$|f|_n = \|f\|_{L^\infty(\mathbb{R}^d)} + \sup_{x\neq y} L_n^\beta \frac{|f(x) - f(y)|}{|x-y|^\beta}. \tag{49}$$

See, for instance, the introductions of [9,20] for a more complete discussion concerning the necessity of these norms as opposed, perhaps, to attempting a generically false L^∞-contraction.

The following estimate is the statement propagated by the arguments of [20], and expresses a comparison between solutions of (43) and (44). Observe that this statement is not true, in general, for all triples $x \in \mathbb{R}^d$, $\omega \in \Omega$ and $n \geq 0$. However, as described in Theorem 8 below, it is shown in [20, Proposition 5.1] that such controls are available for large n, with high probability and on a large portion of space.

Control 6. *Fix $x \in \mathbb{R}^d$, $\omega \in \Omega$ and $n \geq 0$. Then, for each $f \in C^{0,\beta}(\mathbb{R}^d)$,*

$$|\chi_{n,x} S_n f|_n \leq L_n^{-\delta} |f|_n.$$

In order to account for the error introduced by localization, it is necessary to obtain tail-estimates for the diffusion in random environment. The type of control propagated in [20] is an exponential estimate for the probability under $P_{x,\omega}$ that the maximal excursion $X^*_{L_n^2}$ defined in (42) is large with respect to the time elapsed. As with Control 6, it is simply false in general that this type of estimate is satisfied for every triple (x, ω, n). However, it is shown in [20, Proposition 2.2] that such controls are available for large n, with high probability, on a large portion of space.

Control 7. *Fix $x \in \mathbb{R}^d$, $\omega \in \Omega$ and $n \geq 0$. For each $v \geq D_n$, for all $|y - x| \leq 30\sqrt{d} L_n$,*

$$P_{y,\omega}\big(X^*_{L_n^2} \geq v\big) \leq \exp\left(-\frac{v}{D_n}\right).$$

The primary result of [20] proved that, provided the perturbation η_0 is sufficiently small, Controls 6 and 7 are available with high probability on a large portion of space. Precisely, for each $n \geq 0$ and $x \in \mathbb{R}^d$, define the event

$$B_n(x) = \big\{\omega \in \Omega \mid \text{Controls 6 and 7 hold for the triple } (x, \omega, n)\big\}, \tag{50}$$

and notice that, in view of (23), for all $x \in \mathbb{R}^d$ and $n \geq 0$,

$$\mathbb{P}\big(B_n(x)\big) = \mathbb{P}\big(B_n(0)\big). \tag{51}$$

Furthermore, observe that $B_n(0)$ does not include the control of traps described in [20, Proposition 3.3], which play an important role in propagating Control 6, and from which the arguments of this paper have no further need. The following theorem proves that the complement of $B_n(0)$ approaches zero as n tends to infinity, see [20, Theorem 1.1].

Theorem 8. *Assume (32). There exist L_0 and c_0 sufficiently large and $\eta_0 > 0$ sufficiently small such that, for each $n \geq 0$,*

$$\mathbb{P}\big(\Omega \setminus B_n(0)\big) \leq L_n^{-M_0}.$$

Henceforth, the constants L_0, c_0 and η_0 are fixed to satisfy the requirements of Theorems 5 and 8.

$$\text{Fix constants } L_0, c_0 \text{ and } \eta_0 \text{ satisfying the hypothesis of Theorems 5 and 8.} \tag{52}$$

The events which come to define, following an application of the Borel–Cantelli lemma, the event on which Theorem 1 is obtained are chosen to ensure that Controls 6 and 7 are available at a sufficiently small scale in comparison to $\frac{1}{\epsilon}$. Fix the smallest integer $\overline{m} > 0$ satisfying

$$\overline{m} > 1 - \frac{\log(1 - 12a - a^2)}{\log(1 + a)}, \tag{53}$$

and notice that the definition of L_n in (37) implies that, for $C > 0$ independent of $n \geq \overline{m}$,

$$L_{n+1} \tilde{D}_{n-\overline{m}} \leq C L_{n-1}^{2-10a}.$$

Observe as well that this definition is stronger than was necessary for the arguments of [8].

Theorem 8 is now used to obtain Control 6 and Control 7 at scale $L_{n-\overline{m}}$ on the entirety of the rescaled domain U/ϵ whenever $L_n \leq \frac{1}{\epsilon} < L_{n+1}$. It follows from the boundedness of U and the definition of L_n that, for all $n \geq 0$ sufficiently large, whenever $L_n \leq \frac{1}{\epsilon} < L_{n+1}$, the rescaled domain U/ϵ is contained in what becomes the considerably larger set $[-\frac{1}{2}L_{n+2}^2, \frac{1}{2}L_{n+2}^2]^d$. Therefore, for each $n \geq \overline{m}$, define

$$A_n = \big\{\omega \in \Omega \mid \omega \in B_m(x) \text{ for all } x \in L_m\mathbb{Z}^d \cap \big[-L_{n+2}^2, L_{n+2}^2\big]^d \text{ and } n - \overline{m} \leq m \leq n + 2\big\}. \tag{54}$$

The following proposition proves that, as $n \to \infty$, the probability of the events $\{A_n\}_{n=1}^\infty$ rapidly approaches one, since the exponent

$$2d(1+a)^2 - \frac{M_0}{2} < 0,$$

owing to (36) and (41). Up to a change of exponent, the proof is identical to [8, Proposition 3.5].

Proposition 9. *Assume* (32) *and* (52). *For each* $n \geq \overline{m}$, *for* $C > 0$ *independent of* n,

$$\mathbb{P}(\Omega \setminus A_n) \leq C L_n^{2d(1+a)^2 - \frac{1}{2}M_0}.$$

4. A review of [8]

In this section, some results from [8] are briefly recalled and adapted to the arguments of this paper.

4.1. *The global coupling*

The purpose of this section is to construct with high probability a coupling between the diffusion in random environment and a Brownian motion with variance $\alpha_{n-\overline{m}}$. This will be achieved along the discrete sequence of time steps $\{kL_{n-\overline{m}}^2\}_{k=0}^\infty$ through the comparison implied by Control 6.

The choice of $\overline{m}$ from (53) is made to ensure that the discretization scale $L_{n-\overline{m}}$ is sufficiently fine with respect to scales ϵ satisfying $L_n \leq \frac{1}{\epsilon} < L_{n+1}$. It is not obvious a priori that such a discretization exists, since the trapping and ballistic effects of the drift make it necessary in general to employ a discretization on a unit scale in order to accurately represent the process in the asymptotic limit.

Since the construction of the coupling follows closely [20, Proposition 3.1] and [8, Section 5], only the final conclusions will be summarized here. Recall that solutions of (43) with initial condition $f(x)$ admit a representation using the Green's function

$$p_{t,\omega}(x,y) : [0,\infty) \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R},$$

which represents the density of the diffusion beginning from x in environment ω at time t. See [10, Chapter 1] for a detailed discussion of the existence and regularity of these densities, which follow from assumptions (24), (25) and (26). The corresponding representation then takes the form

$$u(x,t) = E_{x,\omega}\big(f(X_t)\big) = \int_{\mathbb{R}^d} p_{t,\omega}(x,y) f(y)\,dy.$$

Analogously, solutions of (44) with initial data $f(x)$ admit the heat kernel representation

$$\overline{u}_n(x,t) = E^{W_x^{n-\overline{m}}}\big(f(X_t)\big) = \int_{\mathbb{R}^d} (4\pi\alpha_{n-\overline{m}}t)^{-\frac{d}{2}} \exp\left(-\frac{|y-x|^2}{4\alpha_{n-\overline{m}}t}\right) f(y)\,dy.$$

For each $n \geq \overline{m}$, the Green's function on scale $L_{n-\overline{m}}$ will be denoted by

$$p_{n-\overline{m},\omega}(x,y) = p_{L_{n-\overline{m}}^2,\omega}(x,y),$$

and the heat kernel on scale $L_{n-\overline{m}}$ will be denoted by

$$\overline{p}_{n-\overline{m}}(x,y) = \big(4\pi\alpha_{n-\overline{m}}L_{n-\overline{m}}^2\big)^{-\frac{d}{2}} \exp\left(-\frac{|y-x|^2}{4\alpha_{n-\overline{m}}L_{n-\overline{m}}^2}\right).$$

The following proposition constructs a Markov process $(X_k, \overline{X}_k)$ on the state space $(\mathbb{R}^d \times \mathbb{R}^d)^{\mathbb{N}}$ such that the transition probabilities of the first coordinate X_k are determined by $p_{n-\overline{m},\omega}(\cdot,\cdot)$ and, such that those of the second coordinate $\overline{X}_k$ are determined by $\overline{p}_{n-\overline{m}}(\cdot,\cdot)$. The proof follows closely [20, Proposition 3.1] and can be found as [8, Proposition 5.1].

Proposition 10. *Assume* (32) *and* (52). *For every* $\omega \in \Omega$, *for every* $x \in \mathbb{R}^d$, *there exists a measure* $Q_{n,x}$ *on the canonical sigma algebra of the space* $(\mathbb{R}^d \times \mathbb{R}^d)^{\mathbb{N}}$ *such that, under* $Q_{n,x}$, *the coordinate processes* X_k *and* $\overline{X}_k$ *respectively have the law of a Markov chain on* $\mathbb{R}^d$, *starting from* x, *with transition kernels* $p_{n-\overline{m},\omega}(\cdot,\cdot)$ *and* $\overline{p}_{n-\overline{m}}(\cdot,\cdot)$.

Furthermore, for every $n \geq \overline{m}$, $\omega \in A_n$ *and* $x \in [-\frac{1}{2}L_{n+2}^2, \frac{1}{2}L_{n+2}^2]^d$, *for* $C > 0$ *independent of* n,

$$Q_{n,x}\left(|X_k - \overline{X}_k| \geq \gamma \,|\, \text{for some } 0 \leq k \leq 2\left(\frac{L_{n+2}}{L_{n-\overline{m}}}\right)^2\right) \leq C\left(\frac{L_{n-\overline{m}}}{\gamma}\right)^\beta \left(\frac{L_{n+2}}{L_{n-\overline{m}}}\right)^4 \tilde{\kappa}_{n-\overline{m}} L_{n-\overline{m}}^{-\delta}. \tag{55}$$

The following corollary follows immediately by choosing $\gamma = L_{n-\overline{m}}$ in Proposition 10. Notice that the exponent $16a - \delta < 0$ owing to definitions (36) and (41).

Corollary 11. *Assume* (32) *and* (52). *For every* $n \geq \overline{m}$, $\omega \in A_n$ *and* $x \in [-\frac{1}{2}L_{n+2}^2, \frac{1}{2}L_{n+2}^2]^d$, *for* $C > 0$ *independent of* n,

$$Q_{n,x}\left(|X_k - \overline{X}_k| \geq L_{n-\overline{m}} \,|\, \text{for some } 0 \leq k \leq 2\left(\frac{L_{n+2}}{L_{n-\overline{m}}}\right)^2\right) \leq C\tilde{\kappa}_{n-\overline{m}} L_{n-\overline{m}}^{16a-\delta}.$$

4.2. *Tail estimates and an upper bound in probability for the exit time*

The purpose of this section is to obtain certain tail estimates for the exit time in probability. Namely, whenever the scale ϵ satisfies $L_n \leq \frac{1}{\epsilon} < L_{n+1}$, the diffusion associated to the generator

$$\frac{1}{2}\sum_{i,j=1}^d a_{ij}(x,\omega)\frac{\partial^2}{\partial x_i \, \partial x_j} + \sum_{i=1}^d b_i(x,\omega)\frac{\partial}{\partial x_i}$$

is shown to exit the rescaled domain U/ϵ prior to time L_{n+2}^2 in overwhelming fashion. The corresponding estimate is then propagated inductively forward in time. Observe, however, that these estimates remain far from the ultimate goal, since the exit time of a Brownian motion from the rescaled domain U/ϵ is expected to be of order $\frac{1}{\epsilon^2}$ which, as n approaches infinity, is much smaller than L_{n+2}^2. Therefore, Proposition 12 alone does not imply the boundedness of solutions to the rescaled equation (8), and this will not be achieved until Theorem 19 of Section 5.

The following argument is virtually identical to [8, Proposition 4.1]. However, for the arguments of this paper it is necessary to make the estimate from [8] more precise in ϵ and then to iterate it inductively forward in time. Only these changes are explained in the proof.

Proposition 12. *Assume* (32) *and* (52). *For all* n *sufficiently large, for every* $\omega \in A_n$, *for all* $\epsilon > 0$ *satisfying* $L_n \leq \frac{1}{\epsilon} < L_{n+1}$, *for* $C > 0$ *independent of* n,

$$\sup_{x \in \overline{U}} P_{\frac{x}{\epsilon},\omega}\left(\tau^\epsilon > L_{n+2}^2\right) \leq C(\epsilon L_{n+2})^{-3}.$$

And, for each $k \geq 0$,

$$\sup_{x \in \overline{U}} P_{\frac{x}{\epsilon},\omega}\left(\tau^\epsilon > kL_{n+2}^2\right) \leq C(\epsilon L_{n+2})^{-3k}.$$

Proof. Using the boundedness of the domain, choose $R \geq 1$ such that $\overline{U} \subset B_R$ and choose $n_1 \geq 0$ so that, whenever $n \geq n_1$,

$$L_{n+1}\overline{U} \subset L_{n+1}B_R \subset \left[-\frac{1}{2}L_{n+2}^2, \frac{1}{2}L_{n+2}^2\right]^d. \tag{56}$$

Henceforth, fix $n \geq n_1$, $\omega \in A_n$ and $L_n \leq \frac{1}{\epsilon} < L_{n+1}$.

It follows from the line preceding [8, Line (4.7)] and [8, Lines (4.6), (4.7)] that

$$\sup_{x\in\overline{U}} P_{\frac{x}{\epsilon},\omega}\big(\tau^{\epsilon} > L_{n+2}^2\big) \le C(\epsilon L_{n+2})^{-3}, \tag{57}$$

which completes the argument for the first statement.

The final statement is a consequence of the Markov property and induction. The case $k = 0$ is immediate, and the case $k = 1$ is (57). For the inductive step, assume that, for $k \ge 1$,

$$\sup_{x\in\overline{U}} P_{\frac{x}{\epsilon},\omega}\big(\tau^{\epsilon} > kL_{n+2}^2\big) \le C(\epsilon L_{n+2})^{-3k}.$$

Then by the Markov property, for each $x \in \overline{U}$,

$$P_{\frac{x}{\epsilon},\omega}\big(\tau^{\epsilon} > (k+1)L_{n+2}^2\big) = E_{\frac{x}{\epsilon},\omega}\big(P_{X_{kL_{n+2}^2},\omega}\big(\tau^{\epsilon} > L_{n+2}^2\big), \tau^{\epsilon} > kL_{n+2}^2\big).$$

Therefore, from the inductive hypothesis and (57), for each $x \in \overline{U}$,

$$P_{\frac{x}{\epsilon},\omega}\big(\tau^{\epsilon} > (k+1)L_{n+2}^2\big) \le \Big(\sup_{x\in\overline{U}} P_{\frac{x}{\epsilon},\omega}\big(\tau^{\epsilon} > L_{n+2}^2\big)\Big) P_{x,\omega}\big(\tau^{\epsilon} > kL_{n+2}^2\big) \le C(\epsilon L_{n+2})^{-3(k+1)},$$

which completes the argument. □

4.3. *Estimates for the exit time of Brownian motion near the boundary*

In this section estimates are obtained, in expectation and near the boundary of the domain, for the exit time of Brownian motion. These estimates will be shown in Section 5 to be inherited with high probability by the diffusion in random environment using the coupling developed in Section 4.1. Since the results of this section are identical to [8, Section 6], the details are omitted. Observe, however, that they are the only such that rely directly upon the exterior ball condition imposed on the domain.

Define, for each $\delta > 0$, the enlargement

$$U_\delta = \big\{x \in \mathbb{R}^d \,|\, d(x, U) < \delta\big\}, \tag{58}$$

and, for each $\delta > 0$, define the $\mathrm{C}([0,\infty);\mathbb{R}^d)$ exit times

$$\tau = \inf\{t \ge 0 \,|\, X_t \notin U\} \quad \text{and} \quad \tau^{\delta} = \inf\{t \ge 0 \,|\, X_t \notin U_\delta\}. \tag{59}$$

The following Proposition controls the expectation of τ and τ^δ in what is essentially the δ-neighborhood of the respective boundaries of U and U_δ. Recall from assumption (31) the radius $r_0 > 0$ quantifying the exterior ball condition. The proof can be found as [8, Corollary 6.2].

Proposition 13. *Assume* (32) *and* (52). *For every* $0 < \delta < \frac{r_0}{2}$, *for every* $n \ge 0$, *for* $C > 0$ *independent of* n *and* δ,

$$\sup_{d(x,\partial U)\le\delta} E^{W_x^n}(\tau) \le C\delta \quad \textit{and} \quad \sup_{d(x,\partial U_\delta)\le 2\delta} E^{W_x^n}\big(\tau^\delta\big) < C\delta.$$

The analogous estimates for the rescaled the domains U/ϵ and U_δ/ϵ are then immediate. For each $\epsilon > 0$, define the $\mathrm{C}([0,\infty);\mathbb{R}^d)$ exit time

$$\tau^{\epsilon} = \inf\{t \ge 0 \,|\, X_t \notin U/\epsilon\} = \inf\{t \ge 0 \,|\, \epsilon X_t \notin U\}, \tag{60}$$

whose expectation $E^{W_x^n}(\tau^\epsilon)$ can be obtained as the rescaling

$$E^{W_x^n}\big(\tau^\epsilon\big) = \epsilon^{-2} E^{W_{\epsilon x}^n}(\tau) \quad \text{on } \overline{U}/\epsilon.$$

And, for each $\epsilon > 0$ and $\delta > 0$, define the exit time

$$\tau^{\epsilon,\delta} = \inf\{t \geq 0 | X_t \notin U_\delta/\epsilon\} = \inf\{t \geq 0 | \epsilon X_t \notin U_\delta\},$$

which has expectation equal to the rescaling

$$E^{W_x^n}\left(\tau^{\epsilon,\delta}\right) = \epsilon^{-2} E^{W_{\epsilon x}^n}\left(\tau^{\delta}\right) \quad \text{on } \overline{U}_\delta/\epsilon.$$

These two equalities and Proposition 13 then immediately yield the following corollary.

Corollary 14. *Assume* (32) *and* (52). *For every* $\epsilon > 0$, $0 < \delta < \frac{r_0}{2\epsilon}$ *and* $n \geq 0$, *for* $C > 0$ *independent of* ϵ, δ *and* n,

$$\sup_{d(x,\partial U/\epsilon) \leq \delta} E^{W_x^n}\left(\tau^\epsilon\right) \leq C\epsilon^{-1}\delta \quad \text{and} \quad \sup_{d(x,\partial U_\delta/\epsilon) \leq 2\delta} E^{W_x^n}\left(\tau^{\epsilon,\delta}\right) < C\epsilon^{-1}\delta.$$

5. The discrete approximation and proof of homogenization

In this section, stochastic homogenization is established for solutions

$$\begin{cases} \frac{1}{2}\operatorname{tr}(A(\frac{x}{\epsilon},\omega)D^2u^\epsilon) + \frac{1}{\epsilon}b(\frac{x}{\epsilon},\omega)\cdot Du^\epsilon = g(x) & \text{on } U, \\ u^\epsilon = f(x) & \text{on } \partial U, \end{cases} \tag{61}$$

which are, on a subset of full probability, shown to converge uniformly, as $\epsilon \to 0$, to the solution

$$\begin{cases} \frac{\overline{\alpha}}{2}\Delta\overline{u} = g(x) & \text{on } U, \\ \overline{u} = f(x) & \text{on } \partial U. \end{cases} \tag{62}$$

The result will be obtained by analyzing the lifetime of the diffusion process associated to the generator

$$\frac{1}{2}\sum_{i,j=1}^d a_{ij}(x,\omega)\frac{\partial^2}{\partial x_i\,\partial x_j} + \sum_{i=1}^d b_i(x,\omega)\frac{\partial}{\partial x_i} \tag{63}$$

in the large domains U/ϵ.

The discrete coupling developed in Section 4.1 will play an essential role in the proof, and suggests the introduction of a discretely stopped version of the diffusion. Namely, whenever the scale satisfies $L_n \leq \frac{1}{\epsilon} < L_{n+1}$, a discrete version of the process with time steps $L_{n-\overline{m}}^2$ will be introduced, and stopped as soon as it hits the $\tilde{D}_{n-\overline{m}}$ neighborhood of the complement of the dilated domain U/ϵ.

Note carefully, however, that this type of discrete approximation does not generally provide an accurate description of processes associated to generators like (1), since a continuous diffusion beginning in the $\tilde{D}_{n-\overline{m}}$ neighborhood of the boundary may be compelled by the drift to exit the domain in a region far removed from the stopping point of its discrete proxy. This fact can be readily observed by considering a locally constant, nonzero drift, which is a situation that can occur within the framework of this paper with a rapidly vanishing but nonzero probability on all scales. In essence, therefore, Propositions 15 and 17 effectively establish a boundary barrier for equation (61) of a quality which is generically impossible to obtain.

The discrete $\mathrm{C}([0,\infty);\mathbb{R}^d)$ stopping time is defined, for each $\epsilon > 0$ and $n \geq \overline{m}$, by

$$\tau_1^{\epsilon,n} = \inf\left\{kL_{n-\overline{m}}^2 \geq 0 \middle| d\left(X_{kL_{n-\overline{m}}^2}, (U/\epsilon)^c\right) \leq \tilde{D}_{n-\overline{m}}\right\}, \tag{64}$$

and represents the first time $X_{kL_{n-\overline{m}}^2}$ enters the $\tilde{D}_{n-\overline{m}}$ neighborhood of the complement of U/ϵ. Since it is not true that $\tau^{\epsilon,n} \leq \tau^\epsilon$ for every path X_t, the failure of this inequality will need to be controlled in probability with respect to $P_{x,\omega}$ by the exponential localization estimate implied by Control 7.

Similarly, for each $\epsilon > 0$, and for each $n \geq \overline{m}$, define the $C([0,\infty);\mathbb{R}^d)$ stopping times

$$\tau_2^{\epsilon,n} = \inf\{kL_{n-\overline{m}}^2 \geq 0 | d(X_{kL_{n-\overline{m}}^2}, (U/\epsilon)) \geq \tilde{D}_{n-\overline{m}}\}. \tag{65}$$

These stopping times quantify the first time that the discrete process $X_{kL_{n-\overline{m}}^2}$ exits the $\tilde{D}_{n-\overline{m}}$ neighborhood of (U/ϵ). The definitions imply $\tau_1^{\epsilon,n} \leq \tau_2^{\epsilon,n}$ and, whenever $\tau_1^{\epsilon,n} \leq \tau^\epsilon$, it is immediate that $\tau_1^{\epsilon,n} \leq \tau^\epsilon \leq \tau_2^{\epsilon,n}$.

Proposition 15 will use Corollary 14 to obtain an effective tail estimate with respect to the Wiener measure W_x^n for $\tau_2^{\epsilon,n}$ near the boundary of U/ϵ. This estimate, together with the coupling constructed in Proposition 10, then yield on the event A_n an upper bound for the probability

$$P_{x,\omega}(\tau^\epsilon - \tau_1^{\epsilon,n} \geq L_{n-1}^2) \quad \text{for } x \in \overline{U}/\epsilon.$$

It is this estimate that effectively acts as a barrier by ensuring that, with high probability and following an application of the exponential estimate implied by Control 7, a diffusion beginning in the $\tilde{D}_{n-\overline{m}}$ neighborhood of the complement $(U/\epsilon)^c$ exits the true domain U/ϵ in a small neighborhood of its starting position, with respect to the $\frac{1}{\epsilon}$ scale.

In what follows, recall that $\overline{m}$ is the smallest integer satisfying

$$\overline{m} > 1 - \frac{\log(1-12a-a^2)}{\log(1+a)}, \tag{66}$$

which ensures that, by the choice of constants L_n in (37) and $\tilde{D}_n$ in (39), for $C > 0$ independent of $n \geq \overline{m}$,

$$L_{n+1}\tilde{D}_{n-\overline{m}} \leq CL_{n-1}^{2-10a}. \tag{67}$$

Further, observe by using the definitions of L_n in (37) and $\tilde{\kappa}_n$ in (38) that there exists $C > 0$ independent of $n \geq \overline{m}$ satisfying

$$\tilde{\kappa}_{n-\overline{m}} L_{n-\overline{m}}^{16a-\delta} \leq CL_{n-1}^{-10a}. \tag{68}$$

The following proposition is the control of the second discrete exit time, in terms of Brownian motion and near the boundary of U/ϵ. The proof is a refinement of the estimate obtained in [8, Proposition 7.1], and only the new elements are explained.

Proposition 15. *Assume* (32) *and* (52). *For all $n \geq 0$ sufficiently large, for every $\epsilon > 0$ satisfying $L_n \leq \frac{1}{\epsilon} < L_{n+1}$, for $C > 0$ independent of n,*

$$\sup_{d(x,(U/\epsilon)^c) \leq 2\tilde{D}_{n-\overline{m}}} W_x^{n-\overline{m}}(\tau_2^{\epsilon,n} \geq L_{n-1}^2) \leq CL_{n-1}^{-10a}.$$

Proof. Fix $n_1 \geq 0$ such that, whenever $n \geq n_1$, for r_0 from the exterior ball condition (31),

$$2\tilde{D}_{n-\overline{m}} \leq \frac{r_0 L_n}{2}. \tag{69}$$

And, therefore, whenever $n \geq n_1$ and $d(x,(U/\epsilon)^c) \leq 2\tilde{D}_{n-\overline{m}}$, the conditions of Proposition 14 are satisfied.

Let $n \geq n_1$, $\epsilon > 0$ satisfying $L_n \leq \frac{1}{\epsilon} < L_{n+1}$ and $x \in \mathbb{R}^d$ such that $d(x,(U/\epsilon)^c) \leq 2\tilde{D}_{n-\overline{m}}$. The stopping time $\tau^{\epsilon,\delta}$ is the exit time from the δ-neighborhood of (U/ϵ), and for $\delta = 2\tilde{D}_{n-\overline{m}}$ Proposition 14 states, for $C > 0$ independent of n,

$$E^{W_x^n}(\tau^{\epsilon,2\tilde{D}_{n-\overline{m}}}) \leq C(2\tilde{D}_{n-\overline{m}})\epsilon^{-1} < C\tilde{D}_{n-\overline{m}}L_{n+1}.$$

So, owing to (67), for $C > 0$ independent of n,

$$E^{W_x^n}(\tau^{\epsilon,2\tilde{D}_{n-\overline{m}}}) \leq CL_{n-1}^{2-10a}.$$

Therefore, by Chebyshev's inequality, for $C>0$ independent of n,

$$W_x^n\left(\tau^{\epsilon,2\tilde{D}_{n-\overline{m}}}\geq\frac{1}{2}L_{n-1}^2\right)\leq CL_{n-1}^{-10a}. \tag{70}$$

The remainder of the proof follows exactly [8, Proposition 7.1] beginning from [8, Line (7.11)]. □

Before proceeding, recall the events $\{A_n\}_{n=1}^{\infty}$ defined, for each $n\geq 0$, as

$$A_n=\left\{\omega\in\Omega|\omega\in B_m(x)\text{ for all }x\in L_m\mathbb{Z}^d\cap\left[-L_{n+2}^2,L_{n+2}^2\right]^d\text{ and }n-\overline{m}\leq m\leq n+2\right\}. \tag{71}$$

In particular, for every environment $\omega\in A_n$ and point $x\in[-L_{n+2}^2,L_{n+2}^2]^d$, the following localization estimates is satisfied.

Control 16. *Fix $x\in\mathbb{R}^d$, $\omega\in\Omega$ and $n\geq 0$. For each $v\geq D_n$, for all $|y-x|\leq 30\sqrt{d}L_n$,*

$$P_{y,\omega}\left(X^*_{L_n^2}\geq v\right)\leq\exp\left(-\frac{v}{D_n}\right).$$

The following establishes, on the event A_n with respect to $P_{x,\omega}$, a comparison between the continuous exit time τ^ϵ and discrete stopping time $\tau_1^{\epsilon,n}$. The estimate will be achieved on scales ϵ satisfying $L_n\leq\frac{1}{\epsilon}<L_{n+1}$ for all n sufficiently large. The proof is a consequence of the global coupling from Corollary 11 and the estimates for Brownian motion from Proposition 15. The estimate improves upon [8, Proposition 7.3] by using Proposition 12, which is itself an improvement of [8, Proposition 4.1]. However, since aside from the application of Proposition 12 (respectively, [8, Proposition 4.1]) the proof is identical to the proof of [8, Proposition 7.3], the details are omitted.

Proposition 17. *Assume (32) and (52). For each $n\geq\overline{m}$ sufficiently large, for every $\epsilon>0$ satisfying $L_n\leq\frac{1}{\epsilon}<L_{n+1}$ and for every $\omega\in A_n$, for $C>0$ independent of n,*

$$\sup_{x\in\overline{U}/\epsilon}P_{x,\omega}\left(\tau^\epsilon-\tau_1^{\epsilon,n}\geq L_{n-1}^2\right)\leq CL_{n-1}^{-10a}+C(\epsilon L_{n+2})^{-3}.$$

Stochastic homogenization for solutions of (61) is now established. Because the case of zero right-hand side and nonzero boundary data was considered in [8], by linearity it remains only to prove homogenization for solutions of (61) with nonzero righ-thand side which vanish along the boundary. Precisely, it will first be shown that, on a subset of full probability, the solutions

$$\begin{cases}\frac{1}{2}\operatorname{tr}(A(\frac{x}{\epsilon},\omega)D^2u^\epsilon)+\frac{1}{\epsilon}b(\frac{x}{\epsilon},\omega)\cdot Du^\epsilon=g(x) & \text{on }U,\\ u^\epsilon=0 & \text{on }\partial U,\end{cases} \tag{72}$$

converge uniformly, as $\epsilon\to 0$, to the solution

$$\begin{cases}\frac{\overline{\alpha}}{2}\Delta\overline{u}=g(x) & \text{on }U,\\ \overline{u}=0 & \text{on }\partial U.\end{cases} \tag{73}$$

The proof will analyze the behavior of solutions to the rescaled equation

$$\begin{cases}\frac{1}{2}\operatorname{tr}(A(x,\omega)D^2v^\epsilon)+b(x,\omega)\cdot Dv^\epsilon=\epsilon^2g(\epsilon x) & \text{on }U/\epsilon,\\ v^\epsilon=0 & \text{on }\partial U/\epsilon,\end{cases} \tag{74}$$

which admit the representation

$$u^\epsilon(x)=v^\epsilon\left(\frac{x}{\epsilon}\right)=E_{\frac{x}{\epsilon},\omega}\left(-\epsilon^2\int_0^{\tau^\epsilon}g(\epsilon X_s)\,ds\right)\quad\text{on }\overline{U},$$

for the exit time τ^ϵ from U/ϵ. The first step will be to apply a sub-optimal bound for the exit time through the use of Proposition 12. Precisely, for scales $L_n \le \frac{1}{\epsilon} < L_{n+1}$, it will be shown on the event A_n that, up to an error vanishing with ϵ, the solution v^ϵ is well-approximated by the quantity

$$v^\epsilon\left(\frac{x}{\epsilon}\right) \simeq E_{\frac{x}{\epsilon},\omega}\left(-\epsilon^2 \int_0^{\tau^\epsilon} g(\epsilon X_s)\,ds, \tau^\epsilon \le L_{n+2}^2\right). \tag{75}$$

Since the exit time of a corresponding Brownian motion is expected to be of order $\frac{1}{\epsilon^2}$ which, as $n \to \infty$, is significantly smaller than L_{n+2}^2, this estimate does not imply an effective upper bound for the exit time of the diffusion in random environment. However, it does allow for the application of the global coupling established in Corollary 11.

The second step replaces the continuous exit time τ^ϵ with its discrete proxy $\tau_1^{\epsilon,n}$, where the exponential estimates guaranteed by Control 16 and Proposition 17 will be used to show that the discretely stopped version of (75) is a good approximation for the solution v^ϵ. Namely,

$$E_{\frac{x}{\epsilon},\omega}\left(-\epsilon^2 \int_0^{\tau^\epsilon} g(\epsilon X_s)\,ds, \tau^\epsilon \le L_{n+2}^2\right) \simeq E_{\frac{x}{\epsilon},\omega}\left(-\epsilon^2 \int_0^{\tau_1^{\epsilon,n}} g(\epsilon X_s)\,ds, \tau_1^{\epsilon,n} \le L_{n+2}^2\right). \tag{76}$$

The integral will then be shown to be accurately represented by its discrete approximation on scale $L_{n-\overline{m}}^2$ in the sense that, up to an error vanishing with ϵ,

$$\begin{aligned} &E_{\frac{x}{\epsilon},\omega}\left(-\epsilon^2 \int_0^{\tau_1^{\epsilon,n}} g(\epsilon X_s)\,ds, \tau_1^{\epsilon,n} \le L_{n+2}^2\right) \\ &\quad \simeq E_{\frac{x}{\epsilon},\omega}\left(-\epsilon^2 \sum_{k=0}^{(\tau_1^{\epsilon,n} - L_{n-\overline{m}}^2)/L_{n-\overline{m}}^2} L_{n-\overline{m}}^2 g(\epsilon X_{kL_{n-\overline{m}}^2}), \tau_1^{\epsilon,n} \le L_{n+2}^2\right). \end{aligned} \tag{77}$$

This will follow from the localization estimates contained in Control 16.

The global coupling established in Section 4.1 now plays its role. It follows from the definition of the measure $Q_{n,\frac{x}{\epsilon}}$ and process $(X_k, \overline{X}_k)$ on $(\mathbb{R}^d \times \mathbb{R}^d)^{\mathbb{N}}$ that, writing $E^{Q_{n,\frac{x}{\epsilon}}}$ for the expectation with respect to the measure $Q_{n,\frac{x}{\epsilon}}$,

$$\begin{aligned} &E_{\frac{x}{\epsilon},\omega}\left(-\epsilon^2 \sum_{k=0}^{(\tau_1^{\epsilon,n} - L_{n-\overline{m}}^2)/L_{n-\overline{m}}^2} L_{n-\overline{m}}^2 g(\epsilon X_{kL_{n-\overline{m}}^2}), \tau_1^{\epsilon,n} \le L_{n+2}^2\right) \\ &\quad = E^{Q_{n,\frac{x}{\epsilon}}}\left(-\epsilon^2 \sum_{k=0}^{T_1^{\epsilon,n}-1} L_{n-\overline{m}}^2 g(\epsilon X_k), T_1^{\epsilon,n} \le \left(\frac{L_{n+2}}{L_{n-\overline{m}}}\right)^2\right), \end{aligned} \tag{78}$$

for the discrete stopping time

$$T_1^{\epsilon,n} = \inf\bigl\{k \ge 0 \mid (X_k, \overline{X}_k) \text{ satisfies } d\bigl(X_k, (U/\epsilon)^c\bigr) \le \tilde{D}_{n-\overline{m}}\bigr\},$$

which is the analogue of $\tau_1^{\epsilon,n}$ for the first coordinate of $(X_k, \overline{X}_k)$. The coupling estimates stated in Corollary 11 are then used to obtain a comparison with Brownian motion of variance $\alpha_{n-\overline{m}}$ and to prove, up to an error vanishing with ϵ,

$$\begin{aligned} &E^{Q_{n,\frac{x}{\epsilon}}}\left(-\epsilon^2 \sum_{k=0}^{T_1^{\epsilon,n}-1} L_{n-\overline{m}}^2 g(\epsilon X_k), T_1^{\epsilon,n} \le \left(\frac{L_{n+2}}{L_{n-\overline{m}}}\right)^2\right) \\ &\quad \simeq E^{Q_{n,\frac{x}{\epsilon}}}\left(-\epsilon^2 \sum_{k=0}^{T_1^{\epsilon,n}-1} L_{n-\overline{m}}^2 g(\epsilon \overline{X}_k), T_1^{\epsilon,n} \le \left(\frac{L_{n+2}}{L_{n-\overline{m}}}\right)^2\right). \end{aligned} \tag{79}$$

The remainder of the proof is then essentially an unwinding of the above outline in terms of Brownian motion. For the discrete stopping time

$$\overline{T}_2^{\epsilon,n} = \inf\big\{k \geq 0 | (X_k, \overline{X}_k) \text{ satisfies } d\big(\overline{X}_k, (U/\epsilon)\big) \geq \tilde{D}_{n-\overline{m}}\big\},$$

which acts as $\tau_2^{\epsilon,n}$ defined for the second coordinate of the process $(X_k, \overline{X}_k)$, it is first shown that, up to an error vanishing with ϵ,

$$\begin{aligned} &E^{Q_{n,\frac{x}{\epsilon}}}\left(-\epsilon^2 \sum_{k=0}^{T_1^{\epsilon,n}-1} L_{n-\overline{m}}^2 g(\epsilon\overline{X}_k), T_1^{\epsilon,n} \leq \left(\frac{L_{n+2}}{L_{n-\overline{m}}}\right)^2\right) \\ &\quad \simeq E^{Q_{n,\frac{x}{\epsilon}}}\left(-\epsilon^2 \sum_{k=0}^{\overline{T}_2^{\epsilon,n}-1} L_{n-\overline{m}}^2 g(\epsilon\overline{X}_k), \overline{T}_2^{\epsilon,n} \leq \left(\frac{L_{n+2}}{L_{n-\overline{m}}}\right)^2\right), \end{aligned} \tag{80}$$

where, by the definition of $Q_{n,\frac{x}{\epsilon}}$,

$$\begin{aligned} &E^{Q_{n,\frac{x}{\epsilon}}}\left(-\epsilon^2 \sum_{k=0}^{\overline{T}_2^{\epsilon,n}-1} L_{n-\overline{m}}^2 g(\epsilon\overline{X}_k), \overline{T}_2^{\epsilon,n} \leq \left(\frac{L_{n+2}}{L_{n-\overline{m}}}\right)^2\right) \\ &\quad = E^{W_{\frac{x}{\epsilon}}^{n-\overline{m}}}\left(-\epsilon^2 \sum_{k=0}^{(\tau_2^{\epsilon,n}-L_{n-\overline{m}}^2)/L_{n-\overline{m}}^2} L_{n-\overline{m}}^2 g(\epsilon X_{kL_{n-\overline{m}}^2}), \tau_2^{\epsilon,n} \leq L_{n+2}^2\right). \end{aligned} \tag{81}$$

It will then be shown that, in view of standard exponential estimates for Brownian motion, the control of the $\{\alpha_n\}_{n=0}^\infty$ from Theorem 5 and the upper bound for the exit time in probability obtained in Proposition 15, up to an error vanishing with ϵ,

$$\begin{aligned} &E^{W_{\frac{x}{\epsilon}}^{n-\overline{m}}}\left(-\epsilon^2 \sum_{k=0}^{(\tau_2^{\epsilon,n}-L_{n-\overline{m}}^2)/L_{n-\overline{m}}^2} L_{n-\overline{m}}^2 g\big(\epsilon\overline{X}_{kL_{n-\overline{m}}}^2\big), \tau_2^{\epsilon,n} \leq L_{n+2}^2\right) \\ &\quad \simeq E^{W_{\frac{x}{\epsilon}}^{n-\overline{m}}}\left(-\epsilon^2 \int_0^{\tau_2^{\epsilon,n}} g(\epsilon X_s)\,ds, \tau_2^{\epsilon,n} \leq L_{n+2}^2\right). \end{aligned} \tag{82}$$

The same estimates then replace $\tau_2^{\epsilon,n}$ with τ^ϵ and remove the cutoff to provide

$$E^{W_{\frac{x}{\epsilon}}^{n-\overline{m}}}\left(-\epsilon^2 \int_0^{\tau_2^{\epsilon,n}} g(\epsilon X_s)\,ds, \tau_2^{\epsilon,n} \leq L_{n+2}^2\right) \simeq E^{W_{\frac{x}{\epsilon}}^{n-\overline{m}}}\left(-\epsilon^2 \int_0^{\tau^\epsilon} g(\epsilon X_s)\,ds\right). \tag{83}$$

The final step comes in approximating the solution of the homogenized equation (73) by solutions of the approximate equations

$$\begin{cases} \frac{\alpha_{n-\overline{m}}}{2}\Delta\overline{u}_{n-\overline{m}} = g(x) & \text{on } U, \\ \overline{u}_{n-\overline{m}} = 0 & \text{on } \partial U, \end{cases} \tag{84}$$

which admit the representation

$$\overline{u}_{n-\overline{m}}(x) = E^{W_{\frac{x}{\epsilon}}^{n-\overline{m}}}\left(-\epsilon^2 \int_0^{\tau^\epsilon} g(\epsilon X_s)\,ds\right) = E^{W_{\frac{x}{\epsilon}}^{n-\overline{m}}}\left(-\int_0^{\epsilon^2\tau^\epsilon} g\big(\epsilon X_{\frac{s}{\epsilon^2}}\big)\,ds\right) \quad \text{on } \overline{U}, \tag{85}$$

and coincide with the right-hand side of (83).

Proposition 18. *For each $n \geq \overline{m}$, for $C = C(U, \overline{\alpha}) > 0$ independent of n, the solutions of* (73) *and* (84) *satisfy*

$$\|\overline{u} - \overline{u}_{n-\overline{m}}\|_{L^\infty(\overline{U})} \leq C\|g\|_{L^\infty(\overline{U})} L_{n-\overline{m}}^{-(1+\frac{9}{10})\delta}.$$

Proof. Fix $n \geq \overline{m}$. For the respective solutions $\overline{u}$ and $\overline{u}_{n-\overline{m}}$ of (73) and (84), the difference

$$w_{n-\overline{m}} = \overline{u} - \overline{u}_{n-\overline{m}}$$

solves the equation

$$\begin{cases} \frac{\overline{\alpha}}{2}\Delta w_{n-\overline{m}} = (1 - \frac{\overline{\alpha}}{\alpha_{n-\overline{m}}})g(x) & \text{on } U, \\ w_{n-\overline{m}} = 0 & \text{on } \partial U. \end{cases}$$

Therefore, using Theorem 5, for $C > 0$ independent of n, writing W_x^∞ for the Wiener measure defining Brownian motion beginning from x with variance $\overline{\alpha}$, and writing τ for the exit time from U,

$$\|w_{n-\overline{m}}\|_{L^\infty(\overline{U})} \leq C\|g\|_{L^\infty(\overline{U})}|\alpha_{n-\overline{m}} - \overline{\alpha}| \sup_{x\in\overline{U}} E^{W_x^\infty}(\tau) \leq C\|g\|_{L^\infty(\overline{U})} L_{n-\overline{m}}^{-(1+\frac{9}{10})\delta},$$

which completes the argument. □

Therefore, returning to the representation (85) and Proposition 18, up to an error vanishing with ϵ,

$$E^{W_{\frac{x}{\epsilon}}^{n-\overline{m}}}\left(-\epsilon^2 \int_0^{\tau^\epsilon} g(\epsilon X_s)\,ds\right) \simeq E^{W_{\frac{x}{\epsilon}}^\infty}\left(-\epsilon^2 \int_0^{\tau^\epsilon} g(\epsilon X_s)\,ds\right). \tag{86}$$

In combination, lines (75)–(86) complete the proof. Furthermore, because the estimates in each step are quantified, they will yield a rate for the homogenization in Section 6.

The stochastic homogenization will be obtained on a subset of full probability defined by the events $\{A_n\}_{n=1}^\infty$ and an application of the Borel–Cantelli lemma. Since Proposition 9 implies that, for each $n \geq \overline{m}$, for $C \geq 0$ independent of n,

$$\mathbb{P}(\Omega \setminus A_n) \leq C L_n^{2d(1+a)^2 - \frac{1}{2}M_0},$$

the definition of L_n in (37) and the negative exponent $2d(1+a)^2 - \frac{1}{2}M_0 < 0$ guarantee the sum

$$\sum_{n=\overline{m}}^\infty \mathbb{P}(\Omega \setminus A_n) \leq C \sum_{n=\overline{m}}^\infty L_n^{2d(1+a)^2 - \frac{1}{2}M_0} < \infty.$$

The Borel–Cantelli lemma therefore implies the event

$$\Omega_0 = \{\omega \in \Omega \mid \text{There exists } \overline{n} = \overline{n}(\omega) \text{ such that } \omega \in A_n \text{ for all } n \geq \overline{n}\} \tag{87}$$

satisfies $\mathbb{P}(\Omega_0) = 1$. Note particularly that the subset of full probability Ω_0 is independent of the domain and the right-hand side. It is on this event that homogenization is achieved following the outline presented between lines (75) to (86).

The result is first established for functions which are the restriction of a smooth, compactly supported function on $\mathbb{R}^d$.

$$\text{Assume } g(x) \in C_c^\infty(\mathbb{R}^d). \tag{88}$$

This assumption is removed by a standard approximation argument in Theorem 20.

Theorem 19. *Assume* (32), (52) *and* (88). *For every $\omega \in \Omega_0$, the respective solutions u^ϵ and $\overline{u}$ of* (72) *and* (73) *satisfy*

$$\lim_{\epsilon\to 0}\|u^\epsilon - \overline{u}\|_{L^\infty(\overline{U})} = 0.$$

Proof. Fix $\omega \in \Omega_0$ and $n_0 \geq \overline{m}$ such that, for all $n \geq n_0$, for r_0 the constant quantifying the exterior ball condition,

$$2\tilde{D}_{n-\overline{m}} < \frac{r_0 L_n}{2} \quad \text{and} \quad \omega \in A_n.$$

Then, fix $\epsilon_0 > 0$ sufficiently small so that, whenever $0 < \epsilon < \epsilon_0$ satisfies $L_n \leq \frac{1}{\epsilon} < L_{n+1}$ it follows that $n \geq n_0$. Furthermore, using the boundedness of the domain U, choose $0 < \epsilon_1 < \epsilon_0$ such that, whenever $0 < \epsilon < \epsilon_1$ and $L_n \leq \frac{1}{\epsilon} < L_{n+1}$,

$$L_{n+1}U \subset \left[-\frac{1}{2}L_{n+2}^2, \frac{1}{2}L_{n+2}^2\right]^d.$$

These conditions guarantee that whenever $0 < \epsilon < \epsilon_1$ the conclusions of Propositions 15 and 17 are satisfied, and that Controls 6 and 16 are available on scales $L_{n-\overline{m}}$ to L_{n+2} for the entirety of the domain U/ϵ.

Henceforth, fix $x \in \overline{U}$ and $0 < \epsilon < \epsilon_1$. Write u^ϵ for the solution of (72) and v^ϵ for the solution of the rescaled (74), and recall the representation

$$u^\epsilon(x) = v^\epsilon\left(\frac{x}{\epsilon}\right) = E_{\frac{x}{\epsilon},\omega}\left(-\epsilon^2\int_0^{\tau^\epsilon} g(\epsilon X_s)\,ds\right). \tag{89}$$

In order to apply the coupling estimates obtained in Section 4.1, it is necessary to restrict the above integral to the event $\{\tau^\epsilon \leq L_{n+2}^2\}$.

The proof of (75). First, observe that

$$\begin{aligned}
&\left|E_{\frac{x}{\epsilon},\omega}\left(-\epsilon^2\int_0^{\tau^\epsilon} g(\epsilon X_s)\,ds\right) - E_{\frac{x}{\epsilon},\omega}\left(-\epsilon^2\int_0^{\tau^\epsilon} g(\epsilon X_s)\,ds, \tau^\epsilon \leq L_{n+2}^2\right)\right| \\
&\quad \leq \epsilon^2\|g\|_{L^\infty(\mathbb{R}^d)}\sum_{k=1}^\infty (k+1)L_{n+2}^2 P_{\frac{x}{\epsilon},\omega}\left(kL_{n+2}^2 < \tau^\epsilon \leq (k+1)L_{n+2}^2\right) \\
&\quad \leq \epsilon^2\|g\|_{L^\infty(\mathbb{R}^d)}\sum_{k=1}^\infty (k+1)L_{n+2}^2 P_{\frac{x}{\epsilon},\omega}\left(\tau^\epsilon > kL_{n+2}^2\right).
\end{aligned} \tag{90}$$

Therefore, since $L_n \leq \frac{1}{\epsilon} < L_{n+1}$, and because Proposition 12 proved that, on the event A_n, for each $k \geq 0$, for $C > 0$ independent of n and k,

$$P_{\frac{x}{\epsilon},\omega}\left(\tau^\epsilon > kL_{n+2}^2\right) \leq C(\epsilon L_{n+2})^{-3k},$$

it follows from the definition of L_n in (37) and properties of the geometric series that, for $C > 0$ independent of n,

$$\begin{aligned}
\epsilon^2\|g\|_{L^\infty(\overline{U})}\sum_{k=1}^\infty (k+1)L_{n+2}^2 P_{\frac{x}{\epsilon},\omega}\left(\tau^\epsilon > kL_{n+2}^2\right) &\leq C\|g\|_{L^\infty(\mathbb{R}^d)}(\epsilon L_{n+2})^2\sum_{k=1}^\infty (k+1)(\epsilon L_{n+2})^{-3k} \\
&\leq C\|g\|_{L^\infty(\mathbb{R}^d)}(\epsilon L_{n+2})^{-1}.
\end{aligned} \tag{91}$$

Notice that this estimate relies upon the assumption $d \geq 3$. Furthermore, since $L_n \leq \frac{1}{\epsilon} < L_{n+1}$, for $C > 0$ independent of n,

$$\left|E_{\frac{x}{\epsilon},\omega}\left(-\epsilon^2\int_0^{\tau^\epsilon} g(\epsilon X_s)\,ds\right) - E_{\frac{x}{\epsilon},\omega}\left(-\epsilon^2\int_0^{\tau^\epsilon} g(\epsilon X_s)\,ds, \tau^\epsilon < L_{n+2}^2\right)\right| \leq C\|g\|_{L^\infty(\mathbb{R}^d)}\frac{L_{n+1}}{L_{n+2}}, \tag{92}$$

which completes the proof of (75).

The proof of (76). Recall the discrete $\mathrm{C}([0,\infty);\mathbb{R}^d)$ stopping time

$$\tau_1^{\epsilon,n} = \inf\big\{kL_{n-\overline{m}}^2 \geq 0 \big| d\big(X_{kL_{n-\overline{m}}^2}, (U/\epsilon)^c\big) \leq \tilde{D}_{n-\overline{m}}\big\}.$$

First, decompose the second term of (92) as

$$\begin{aligned}
&E_{\frac{x}{\epsilon},\omega}\left(-\epsilon^2\int_0^{\tau^\epsilon} g(\epsilon X_s)\,ds, \tau^\epsilon < L_{n+2}^2\right)\\
&\quad= E_{\frac{x}{\epsilon},\omega}\left(-\epsilon^2\int_0^{\tau^\epsilon} g(\epsilon X_s)\,ds, \tau^\epsilon < L_{n+2}^2, \tau^\epsilon + L_{n-\overline{m}}^2 \leq \tau_1^{\epsilon,n}\right)\\
&\qquad+ E_{\frac{x}{\epsilon},\omega}\left(-\epsilon^2\int_0^{\tau^\epsilon} g(\epsilon X_s)\,ds, \tau^\epsilon \leq L_{n+2}^2, \tau^\epsilon + L_{n-\overline{m}}^2 > \tau_1^{\epsilon,n}\right).
\end{aligned}\tag{93}$$

Since $\omega \in A_n$ and because the definitions imply that on the event

$$\big\{\tau^\epsilon + L_{n-\overline{m}}^2 \leq \tau_1^{\epsilon,n}\big\}$$

the diffusion undergoes an excursion of size at least $\tilde{D}_{n-\overline{m}}$ in time $L_{n-\overline{m}}^2$, the exponential estimates guaranteed by Control 16 act to bound the first term of this equality, and yield

$$\begin{aligned}
&\left|E_{\frac{x}{\epsilon},\omega}\left(-\epsilon^2\int_0^{\tau^\epsilon} g(\epsilon X_s)\,ds, \tau^\epsilon \leq L_{n+2}^2, \tau^\epsilon + L_{n-\overline{m}}^2 \leq \tau_1^{\epsilon,n}\right)\right|\\
&\quad\leq \epsilon^2 L_{n+2}^2 \|g\|_{L^\infty(\overline{U})} E_{\frac{x}{\epsilon},\omega}\big(P_{X_{\tau^\epsilon},\omega}\big(X^*_{L_{n-\overline{m}}^2} \geq \tilde{D}_{n-\overline{m}}\big)\big)\\
&\quad\leq \left(\frac{L_{n+2}}{L_n}\right)^2 \|g\|_{L^\infty(\mathbb{R}^d)} \exp(-\tilde{\kappa}_{n-\overline{m}}).
\end{aligned}\tag{94}$$

The second term of (93) is further decomposed according to

$$\begin{aligned}
&E_{\frac{x}{\epsilon},\omega}\left(-\epsilon^2\int_0^{\tau^\epsilon} g(\epsilon X_s)\,ds, \tau^\epsilon \leq L_{n+2}^2, \tau^\epsilon + L_{n-\overline{m}}^2 > \tau_1^{\epsilon,n}\right)\\
&\quad= E_{\frac{x}{\epsilon},\omega}\left(-\epsilon^2\int_0^{\tau^\epsilon} g(\epsilon X_s)\,ds, \tau^\epsilon \leq L_{n+2}^2, 0 \leq \tau_1^{\epsilon,n} - \tau^\epsilon < L_{n-\overline{m}}^2\right)\\
&\qquad+ E_{\frac{x}{\epsilon},\omega}\left(-\epsilon^2\int_0^{\tau^\epsilon} g(\epsilon X_s)\,ds, \tau^\epsilon \leq L_{n+2}^2, \tau^\epsilon - \tau_1^{\epsilon,n} > 0\right).
\end{aligned}\tag{95}$$

In comparing the left-hand side of (95) with the discretely stopped version

$$E_{\frac{x}{\epsilon},\omega}\left(-\epsilon^2\int_0^{\tau_1^{\epsilon,n}} g(\epsilon X_s)\,ds, \tau^\epsilon \leq L_{n+2}^2, \tau^\epsilon + L_{n-\overline{m}}^2 > \tau_1^{\epsilon,n}\right),\tag{96}$$

the decomposition (95) implies that the difference is bounded by

$$\begin{aligned}
&\left|E_{\frac{x}{\epsilon},\omega}\left(-\epsilon^2\int_0^{\tau^\epsilon} g(\epsilon X_s)\,ds + \epsilon^2\int_0^{\tau_1^{\epsilon,n}} g(\epsilon X_s)\,ds, \tau^\epsilon \leq L_{n+2}^2, \tau^\epsilon + L_{n-\overline{m}}^2 > \tau_1^{\epsilon,n}\right)\right|\\
&\quad\leq \epsilon^2\|g\|_{L^\infty(\mathbb{R}^d)} E_{\frac{x}{\epsilon},\omega}\big(\tau_1^{\epsilon,n} - \tau^\epsilon, \tau^\epsilon \leq L_{n+2}^2, 0 \leq \tau_1^{\epsilon,n} - \tau^\epsilon < L_{n-\overline{m}}^2\big)\\
&\qquad+ \epsilon^2\|g\|_{L^\infty(\mathbb{R}^d)} E_{\frac{x}{\epsilon},\omega}\big(\tau^\epsilon - \tau_1^{\epsilon,n}, \tau^\epsilon \leq L_{n+2}^2, \tau^\epsilon - \tau_1^{\epsilon,n} > 0\big).
\end{aligned}\tag{97}$$

The event describing the first term of the right-hand side of (97) allows for the immediate L^∞-estimate of the integrand

$$\epsilon^2 E_{\frac{x}{\epsilon},\omega}\big(\tau_1^{\epsilon,n}-\tau^\epsilon, \tau^\epsilon \le L_{n+2}^2, 0 \le \tau_1^{\epsilon,n}-\tau^\epsilon < L_{n-\overline{m}}^2\big) \le \epsilon^2 L_{n-\overline{m}}^2 \le \left(\frac{L_{n-\overline{m}}}{L_n}\right)^2. \tag{98}$$

The second term of the right-hand side of (97) is bounded using Proposition 17. Form the decomposition

$$\begin{aligned}
&\epsilon^2 E_{\frac{x}{\epsilon},\omega}\big(\tau^\epsilon-\tau_1^{\epsilon,n}, \tau^\epsilon \le L_{n+2}^2, \tau^\epsilon-\tau_1^{\epsilon,n} > 0\big)\\
&\quad= \epsilon^2 E_{\frac{x}{\epsilon},\omega}\big(\tau^\epsilon-\tau_1^{\epsilon,n}, \tau^\epsilon \le L_{n+2}^2, 0 < \tau^\epsilon-\tau_1^{\epsilon,n} \le L_{n-1}^2\big)\\
&\qquad+ \epsilon^2 E_{\frac{x}{\epsilon},\omega}\big(\tau^\epsilon-\tau_1^{\epsilon,n}, \tau^\epsilon \le L_{n+2}^2, \tau^\epsilon-\tau_1^{\epsilon,n} > L_{n-1}^2\big).
\end{aligned} \tag{99}$$

The event defining the first term of the right-hand side of (99) admits the immediate L^∞-estimate

$$\epsilon^2 E_{\frac{x}{\epsilon},\omega}\big(\tau^\epsilon-\tau_1^{\epsilon,n}, \tau^\epsilon \le L_{n+2}^2, 0 < \tau^\epsilon-\tau_1^{\epsilon,n} \le L_{n-1}^2\big) \le \epsilon^2 L_{n-1}^2 \le \left(\frac{L_{n-1}}{L_n}\right)^2. \tag{100}$$

Then, Proposition 17 is applied to the second term of (99) to obtain by estimating the integrand in L^∞, for $C>0$ independent of n,

$$\begin{aligned}
\epsilon^2 E_{\frac{x}{\epsilon},\omega}\big(\tau^\epsilon-\tau_1^{\epsilon,n}, \tau^\epsilon \le L_{n+2}^2, \tau^\epsilon-\tau_1^{\epsilon,n} > L_{n-1}^2\big) &\le \epsilon^2 L_{n+2}^2 P_{\frac{x}{\epsilon},\omega}\big(\tau^\epsilon-\tau_1^{\epsilon,n} \ge L_{n-1}^2\big)\\
&\le C(\epsilon L_{n+2})^2\big(L_{n-1}^{-10a}+(\epsilon L_{n+2})^{-3}\big).
\end{aligned} \tag{101}$$

Therefore, owing to the definition of L_n in (37), since $L_n \le \frac{1}{\epsilon} < L_{n+1}$, for $C>0$ independenet of n,

$$\epsilon^2 E_{\frac{x}{\epsilon},\omega}\big(\tau^\epsilon-\tau_1^{\epsilon,n}, \tau^\epsilon \le L_{n+2}^2, \tau^\epsilon-\tau_1^{\epsilon,n} > L_{n-1}^2\big) \le C\left(L_n^{4a+2a^2-\frac{10a}{1+a}} + \frac{L_{n+1}}{L_{n+2}}\right), \tag{102}$$

where the exponent

$$4a+2a^2-\frac{10a}{1+a} < 0$$

owing to definition (36).

In combination, (98), (100) and (102) imply using (97) the bound, for $C>0$ independent of n,

$$\begin{aligned}
&\left|E_{\frac{x}{\epsilon},\omega}\left(-\epsilon^2\int_0^{\tau^\epsilon} g(\epsilon X_s)\,ds + \epsilon^2\int_0^{\tau_1^{\epsilon,n}} g(\epsilon X_s)\,ds, \tau^\epsilon \le L_{n+2}^2, \tau^\epsilon + L_{n-\overline{m}}^2 > \tau_1^{\epsilon,n}\right)\right|\\
&\quad\le C\|g\|_{L^\infty(\mathbb{R}^d)}\left(\left(\frac{L_{n-1}}{L_n}\right)^2 + L_n^{4a+2a^2-\frac{10a}{1+a}} + \frac{L_{n+1}}{L_{n+2}}\right).
\end{aligned} \tag{103}$$

And, since the definitions (37) and (38) imply that, for $C>0$ independent of n,

$$\left(\frac{L_{n+2}}{L_n}\right)^2 \exp(-\tilde{\kappa}_{n-\overline{m}}) \le C\left(\left(\frac{L_{n-1}}{L_n}\right)^2 + L_n^{4a+2a^2-\frac{10a}{1+a}} + \frac{L_{n+1}}{L_{n+2}}\right),$$

equation (93) and estimates (94) and (103) combine to yield, for $C>0$ independent of n,

$$\begin{aligned}
&\left|E_{\frac{x}{\epsilon},\omega}\left(-\epsilon^2\int_0^{\tau^\epsilon} g(\epsilon X_s)\,ds, \tau^\epsilon \le L_{n+2}^2\right) - E_{\frac{x}{\epsilon},\omega}\left(-\epsilon^2\int_0^{\tau_1^{\epsilon,n}} g(\epsilon X_s)\,ds, \tau^\epsilon \le L_{n+2}^2, \tau^\epsilon + L_{n-\overline{m}}^2 > \tau_1^{\epsilon,n}\right)\right|\\
&\quad\le C\|g\|_{L^\infty(\mathbb{R}^d)}\left(\left(\frac{L_{n-1}}{L_n}\right)^2 + L_n^{4a+2a^2-\frac{10a}{1+a}} + \frac{L_{n+1}}{L_{n+2}}\right).
\end{aligned} \tag{104}$$

To obtain (76), it remains only to estimate the difference between the discretely stopped quantity within the absolute value of (104) and

$$E_{\frac{x}{\epsilon},\omega}\left(-\epsilon^2\int_0^{\tau_1^{\epsilon,n}} g(\epsilon X_s)\,ds,\ \tau_1^{\epsilon,n}\leq L_{n+2}^2\right). \tag{105}$$

First, notice with the aid of Proposition 12 that, for $C>0$ independent of n,

$$\begin{aligned}&\left|E_{\frac{x}{\epsilon},\omega}\left(-\epsilon^2\int_0^{\tau_1^{\epsilon,n}} g(\epsilon X_s)\,ds,\ \tau^\epsilon\leq L_{n+2}^2,\ \tau^\epsilon+L_{n-\overline{m}}^2>\tau_1^{\epsilon,n}\right)\right.\\&\qquad\left.-E_{\frac{x}{\epsilon},\omega}\left(-\epsilon^2\int_0^{\tau_1^{\epsilon,n}} g(\epsilon X_s)\,ds,\ \tau_1^{\epsilon,n}<L_{n+2}^2+L_{n-\overline{m}}^2\right)\right|\\&\quad\leq C\epsilon^2\|g\|_{L^\infty(\mathbb{R}^d)}L_{n+2}^2P_{\frac{x}{\epsilon},\omega}\left(\tau^\epsilon>L_{n+2}^2\right)\leq C\|g\|_{L^\infty(\mathbb{R}^d)}(\epsilon L_{n+2})^{-1}\leq C\|g\|_{L^\infty(\mathbb{R}^d)}\frac{L_{n+1}}{L_{n+2}}.\end{aligned} \tag{106}$$

Then, again using Proposition 12 and in particular [8, Line (4.7)] which applies equally to the discrete sequence, it follows that, for $C>0$ independent of n,

$$\begin{aligned}&\left|E_{\frac{x}{\epsilon},\omega}\left(-\epsilon^2\int_0^{\tau_1^{\epsilon,n}} g(\epsilon X_s)\,ds,\ \tau_1^{\epsilon,n}<L_{n+2}^2+L_{n-\overline{m}}^2\right)-E_{\frac{x}{\epsilon},\omega}\left(-\epsilon^2\int_0^{\tau_1^{\epsilon,n}} g(\epsilon X_s)\,ds,\ \tau_1^{\epsilon,n}\leq L_{n+2}^2\right)\right|\\&\quad\leq C\epsilon^2\|g\|_{L^\infty(\mathbb{R}^d)}L_{n+2}^2P_{\frac{x}{\epsilon},\omega}\left(\tau_1^{\epsilon,n}>L_{n+2}^2\right)\leq C\|g\|_{L^\infty(\mathbb{R}^d)}(\epsilon L_{n+2})^{-1}\leq C\|g\|_{L^\infty(\mathbb{R}^d)}\frac{L_{n+1}}{L_{n+2}}.\end{aligned} \tag{107}$$

Therefore, in view of (104), (106) and (107), for $C>0$ independent of n,

$$\begin{aligned}&\left|E_{\frac{x}{\epsilon},\omega}\left(-\epsilon^2\int_0^{\tau^\epsilon} g(\epsilon X_s)\,ds,\ \tau^\epsilon\leq L_{n+2}^2\right)-E_{\frac{x}{\epsilon},\omega}\left(-\epsilon^2\int_0^{\tau_1^{\epsilon,n}} g(\epsilon X_s)\,ds,\ \tau_1^{\epsilon,n}\leq L_{n+2}^2\right)\right|\\&\quad\leq C\|g\|_{L^\infty(\mathbb{R}^d)}\left(\left(\frac{L_{n-1}}{L_n}\right)^2+L_n^{4a+2a^2-\frac{10a}{1+a}}+\frac{L_{n+1}}{L_{n+2}}\right),\end{aligned} \tag{108}$$

which completes the proof of (76).

The proof of (77). The discrete approximation of the integral is a result of the Lipschitz continuity of g and the exponential estimates implied by Control 16. Observe that, for $C>0$ independent of n,

$$\begin{aligned}&\left|E_{\frac{x}{\epsilon},\omega}\left(-\epsilon^2\int_0^{\tau_1^{\epsilon,n}} g(\epsilon X_s)\,ds,\ \tau_1^{\epsilon,n}\leq L_{n+2}^2\right)\right.\\&\qquad\left.-E_{\frac{x}{\epsilon},\omega}\left(-\epsilon^2\sum_{k=0}^{(\tau_1^{\epsilon,n}-L_{n-\overline{m}}^2)/L_{n-\overline{m}}^2}L_{n-\overline{m}}^2 g(\epsilon X_{kL_{n-\overline{m}}^2}),\ \tau_1^{\epsilon,n}\leq L_{n+2}^2\right)\right|\\&\quad\leq C\|g\|_{L^\infty(\mathbb{R}^d)}(\epsilon L_{n-\overline{m}})^2E_{\frac{x}{\epsilon},\omega}\left(I_1(X_\cdot)\right)+C(\epsilon L_{n-\overline{m}})^2\|Dg\|_{L^\infty(\mathbb{R}^d)}\epsilon\tilde{D}_{n-\overline{m}}E_{\frac{x}{\epsilon},\omega}\left(I_2(X_\cdot)\right),\end{aligned} \tag{109}$$

for the random variables

$$I_1(X_\cdot)=\left(\sum_{k=0}^{(\tau_1^{\epsilon,n}-L_{n-\overline{m}}^2)/L_{n-\overline{m}}^2}P_{X_{kL_{n-\overline{m}}^2},\omega}\left(X_{L_{n-\overline{m}}^2}^*>\tilde{D}_{n-\overline{m}}\right)\right)\mathbf{1}_{\{\tau_1^{\epsilon,n}\leq L_{n+2}^2\}}$$

and

$$I_2(X_\cdot)=\left(\sum_{k=0}^{(\tau_1^{\epsilon,n}-L_{n-\overline{m}}^2)/L_{n-\overline{m}}^2}P_{X_{kL_{n-\overline{m}}^2},\omega}\left(X_{L_{n-\overline{m}}^2}^*\leq\tilde{D}_{n-\overline{m}}\right)\right)\mathbf{1}_{\{\tau_1^{\epsilon,n}\leq L_{n+2}^2\}},$$

where $\mathbf{1}_A$ denotes the indicator function of a subset $A \subset \mathrm{C}([0,\infty);\mathbb{R}^d)$. Therefore, Control 16, the event $\{\tau_1^{\epsilon,n} \le L_{n+2}^2\}$, and $L_n \le \frac{1}{\epsilon} < L_{n+1}$ imply that, for $C>0$ independent of n,

$$\begin{aligned}
&\bigg| E_{\frac{x}{\epsilon},\omega}\bigg(-\epsilon^2\int_0^{\tau_1^{\epsilon,n}} g(\epsilon X_s)\,ds, \tau_1^{\epsilon,n} \le L_{n+2}^2\bigg) \\
&\quad - E_{\frac{x}{\epsilon},\omega}\Bigg(-\epsilon^2 \sum_{k=0}^{(\tau_1^{\epsilon,n}-L_{n-\overline{m}}^2)/L_{n-\overline{m}}^2} L_{n-\overline{m}}^2 g(\epsilon X_{kL_{n-\overline{m}}^2}), \tau_1^{\epsilon,n} \le L_{n+2}^2\Bigg)\Bigg| \\
&\quad \le C\|g\|_{L^\infty(\mathbb{R}^d)}\bigg(\frac{L_{n+2}}{L_n}\bigg)^2 \exp(-\tilde{\kappa}_{n-\overline{m}}) + C\|Dg\|_{L^\infty(\mathbb{R}^d)}\epsilon^3 L_{n+2}^2 \tilde{D}_{n-\overline{m}}.
\end{aligned} \tag{110}$$

Since the definitions (37), (38) and (39), the choice of $\overline{m}$ in (66), and $L_n \le \frac{1}{\epsilon} < L_{n+1}$ ensure that, for $C>0$ independent of n,

$$\bigg(\frac{L_{n+2}}{L_n}\bigg)^2 \exp(-\tilde{\kappa}_{n-\overline{m}}) \le C\epsilon^3 L_{n+2}^2 \tilde{D}_{n-\overline{m}} \le C L_n^{-3+2(1+a)^2+(1+a)^{-\overline{m}}} \le C L_n^{-5a},$$

the left-hand side of (110) is bounded, for $C>0$ independent of n, by

$$C\big(\|g\|_{L^\infty(\mathbb{R}^d)} + \|Dg\|_{L^\infty(\mathbb{R}^d)}\big) L_n^{-5a}, \tag{111}$$

which completes the proof of (77).

The proof of (79). Recall the stopping time

$$T_1^{\epsilon,n} = \inf\big\{k \ge 0 | (X_k, \overline{X}_k) \text{ satisfies } d\big(X_k, (U/\epsilon)^c\big) \le \tilde{D}_{n-\overline{m}}\big\},$$

which is the discrete version of $\tau_1^{\epsilon,n}$ defined for the first coordinate of the process $(X_k, \overline{X}_k)$ described by the measure $Q_{n,\frac{x}{\epsilon}}$ constructed in Section 4.1. The definition of $Q_{n,\frac{x}{\epsilon}}$ and the Markov property imply that

$$\begin{aligned}
&E_{\frac{x}{\epsilon},\omega}\Bigg(-\epsilon^2 \sum_{k=0}^{(\tau_1^{\epsilon,n}-L_{n-\overline{m}}^2)/L_{n-\overline{m}}^2} L_{n-\overline{m}}^2 g(\epsilon X_{kL_{n-\overline{m}}^2}), \tau_1^{\epsilon,n} \le L_{n+2}^2\Bigg) \\
&\quad = E^{Q_{n,\frac{x}{\epsilon}}}\Bigg(-\epsilon^2 \sum_{k=0}^{T_1^{\epsilon,n}-1} L_{n-\overline{m}}^2 g(\epsilon X_k), T_1^{\epsilon,n} \le \bigg(\frac{L_{n+2}}{L_{n-\overline{m}}}\bigg)^2\Bigg),
\end{aligned} \tag{112}$$

and therefore, to recap the progress to this point, in combination (89), (92), (104), (108) (111) imply that, for $C>0$ independent of n,

$$\begin{aligned}
&\Bigg| u^\epsilon(x) - E^{Q_{n,\frac{x}{\epsilon}}}\Bigg(-\epsilon^2 \sum_{k=0}^{T_1^{\epsilon,n}-1} L_{n-\overline{m}}^2 g(\epsilon X_k), T_1^{\epsilon,n} \le \bigg(\frac{L_{n+2}}{L_{n-\overline{m}}}\bigg)^2\Bigg)\Bigg| \\
&\quad \le C\|g\|_{L^\infty(\overline{U})}\bigg(\frac{L_{n+1}}{L_{n+2}} + \bigg(\frac{L_{n-1}}{L_n}\bigg)^2 + L_n^{4a+2a^2-\frac{10a}{1+a}} + L_n^{-5a}\bigg) + C\|Dg\|_{L^\infty(\mathbb{R}^d)} L_n^{-5a}.
\end{aligned} \tag{113}$$

This estimate effectively proves the efficacy of the discrete approximation scheme. The next step in the proof will follow from the global coupling estimates established by Corollary 11 and standard estimates for Brownian motion.

Define the event

$$C_n = \bigg\{|X_k - \overline{X}_k| \ge L_{n-\overline{m}}| \text{ for some } 0 \le k \le 2\bigg(\frac{L_{n+2}}{L_{n-\overline{m}}}\bigg)^2\bigg\},$$

and observe by Corollary 11 that, for $C > 0$ independent of n,

$$Q_{n,\frac{x}{\epsilon}}(C_n) \le C\tilde{\kappa}_{n-\overline{m}} L_{n-\overline{m}}^{16a-\delta}. \tag{114}$$

The goal now is to estimate the magnitude of the difference

$$\left| E^{Q_{n,\frac{x}{\epsilon}}}\left(-\epsilon^2 \sum_{k=0}^{T_1^{\epsilon,n}-1} L_{n-\overline{m}}^2\big(g(\epsilon X_k) - g(\epsilon \overline{X}_k)\big), T_1^{\epsilon,n} \le \left(\frac{L_{n+2}}{L_{n-\overline{m}}}\right)^2 \right) \right|. \tag{115}$$

Form the decomposition with respect to the event C_n and use the triangle inequality to bound (115) by the sum of the differences

$$\left| E^{Q_{n,\frac{x}{\epsilon}}}\left(-\epsilon^2 \sum_{k=0}^{T_1^{\epsilon,n}-1} L_{n-\overline{m}}^2\big(g(\epsilon X_k) - g(\epsilon \overline{X}_k)\big), T_1^{\epsilon,n} \le \left(\frac{L_{n+2}}{L_{n-\overline{m}}}\right)^2, C_n \right) \right|$$
$$+ \left| E^{Q_{n,\frac{x}{\epsilon}}}\left(-\epsilon^2 \sum_{k=0}^{T_1^{\epsilon,n}-1} L_{n-\overline{m}}^2\big(g(\epsilon X_k) - g(\epsilon \overline{X}_k)\big), T_1^{\epsilon,n} \le \left(\frac{L_{n+2}}{L_{n-\overline{m}}}\right)^2, C_n^c \right) \right|. \tag{116}$$

The first term of (116) is bounded using (114) and the event

$$\left\{ T_1^{\epsilon,n} \le \left(\frac{L_{n+2}}{L_{n-\overline{m}}}\right)^2 \right\},$$

which imply, for $C > 0$ independent of n, using the definitions of L_n in (37) and $\tilde{\kappa}_{n-\overline{m}}$ in (38),

$$\left| E^{Q_{n,\frac{x}{\epsilon}}}\left(-\epsilon^2 \sum_{k=0}^{T_1^{\epsilon,n}-1} L_{n-\overline{m}}^2\big(g(\epsilon X_k) - g(\epsilon \overline{X}_k)\big), T_1^{\epsilon,n} \le \left(\frac{L_{n+2}}{L_{n-\overline{m}}}\right)^2, C_n \right) \right|$$
$$\le C\|g\|_{L^\infty(\mathbb{R}^d)} (\epsilon L_{n+2})^2 \tilde{\kappa}_{n-\overline{m}} L_{n-\overline{m}}^{16a-\delta} \le C\|g\|_{L^\infty(\mathbb{R}^d)} L_n^{21a-\frac{\delta}{2}}, \tag{117}$$

where the exponent

$$21a - \frac{\delta}{2} < 0$$

owing to definitions (36) and (41).

The second term of (116) is bounded using the Lipschitz continuity of g, the upper bound for $T_1^{\epsilon,n}$ and the definition of C_n^c. Namely, for $C > 0$ independent of n,

$$\left| E^{Q_{n,\frac{x}{\epsilon}}}\left(-\epsilon^2 \sum_{k=0}^{T_1^{\epsilon,n}-1} L_{n-\overline{m}}^2\big(g(\epsilon X_k) - g(\epsilon \overline{X}_k)\big), T_1^{\epsilon,n} \le \left(\frac{L_{n+2}}{L_{n-\overline{m}}}\right)^2, C_n^c \right) \right|$$
$$\le C\|Dg\|_{L^\infty(\mathbb{R}^d)} (\epsilon L_{n+2})^2 \epsilon L_{n-\overline{m}} \le C\|Dg\|_{L^\infty(\mathbb{R}^d)} L_n^{-5a}, \tag{118}$$

where the final inequality is obtained as in the arguments leading from (110) to (111). Therefore, in view of (116), estimates (117) and (118) combine to form the estimate, for $C > 0$ independent of n,

$$\left| E^{Q_{n,\frac{x}{\epsilon}}}\left(-\epsilon^2 \sum_{k=0}^{T_1^{\epsilon,n}-1} L_{n-\overline{m}}^2\big(g(\epsilon X_k) - g(\epsilon \overline{X}_k)\big), T_1^{\epsilon,n} \le \left(\frac{L_{n+2}}{L_{n-\overline{m}}}\right)^2 \right) \right|$$
$$\le C\|g\|_{L^\infty(\mathbb{R}^d)} L_n^{21a-\frac{\delta}{2}} + C\|Dg\|_{L^\infty(\mathbb{R}^d)} L_n^{-5a}, \tag{119}$$

and to complete the proof of (79).

The proof of (80). Recall the discrete exit time

$$\overline{T}_2^{\epsilon,n} = \inf\{k \geq 0 | (X_k, \overline{X}_k) \text{ satisfies } d(\overline{X}_k, (U/\epsilon)) \geq \tilde{D}_{n-\overline{m}}\},$$

which is acts as $\tau_2^{\epsilon,n}$ for the second coordinate of the process $(X_k, \overline{X}_k)$. The purpose now is to replace $T_1^{\epsilon,n}$ with $\overline{T}_2^{\epsilon,n}$ for the second term of the difference (119). First, an upper bound is imposed for $\overline{T}_2^{\epsilon,n}$, and the difference is bounded, for $C > 0$ independent of n, by

$$\begin{aligned}&\left| E^{Q_{n,\frac{x}{\epsilon}}}\left(-\epsilon^2 \sum_{k=0}^{T_1^{\epsilon,n}-1} L_{n-\overline{m}}^2 g(\epsilon \overline{X}_k), T_1^{\epsilon,n} \leq \left(\frac{L_{n+2}}{L_{n-\overline{m}}}\right)^2\right)\right.\\ &\quad \left. - E^{Q_{n,\frac{x}{\epsilon}}}\left(-\epsilon^2 \sum_{k=0}^{T_1^{\epsilon,n}-1} L_{n-\overline{m}}^2 g(\epsilon \overline{X}_k), T_1^{\epsilon,n} \leq \left(\frac{L_{n+2}}{L_{n-\overline{m}}}\right)^2, \overline{T}_2^{\epsilon,n} \leq \left(\frac{L_{n+2}}{L_{n-\overline{m}}}\right)^2\right)\right|\\ &\quad \leq C\|g\|_{L^\infty(\mathbb{R}^d)}(\epsilon L_{n+2})^2 Q_{n,\frac{x}{\epsilon}}\left(\overline{T}_2^{\epsilon,n} > \left(\frac{L_{n+2}}{L_{n-\overline{m}}}\right)^2\right). \end{aligned} \tag{120}$$

Since Proposition 12 applies to the discrete sequence and stopping time after increasing R to $2R$ and increasing the constant, it follows from the definition of $Q_{n,\frac{x}{\epsilon}}$ and the stopping times that, for $C > 0$ independent of n,

$$Q_{n,\frac{x}{\epsilon}}\left(\overline{T}_2^{\epsilon,n} > \left(\frac{L_{n+2}}{L_{n-\overline{m}}}\right)^2\right) = W_{\frac{x}{\epsilon}}^{n-\overline{m}}\left(\tau_2^{\epsilon,n} > L_{n+2}^2\right) \leq C(\epsilon L_{n+2})^{-3}. \tag{121}$$

Therefore, for $C > 0$ independent of n, the left-hand side of (120) is bounded by

$$C\|g\|_{L^\infty(\mathbb{R}^d)}(\epsilon L_{n+2})^{-1} \leq C\|g\|_{L^\infty(\mathbb{R}^d)}\frac{L_{n+1}}{L_{n+2}}. \tag{122}$$

Note that a better estimate can be achieved in (121) for Brownian motion, however any improvement at this stage will not improve the overall rate of homogenization.

The next step replaces $T_1^{\epsilon,n}$ in the sum with $\overline{T}_2^{\epsilon,n}$ following a decomposition in terms of the event C_n and an application of the triangle inequality. Using (114) and the bounds for the exit times, on the event C_n the expectation of the difference

$$\begin{aligned}&\left| E^{Q_{n,\frac{x}{\epsilon}}}\left(-\epsilon^2 \sum_{k=0}^{T_1^{\epsilon,n}-1} L_{n-\overline{m}}^2 g(\epsilon \overline{X}_k), T_1^{\epsilon,n} \leq \left(\frac{L_{n+2}}{L_{n-\overline{m}}}\right)^2, \overline{T}_2^{\epsilon,n} \leq \left(\frac{L_{n+2}}{L_{n-\overline{m}}}\right)^2, C_n\right)\right.\\ &\quad \left. - E^{Q_{n,\frac{x}{\epsilon}}}\left(-\epsilon^2 \sum_{k=0}^{\overline{T}_2^{\epsilon,n}-1} L_{n-\overline{m}}^2 g(\epsilon \overline{X}_k), T_1^{\epsilon,n} \leq \left(\frac{L_{n+2}}{L_{n-\overline{m}}}\right)^2, \overline{T}_2^{\epsilon,n} \leq \left(\frac{L_{n+2}}{L_{n-\overline{m}}}\right)^2, C_n\right)\right| \end{aligned} \tag{123}$$

is bounded, for $C > 0$ independent of n, by

$$C(\epsilon L_{n+2})^2\|g\|_{L^\infty(\mathbb{R}^d)} Q_{n,\frac{x}{\epsilon}}(C_n) \leq C\|g\|_{L^\infty(\mathbb{R}^d)}\left(\frac{L_{n+2}}{L_n}\right)^2 \tilde{\kappa}_{n-\overline{m}} L_{n-\overline{m}}^{16a-\delta} \leq C\|g\|_{L^\infty(\mathbb{R}^d)} L_n^{21a-\frac{\delta}{2}}, \tag{124}$$

where the final inequality is obtained identically to (117).

It follows immediately from the definitions that $T_1^{\epsilon,n} \leq \overline{T}_2^{\epsilon,n}$ on the event

$$\left\{T_1^{\epsilon,n} \leq \left(\frac{L_{n+2}}{L_{n-\overline{m}}}\right)^2, \overline{T}_2^{\epsilon,n} \leq \left(\frac{L_{n+2}}{L_{n-\overline{m}}}\right)^2, C_n^c\right\}. \tag{125}$$

Therefore, on the event C_n^c, the expectation of the difference

$$\left| E^{Q_{n,\frac{x}{\epsilon}}}\left(\left(-\epsilon^2 \sum_{k=0}^{T_1^{\epsilon,n}-1} L_{n-\overline{m}}^2 g(\epsilon \overline{X}_k) + \epsilon^2 \sum_{k=0}^{\overline{T}_2^{\epsilon,n}-1} L_{n-\overline{m}}^2 g(\epsilon \overline{X}_k)\right), T_1^{\epsilon,n} \leq \left(\frac{L_{n+2}}{L_{n-\overline{m}}}\right)^2, \right.\right.$$

$$\left.\left. \overline{T}_2^{\epsilon,n} \leq \left(\frac{L_{n+2}}{L_{n-\overline{m}}}\right)^2, C_n^c\right)\right| \tag{126}$$

is bounded by

$$(\epsilon L_{n-\overline{m}})^2 \|g\|_{L^\infty(\mathbb{R}^d)} E^{Q_{n,\frac{x}{\epsilon}}}\left(\overline{T}_2^{\epsilon,n} - T_1^{\epsilon,n}, T_1^{\epsilon,n} \leq \left(\frac{L_{n+2}}{L_{n-\overline{m}}}\right)^2, \overline{T}_2^{\epsilon,n} \leq \left(\frac{L_{n+2}}{L_{n-\overline{m}}}\right)^2, C_n^c\right). \tag{127}$$

To estimate the right-hand side of (127), first notice that on the event (125) the definitions of $T_1^{\epsilon,n}$ and C_n^c imply

$$d\left(X_{T_1^{\epsilon,n}}, (U/\epsilon)^c\right) \leq \tilde{D}_{n-\overline{m}} \qquad \text{which guarantees } d\left(\overline{X}_{T_1^{\epsilon,n}}, (U/\epsilon)^c\right) \leq 2\tilde{D}_{n-\overline{m}}.$$

Therefore, the definition of $Q_{n,\frac{x}{\epsilon}}$, the Markov property, standard exponential estimates for Brownian motion [18, Chapter 2, Proposition 1.8], and Proposition 14 imply that, using the definitions of (37), (38) and (39), for $C > 0$ independent of n,

$$E^{Q_{n,\frac{x}{\epsilon}}}\left(\overline{T}_2^{\epsilon,n} - T_1^{\epsilon,n}, T_1^{\epsilon,n} \leq \left(\frac{L_{n+2}}{L_{n-\overline{m}}}\right)^2, \overline{T}_2^{\epsilon,n} \leq \left(\frac{L_{n+2}}{L_{n-\overline{m}}}\right)^2, C_n^c\right)$$

$$\leq \sup_{d(y,(U/\epsilon)^c) \leq 2\tilde{D}_{n-\overline{m}}} E^{W_y^{n-\overline{m}}}\left(\frac{\tau_2^{\epsilon,n}}{L_{n-\overline{m}}^2}\right) \leq C \frac{\tilde{\kappa}_{n-\overline{m}}}{\epsilon L_{n-\overline{m}}}. \tag{128}$$

Therefore, combining (126), (127) and (128) and using $L_n \leq \frac{1}{\epsilon} < L_{n+1}$, for $C > 0$ independent of n,

$$\left| E^{Q_{n,\frac{x}{\epsilon}}}\left(-\epsilon^2 \sum_{k=0}^{T_1^{\epsilon,n}-1} L_{n-\overline{m}}^2 g(\epsilon \overline{X}_k) + \epsilon^2 \sum_{k=0}^{\overline{T}_2^{\epsilon,n}-1} L_{n-\overline{m}}^2 g(\epsilon \overline{X}_k)\right), \right.$$

$$\left. T_1^{\epsilon,n} \leq \left(\frac{L_{n+2}}{L_{n-\overline{m}}}\right)^2, \overline{T}_2^{\epsilon,n} \leq \left(\frac{L_{n+2}}{L_{n-\overline{m}}}\right)^2, C_n^c \right|$$

$$\leq C\|g\|_{L^\infty(\mathbb{R}^d)} \frac{\tilde{D}_{n-\overline{m}}}{L_n}. \tag{129}$$

So, with (123), (124) and (129), conclude using the triangle inequality that, for $C > 0$ independent of n,

$$\left| E^{Q_{n,\frac{x}{\epsilon}}}\left(-\epsilon^2 \sum_{k=0}^{T_1^{\epsilon,n}-1} L_{n-\overline{m}}^2 g(\epsilon \overline{X}_k), T_1^{\epsilon,n} \leq \left(\frac{L_{n+2}}{L_{n-\overline{m}}}\right)^2, \overline{T}_2^{\epsilon,n} \leq \left(\frac{L_{n+2}}{L_{n-\overline{m}}}\right)^2\right) \right.$$

$$\left. - E^{Q_{n,\frac{x}{\epsilon}}}\left(-\epsilon^2 \sum_{k=0}^{\overline{T}_2^{\epsilon,n}-1} L_{n-\overline{m}}^2 g(\epsilon \overline{X}_k), T_1^{\epsilon,n} \leq \left(\frac{L_{n+2}}{L_{n-\overline{m}}}\right)^2, \overline{T}_2^{\epsilon,n} \leq \left(\frac{L_{n+2}}{L_{n-\overline{m}}}\right)^2\right)\right|$$

$$\leq C\|g\|_{L^\infty(\mathbb{R}^d)}\left(L_n^{21a-\frac{\delta}{2}} + \frac{\tilde{D}_{n-\overline{m}}}{L_n}\right). \tag{130}$$

Finally, analogously to the arguments (120) to (122), for $C > 0$ independent of n,

$$
\begin{aligned}
&\left| E^{Q_{n,\frac{x}{\epsilon}}}\left(-\epsilon^2 \sum_{k=0}^{\overline{T}_2^{\epsilon,n}-1} L_{n-\overline{m}}^2 g(\epsilon \overline{X}_k), T_1^{\epsilon,n} \leq \left(\frac{L_{n+2}}{L_{n-\overline{m}}}\right)^2, \overline{T}_2^{\epsilon,n} \leq \left(\frac{L_{n+2}}{L_{n-\overline{m}}}\right)^2\right)\right. \\
&\qquad \left. - E^{Q_{n,\frac{x}{\epsilon}}}\left(-\epsilon^2 \sum_{k=0}^{\overline{T}_2^{\epsilon,n}-1} L_{n-\overline{m}}^2 g(\epsilon \overline{X}_k), \overline{T}_2^{\epsilon,n} \leq \left(\frac{L_{n+2}}{L_{n-\overline{m}}}\right)^2\right)\right| \\
&\quad \leq C\|g\|_{L^\infty(\mathbb{R}^d)} (\epsilon L_{n+2})^2 Q_{n,\frac{x}{\epsilon}}\left(T_1^{\epsilon,n} > \left(\frac{L_{n+2}}{L_{n-\overline{m}}}\right)^2\right).
\end{aligned}
\tag{131}
$$

Since the exponential estimates implied by Control 16 and Proposition 12 yield, for $C > 0$ independent of n,

$$
Q_{n,\frac{x}{\epsilon}}\left(T_1^{\epsilon,n} > \left(\frac{L_{n+2}}{L_{n-\overline{m}}}\right)^2\right) \leq C(\epsilon L_{n+2})^{-3},
$$

the left-hand side of (131) is bounded, for $C > 0$ independent of n, by

$$
C\|g\|_{L^\infty(\mathbb{R}^d)} (\epsilon L_{n+2})^{-1} \leq \|g\|_{L^\infty(\mathbb{R}^d)} \frac{L_{n+1}}{L_{n+2}}.
\tag{132}
$$

In total then, the collection (122), (130) and (132) yield, for $C > 0$ independent of n,

$$
\begin{aligned}
&\left| E^{Q_{n,\frac{x}{\epsilon}}}\left(-\epsilon^2 \sum_{k=0}^{T_1^{\epsilon,n}-1} L_{n-\overline{m}}^2 g(\epsilon \overline{X}_k), T_1^{\epsilon,n} \leq \left(\frac{L_{n+2}}{L_{n-\overline{m}}}\right)^2\right)\right. \\
&\qquad \left. - E^{Q_{n,\frac{x}{\epsilon}}}\left(-\epsilon^2 \sum_{k=0}^{\overline{T}_2^{\epsilon,n}-1} L_{n-\overline{m}}^2 g(\epsilon \overline{X}_k), \overline{T}_2^{\epsilon,n} \leq \left(\frac{L_{n+2}}{L_{n-\overline{m}}}\right)^2\right)\right| \\
&\quad \leq C\|g\|_{L^\infty(\mathbb{R}^d)} \left(L_n^{21a-\frac{\delta}{2}} + \frac{\tilde{D}_{n-\overline{m}}}{L_n} + \frac{L_{n+1}}{L_{n+2}}\right),
\end{aligned}
\tag{133}
$$

and complete the proof of (80).

The definition of $Q_{n,\frac{x}{\epsilon}}$ and the Markov property imply

$$
\begin{aligned}
&E^{Q_{n,\frac{x}{\epsilon}}}\left(-\epsilon^2 \sum_{k=0}^{\overline{T}_2^{\epsilon,n}-1} L_{n-\overline{m}}^2 g(\epsilon \overline{X}_k), \overline{T}_2^{\epsilon,n} \leq \left(\frac{L_{n+2}}{L_{n-\overline{m}}}\right)^2\right) \\
&\quad = E^{W^{n-\overline{m}}_{\frac{x}{\epsilon}}}\left(-\epsilon^2 \sum_{k=0}^{(\tau_2^{\epsilon,n}-L_{n-\overline{m}}^2)/L_{n-\overline{m}}^2} L_{n-\overline{m}}^2 g(\epsilon X_{kL_{n-\overline{m}}^2}), \tau_2^{\epsilon,n} \leq L_{n+2}^2\right).
\end{aligned}
\tag{134}
$$

Therefore, to recap the progress, the collection of estimates (113), (119), (133) and (134) produce the bound, for $C > 0$ independent of n,

$$
\begin{aligned}
&\left| u^\epsilon(x) - E^{W^{n-\overline{m}}_{\frac{x}{\epsilon}}}\left(-\epsilon^2 \sum_{k=0}^{(\tau_2^{\epsilon,n}-L_{n-\overline{m}}^2)/L_{n-\overline{m}}^2} L_{n-\overline{m}}^2 g(\epsilon X_{kL_{n-\overline{m}}^2}), \tau_2^{\epsilon,n} \leq L_{n+2}^2\right)\right| \\
&\quad \leq C\|g\|_{L^\infty(\mathbb{R}^d)} \left(\frac{L_{n+1}}{L_{n+2}} + \left(\frac{L_{n-1}}{L_n}\right)^2 + L_n^{4a+2a^2-\frac{10a}{1+a}} + L_n^{21a-\frac{\delta}{2}} + \frac{\tilde{D}_{n-\overline{m}}}{L_n} + L_n^{-5a}\right) \\
&\qquad + C\|Dg\|_{L^\infty(\mathbb{R}^d)} L_n^{-5a}.
\end{aligned}
\tag{135}
$$

It remains to recover the integral with respect to Brownian motion from its discrete approximation.

The proof of (82). The proof follows, in reverse order, the arguments leading to the proof of (79) from (75). Observe that, for $C>0$ independent of n,

$$\begin{aligned}&\left|E^{W^{n-\overline{m}}_{\frac{x}{\epsilon}}}\left(-\epsilon^2\sum_{k=0}^{(\tau_2^{\epsilon,n}-L^2_{n-\overline{m}})/L^2_{n-\overline{m}}}L^2_{n-\overline{m}}g(\epsilon X_{kL^2_{n-\overline{m}}}),\tau_2^{\epsilon,n}\le L^2_{n+2}\right)\right.\\&\qquad\left.-E^{W^{n-\overline{m}}_{\frac{x}{\epsilon}}}\left(-\epsilon^2\int_0^{\tau_2^{\epsilon,n}}g(\epsilon X_s)\,ds,\tau_2^{\epsilon,n}\le L^2_{n+2}\right)\right|\\&\quad\le C(\epsilon L_{n-\overline{m}})^2\|g\|_{L^\infty(\mathbb{R}^d)}E^{W^{n-\overline{m}}_{\frac{x}{\epsilon}}}\left(\overline{I}_1(X.)\right)\\&\qquad+(\epsilon L_{n-\overline{m}})^2\|Dg\|_{L^\infty(\mathbb{R}^d)}\epsilon\tilde{D}_{n-\overline{m}}E^{W^{n-\overline{m}}_{\frac{x}{\epsilon}}}\left(\overline{I}_2(X.)\right),\end{aligned}\tag{136}$$

for the random variables

$$\overline{I}_1(X.)=\left(\sum_{k=0}^{(\tau_2^{\epsilon,n}-L^2_{n-\overline{m}})/L^2_{n-\overline{m}}}W^{n-\overline{m}}_{X_{kL^2_{n-\overline{m}}},\omega}\left(X^*_{L^2_{n-\overline{m}}}>\tilde{D}_{n-\overline{m}}\right)\right)\mathbf{1}_{\{\tau_2^{\epsilon,n}\le L^2_{n+2}\}}$$

and

$$\overline{I}_2(X.)=\left(\sum_{k=0}^{(\tau_2^{\epsilon,n}-L^2_{n-\overline{m}})/L^2_{n-\overline{m}}}W^{n-\overline{m}}_{X_{kL^2_{n-\overline{m}}},\omega}\left(X^*_{L^2_{n-\overline{m}}}\le\tilde{D}_{n-\overline{m}}\right)\right)\mathbf{1}_{\{\tau_2^{\epsilon,n}\le L^2_{n+2}\}}.$$

Standard exponential estimates for Brownian motion, see [18, Chapter 2, Proposition 1.8], and $L_n\le\frac{1}{\epsilon}<L_{n+1}$ imply that the first term of (136) is bounded, for $C>0$ independent of n, by

$$C\|g\|_{L^\infty(\mathbb{R}^d)}(\epsilon L_{n+2})^2\exp(-\tilde{\kappa}_{n-\overline{m}})\le C\|g\|_{L^\infty(\mathbb{R}^d)}\left(\frac{L_{n+2}}{L_n}\right)^2\exp(-\tilde{\kappa}_{n-\overline{m}}).\tag{137}$$

The L^∞-estimate implied by the upper bound on $\tau_2^{\epsilon,n}$ ensures that the second term of (136) is bounded, for $C>0$ independent of n, by

$$C\|Dg\|_{L^\infty(\mathbb{R}^d)}\epsilon^3L^2_{n+2}\tilde{D}_{n-\overline{m}}\le C\|Dg\|_{L^\infty(\mathbb{R}^d)}L_n^{-5a},\tag{138}$$

where the final inequality is obtained identically as in the arguments leading from (110) to (111).

In combination, lines (137) and (138) bound the left-hand side of (136), for $C>0$ independent of n, by

$$\begin{aligned}&\left|E^{W^{n-\overline{m}}_{\frac{x}{\epsilon}}}\left(-\epsilon^2\sum_{k=0}^{(\tau_2^{\epsilon,n}-L^2_{n-\overline{m}})/L^2_{n-\overline{m}}}L^2_{n-\overline{m}}g(\epsilon X_{kL^2_{n-\overline{m}}}),\tau_2^{\epsilon,n}\le L^2_{n+2}\right)\right.\\&\qquad\left.-E^{W^{n-\overline{m}}_{\frac{x}{\epsilon}}}\left(-\epsilon^2\int_0^{\tau_2^{\epsilon,n}}g(\epsilon X_s)\,ds,\tau_2^{\epsilon,n}\le L^2_{n+2}\right)\right|\\&\quad\le C\|g\|_{L^\infty(\mathbb{R}^d)}\left(\frac{L_{n+2}}{L_n}\right)^2\exp(-\tilde{\kappa}_{n-\overline{m}})+C\|Dg\|_{L^\infty(\mathbb{R}^d)}L_n^{-5a},\end{aligned}\tag{139}$$

and complete the proof of (82).

The proof of (83). Recall that τ^ϵ denotes the exit time from U/ϵ. It follows form Theorem 5, Proposition 12 and $L_n \leq \frac{1}{\epsilon} < L_{n+1}$ that, for $C>0$ independent of n,

$$\begin{aligned}
&\bigg| E^{W^{n-\overline{m}}_{\frac{x}{\epsilon}}}\bigg(-\epsilon^2\int_0^{\tau_2^{\epsilon,n}} g(\epsilon X_s)\,ds,\ \tau_2^{\epsilon,n}\leq L^2_{n+2}\bigg)\\
&\quad - E^{W^{n-\overline{m}}_{\frac{x}{\epsilon}}}\bigg(-\epsilon^2\int_0^{\tau_2^{\epsilon,n}} g(\epsilon X_s)\,ds,\ \tau_2^{\epsilon,n}\leq L^2_{n+2},\ \tau^\epsilon\leq L^2_{n+2}\bigg)\bigg|\\
&\quad\leq C\|g\|_{L^\infty(\mathbb{R}^d)}(\epsilon L_{n+2})^2 W^{n-\overline{m}}_{\frac{x}{\epsilon}}\big(\tau^\epsilon > L^2_{n+2}\big)\leq C\|g\|_{L^\infty(\mathbb{R}^d)}(\epsilon L_{n+2})^{-1}\\
&\quad\leq C\|g\|_{L^\infty(\mathbb{R}^d)}\frac{L_{n+1}}{L_{n+2}}.
\end{aligned}\tag{140}$$

Of course, estimate (140) can be improved for Brownian motion, but what is written is sufficient and does not negatively effect the rate to be obtained in Section 6.

Since the definitions imply $\tau^\epsilon \leq \tau_2^{\epsilon,n}$, the Markov property, Corollary 14, standard exponential estimates for Brownian motion, see [18, Chapter 2, Proposition 1.8], and $L_n \leq \frac{1}{\epsilon} < L_{n+1}$ then produce the estimate, for $C>0$ independent of n,

$$\begin{aligned}
&\bigg| E^{W^{n-\overline{m}}_{\frac{x}{\epsilon}}}\bigg(-\epsilon^2\int_{\tau^\epsilon}^{\tau_2^{\epsilon,n}} g(\epsilon X_s)\,ds,\ \tau_2^{\epsilon,n}\leq L^2_{n+2},\ \tau^\epsilon\leq L^2_{n+2}\bigg)\bigg|\\
&\quad\leq \epsilon^2\|g\|_{L^\infty(\mathbb{R}^d)}\sup_{y\in\partial U} E^{W^{n-\overline{m}}_y}\big(\tau_2^{\epsilon,n}\big)\leq C\|g\|_{L^\infty(\mathbb{R}^d)}\epsilon\tilde{D}_{n-\overline{m}}\leq C\|g\|_{L^\infty(\mathbb{R}^d)}\frac{\tilde{D}_{n-\overline{m}}}{L_n}.
\end{aligned}\tag{141}$$

Then, again using Proposition 12 (after replacing R with $2R$ and increasing the constant) and standard exponential estimates for Brownian motion, see [18, Chapter 2, Proposition 1.8],

$$\begin{aligned}
&\bigg| E^{W^{n-\overline{m}}_{\frac{x}{\epsilon}}}\bigg(-\epsilon^2\int_0^{\tau^\epsilon} g(\epsilon X_s)\,ds,\ \tau_2^{\epsilon,n}\leq L^2_{n+2},\ \tau^\epsilon\leq L^2_{n+2}\bigg) - E^{W^{n-\overline{m}}_{\frac{x}{\epsilon}}}\bigg(-\epsilon^2\int_0^{\tau^\epsilon} g(\epsilon X_s)\,ds,\ \tau^\epsilon\leq L^2_{n+2}\bigg)\bigg|\\
&\quad\leq C\|g\|_{L^\infty(\mathbb{R}^d)}(\epsilon L_{n+2})^2 W^{n-\overline{m}}_{\frac{x}{\epsilon}}\big(\tau_2^{\epsilon,n} > L^2_{n+2}\big)\leq C\|g\|_{L^\infty(\mathbb{R}^d)}(\epsilon L_{n+2})^{-1}\leq C\|g\|_{L^\infty(\mathbb{R}^d)}\frac{L_{n+1}}{L_{n+2}}.
\end{aligned}\tag{142}$$

As before, estimate (141) is not optimal for Brownian motion, but is sufficient and does not negatively impact the rate.

It remains only to estimate the difference

$$\bigg| E^{W^{n-\overline{m}}_{\frac{x}{\epsilon}}}\bigg(-\epsilon^2\int_0^{\tau^\epsilon} g(\epsilon X_s)\,ds,\ \tau^\epsilon\leq L^2_{n+2}\bigg) - E^{W^{n-\overline{m}}_{\frac{x}{\epsilon}}}\bigg(-\epsilon^2\int_0^{\tau^\epsilon} g(\epsilon X_s)\,ds\bigg)\bigg|.\tag{143}$$

Since the control of the α_n implied by Theorem 5 implies Proposition 12 applies equally to Brownian motion (though, as before, a better estimate can be obtained to no effect on the rate), it follows as in (90) that, for $C>0$ independent of n,

$$\begin{aligned}
&\bigg| E^{W^{n-\overline{m}}_{\frac{x}{\epsilon}}}\bigg(-\epsilon^2\int_0^{\tau^\epsilon} g(\epsilon X_s)\,ds,\ \tau^\epsilon\leq L^2_{n+2}\bigg) - E^{W^{n-\overline{m}}_{\frac{x}{\epsilon}}}\bigg(-\epsilon^2\int_0^{\tau^\epsilon} g(\epsilon X_s)\,ds\bigg)\bigg|\\
&\quad\leq C\epsilon^2\|g\|_{L^\infty(\mathbb{R}^d)}\sum_{k=1}^\infty (k+1)L^2_{n+2}W^{n-\overline{m}}_{\frac{x}{\epsilon}}\big(\tau^\epsilon > kL^2_{n+2}\big)\leq C\|g\|_{L^\infty(\mathbb{R}^d)}(\epsilon L_{n+2})^{-1}\\
&\quad\leq C\|g\|_{L^\infty(\mathbb{R}^d)}\frac{L_{n+1}}{L_{n+2}}.
\end{aligned}\tag{144}$$

Therefore, in view of (141), (142) and (144), for $C > 0$ independent of n,

$$\left|E^{W^{n-\overline{m}}_{\frac{x}{\epsilon}}}\left(-\epsilon^2\int_0^{\tau_2^{\epsilon,n}} g(\epsilon X_s)\,ds,\, \tau_2^{\epsilon,n} \le L_{n+2}^2\right) - E^{W^{n-\overline{m}}_{\frac{x}{\epsilon}}}\left(-\epsilon^2\int_0^{\tau^\epsilon} g(\epsilon X_s)\,ds\right)\right|$$
$$\le C\|g\|_{L^\infty(\mathbb{R}^d)}\left(\frac{\tilde{D}_{n-\overline{m}}}{L_n} + \frac{L_{n+1}}{L_{n+2}}\right), \tag{145}$$

which completes the proof of (83).

Conclusion. Finally, writing $\overline{u}$ and $\overline{u}_{n-\overline{m}}$ for the respective solutions of (73) and (84), Proposition 18 implies that, for $C > 0$ independent of n,

$$\left|E^{W^{n-\overline{m}}_{\frac{x}{\epsilon}}}\left(-\epsilon^2\int_0^{\tau^\epsilon} g(\epsilon X_s)\,ds\right) - E^{W^{\infty}_{\frac{x}{\epsilon}}}\left(-\epsilon^2\int_0^{\tau^\epsilon} g(\epsilon X_s)\,ds\right)\right|$$
$$= \left|\overline{u}_{n-\overline{m}}(x) - \overline{u}(x)\right| \le C\|g\|_{L^\infty(\overline{U})} L_{n-\overline{m}}^{-(1+\frac{9}{10})\delta}. \tag{146}$$

Since there exists $C > 0$ independent of $n \ge \overline{m}$ such that

$$\left(\frac{L_{n+2}}{L_n}\right)^2 \exp(-\tilde{\kappa}_{n-\overline{m}}) \le C L_{n-\overline{m}}^{-(1+\frac{9}{10})\delta},$$

the combination of (135), (139), (145) and (146) results in the estimate, for $C > 0$ independent of n,

$$\left|u^\epsilon(x) - \overline{u}(x)\right|$$
$$\le C\|g\|_{L^\infty(\mathbb{R}^d)}\left(L_{n-\overline{m}}^{-(1+\frac{9}{10})\delta} + \frac{L_{n+1}}{L_{n+2}} + \left(\frac{L_{n-1}}{L_n}\right)^2 + L_n^{4a+2a^2-\frac{10a}{1+a}} + L_n^{21a-\frac{\delta}{2}} + \frac{\tilde{D}_{n-\overline{m}}}{L_n} + L_n^{-5a}\right)$$
$$+ C\|Dg\|_{L^\infty(\mathbb{R}^d)} L_n^{-5a}. \tag{147}$$

Since definitions (36), (37), (39) and (41) imply the right-hand side of (147) vanishes as n approaches infinity, since n approaches infinity and ϵ approaches zero, and since $\omega \in \Omega_0$ and $x \in \overline{U}$ were arbitrary, this completes the argument. □

It remains to extend Theorem 19 to a general continuous right-hand side, which follows from a standard extension argument.

$$\text{Assume } g \in \mathrm{C}(\overline{U}). \tag{148}$$

Notice that the approximation argument relies upon the result of Theorem 19 for $g = -1$. That is, it relies upon the fact that Theorem 19 already contains an almost sure control of the exit time in expectation.

Theorem 20. *Assume* (32), (52) *and* (148). *For every* $\omega \in \Omega_0$, *the respective solutions* u^ϵ *and* $\overline{u}$ *of* (72) *and* (73) *satisfy*

$$\lim_{\epsilon\to 0}\left\|u^\epsilon - \overline{u}\right\|_{L^\infty(\overline{U})} = 0.$$

Proof. Use the Tietze Extension Theorem, see for example Armstrong [1, Theorem 2.15], to construct an extension with compact support

$$\tilde{g} \in \mathrm{BUC}(\mathbb{R}^d) \quad \text{satisfying } \tilde{g}|_{\overline{U}} = g.$$

By convolution construct, for each $\delta > 0$, a $\tilde{g}^\delta \in \mathrm{C}_c^\infty(\mathbb{R}^d)$ such that

$$\left\|\tilde{g}^\delta - \tilde{g}\right\|_{L^\infty(\mathbb{R}^d)} \le \delta,$$

and write $u^{\epsilon,\delta}$ for the solution

$$\begin{cases} \frac{1}{2}\operatorname{tr}(A(\frac{x}{\epsilon},\omega)D^2u^{\epsilon,\delta})+\frac{1}{\epsilon}b(\frac{x}{\epsilon},\omega)\cdot Du^{\epsilon,\delta}=\tilde{g}^{\delta}(x) & \text{on } U,\\ u^{\epsilon,\delta}=0 & \text{on } \partial U.\end{cases}$$

Similarly, write $\overline{u}^{\delta}$ for the solution

$$\begin{cases} \frac{\overline{\alpha}}{2}\Delta\overline{u}^{\delta}=\tilde{g}^{\delta}(x) & \text{on } U,\\ \overline{u}^{\delta}=0 & \text{on } \partial U.\end{cases}$$

The representation formula for the solutions, the comparison principle and the triangle inequality imply that, for the exit time τ from U, for each $\omega\in\Omega_0$, $\delta>0$, and $\epsilon>0$,

$$\begin{aligned}\|u^{\epsilon}-\overline{u}\|_{L^\infty(\overline{U})} &\le \|u^{\epsilon}-u^{\epsilon,\delta}\|_{L^\infty(\overline{U})}+\|u^{\epsilon,\delta}-\overline{u}^{\delta}\|_{L^\infty(\overline{U})}+\|\overline{u}^{\delta}-\overline{u}\|_{L^\infty(\overline{U})}\\ &\le \delta\sup_{x\in\overline{U}}E_{\frac{x}{\epsilon},\omega}\big(\epsilon^2\tau^{\epsilon}\big)+\delta\sup_{x\in\overline{U}}E^{W_x^\infty}(\tau)+\|u^{\epsilon,\delta}-\overline{u}^{\delta}\|_{L^\infty(\overline{U})}.\end{aligned}$$

Therefore, since Theorem 19 implies that

$$\lim_{\epsilon\to 0}\Big(\sup_{x\in\overline{U}}E_{\frac{x}{\epsilon},\omega}\big(\epsilon^2\tau^{\epsilon}\big)\Big)=\sup_{x\in\overline{U}}E^{W_x^\infty}(\tau),$$

and because $\tilde{g}^{\delta}$ satisfies the conditions of Theorem 19, for every $\omega\in\Omega_0$ and $\delta>0$,

$$\limsup_{\epsilon\to 0}\|u^{\epsilon}-\overline{u}\|_{L^\infty(\overline{U})}\le 2\delta\sup_{x\in\overline{U}}E^{W_x^\infty}(\tau)+\limsup_{\epsilon\to 0}\|u^{\epsilon,\delta}-\overline{u}^{\delta}\|_{L^\infty(\overline{U})}=2\delta\sup_{x\in\overline{U}}E^{W_x^\infty}(\tau).$$

Since $\delta>0$ is arbitrary, this completes the argument. $\square$

The general homogenization statement for nonzero boundary data is now presented, after recalling the result of [8]. The purpose will be to show that, on the event Ω_0, solutions

$$\begin{cases} \frac{1}{2}\operatorname{tr}(A(\frac{x}{\epsilon},\omega)D^2u^{\epsilon})+\frac{1}{\epsilon}b(\frac{x}{\epsilon},\omega)\cdot Du^{\epsilon}=g(x) & \text{on } U,\\ u^{\epsilon}=f(x) & \text{on } \partial U,\end{cases} \tag{149}$$

converge uniformly, as $\epsilon\to 0$, to the solution

$$\begin{cases} \frac{\overline{\alpha}}{2}\Delta\overline{u}=g(x) & \text{on } U,\\ \overline{u}=f(x) & \text{on } \partial U,\end{cases} \tag{150}$$

whenever the right-hand side and boundary data are continuous.

$$\text{Assume } g\in \mathrm{C}(\overline{U}) \text{ and } f\in \mathrm{C}(\partial U). \tag{151}$$

Notice that, in the case $g=0$, the variance $\overline{\alpha}$ does not effect the exit distribution because it reflects only a time change of the underlying Brownian motion. Or, in terms of the equation, for each $n\ge 0$, the solution to the approximate homogenized problem

$$\begin{cases} \frac{\alpha_n}{2}\Delta\overline{u}_n=g(x) & \text{on } U,\\ \overline{u}_n=f(x) & \text{on } \partial U,\end{cases} \tag{152}$$

satisfies (150).

Proposition 21. *Assume* (32), (52), *and* $g = 0$. *For each* $n \geq 0$, *for* $\overline{u}$ *and* $\overline{u}_n$ *the respective solutions of* (149) *and* (152),

$$\overline{u} = \overline{u}_n \quad on\ \overline{U}.$$

The following theorem is then an immediate consequence of [8, Theorem 7.5], since the event on which the statement was obtained in [8] contains the Ω_0 defined in (87) as a subset.

Theorem 22. *Assume* (32), (52), (151) *and* $g = 0$. *For every* $\omega \in \Omega_0$, *the respective solutions* u^ϵ *and* $\overline{u}$ *of* (149) *and* (150) *satisfy*

$$\lim_{\epsilon \to 0} \left\| u^\epsilon - \overline{u} \right\|_{L^\infty(\overline{U})} = 0.$$

The final theorem is then an immediate consequence of Theorem 20, Theorem 22, and linearity.

Theorem 23. *Assume* (32), (52), (151). *For every* $\omega \in \Omega_0$, *the respective solutions* u^ϵ *and* $\overline{u}$ *of* (149) *and* (150) *satisfy*

$$\lim_{\epsilon \to 0} \left\| u^\epsilon - \overline{u} \right\|_{L^\infty(\overline{U})} = 0.$$

6. The rate of homogenization

An algebraic rate for the convergence established in Theorem 23 is now obtained. The result will be shown first for boundary data which is the restriction of a bounded, uniformly continuous function and interior data which is the restriction of a bounded, Lipschitz function.

$$\text{Assume } f \in \mathrm{BUC}(\mathbb{R}^d) \text{ and } g \in \mathrm{Lip}(\mathbb{R}^d). \tag{153}$$

The moduli of continuity will be denoted σ_f and Dg. Namely, for each $x, y \in \mathbb{R}^d$,

$$\left| f(x) - f(y) \right| \leq \sigma_f\big(|x - y|\big) \quad \text{and} \quad \left| g(x) - g(y) \right| \leq \| Dg \|_{L^\infty(\mathbb{R}^d)} |x - y|. \tag{154}$$

A rate for the convergence in the case $g = 0$ was established in [8, Theorem 8.1].

Theorem 24. *Assume* (32), (52), (153) *and* $g = 0$. *There exists* $C > 0$ *and* $c_1, c_2 > 0$ *such that, for every* $\omega \in \Omega_0$, *for all* $\epsilon > 0$ *sufficiently small depending on* ω, *the respective solutions* u^ϵ *and* $\overline{u}$ *of* (149) *and* (150) *satisfy*

$$\left\| u^\epsilon - \overline{u} \right\|_{L^\infty(\overline{U})} \leq C \| f \|_{L^\infty(\mathbb{R}^d)} \epsilon^{c_1} + C \sigma_f\big(\epsilon^{c_2}\big).$$

The following establishes a similar result in the case $f = 0$, and follows quickly from the analysis carried out in Theorem 19.

Theorem 25. *Assume* (32), (52), (153) *and* $f = 0$. *There exists* $C > 0$ *and* $c_3, c_4 > 0$ *such that, for every* $\omega \in \Omega_0$, *for all* $\epsilon > 0$ *sufficiently small depending on* ω, *the respective solutions* u^ϵ *and* $\overline{u}$ *of* (149) *and* (150) *satisfy*

$$\left\| u^\epsilon - \overline{u} \right\|_{L^\infty(\overline{U})} \leq C \| g \|_{L^\infty(\mathbb{R}^d)} \epsilon^{c_3} + C \| Dg \|_{L^\infty(\mathbb{R}^d)} \epsilon^{c_4}.$$

Proof. Fix $\omega \in \Omega_0$ and $n_1 \geq \overline{m}$ such that, for all $n \geq n_1$ and for $r_0 > 0$ the constant quantifying the exterior ball condition,

$$2\tilde{D}_{n-\overline{m}} < \frac{r_0 L_n}{2} \quad \text{and} \quad \omega \in A_n.$$

Then, fix $\epsilon_1 > 0$ sufficiently small so that, whenever $0 < \epsilon < \epsilon_1$ satisfies $L_n \leq \frac{1}{\epsilon} < L_{n+1}$ it follows that $n \geq n_1$. Furthermore, using the boundedness of the domain U, choose $0 < \epsilon_2 < \epsilon_1$ such that, whenever $0 < \epsilon < \epsilon_2$ and $L_n \leq \frac{1}{\epsilon} < L_{n+1}$,

$$L_{n+1}U \subset \left[-\frac{1}{2}L_{n+2}^2, \frac{1}{2}L_{n+2}^2\right]^d.$$

These conditions guarantee, whenever $0 < \epsilon < \epsilon_2$, the conclusion of line (147) of Theorem 19.

Precisely, for every $0 < \epsilon < \epsilon_2$ satisfying $L_n \leq \frac{1}{\epsilon} < L_{n+1}$, for $C > 0$ independent of n,

$$\begin{aligned}
&\left\|u^\epsilon(x) - \overline{u}(x)\right\|_{L^\infty(\overline{U})} \\
&\quad \leq C\|g\|_{L^\infty(\mathbb{R}^d)}\left(L_{n-\overline{m}}^{-(1+\frac{9}{10})\delta} + \frac{L_{n+1}}{L_{n+2}} + \left(\frac{L_{n-1}}{L_n}\right)^2 + L_n^{4a+2a^2-\frac{10a}{1+a}} + L_n^{21a-\frac{\delta}{2}} + \frac{\tilde{D}_{n-\overline{m}}}{L_n} + L_n^{-5a}\right) \\
&\qquad + C\|Dg\|_{L^\infty(\mathbb{R}^d)}L_n^{-5a}.
\end{aligned} \tag{155}$$

The definitions (36), (37), (39) imply that, since $L_n \leq \frac{1}{\epsilon} < L_{n+1}$, there exists $c_3 > 0$ satisfying, for $C > 0$ independent of n,

$$L_{n-\overline{m}}^{-(1+\frac{9}{10})\delta} + \frac{L_{n+1}}{L_{n+2}} + \left(\frac{L_{n-1}}{L_n}\right)^2 + L_n^{4a+2a^2-\frac{10a}{1+a}} + L_n^{21a-\frac{\delta}{2}} + \frac{\tilde{D}_{n-\overline{m}}}{L_n} + L_n^{-5a} \leq C\epsilon^{c_3}. \tag{156}$$

Definitions (36) and (37) then imply, using $L_n \leq \frac{1}{\epsilon} < L_{n+1}$, the existence of $c_4 > 0$ satisfying, for $C > 0$ independent of n,

$$L_n^{-5a} \leq C\epsilon^{c_4}. \tag{157}$$

Therefore, in combination (155), (156) and (157) yield, for all $0 < \epsilon < \epsilon_2$, for $C > 0$ independent of $0 < \epsilon < \epsilon_2$,

$$\left\|u^\epsilon(x) - \overline{u}(x)\right\|_{L^\infty(\overline{U})} \leq C\|g\|_{L^\infty(\mathbb{R}^d)}\epsilon^{c_3} + C\|Dg\|_{L^\infty(\mathbb{R}^d)}\epsilon^{c_4},$$

which completes the argument. □

The following statement establishes an algebraic rate of convergence for boundary and interior data which are respectively the restrictions of a bounded, uniformly continuous function and a bounded, Lipschitz function. This requirement is removed for smooth domains in Theorem 27. The proof follows immediately from Theorem 24, Theorem 25, and linearity.

Theorem 26. *Assume* (32), (52) *and* (153). *There exists $C > 0$ and $c_1, c_2, c_3, c_4 > 0$ such that, for every $\omega \in \Omega_0$, for all $\epsilon > 0$ sufficiently small depending on ω, the respective solutions u^ϵ and $\overline{u}$ of* (149) *and* (150) *satisfy*

$$\left\|u^\epsilon - \overline{u}\right\|_{L^\infty(\overline{U})} \leq C\left(\|f\|_{L^\infty(\mathbb{R}^d)}\epsilon^{c_1} + \sigma_f\left(\epsilon^{c_2}\right) + \|g\|_{L^\infty(\mathbb{R}^d)}\epsilon^{c_3} + \|Dg\|_{L^\infty(\mathbb{R}^d)}\epsilon^{c_4}\right).$$

Theorem 26 is now extended to general smooth domains up to a domain dependent factor. Observe that, in the case of the ball $U = B_r$, it follows by an explicit radial extension or, in the case that the domain U is smooth, it follows from the Product Neighborhood Theorem, see Milnor [12, Page 46], that every continuous function $f \in \mathrm{C}(\partial U)$ and Lipschitz function $g \in \mathrm{Lip}(\overline{U})$ admit extensions

$$\tilde{f} \in \mathrm{BUC}\left(\mathbb{R}^d\right) \quad \text{and} \quad \tilde{g} \in \mathrm{Lip}\left(\mathbb{R}^d\right)$$

satisfying, for a constant $C = C(U)$ depending only upon the domain,

$$\sigma_{\tilde{f}}(s) \leq \sigma_f(Cs) \quad \text{for all } s \geq 0 \text{ sufficiently small}$$

and

$$|D\tilde{g}| \leq C|Dg| \quad \text{in a neighborhood of } U.$$

That is, for smooth domains, assumption (153) can always be achieved up to a domain dependent factor.

$$\text{Assume } f \in \mathrm{C}(\partial U), g \in \mathrm{Lip}(\overline{U}), \text{ and that the domain } U \text{ is smooth.} \tag{158}$$

The following statement is then an immediate consequence of Theorem 26 and the preceding remarks.

Theorem 27. *Assume* (32), (52) *and* (158). *There exists* $C > 0$, $c_1, c_2, c_3, c_4 > 0$ *and* $C_1 = C_1(U) > 0$ *such that, for every* $\omega \in \Omega_0$, *for all* $\epsilon > 0$ *sufficiently small depending on* ω, *the respective solutions* u^ϵ *and* $\overline{u}$ *of* (149) *and* (150) *satisfy*

$$\left\|u^\epsilon - \overline{u}\right\|_{L^\infty(\overline{U})} \leq C\left(\|f\|_{L^\infty(\partial U)}\epsilon^{c_1} + \sigma_f\left(C_1\epsilon^{c_2}\right) + \|g\|_{L^\infty(\overline{U})}\epsilon^{c_3} + C_1\|Dg\|_{L^\infty(\overline{U})}\epsilon^{c_4}\right).$$

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