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Functional limit theorem for the self-intersection local time of the fractional Brownian motion

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Abstract. Let $\{B_t\}_{t\geq 0}$ be a d-dimensional fractional Brownian motion with Hurst parameter $0<H<1$, where $d\geq 2$. Consider the approximation of the self-intersection local time of B, defined as

$$I_T^\varepsilon = \int_0^T \int_0^t p_\varepsilon(B_t - B_s)\,ds\,dt,$$

where $p_\varepsilon(x)$ is the heat kernel. We prove that the process $\{I_T^\varepsilon - \mathbb{E}[I_T^\varepsilon]\}_{T\geq 0}$, rescaled by a suitable normalization, converges in law to a constant multiple of a standard Brownian motion for $\frac{3}{2d} < H \leq \frac{3}{4}$ and to a multiple of a sum of independent Hermite processes for $\frac{3}{4} < H < 1$, in the space $C[0,\infty)$, endowed with the topology of uniform convergence on compacts.

Résumé. Soit $\{B_t\}_{t\geq 0}$ un mouvement brownien fractionnaire d-dimensionel avec paramètre de Hurst $0<H<1$, où $d\geq 2$. On considère l'approximation du temps local d'auto-intersection du processus B, défini comme

$$I_T^\varepsilon = \int_0^T \int_0^t p_\varepsilon(B_t - B_s)\,ds\,dt,$$

où $p_\varepsilon(x)$ est le noyau de la chaleur. Nous démontrons que le processus $\{I_T^\varepsilon - \mathbb{E}[I_T^\varepsilon]\}_{T\geq 0}$, rééchelonné avec une normalisation convenable, converge en loi vers un mouvement brownien multiplié par une constante si $\frac{3}{2d} < H \leq \frac{3}{4}$ et vers une somme de processus de Hermite indépendants multipliée par une constante si $\frac{3}{4} < H < 1$, dans l'espace $C[0,\infty)$, muni de la topologie de la convergence uniforme sur les compacts.

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Keywords: Fractional Brownian motion; self-intersection local time; Wiener chaos expansion; central limit theorem

1. Introduction

Let $B=\{B_t\}_{t\geq 0}$ be a d-dimensional fractional Brownian motion of Hurst parameter $H\in(0,1)$. Fix $T>0$. The self-intersection local time of B in the interval $[0,T]$ is formally defined by

$$I := \int_0^T \int_0^t \delta(B_t - B_s)\,ds\,dt,$$

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where δ denotes the Dirac delta function. A rigorous definition of this random variable may be obtained by approximating the delta function by the heat kernel

$$p_\varepsilon(x) := (2\pi\varepsilon)^{-\frac{d}{2}} \exp\left\{-\frac{1}{2\varepsilon}\|x\|^2\right\}, \quad x \in \mathbb{R}^d.$$

In the case $H = \frac{1}{2}$, B is a classical Brownian motion, and its self-intersection local time has been studied by many authors (see Albeverio, Hu and Zhou [1], Hu [4], Imkeller, Pérez-Abreu and Vives [6], Varadhan [14], Yor [15] and the references therein). In the case $H \neq \frac{1}{2}$, the self-intersection local time for B was first studied by Rosen in [13] in the planar case and it was further investigated using techniques from Malliavin calculus by Hu and Nualart in [5]. In particular, it was proved that the approximation of the self-intersection local time of B in $[0, T]$, defined by

$$I_T^\varepsilon := \int_0^T \int_0^t p_\varepsilon(B_t - B_s)\, ds\, dt, \tag{1.1}$$

converges in $L^2(\Omega)$ when $H < \frac{1}{d}$. Furthermore, it was shown that when $\frac{1}{d} \leq H < \frac{3}{2d}$, $I_T^\varepsilon - \mathbb{E}[I_T^\varepsilon]$ to converges in $L^2(\Omega)$, and for the case $\frac{3}{2d} < H < \frac{3}{4}$, the following limit theorem holds (see [5, Theorem 2]).

Theorem 1.1. *If $\frac{3}{2d} < H < \frac{3}{4}$, then $\varepsilon^{\frac{d}{2}-\frac{3}{4H}}(I_T^\varepsilon - \mathbb{E}[I_T^\varepsilon])$ converges in law to a centered Gaussian distribution with variance $\sigma^2 T$, as $\varepsilon \to 0$, where the constant σ^2 is given by* (3.3).

The case $H = \frac{3}{2d}$ was addressed as well in [5], where it was shown that the sequence $(\log(1/\varepsilon))^{-\frac{1}{2}}(I_T^\varepsilon - \mathbb{E}[I_T^\varepsilon])$ converges in law to a centered Gaussian distribution with variance $\sigma_{\log}^2$, as $\varepsilon \to 0$, where $\sigma_{\log}^2$ is the constant given by [5, Equation (42)].

The aim of this paper is to prove a functional version of Theorem 1.1, and extend it to the case $\frac{3}{4} \leq H < 1$. Our main results are Theorems 1.2, 1.3 and 1.4.

Theorem 1.2. *Let $\frac{3}{2d} < H < \frac{3}{4}$, $d \geq 2$ be fixed. Then,*

$$\left\{\varepsilon^{\frac{d}{2}-\frac{3}{4H}}\left(I_T^\varepsilon - \mathbb{E}\left[I_T^\varepsilon\right]\right)\right\}_{T\geq 0} \overset{\text{Law}}{\to} \{\sigma W_T\}_{T\geq 0}, \tag{1.2}$$

in the space $C[0,\infty)$, endowed with the topology of uniform convergence on compact sets, where W is a standard Brownian motion, and the constant σ^2 is given by (3.3).

We briefly outline the proof of (1.2). The proof of the convergence of the finite-dimensional distributions, is based on the application of a multivariate central limit theorem established by Peccati and Tudor in [12] (see Section 2.3), and follows ideas similar to those presented in [5]. On the other hand, proving the tightness property for the process

$$\widetilde{I}_T^\varepsilon := \varepsilon^{\frac{d}{2}-\frac{3}{4H}}\left(I_T^\varepsilon - \mathbb{E}\left[I_T^\varepsilon\right]\right),$$

presents a great technical difficulty. In fact, by the Billingsley criterion (see [2, Theorem 12.3]), the tightness property can be obtained by showing that there exists $p > 2$, such that for every $0 \leq T_1 \leq T_2$,

$$\mathbb{E}\left[\left|\widetilde{I}_{T_2}^\varepsilon - \widetilde{I}_{T_1}^\varepsilon\right|^p\right] \leq C|T_2 - T_1|^{\frac{p}{2}}, \tag{1.3}$$

for some constant $C > 0$ independent of T_1, T_2 and ε. The problem of finding a bound like (1.3) comes from the fact that the smallest even integer such that $p > 2$ is $p = 4$, and a direct computation of the moment of order four $\mathbb{E}[|\widetilde{I}_{T_2}^\varepsilon - \widetilde{I}_{T_1}^\varepsilon|^4]$ is too complicated to be handled. To overcome this difficulty, in this paper we introduce a new approach to prove tightness based on the techniques of Malliavin calculus. Let us describe the main ingredients of this approach.

First, we write the centered random variable $Z := \widetilde{I}_{T_2}^\varepsilon - \widetilde{I}_{T_1}^\varepsilon$ as

$$Z = -\delta D L^{-1} Z,$$

where δ, D and L are the basic operators in Malliavin calculus. Then, taking into consideration that $\mathbb{E}[DL^{-1}Z]=0$ we apply Meyer's inequalities to obtain a bound of the type

$$\|Z\|_{L^p(\Omega)} \le c_p \big\| D^2 L^{-1} Z \big\|_{L^p(\Omega;(\mathfrak{H}^d)^{\otimes 2})}, \tag{1.4}$$

for any $p>1$, where the Hilbert space $\mathfrak{H}$ is defined in Section 2.1. Notice that

$$Z = \varepsilon^{\frac{d}{2}-\frac{3}{4H}} \int_{0\le s\le t, T_1\le t\le T_2} \big(p_\varepsilon(B_t-B_s) - \mathbb{E}\big[p_\varepsilon(B_t-B_s)\big]\big)\, ds\, dt.$$

Applying Minkowski's inequality and (1.4), we obtain

$$\|Z\|_{L^p(\Omega)} \le c_p \varepsilon^{\frac{d}{2}-\frac{3}{4H}} \int_{0\le s\le t, T_1\le t\le T_2} \big\| D^2 L^{-1} p_\varepsilon(B_t-B_s)\big\|_p\, ds\, dt.$$

Then, we get the desired estimate by choosing $p>2$ close to 2, using the self-similarity of the fractional Brownian motion, the expression of the operator L^{-1} in terms of the Ornstein–Uhlenbeck semigroup, Mehler's formula and Gaussian computations. In this way, we reduce the problem to showing the finiteness of an integral (see Lemma 5.3), similar to the integral appearing in the proof of the convergence of the variances. It is worth mentioning that this approach for proving tightness has not been used before, and has its own interest.

In the case $H>\frac{3}{4}$, the process $\varepsilon^{\frac{d}{2}-\frac{3}{2H}+1}(I_T^\varepsilon - \mathbb{E}[I_T^\varepsilon])$ also converges in law, in the topology of $C[0,\infty)$, but the limit is no longer a multiple of a Brownian motion, but a multiple of a sum of independent Hermite processes of order two. More precisely, if $\{X_T^j\}_{T\ge 0}$ denotes the second order Hermite process, with respect to $\{B_t^{(j)}\}_{t\ge 0}$, defined in Section 2.1, then $\{\widetilde{I}^\varepsilon\}_{\varepsilon\in(0,1)}$ satisfies the following limit theorem

Theorem 1.3. *Let $H>\frac{3}{4}$, and $d\ge 2$ be fixed. Then, for every $T>0$,*

$$\varepsilon^{\frac{d}{2}-\frac{3}{2H}+1}\big(I_T^\varepsilon - \mathbb{E}\big[I_T^\varepsilon\big]\big) \overset{L^2(\Omega)}{\to} -\Lambda \sum_{j=1}^d X_T^j, \tag{1.5}$$

where the constant Λ is defined by

$$\Lambda := \frac{(2\pi)^{-\frac{d}{2}}}{2} \int_0^\infty \big(1+u^{2H}\big)^{-\frac{d}{2}-1} u^2\, du. \tag{1.6}$$

In addition,

$$\big\{\varepsilon^{\frac{d}{2}-\frac{3}{2H}+1}\big(I_T^\varepsilon - \mathbb{E}\big[I_T^\varepsilon\big]\big)\big\}_{T\ge 0} \overset{\text{Law}}{\to} \left\{-\Lambda \sum_{j=1}^d X_T^j\right\}_{T\ge 0}, \tag{1.7}$$

in the space $C[0,\infty)$, endowed with the topology of uniform convergence on compact sets.

We briefly outline the proof of Theorem 1.3. The convergence (1.5) is obtained from the chaotic decomposition of I_T^ε. It turns out that the chaos of order two completely determines the asymptotic behavior of $\varepsilon^{\frac{d}{2}-\frac{3}{2H}+1}(I_T^\varepsilon - \mathbb{E}[I_T^\varepsilon])$, and consequently, (1.5) can be obtained by the characterization of the Hermite processes presented in [8], applied to the second chaotic component of I_T^ε. Similarly to the case $\frac{3}{2d}<H<\frac{3}{4}$, we show that the sequence $\varepsilon^{\frac{d}{2}-\frac{3}{2H}+1}(I_T^\varepsilon - \mathbb{E}[I_T^\varepsilon])$ is tight, which proves the convergence in law (1.7).

The technique we use to prove tightness doesn't work for the case $Hd\le\frac{3}{2}$, so the convergence in law of $\{\log(1/\varepsilon)^{-\frac{1}{2}}(I_T^\varepsilon - \mathbb{E}[I_T^\varepsilon])\}_{T\ge 0}$ to a scalar multiple of a Brownian motion for the case $Hd=\frac{3}{2}$ still remains open. Nevertheless, for the critical case $H=\frac{3}{4}$ and $d\ge 3$, the technique does work, and we prove the following limit theorem

Theorem 1.4. *Suppose $H = \frac{3}{4}$ and $d \geq 3$. Then,*

$$\left\{ \frac{\varepsilon^{\frac{d}{2}-1}}{\sqrt{\log(1/\varepsilon)}} \left(I_T^\varepsilon - \mathbb{E}\left[I_T^\varepsilon \right] \right) \right\}_{T \geq 0} \overset{\text{Law}}{\to} \{\rho W_T\}_{T\geq 0}, \tag{1.8}$$

in the space $C[0,\infty)$, endowed with the topology of uniform convergence on compact sets, where W is a standard Brownian motion, and the constant ρ is defined by (3.51).

Remark. We impose the stronger condition $d \geq 3$ instead of $d \geq 2$, since the choice $H = \frac{3}{4}$, $d = 2$ gives $Hd = \frac{3}{2}$, and as mentioned before, it is not clear how to prove tightness for this case.

We briefly outline the proof of Theorem 1.4. The proof of the tightness property is analogous to the case $\frac{3}{2d} < H < \frac{3}{4}$. On the other hand, the proof of the convergence of the finite dimensional distributions requires a new approach. First we show that, as in the case $H > \frac{3}{4}$, the chaos of order two determines the asymptotic behavior of $\{I_T^\varepsilon\}_{T\geq 0}$. Then we describe the behavior of the second chaotic component of I_T^ε, which is given by

$$-\frac{(2\pi)^{-\frac{d}{2}}\varepsilon^{\frac{2}{3}-\frac{d}{2}}}{2} \sum_{j=1}^{d} \int_0^T \int_0^{\varepsilon^{-\frac{2}{3}}(T-s)} \frac{u^{\frac{3}{2}}}{(1+u^{\frac{3}{2}})^{\frac{d}{2}+1}} H_2\left(\frac{B^{(j)}_{s+\varepsilon^{\frac{2}{3}}u} - B^{(j)}_s}{\sqrt{\varepsilon} u^{\frac{3}{4}}} \right) du\, ds, \tag{1.9}$$

where H_2 denotes the Hermite polynomial of order 2. Then we show that we can replace the domain of integration of u by $[0,\infty)$, and this integral can be approximated by Riemann sums of the type

$$-\frac{1}{2^M} \sum_{k=2}^{M2^M} \frac{u(k)^{\frac{3}{2}}}{(1+u(k)^{\frac{2}{3}})^{\frac{d}{2}+1}} \int_0^T H_2\left(\frac{B^{(j)}_{s+\varepsilon^{2^M}u(k)} - B^{(j)}_s}{\sqrt{\varepsilon} u(k)^{\frac{3}{4}}} \right) ds, \tag{1.10}$$

where $u(k) = \frac{k}{2^M}$, and M is some fixed positive number. By [3, Equation (1.4)], we have that, for k fixed, the random variable

$$\xi_k^\varepsilon(T) := \frac{\varepsilon^{-\frac{1}{3}}}{\sqrt{\log(1/\varepsilon)}} \int_0^T H_2\left(\frac{B^{(j)}_{s+\varepsilon^{\frac{2}{3}}u(k)} - B^{(j)}_s}{\sqrt{\varepsilon} u(k)^{\frac{3}{4}}} \right) ds$$

converges in law to a Gaussian distribution as $\varepsilon \to 0$. Hence, after a suitable analysis of the covariances of the process $\{\xi_k^\varepsilon(T) \mid 2 \leq k \leq M2^M, \text{ and } T \geq 0\}$ and an application of the Peccati–Tudor criterion (see [12]), we obtain that the process (1.10) multiplied by the factor $\frac{(2\pi)^{-\frac{d}{2}}\varepsilon^{-\frac{1}{3}}}{2\sqrt{\log(1/\varepsilon)}}$ converges to a constant multiple of a Brownian motion $\rho_M W$, for some $\rho_M > 0$. The result then follows by proving that the approximations (1.10) to the integral in (1.9), are uniform over $\varepsilon \in (0, 1/e)$ as $M \to \infty$, and that $\rho_M \to \rho$ as $M \to \infty$.

The paper is organized as follows. In Section 2 we present some preliminary results on the fractional Brownian motion and the chaotic decomposition of I_T^ε. In Section 3, we compute the asymptotic behavior of the variances of the chaotic components of I_T^ε as $\varepsilon \to 0$. The proofs of the main results are presented in Section 4. Finally, in Section 5 we prove some technical lemmas.

2. Preliminaries and main results

2.1. *Some elements of Malliavin calculus for the fractional Brownian motion*

Throughout the paper, $B = \{(B_t^{(1)}, \dots, B_t^{(d)})\}_{t\geq 0}$ will denote a d-dimensional fractional Brownian motion with Hurst parameter $H \in (0,1)$, defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. That is, B is a centered, $\mathbb{R}^d$-valued Gaussian process with covariance function

$$\mathbb{E}\big[B_t^{(i)} B_s^{(j)}\big] = \frac{\delta_{i,j}}{2}\big(t^{2H} + s^{2H} - |t-s|^{2H}\big).$$

We will denote by $\mathfrak{H}$ the Hilbert space obtained by taking the completion of the space of step functions on $[0,\infty)$, endowed with the inner product

$$\langle \mathbb{1}_{[a,b]}, \mathbb{1}_{[c,d]}\rangle_{\mathfrak{H}} := \mathbb{E}\big[\big(B_b^{(1)} - B_a^{(1)}\big)\big(B_d^{(1)} - B_c^{(1)}\big)\big], \quad \text{for } 0 \le a \le b, \text{ and } 0 \le c \le d.$$

For every $1 \le j \le d$ fixed, the mapping $\mathbb{1}_{[0,t]} \mapsto B_t^{(j)}$ can be extended to linear isometry between $\mathfrak{H}$ and the Gaussian subspace of $L^2(\Omega)$ generated by the process $B^{(j)}$. We will denote this isometry by $B^{(j)}(f)$, for $f \in \mathfrak{H}$. If $f \in \mathfrak{H}^d$ is of the form $f = (f_1, \dots, f_d)$, with $f_j \in \mathfrak{H}$, we set $B(f) := \sum_{j=1}^d B^{(j)}(f_j)$. Then $f \mapsto B(f)$ is a linear isometry between $\mathfrak{H}^d$ and the Gaussian subspace of $L^2(\Omega)$ generated by B.

For any integer $q \ge 1$, we denote by $(\mathfrak{H}^d)^{\otimes q}$ and $(\mathfrak{H}^d)^{\odot q}$ the qth tensor product of $\mathfrak{H}^d$, and the qth symmetric tensor product of $\mathfrak{H}^d$, respectively. The qth Wiener chaos of $L^2(\Omega)$, denoted by $\mathcal{H}_q$, is the closed subspace of $L^2(\Omega)$ generated by the variables

$$\left\{ \prod_{j=1}^d H_{q_j}\big(B^{(j)}(f_j)\big) \,\Big|\, \sum_{j=1}^d q_j = q, \text{ and } f_1, \dots, f_d \in \mathfrak{H}, \|f_j\|_{\mathfrak{H}} = 1 \right\},$$

where H_q is the qth Hermite polynomal, defined by

$$H_q(x) := (-1)^q e^{\frac{x^2}{2}} \frac{d^q}{dx^q} e^{-\frac{x^2}{2}}.$$

For $q \in \mathbb{N}$, with $q \ge 1$, and $f \in \mathfrak{H}^d$ of the form $f = (f_1, \dots, f_d)$, with $\|f_j\|_{\mathfrak{H}} = 1$, we can write

$$f^{\otimes q} = \sum_{i_1, \dots, i_q = 1}^d f_{i_1} \otimes \cdots \otimes f_{i_q}.$$

For such f, we define the mapping

$$I_q\big(f^{\otimes q}\big) := \sum_{i_1, \dots, i_q = 1}^d \prod_{j=1}^d H_{q_j(i_1, \dots, i_q)}\big(B^{(j)}(f_j)\big),$$

where $q_j(i_1, \dots, i_q)$ denotes the number of indices in $(i_1, \dots, i_q)$ equal to j. The range of I_q is contained in $\mathcal{H}_q$. Furthermore, this mapping can be extended to a linear isometry between $\mathfrak{H}^{\odot q}$ (equipped with the norm $\sqrt{q!}\|\cdot\|_{(\mathfrak{H}^d)^{\otimes q}}$) and $\mathcal{H}_q$ (equipped with the $L^2(\Omega)$-norm).

Denote by $\mathcal{G}$ the σ-algebra generated by B. It is well known that every square integrable random variable $\mathcal{G}$-measurable, has a chaos decomposition of the type

$$F = \mathbb{E}[F] + \sum_{q=1}^\infty I_q(f_q), \tag{2.1}$$

for some $f_q \in (\mathfrak{H}^d)^{\odot q}$.

Let $\mathcal{S}$ denote the set of all cylindrical random variables of the form

$$F = g\big(B(h_1), \dots, B(h_n)\big),$$

where $g : \mathbb{R}^n \to \mathbb{R}$ is an infinitely differentiable function with compact support, and $h_j \in \mathfrak{H}^d$. The Malliavin derivative of F with respect to B, is the element of $L^2(\Omega; \mathfrak{H}^d)$, defined by

$$DF = \sum_{i=1}^{n} \frac{\partial g}{\partial x_i}\big(B(h_1), \dots, B(h_n)\big)h_i.$$

By iteration, one can define the rth derivative D^r for every $r \geq 2$, which is an element of $L^2(\Omega; (\mathfrak{H}^d)^{\otimes r})$.

For $p \geq 1$ and $r \geq 1$, the space $\mathbb{D}^{r,p}$ denotes the closure of $\mathcal{S}$ with respect to the norm $\|\cdot\|_{\mathbb{D}^{r,p}}$, defined by

$$\|F\|_{\mathbb{D}^{r,p}} := \left(\mathbb{E}\big[|F|^p\big] + \sum_{i=1}^{r} \mathbb{E}\big[\big\|D^i F\big\|^p_{(\mathfrak{H}^d)^{\otimes i}}\big]\right)^{\frac{1}{p}}.$$

The operator D^r can be consistently extended to the space $\mathbb{D}^{r,p}$. We denote by δ the adjoint of the operator D, also called the divergence operator. A random element $u \in L^2(\Omega; \mathfrak{H}^d)$ belongs to the domain of δ, denoted by $\operatorname{Dom}\delta$, if and only if satisfies

$$\big|\mathbb{E}\big[\langle DF, u\rangle_{\mathfrak{H}^d}\big]\big| \leq C_u \mathbb{E}\big[F^2\big]^{\frac{1}{2}}, \quad \text{for every } F \in \mathbb{D}^{1,2},$$

where C_u is a constant only depending on u. If $u \in \operatorname{Dom}\delta$, then the random variable $\delta(u)$ is defined by the duality relationship

$$\mathbb{E}\big[F\delta(u)\big] = \mathbb{E}\big[\langle DF, u\rangle_{\mathfrak{H}^d}\big],$$

which holds for every $F \in \mathbb{D}^{1,2}$. The operator L is defined on a random variable F of the form (2.1), by

$$LF := \sum_{q=1}^{\infty} -q I_q(f_q),$$

provided the series converges in $L^2(\Omega)$. Then, L coincides with the infinitesimal generator of the Ornstein–Uhlenbeck semigroup $\{P_\theta\}_{\theta\geq 0}$, which is defined, for F of the form (2.1), by

$$P_\theta F := \sum_{q=0}^{\infty} e^{-q\theta} I_q(f_q).$$

A random variable F belongs to the domain of L if and only if $F \in \mathbb{D}^{1,2}$, and $DF \in \operatorname{Dom}\delta$, in which case

$$\delta DF = -LF.$$

We also define the operator L^{-1}, on F of the form (2.1), by

$$L^{-1}F = \sum_{q=1}^{\infty} -\frac{1}{q} I_q(f_q).$$

Notice that L^{-1} is a bounded operator and satisfies $LL^{-1}F = F - \mathbb{E}[F]$ for every $F \in L^2(\Omega)$, so that L^{-1} acts as a pseudo-inverse of L. The operator L^{-1} satisfies the following contraction property for every $F \in L^2(\Omega)$ with $\mathbb{E}[F] = 0$,

$$\mathbb{E}\big[\big\|DL^{-1}F\big\|^2_{\mathfrak{H}^d}\big] \leq \mathbb{E}\big[F^2\big].$$

In addition, by Meyer's inequalities (see [10, Proposition 1.5.8]), for every $p > 1$, there exists a constant $c_p > 0$ such that the following relation holds for every $F \in \mathbb{D}^{2,p}$, with $\mathbb{E}[F] = 0$

$$\|\delta(DL^{-1}F)\|_{L^p(\Omega)} \le c_p\big(\|D^2L^{-1}F\|_{L^p(\Omega;(\mathfrak{H}^d)^{\otimes 2})} + \|\mathbb{E}[DL^{-1}F]\|_{(\mathfrak{H})^d}\big). \tag{2.2}$$

Assume that $\widetilde{B}$ is an independent copy of B, and such that B, $\widetilde{B}$ are defined in the product space $(\Omega \times \widetilde{\Omega}, \mathcal{F} \otimes \widetilde{\mathcal{F}}, \mathbb{P} \otimes \widetilde{\mathbb{P}})$. Given a random variable $F \in L^2(\Omega)$, measurable with respect to the σ-algebra generated by B, we can write $F = \Psi_F(B)$, where Ψ_F is a measurable mapping from $\mathbb{R}^{\mathfrak{H}^d}$ to $\mathbb{R}$, determined $\mathbb{P}$-a.s. Then, for every $\theta \ge 0$ we have the Mehler formula

$$P_\theta F = \widetilde{\mathbb{E}}\big[\Psi_F\big(e^{-\theta}B + \sqrt{1 - e^{-2\theta}}\widetilde{B}\big)\big], \tag{2.3}$$

where $\widetilde{\mathbb{E}}$ denotes the expectation with respect to $\widetilde{\mathbb{P}}$. The operator L^{-1} can be expressed in terms of P_θ, as follows

$$L^{-1}F = \int_0^\infty P_\theta F\, d\theta, \quad \text{for } F \text{ such that } \mathbb{E}[F] = 0. \tag{2.4}$$

2.2. *Hermite process*

When $H > \frac{1}{2}$, the inner product in the space $\mathfrak{H}$ can be written, for every step functions φ, ϑ on $[0, \infty)$, as

$$\langle \varphi, \vartheta \rangle_{\mathfrak{H}} = H(2H-1)\int_{\mathbb{R}_+^2} \varphi(\xi)\vartheta(\nu)|\xi - \nu|^{2H-2}\, d\xi\, d\nu. \tag{2.5}$$

Following [8], we introduce the Hermite process $\{X_T^j\}_{T\ge 0}$ of order 2, associated to the jth component of B, $\{B_t^{(j)}\}_{t\ge 0}$, and describe some of its properties. The family of kernels $\{\varphi_{j,T}^{\varepsilon} \mid T \ge 0, \varepsilon \in (0,1)\} \subset (\mathfrak{H}^d)^{\otimes 2}$, defined, for every multi-index $\mathbf{i} = (i_1, i_2)$, $1 \le i_1, i_2 \le d$, by

$$\varphi_{j,T}^{\varepsilon}(\mathbf{i}, x_1, x_2) := \varepsilon^{-2}\int_0^T \delta_{j,i_1}\delta_{j,i_2}\mathbb{1}_{[s,s+\varepsilon]}(x_1)\mathbb{1}_{[s,s+\varepsilon]}(x_2)\, ds, \tag{2.6}$$

satisfies the following relation for every $H > \frac{3}{4}$, and $T \ge 0$

$$\lim_{\varepsilon,\eta\to 0}\langle \varphi_{j,T}^{\varepsilon}, \varphi_{j,T}^{\eta}\rangle_{(\mathfrak{H}^d)^{\otimes 2}} = H^2(2H-1)^2\int_{[0,T]^2}|s_1 - s_2|^{4H-4}\, d\vec{s} = c_H T^{4H-2}, \tag{2.7}$$

where $d\vec{s} := ds_1\, ds_2$ and $c_H := \frac{H^2(2H-1)}{4H-3}$. This implies that $\varphi_{j,T}^{\varepsilon}$ converges, as $\varepsilon \to 0$, to an element of $(\mathfrak{H}^d)^{\otimes 2}$, denoted by π_T^j. In particular, for every $K > 0$, $\|\varphi_{j,K}^{\varepsilon}\|_{(\mathfrak{H}^d)^{\otimes 2}}$ is bounded by some constant $C_{K,H}$, only depending on K and H. On the other hand, by (2.5) and (2.6), we deduce that for every $T \in [0, K]$, it holds $\|\varphi_{j,T}^{\varepsilon}\|_{(\mathfrak{H}^d)^{\otimes 2}} \le \|\varphi_{j,K}^{\varepsilon}\|_{(\mathfrak{H}^d)^{\otimes 2}}$, and hence

$$\begin{aligned}\sup_{\substack{T_1,T_2\in(0,K]\\ \varepsilon,\eta\in(0,1)}}\big|\langle \varphi_{j,T_1}^{\varepsilon}, \varphi_{j,T_2}^{\eta}\rangle_{(\mathfrak{H}^d)^{\otimes 2}}\big| &\le \sup_{\substack{T_1,T_2\in(0,K]\\ \varepsilon,\eta\in(0,1)}}\|\varphi_{j,T_1}^{\varepsilon}\|_{(\mathfrak{H}^d)^{\otimes 2}}\|\varphi_{j,T_2}^{\eta}\|_{(\mathfrak{H}^d)^{\otimes 2}} \\ &\le \sup_{\varepsilon\in(0,1)}\|\varphi_{j,K}^{\varepsilon}\|_{(\mathfrak{H}^d)^{\otimes 2}}^2 \le C_{K,H}.\end{aligned} \tag{2.8}$$

The element π_T^j, can be characterized as follows. For any vector of step functions with compact support $f_i = (f_i^{(1)}, \dots, f_i^{(d)}) \in \mathfrak{H}^d$, $i = 1, 2$, we have

$$\langle \pi_t^j, f_1 \otimes f_2\rangle_{(\mathfrak{H}^d)^{\otimes 2}} = \lim_{\varepsilon\to 0}\varepsilon^{-2}H^2(2H-1)^2\int_0^T\prod_{i=1,2}\int_s^{s+\varepsilon}\int_0^T|\xi - \eta|^{2H-2}f_i^{(j)}(\eta)\, d\eta\, d\xi\, ds,$$

and hence

$$\langle \pi_t^j, f_1 \otimes f_2 \rangle_{(\mathfrak{H}^d)^{\otimes 2}} = H^2(2H-1)^2 \int_0^T \prod_{i=1,2} \int_0^T |s-\eta|^{2H-2} f_i^{(j)}(\eta)\, d\eta\, ds. \tag{2.9}$$

We define the second order Hermite process $\{X_T^j\}_{T\geq 0}$, with respect to $\{B_t^{(j)}\}_{t\geq 0}$, as $X_T^j := I_2(\pi_T^j)$.

2.3. *A multivariate central limit theorem*

In the seminal paper [11], Nualart and Peccati established a central limit theorem for sequences of multiple stochastic integrals of a fixed order. In this context, assuming that the variances converge, convergence in distribution to a centered Gaussian law is actually equivalent to convergence of just the fourth moment. Shortly afterwards, in [12], Peccati and Tudor gave a multidimensional version of this characterization. More recent developments on these type of results have been addressed by using Stein's method and Malliavin techniques (see the monograph by Nourdin and Peccati [9] and the references therein). In the sequel, we will use the following multivariate central limit theorem obtained by Peccati and Tudor in [12] (see also Theorems 6.2.3 and 6.3.1 in [9]).

Theorem 2.1. *For $r \in \mathbb{N}$ fixed, consider a sequence $\{F_n\}_{n\geq 1}$ of random vectors of the form $F_n = (F_n^{(1)}, \dots, F_n^{(r)})$. Suppose that for $i = 1, \dots, r$ and $n \in \mathbb{N}$, the random variables $F_n^{(i)}$ belong to $L^2(\Omega)$, and have chaos decomposition*

$$F_n^{(i)} = \sum_{q=1}^{\infty} I_q(f_{q,i,n}),$$

for some $f_{q,i,n} \in (\mathfrak{H}^d)^{\otimes q}$. Suppose, in addition, that for every $q \geq 1$, there is a real symmetric non negative definite matrix $C_q = \{C_q^{i,j} \mid 1 \leq i, j \leq r\}$, such that the following conditions hold:

(i) *For every fixed $q \geq 1$, and $1 \leq i, j \leq r$, we have $q!\langle f_{q,i,n}, f_{q,j,n}\rangle_{(\mathfrak{H}^d)^{\otimes q}} \to C_q^{i,j}$ as $n \to \infty$.*
(ii) *There exists a real symmetric nonnegative definite matrix $C = \{C^{i,j} \mid 1 \leq i, j \leq r\}$, such that $C^{i,j} = \lim_{Q\to\infty} \sum_{q=1}^{Q} C_q^{i,j}$.*
(iii) *For all $q \geq 1$ and $i = 1, \dots, r$, the sequence $\{I_q(f_{q,i,n})\}_{n\geq 1}$ converges in law to a centered Gaussian distribution as $n \to \infty$.*
(iv) $\lim_{Q\to\infty} \sup_{n\geq 1} \sum_{q=Q}^{\infty} q!\|f_{q,i,n}\|^2_{(\mathfrak{H}^d)^{\otimes q}} = 0$, *for all $i = 1, \dots, r$.*

Then, F_n converges in law as $n \to \infty$, to a centered Gaussian vector with covariance matrix C.

2.4. *Chaos decomposition for the self-intersection local time*

In this section we describe the chaos decomposition of the variable I_T^ε defined by (1.1). Let $\varepsilon \in (0,1)$, and $T \geq 0$ be fixed. Define the set

$$\mathcal{R} := \left\{(s,t) \in \mathbb{R}_+^2 \mid s \leq t \leq 1\right\}.$$

For every $\gamma > 0$, we will denote by $\gamma\mathcal{R}$ the set $\gamma\mathcal{R} := \{\gamma v \mid v \in \mathcal{R}\}$. First we write

$$I_T^\varepsilon = \int_{\mathbb{R}_+^2} \mathbb{1}_{T\mathcal{R}}(s,t) p_\varepsilon(B_t - B_s)\, ds\, dt. \tag{2.10}$$

We can determine the chaos decomposition of the random variable $p_\varepsilon(B_t - B_s)$ appearing in (2.10) as follows. Given a multi-index $\mathbf{i}_n = (i_1, \dots, i_n)$, $n \in \mathbb{N}$, $1 \leq i_j \leq d$, we set

$$\alpha(\mathbf{i}_n) := \mathbb{E}[X_{i_1} \cdots X_{i_n}],$$

where the X_i are independent standard Gaussian random variables. Notice that

$$\alpha(\mathbf{i}_{2q}) = \frac{(2q_1)!\cdots(2q_d)!}{(q_1)!\cdots(q_d)!2^q}, \tag{2.11}$$

if $n=2q$ is even and for each $k=1,\dots,d$, the number of components of $\mathbf{i}_{2q}$ equal to k, denoted by $2q_k$, is also even, and $\alpha(\mathbf{i}_n)=0$ otherwise. Proceeding as in [5, Lemma 7], we can prove that

$$p_\varepsilon(B_t-B_s) = \mathbb{E}\big[p_\varepsilon(B_t-B_s)\big] + \sum_{q=1}^{\infty} I_{2q}\big(f^{\varepsilon}_{2q,s,t}\big), \tag{2.12}$$

where $f^{\varepsilon}_{2q,s,t}$ is the element of $(\mathfrak{H}^d)^{\otimes 2q}$, given by

$$f^{\varepsilon}_{2q,s,t}(\mathbf{i}_{2q}, x_1,\dots,x_{2q}) := (-1)^q \frac{(2\pi)^{-\frac{d}{2}}\alpha(\mathbf{i}_{2q})}{(2q)!}\big(\varepsilon+(t-s)^{2H}\big)^{-\frac{d}{2}-q}\prod_{j=1}^{2q}\mathbb{1}_{[s,t]}(x_j), \tag{2.13}$$

and

$$\mathbb{E}\big[p_\varepsilon(B_t-B_s)\big] = (2\pi)^{-\frac{d}{2}}\big(\varepsilon+(t-s)^{2H}\big)^{-\frac{d}{2}}. \tag{2.14}$$

By (2.10), (2.12) and (2.14), it follows that the random variable I^{ε}_T has the chaos decomposition

$$I^{\varepsilon}_T = \mathbb{E}\big[I^{\varepsilon}_T\big] + \sum_{q=1}^{\infty} I_{2q}\big(h^{\varepsilon}_{2q,T}\big), \tag{2.15}$$

where

$$h^{\varepsilon}_{2q,T}(\mathbf{i}_{2q}, x_1,\dots,x_{2q}) := \int_{\mathbb{R}^2_+} \mathbb{1}_{T\mathcal{R}}(s,t) f^{\varepsilon}_{2q,s,t}(\mathbf{i}_{2q}, x_1,\dots,x_{2q})\,ds\,dt, \tag{2.16}$$

and

$$\mathbb{E}\big[I^{\varepsilon}_T\big] = (2\pi)^{-\frac{d}{2}}\int_{\mathbb{R}^2_+} \mathbb{1}_{T\mathcal{R}}(s,t)\big(\varepsilon+(t-s)^{2H}\big)^{-\frac{d}{2}}\,ds\,dt. \tag{2.17}$$

In Section 3, we will describe the behavior as $\varepsilon\to 0$ of the covariance function of the processes $\{I^{\varepsilon}_T\}_{T\ge 0}$ and $\{I_{2q}(h^{\varepsilon}_{2q,T})\}_{T\ge 0}$. In order to address this problem, we will first introduce some notation that will help us to describe the covariance function of the variables $p_\varepsilon(B_t-B_s)$ and its chaotic components, which ultimately will lead to an expresion for the covariance function of I^{ε}_T.

First we describe the inner product $\langle f^{\varepsilon}_{2q,s_1,t_1}, f^{\varepsilon}_{2q,s_2,t_2}\rangle_{(\mathfrak{H}^d)^{\otimes 2q}}$. From (2.13), we can prove that for every $0\le s_1\le t_1$ and $0\le s_2\le t_2$,

$$\begin{aligned}\big\langle f^{\varepsilon}_{2q,s_1,t_1}, f^{\varepsilon}_{2q,s_2,t_2}\big\rangle_{(\mathfrak{H}^d)^{\otimes 2q}} &= \sum_{q_1+\cdots+q_d=q}(2q_1,\dots,2q_d)!\frac{(2\pi)^{-d}\alpha(\mathbf{i}_{2q})^2}{((2q)!)^2}\big(\varepsilon+(t_1-s_1)^{2H}\big)^{-\frac{d}{2}-q}\\ &\quad\times\big(\varepsilon+(t_2-s_2)^{2H}\big)^{-\frac{d}{2}-q}\big\langle\mathbb{1}^{\otimes 2q}_{[s_1,t_1]},\mathbb{1}^{\otimes 2q}_{[s_2,t_2]}\big\rangle_{\mathfrak{H}^{\otimes 2q}},\end{aligned} \tag{2.18}$$

where $(2q_1,\dots,2q_d)!$ denotes the multinomial coefficient $(2q_1,\dots,2q_d)! = \frac{(2q)!}{(2q_1)!\cdots(2q_d)!}$. To compute the term $\langle\mathbb{1}^{\otimes 2q}_{[s_1,t_1]},\mathbb{1}^{\otimes 2q}_{[s_2,t_2]}\rangle_{\mathfrak{H}^{\otimes 2q}}$ appearing in the previous expression, we will introduce the following notation. For every $x,u_1,u_2>0$, define

$$\mu(x,u_1,u_2) := \mathbb{E}\big[B^{(1)}_{u_1}\big(B^{(1)}_{x+u_2}-B^{(1)}_x\big)\big]. \tag{2.19}$$

Define as well $\mu(x,u_1,u_2)$, for $x<0$, by $\mu(x,u_1,u_2):=\mu(-x,u_2,u_1)$. Using the property of stationary increments of B, we can check that for every $s_1,s_2,t_1,t_2\geq 0$, such that $s_1\leq t_1$ and $s_2\leq t_2$, it holds

$$\mathbb{E}\big[\big(B^{(1)}_{t_1}-B^{(1)}_{s_1}\big)\big(B^{(1)}_{t_2}-B^{(1)}_{s_2}\big)\big]=\mu(s_2-s_1,t_1-s_1,t_2-s_2). \tag{2.20}$$

As a consequence, by (2.11) and (2.18),

$$\begin{aligned}\big\langle f^{\varepsilon}_{2q,s_1,t_1},f^{\varepsilon}_{2q,s_2,t_2}\big\rangle_{(\mathfrak{H}^d)^{\otimes 2q}}&=\frac{\alpha_q}{(2\pi)^d(2q)!2^{2q}}\big(\varepsilon+(t_1-s_1)^{2H}\big)^{-\frac{d}{2}-q}\big(\varepsilon+(t_2-s_2)^{2H}\big)^{-\frac{d}{2}-q}\\&\quad\times\mu(s_2-s_1,t_1-s_1,t_2-s_2)^{2q},\end{aligned}$$

where the constant α_q is defined by

$$\alpha_q:=\sum_{q_1+\cdots+q_d=q}\frac{(2q_1)!\cdots(2q_d)!}{(q_1!)^2\cdots(q_d!)^2}. \tag{2.21}$$

From here we can conclude that

$$\big\langle f^{\varepsilon}_{2q,s_1,t_1},f^{\varepsilon}_{2q,s_2,t_2}\big\rangle_{(\mathfrak{H}^d)^{\otimes 2q}}=\frac{\alpha_q}{(2\pi)^d(2q)!2^{2q}}G^{(q)}_{\varepsilon,s_2-s_1}(t_1-s_1,t_2-s_2), \tag{2.22}$$

where $G^{(q)}_{\varepsilon,x}(u_1,u_2)$ is defined by

$$G^{(q)}_{\varepsilon,x}(u_1,u_2):=\big(\varepsilon+u_1^{2H}\big)^{-\frac{d}{2}-q}\big(\varepsilon+u_2^{2H}\big)^{-\frac{d}{2}-q}\mu(x,u_1,u_2)^{2q}. \tag{2.23}$$

Now we describe the covariance $\mathrm{Cov}[p_\varepsilon(B_{t_1}-B_{s_1}),p_\varepsilon(B_{t_2}-B_{s_2})]$. Using the chaos expansion (2.12) and (2.22), we obtain

$$\mathrm{Cov}\big[p_\varepsilon(B_{t_1}-B_{s_1}),p_\varepsilon(B_{t_2}-B_{s_2})\big]=\sum_{q=1}^{\infty}\frac{\alpha_q}{(2\pi)^d2^{2q}}G^{(q)}_{\varepsilon,s_2-s_1}(t_1-s_1,t_2-s_2). \tag{2.24}$$

On the other hand, using once more the property of stationary increments of B, we can prove that for every $s_1\leq t_1$, and $s_2\leq t_2$,

$$\mathrm{Cov}\big[p_\varepsilon(B_{t_1}-B_{s_1}),p_\varepsilon(B_{t_2}-B_{s_2})\big]=F_{\varepsilon,s_2-s_1}(t_1-s_1,t_2-s_2), \tag{2.25}$$

where the function $F_{\varepsilon,x}(u_1,u_2)$, for $u_1,u_2>0$, is defined by

$$F_{\varepsilon,x}(u_1,u_2):=\mathrm{Cov}\big[p_\varepsilon(B_{u_1}),p_\varepsilon(B_{x+u_2}-B_x)\big], \tag{2.26}$$

in the case $x>0$, and by $F_{\varepsilon,x}(u_1,u_2):=F_{\varepsilon,-x}(u_2,u_1)$ in the case $x<0$. Proceeding as in [5], equations (13)–(14), we can prove that for every $u_1,u_2\geq 0$, $x\in\mathbb{R}$,

$$\begin{aligned}F_{\varepsilon,x}(u_1,u_2)&=(2\pi)^{-d}\Big[\big(\big(\varepsilon+u_1^{2H}\big)\big(\varepsilon+u_2^{2H}\big)-\mu(x,u_1,u_2)^2\big)^{-\frac{d}{2}}\\&\quad-\big(\varepsilon+u_1^{2H}\big)^{-\frac{d}{2}}\big(\varepsilon+u_2^{2H}\big)^{-\frac{d}{2}}\Big],\end{aligned} \tag{2.27}$$

and consequently,

$$F_{\varepsilon,x}(u_1,u_2)=(2\pi)^{-d}\big(\varepsilon+u_1^{2H}\big)^{-\frac{d}{2}}\big(\varepsilon+u_2^{2H}\big)^{-\frac{d}{2}}\bigg(\bigg(1-\frac{\mu(x,u_1,u_2)^2}{(\varepsilon+u_1^{2H})(\varepsilon+u_2^{2H})}\bigg)^{-\frac{d}{2}}-1\bigg). \tag{2.28}$$

From (2.24) and (2.25) it follows that the functions $G^{(q)}_{\varepsilon,x}(u_1,u_2)$ and $F_{\varepsilon,x}(u_1,u_2)$ appearing in (2.22) and (2.28) are related in the following manner:

$$F_{\varepsilon,x}(u_1,u_2)=\sum_{q=1}^{\infty}\beta_q G^{(q)}_{\varepsilon,x}(u_1,u_2), \tag{2.29}$$

where β_q is defined by

$$\beta_q:=\frac{\alpha_q}{(2\pi)^d 2^{2q}}. \tag{2.30}$$

The functions $G^{(q)}_{1,x}(u_1,u_2)$ and $F_{1,x}(u_1,u_2)$ satisfy the following useful integrability condition, which was proved in [5, Lemma 13], .

Lemma 2.2. *Let $\frac{3}{2d}<H<\frac{3}{4}$, and $q\in\mathbb{N}$, $q\geq 1$ be fixed. Define $G^{(q)}_{1,x}(u_1,u_2)$ by (2.23) and β_q by (2.30). Then,*

$$\beta_q\int_{\mathbb{R}^3_+}G^{(q)}_{1,x}(u_1,u_2)\,dx\,d\vec{u}\leq\int_{\mathbb{R}^3_+}F_{1,x}(u_1,u_2)\,dx\,d\vec{u}<\infty,$$

where $d\vec{u}:=du_1\,du_2$.

Proof. By (2.29), it follows that $\beta_q G^{(q)}_{1,x}(u_1,u_2)\leq F_{1,x}(u_1,u_2)$. The integrability of the function $F_{1,x}(u_1,u_2)$ over $x,u_1,u_2\geq 0$, written as in (2.27), is proved in [5, Lemma 13] (see equation (40) for notation reference). □

With the notation previously introduced, we can compute the covariance functions of the increments of the processes $\{I^\varepsilon_T\}_{T\geq 0}$ and $\{I_{2q}(h^\varepsilon_{2q,T})\}_{T\geq 0}$ as follows. Define the set $\mathcal{K}_{T_1,T_2}$ by

$$\mathcal{K}_{T_1,T_2}:=\{(s,t)\in\mathbb{R}^2_+ \mid s\leq t,\ \text{and } T_1\leq t\leq T_2\}. \tag{2.31}$$

By (2.10) and (2.16), for every $T_1<T_2$, we can write

$$I^\varepsilon_{T_2}-\mathbb{E}[I^\varepsilon_{T_2}]-(I^\varepsilon_{T_1}-\mathbb{E}[I^\varepsilon_{T_1}])=\int_{\mathbb{R}^2_+}\mathbb{1}_{\mathcal{K}_{T_1,T_2}}(s,t)\big(p_\varepsilon(B_t-B_s)-\mathbb{E}[p_\varepsilon(B_t-B_s)]\big)\,ds\,dt,$$

and

$$I_{2q}(h^\varepsilon_{2q,T_2})-I_{2q}(h^\varepsilon_{2q,T_1})=\int_{\mathbb{R}^2_+}\mathbb{1}_{\mathcal{K}_{T_1,T_2}}(s,t)I_{2q}(f^\varepsilon_{2q,s,t})\,ds\,dt.$$

By (2.25), we deduce the following identity for every $T_1\leq T_2$ and $\widetilde{T}_1\leq\widetilde{T}_2$,

$$\operatorname{Cov}[I^\varepsilon_{T_2}-I^\varepsilon_{T_1},I^\varepsilon_{\widetilde{T}_2}-I^\varepsilon_{\widetilde{T}_1}]=\int_{\mathbb{R}^4_+}\mathbb{1}_{\mathcal{K}_{T_1,T_2}}(s_1,t_1)\mathbb{1}_{\mathcal{K}_{\widetilde{T}_1,\widetilde{T}_2}}(s_2,t_2)F_{\varepsilon,s_2-s_1}(t_1-s_1,t_2-s_2)\,d\vec{s}\,d\vec{t}, \tag{2.32}$$

where $d\vec{s}:=ds_1\,ds_2$ and $d\vec{t}:=dt_1\,dt_2$. Similarly, by (2.22),

$$\begin{aligned}&\mathbb{E}\big[\big(I_{2q}(h^\varepsilon_{2q,T_2})-I_{2q}(h^\varepsilon_{2q,T_1})\big)\big(I_{2q}(h^\varepsilon_{2q,\widetilde{T}_2})-I_{2q}(h^\varepsilon_{2q,\widetilde{T}_1})\big)\big]\\&\quad=\beta_q\int_{\mathbb{R}^4_+}\mathbb{1}_{\mathcal{K}_{T_1,T_2}}(s_1,t_1)\mathbb{1}_{\mathcal{K}_{\widetilde{T}_1,\widetilde{T}_2}}(s_2,t_2)G^{(q)}_{\varepsilon,s_2-s_1}(t_1-s_1,t_2-s_2)\,d\vec{s}\,d\vec{t},\end{aligned} \tag{2.33}$$

where β_q is defined by (2.30).

We end this section by introducing some notation, which will be used throughout the paper to describe expectations of the form $\mathbb{E}[p_\varepsilon(B_{t_1}-B_{s_1})p_\varepsilon(B_{t_2}-B_{s_2})]$. For every n-dimensional non-negative definite matrix A, we will denote by ϕ_A the density function of a Gaussian vector with mean zero and covariance A. In addition, we will denote by $|A|$ the determinant of A, and by I_n the identity matrix of dimension n.

Let Σ be the covariance matrix of the 2-dimensional random vector $(B^{(1)}_{t_1} - B^{(1)}_{s_1}, B^{(1)}_{t_2} - B^{(1)}_{s_2})$. Then, the covariance matrix of the $2d$-dimensional random vector $(B_{t_1} - B_{s_1}, B_{t_2} - B_{s_2})$ can be written as

$$\operatorname{Cov}(B_{t_1} - B_{s_1}, B_{t_2} - B_{s_2}) = I_d \otimes \Sigma,$$

where in the previous identity $\otimes$ denotes the Kronecker product of matrices. Consider the $2d$-dimensional Gaussian density $\phi_{\varepsilon I_{2d}}(x, y) = p_\varepsilon(x)p_\varepsilon(y)$, where $x, y \in \mathbb{R}^d$, and denote by $*$ the convolution operation. Then we have that

$$\begin{aligned}\mathbb{E}\big[p_\varepsilon(B_{t_1} - B_{s_1})p_\varepsilon(B_{t_2} - B_{s_2})\big] &= \int_{\mathbb{R}^{2d}} \phi_{\varepsilon I_{2d}}(x, y)\phi_{I_d\otimes\Sigma}(-x, -y)\,dx\,dy \\ &= \phi_{\varepsilon I_{2d}} * \phi_{I_d\otimes\Sigma}(0, 0) = (2\pi)^{-d}|\varepsilon I_{2d} + I_d \otimes \Sigma|^{-\frac{1}{2}}.\end{aligned}$$

From the previous equation it follows that

$$\mathbb{E}\big[p_\varepsilon(B_{t_1} - B_{s_1})p_\varepsilon(B_{t_2} - B_{s_2})\big] = (2\pi)^{-d}|\varepsilon I_2 + \Sigma|^{-\frac{d}{2}}. \tag{2.34}$$

The right-hand side of the previous identity can be rewritten as follows. Define the function

$$\Theta_\varepsilon(x, u_1, u_2) := \varepsilon^2 + \varepsilon\big(u_1^{2H} + u_2^{2H}\big) + u_1^{2H}u_2^{2H} - \mu(x, u_1, u_2)^2. \tag{2.35}$$

Then, using (2.20), we can easily show that

$$|\varepsilon I_2 + \Sigma| = \Theta_\varepsilon(s_2 - s_1, t_1 - s_1, t_2 - s_2),$$

which, by (2.34), implies that

$$\mathbb{E}\big[p_\varepsilon(B_{t_1} - B_{s_1})p_\varepsilon(B_{t_2} - B_{s_2})\big] = (2\pi)^{-d}\Theta_\varepsilon(s_2 - s_1, t_1 - s_1, t_2 - s_2)^{-\frac{d}{2}}. \tag{2.36}$$

Therefore, we can write $\mathbb{E}[(I^\varepsilon_T)^2]$, as

$$\mathbb{E}\big[\big(I^\varepsilon_T\big)^2\big] = (2\pi)^{-d}\int_{(T\mathcal{R})^2}\Theta_\varepsilon(s_2 - s_1, t_1 - s_1, t_2 - s_2)^{-\frac{d}{2}}\,d\vec{s}\,d\vec{t}. \tag{2.37}$$

Finally, we prove the following inequality, which estimates the function $F_{\varepsilon,x}(u_1, u_2)$, defined in (2.26), in terms of $\Theta_\varepsilon(x, u_1, u_2)$

$$F_{\varepsilon,x}(u_1, u_2) \le (2\pi)^{-d}\left(\frac{d}{2} + 1\right)\frac{\mu(x, u_1, u_2)^2}{u_1^{2H}u_2^{2H}}\Theta_\varepsilon(x, u_1, u_2)^{-\frac{d}{2}}. \tag{2.38}$$

Indeed, using relation (2.28), as well as the binomial theorem, we deduce that

$$\begin{aligned}F_{\varepsilon,x}(u_1, u_2) &= (2\pi)^{-d}\big(\varepsilon + u_1^{2H}\big)^{-\frac{d}{2}-1}\big(\varepsilon + u_2^{2H}\big)^{-\frac{d}{2}-1}\mu(x, u_1, u_2)^2 \\ &\quad\times\sum_{q=0}^{\infty}\frac{(\frac{d}{2})^{\overline{q+1}}}{(q+1)!}\left(\frac{\mu(x, u_1, u_2)^2}{(\varepsilon + u_1^{2H})(\varepsilon + u_2^{2H})}\right)^q,\end{aligned}$$

where $a^{\overline{n}}$ denotes the nth raising factorial of a. Hence, using the fact that

$$\frac{(\frac{d}{2})^{\overline{q+1}}}{(q+1)!} = \frac{(\frac{d}{2} + q)}{q+1}\frac{(\frac{d}{2})^{\overline{q}}}{q!} \le \left(\frac{d}{2} + 1\right)\frac{(\frac{d}{2})^{\overline{q}}}{q!},$$

we deduce that

$$F_{\varepsilon,x}(u_1,u_2) \le (2\pi)^{-d}\left(\frac{d}{2}+1\right)\left(1+u_1^{2H}\right)^{-\frac{d}{2}}\left(1+u_2^{2H}\right)^{-\frac{d}{2}} \frac{\mu(x,u_1,u_2)^2}{(\varepsilon+u_1^{2H})(\varepsilon+u_2^{2H})} \times \sum_{q=0}^{\infty} \frac{(\frac{d}{2})^{\overline{q}}}{q!}\left(\frac{\mu(x,u_1,u_2)^2}{(\varepsilon+u_1^{2H})(\varepsilon+u_2^{2H})}\right)^q,$$

which, by the binomial theorem, implies (2.38).

3. Behavior of the covariances of I_T^ε and its chaotic components

In this section we describe the behavior as $\varepsilon \to 0$ of the covariance of $I_{T_1}^\varepsilon$ and $I_{T_2}^\varepsilon$, as well as the covariance of $I_{2q}(h_{2q,T_1}^\varepsilon)$ and $I_{2q}(h_{2q,T_2}^\varepsilon)$, for $0 \le T_1 \le T_2$.

Theorem 3.1. *Let $T_1, T_2 \ge 0$ be fixed. Then, if $\frac{3}{2d} < H < \frac{3}{4}$,*

$$\lim_{\varepsilon\to 0} \varepsilon^{d-\frac{3}{2H}} \mathbb{E}\left[I_{2q}\left(h_{2q,T_1}^\varepsilon\right) I_{2q}\left(h_{2q,T_2}^\varepsilon\right)\right] = \sigma_q^2 (T_1 \wedge T_2),$$

where

$$\sigma_q^2 := 2\beta_q \int_{\mathbb{R}_+^3} G_{1,x}^{(q)}(u_1,u_2)\,dx\,d\vec{u}, \tag{3.1}$$

β_q is defined by (2.30) and $G_{1,x}^{(q)}(u_1,u_2)$ by (2.23). Moreover, we have

$$\sum_{q=1}^{\infty} \sigma_q^2 = \sigma^2, \tag{3.2}$$

where σ^2 is a finite constant given by

$$\sigma^2 := 2\int_{\mathbb{R}_+^3} F_{1,x}(u_1,u_2)\,dx\,d\vec{u}, \tag{3.3}$$

and $F_{1,x}(u_1,u_2)$ is defined in (2.26).

Proof. To prove the result, it suffices to show that for each $a < b < \alpha < \beta$,

$$\lim_{\varepsilon\to 0} \varepsilon^{d-\frac{3}{2H}} \mathbb{E}\left[\left(I_{2q}\left(h_{2q,b}^\varepsilon\right) - I_{2q}\left(h_{2q,a}^\varepsilon\right)\right)\left(I_{2q}\left(h_{2q,\beta}^\varepsilon\right) - I_{2q}\left(h_{2q,\alpha}^\varepsilon\right)\right)\right] = 0, \tag{3.4}$$

and

$$\lim_{\varepsilon\to 0} \varepsilon^{d-\frac{3}{2H}} \mathbb{E}\left[\left(I_{2q}\left(h_{2q,b}^\varepsilon\right) - I_{2q}\left(h_{2q,a}^\varepsilon\right)\right)^2\right] = \sigma_q^2(b-a). \tag{3.5}$$

First we prove (3.4). Set

$$\Phi^\varepsilon = \mathbb{E}\left[\left(I_{2q}\left(h_{2q,b}^\varepsilon\right) - I_{2q}\left(h_{2q,a}^\varepsilon\right)\right)\left(I_{2q}\left(h_{2q,\beta}^\varepsilon\right) - I_{2q}\left(h_{2q,\alpha}^\varepsilon\right)\right)\right].$$

Define the set $\mathcal{K}_{T_1,T_2}$ by (2.31), and $\gamma := \frac{\alpha-b}{2} > 0$. We can easily check that for every $(s_1,t_1) \in \mathcal{K}_{a,b}$, and $(s_2,t_2) \in \mathcal{K}_{\alpha,\beta}$, it holds that either $t_2 - s_2 > \gamma$, or $s_2 - s_1 \ge \gamma$, and hence, by taking $T_1 = a$, $T_2 = b$, $\widetilde{T}_1 = \alpha$, $\widetilde{T}_2 = \beta$ in (2.33), we get

$$\left|\Phi^{\varepsilon}\right| \leq \beta_q \int_{[0,\beta]^4} \big(\mathbb{1}_{(\gamma,\infty)}(t_2-s_2)+\mathbb{1}_{(\gamma,\infty)}(s_2-s_1)\big) G^{(q)}_{\varepsilon,s_2-s_1}(t_1-s_1,t_2-s_2)\,d\vec{s}\,d\vec{t}.$$

Changing the coordinates (s_1,s_2,t_1,t_2) by $(s:=s_1, x:=s_2-s_1, u_1:=t_1-s_1, u_2:=t_2-s_2)$ for $s_2\geq s_1$, and by $(s:= s_2, x:=s_1-s_2, u_1:=t_1-s_1, u_2:=t_2-s_2)$ for $s_2\leq s_1$, in (3.6), using the fact that $G^{(q)}_{\varepsilon,-x}(u_1,u_2)=G^{(q)}_{\varepsilon,x}(u_2,u_1)$, and integrating the s_1 variable, we can prove that

$$\left|\Phi^{\varepsilon}\right| \leq \beta_q \beta \int_{[0,\beta]^3} \big(\mathbb{1}_{(\gamma,\infty)}(u_1)+\mathbb{1}_{(\gamma,\infty)}(u_2)+\mathbb{1}_{(\gamma,\infty)}(x)\big) G^{(q)}_{\varepsilon,x}(u_1,u_2)\,dx\,d\vec{u}.$$

Next, changing the coordinates (x,u_1,u_2) by $(\varepsilon^{-\frac{1}{2H}}x, \varepsilon^{-\frac{1}{2H}}u_1, \varepsilon^{-\frac{1}{2H}}u_2)$, and using the fact that $G^{(q)}_{\varepsilon,\varepsilon^{\frac{1}{2H}}x}(\varepsilon^{\frac{1}{2H}}u_1, \varepsilon^{\frac{1}{2H}}u_2)=\varepsilon^{-d}G^{(q)}_{1,x}(u_1,u_2)$, we get

$$\left|\Phi^{\varepsilon}\right| \leq \varepsilon^{\frac{3}{2H}-d}\beta_q \beta \int_{[0,\varepsilon^{-\frac{1}{2H}}\beta]^3} \big(\mathbb{1}_{(\gamma,\infty)}\big(\varepsilon^{\frac{1}{2H}}u_1\big)+\mathbb{1}_{(\gamma,\infty)}\big(\varepsilon^{\frac{1}{2H}}u_2\big)+\mathbb{1}_{(\gamma,\infty)}\big(\varepsilon^{\frac{1}{2H}}x\big)\big) G^{(q)}_{1,x}(u_1,u_2)\,dx\,d\vec{u}.$$

Since $\gamma>0$, the arguments in the previous integrals converge to zero pointwise, and are dominated by the function $3\beta_q\beta G^{(q)}_{1,x}(u_1,u_2)$, which is integrable by Lemma 2.2 due to the condition $\frac{3}{2d}<H<\frac{3}{4}$. Hence, by the dominated convergence theorem,

$$\lim_{\varepsilon\to 0}\varepsilon^{d-\frac{3}{2H}}\left|\Phi^{\varepsilon}\right|=0,$$

as required. Next we prove (3.5). By taking $T_1=\widetilde{T}_1=a$, and $T_2=\widetilde{T}_2=b$ in (2.33), we deduce that

$$\mathbb{E}\big[\big(I_{2q}\big(h^{\varepsilon}_{2q,b}\big)-I_{2q}\big(h^{\varepsilon}_{2q,a}\big)\big)^2\big]=2\beta_q\int_{[0,b]^4}\mathbb{1}_{\{s_1\leq s_2\}}\mathbb{1}_{\mathcal{K}_{a,b}}(s_1,t_1)\mathbb{1}_{\mathcal{K}_{a,b}}(s_2,t_2) \times G^{(q)}_{\varepsilon,s_2-s_1}(t_1-s_1,t_2-s_2)\,d\vec{s}\,d\vec{t}.$$

Changing the coordinates (s_1,s_2,t_1,t_2) by $(s_1, x:=s_2-s_1, u_1:=t_1-s_1, u_2:=t_2-s_2)$, we get

$$\begin{aligned}
&\mathbb{E}\big[\big(I_{2q}\big(h^{\varepsilon}_{2q,b}\big)-I_{2q}\big(h^{\varepsilon}_{2q,a}\big)\big)^2\big]\\
&\quad=2\beta_q\int_{[0,b]^4}\mathbb{1}_{\mathcal{K}_{a,b}}(s_1,s_1+u_1)\mathbb{1}_{\mathcal{K}_{a,b}}(s_1+x,s_1+x+u_2)G^{(q)}_{\varepsilon,x}(u_1,u_2)\,ds_1\,dx\,d\vec{u}\\
&\quad=2\beta_q\int_{[0,b]^3}\int_{(a-u_1)_+\vee(a-x-u_2)_+}^{(b-u_1)_+\wedge(b-x-u_2)_+} ds_1\,G^{(q)}_{\varepsilon,x}(u_1,u_2)\,dx\,d\vec{u}.
\end{aligned}\tag{3.6}$$

Notice that $G^{(q)}_{\varepsilon,\varepsilon^{\frac{1}{2H}}x}(\varepsilon^{\frac{1}{2H}}u_1,\varepsilon^{\frac{1}{2H}}u_2)=\varepsilon^{-d}G_{1,x}(u_1,u_2)$. Therefore, integrating the variable s_1, and changing the coordinates (x,u_1,u_2) by $(\varepsilon^{-\frac{1}{2H}}x, \varepsilon^{-\frac{1}{2H}}u_1, \varepsilon^{-\frac{1}{2H}}u_2)$ in (3.6), we conclude that

$$\begin{aligned}
\varepsilon^{d-\frac{3}{2H}}\mathbb{E}\big[\big(I_{2q}\big(h^{\varepsilon}_{2q,b}\big)-I_{2q}\big(h^{\varepsilon}_{2q,a}\big)\big)^2\big]&=2\beta_q\int_{[0,\varepsilon^{-\frac{1}{2H}}b]^3}G^{(q)}_{1,x}(u_1,u_2)\\
&\qquad\times\big[\big(b-\varepsilon^{\frac{1}{2H}}u_1\big)_+\wedge\big(b-\varepsilon^{\frac{1}{2H}}(x+u_2)\big)_+\\
&\qquad-\big(a-\varepsilon^{\frac{1}{2H}}u_1\big)_+\vee\big(a-\varepsilon^{\frac{1}{2H}}(x+u_2)\big)_+\big]\,dx\,d\vec{u}.
\end{aligned}\tag{3.7}$$

The integrand in (3.7) converges increasingly to $2(b-a)G^{(q)}_{1,x}(u_1,u_2)$ as $\varepsilon\to 0$, which is integrable by Lemma 2.2. Identity (3.5) then follows by applying the dominated convergence theorem in (3.7).

Relation (3.2) is obtained by integrating both sides of relation (2.29) over the variables $x, u_1, u_2 \geq 0$, for $\varepsilon = 1$, and then using the monotone convergence theorem. The constant σ^2 is finite by Lemma 2.2. The proof is now complete. □

In order to determine the behavior of the covariances of I_T^ε for the case $H = \frac{3}{4}$, we will first prove that the second chaotic component $I_2(h_{2,T}^\varepsilon)$ characterizes the asymptotic behavior of $I_T^\varepsilon - \mathbb{E}[I_T^\varepsilon]$ as $\varepsilon \to \infty$, for every $H \geq \frac{3}{4}$.

We start by showing that, after a suitable rescaling, the sequence $I_2(h_{2,T}^\varepsilon)$ approximates $I_T^\varepsilon - \mathbb{E}[I_T^\varepsilon]$ in $L^2(\Omega)$ for $H > \frac{3}{4}$. This result will be latter used in the proof of Theorem 1.3.

Lemma 3.2. *Let* $\frac{3}{4} < H < 1$ *be fixed. Then,*

$$\lim_{\varepsilon \to 0} \varepsilon^{\frac{d}{2} - \frac{3}{2H} + 1} \| I_T^\varepsilon - \mathbb{E}[I_T^\varepsilon] - I_2(h_{2,T}^\varepsilon) \|_{L^2(\Omega)} = 0.$$

Proof. For $T > 0$ fixed, define the quantity

$$Q_\varepsilon := \| I_T^\varepsilon - \mathbb{E}[I_T^\varepsilon] - I_2(h_{2,T}^\varepsilon) \|_{L^2(\Omega)}^2.$$

From the chaos decomposition (2.15), we get

$$Q_\varepsilon = \mathbb{E}[(I_T^\varepsilon)^2] - \mathbb{E}[I_T^\varepsilon]^2 - 2 \left\| \int_{T\mathcal{R}} f_{2,s,t}^\varepsilon \, ds \, dt \right\|_{(\mathfrak{H}^d)^{\otimes 2}}^2. \tag{3.8}$$

By (2.17) and (2.37), the first two terms in the right-hand side of the previous identity can be written as

$$\mathbb{E}[(I_T^\varepsilon)^2] = (2\pi)^{-d} \int_{(T\mathcal{R})^2} \Theta_\varepsilon(s_2 - s_1, t_1 - s_1, t_2 - s_2)^{-\frac{d}{2}} \, d\vec{s} \, d\vec{t}, \tag{3.9}$$

and

$$\mathbb{E}[I_T^\varepsilon]^2 = (2\pi)^{-d} \int_{(T\mathcal{R})^2} G_{\varepsilon, s_2 - s_1}^{(0)}(t_1 - s_1, t_2 - s_2) \, d\vec{s} \, d\vec{t}, \tag{3.10}$$

where $G_{\varepsilon,x}^{(q)}(u_1, u_2)$ and $\Theta_\varepsilon(x, u_1, u_2)$ are given by (2.23) and (2.35), respectively. To handle the third term in (3.8), recall that the constants α_q are given by (2.21), and notice that $\alpha_1 = 2d$. Hence, from (2.22), we deduce that

$$\left\| \int_{T\mathcal{R}} f_{2,s,t}^\varepsilon \, ds \, dt \right\|_{(\mathfrak{H}^d)^{\otimes 2}}^2 = \frac{d(2\pi)^{-d}}{4} \int_{(T\mathcal{R})^2} G_{\varepsilon, s_2 - s_1}^{(1)}(t_1 - s_1, t_2 - s_2) \, d\vec{s} \, d\vec{t}. \tag{3.11}$$

From equations (3.8)–(3.11), we conclude that

$$\begin{aligned} Q_\varepsilon = (2\pi)^{-d} \int_{(T\mathcal{R})^2} \bigg(& \Theta_\varepsilon(s_2 - s_1, t_1 - s_1, t_2 - s_2)^{-\frac{d}{2}} \\ & - G_{\varepsilon, s_2 - s_1}^{(0)}(t_1 - s_1, t_2 - s_2) - \frac{d}{2} G_{\varepsilon, s_2 - s_1}^{(1)}(t_1 - s_1, t_2 - s_2) \bigg) \, d\vec{s} \, d\vec{t}. \end{aligned} \tag{3.12}$$

The integrand appearing in the right-hand side is positive. Indeed, if we define

$$\rho_\varepsilon(x, u_1, u_2) := \mu(x, u_1, u_2)^2 (\varepsilon + u_1^{2H})^{-1} (\varepsilon + u_2^{2H})^{-1},$$

then, applying relations (2.23), (2.35) we obtain

$$\begin{aligned} \Theta_\varepsilon(x, u_1, u_2)^{-\frac{d}{2}} - G_{\varepsilon,x}^{(0)}(u_1, u_2) - \frac{d}{2} G_{\varepsilon,x}^{(1)}(u_1, u_1) = 2(2\pi)^{-d} (\varepsilon + u_1^{2H})^{-\frac{d}{2}} (\varepsilon + u_2^{2H})^{-\frac{d}{2}} \\ \times \left((1 - \rho_\varepsilon(x, u_1, u_2))^{-\frac{d}{2}} - 1 - \frac{d}{2} \rho_\varepsilon(x, u_1, u_2) \right) \end{aligned} \tag{3.13}$$

and the right-hand side of the previous identity is positive by the binomial theorem. As a consequence, by changing the coordinates (s_1, s_2, t_1, t_2) by $(s_1, x := s_2 - s_1, u_1 := t_1 - s_1, u_2 := t_2 - s_2)$, and integrating the variable s_1 in (3.12), we get

$$Q_\varepsilon \leq 2(2\pi)^{-d} T \int_{[0,T]^3} \left(\Theta_\varepsilon(x, u_1, u_2)^{-\frac{d}{2}} - G^{(0)}_{\varepsilon,x}(u_1, u_2) - \frac{d}{2} G^{(1)}_{\varepsilon,x}(u_1, u_2) \right) dx\, d\vec{u}.$$

In addition, by the binomial theorem, we have that for every $0 < y < 1$,

$$(1-y)^{-\frac{d}{2}} - 1 - \frac{d}{2} y = \sum_{q=2}^{\infty} (-1)^q \binom{-\frac{d}{2}}{q} y^q = y^2 \sum_{q=0}^{\infty} \frac{(\frac{d}{2})^{\overline{q+2}}}{(q+2)!} y^q,$$

where $(x)^{\overline{q}}$ denotes the raising factorial $(x)^{\overline{q}} := x(x+1)\cdots(x+q-1)$. Hence, by (3.13),

$$Q_\varepsilon \leq 2(2\pi)^{-d} T \int_{[0,T]^3} \left(\varepsilon + u_1^{2H}\right)^{-\frac{d}{2}} \left(\varepsilon + u_2^{2H}\right)^{-\frac{d}{2}} \rho_\varepsilon(x, u_1, u_2)^2 \sum_{q=0}^{\infty} \frac{(\frac{d}{2})^{\overline{q+2}}}{(q+2)!} \rho_\varepsilon(x, u_1, u_2)^q\, dx\, d\vec{u}. \tag{3.14}$$

Since

$$\frac{(\frac{d}{2})^{\overline{q+2}}}{(q+2)!} = \frac{(\frac{d}{2})^{\overline{q}}}{q!} \frac{(\frac{d}{2}+q)(\frac{d}{2}+q+1)}{(q+1)(q+2)} \leq \left(\frac{d}{2}+1\right)^2 \frac{(\frac{d}{2})^{\overline{q}}}{q!},$$

then, by (3.14),

$$\begin{aligned} Q_\varepsilon &\leq 2(2\pi)^{-d} T \left(\frac{d}{2}+1\right)^2 \int_{[0,T]^3} \left(\varepsilon + u_1^{2H}\right)^{-\frac{d}{2}} \left(\varepsilon + u_2^{2H}\right)^{-\frac{d}{2}} \\ &\quad \times \rho_\varepsilon(x, u_1, u_2)^2 \sum_{q=0}^{\infty} \frac{(\frac{d}{2})^{\overline{q}}}{q!} \rho_\varepsilon(x, u_1, u_2)^q\, dx\, d\vec{u}, \end{aligned}$$

which, by the binomial theorem, implies that there exists a constant $C > 0$ only depending on T and d, such that

$$Q_\varepsilon \leq C \int_{[0,T]^3} \frac{\mu(x, u_1, u_2)^4}{(\varepsilon + u_1^{2H})^2 (\varepsilon + u_2^{2H})^2} \Theta_\varepsilon(x, u_1, u_2)^{-\frac{d}{2}}\, dx\, d\vec{u}. \tag{3.15}$$

Hence, to prove the lemma it suffices to show that

$$\lim_{\varepsilon \to 0} \varepsilon^{d - \frac{3}{H} + 2} \int_{[0,T]^3} \Psi_\varepsilon(x, u_1, u_2)\, dx\, d\vec{u} = 0, \tag{3.16}$$

where

$$\Psi_\varepsilon(x, u_1, u_2) := \frac{\mu(x, u_1, u_2)^4}{(\varepsilon + u_1^{2H})^2 (\varepsilon + u_2^{2H})^2} \Theta_\varepsilon(x, u_1, u_2)^{-\frac{d}{2}}. \tag{3.17}$$

In order to prove (3.16), we proceed as follows. First we decompose the domain of integration of (3.16) as $[0, T]^3 = \mathcal{S}_1^T \cup \mathcal{S}_2^T \cup \mathcal{S}_3^T$, where $\mathcal{S}_i^T := \mathcal{S}_i \cap [0, T]^3$, and

$$\begin{aligned} \mathcal{S}_1 &:= \left\{(x, u_1, u_2) \in \mathbb{R}_+^3 \mid x + u_2 - u_1 \geq 0, u_1 - x \geq 0\right\}, \\ \mathcal{S}_2 &:= \left\{(x, u_1, u_2) \in \mathbb{R}_+^3 \mid u_1 - x - u_2 \geq 0\right\}, \\ \mathcal{S}_3 &:= \left\{(x, u_1, u_2) \in \mathbb{R}_+^3 \mid x - u_1 \geq 0\right\}. \end{aligned} \tag{3.18}$$

Then, it suffices to show that for $i = 1, 2, 3$,

$$\lim_{\varepsilon\to 0} \varepsilon^{d-\frac{3}{H}+2} \int_{\mathcal{S}_i^T} \Psi_\varepsilon(x, u_1, u_2)\, dx\, d\vec{u} = 0. \tag{3.19}$$

First prove (3.19) in the cases $i = 1, 2$. Changing the coordinates (x, u_1, u_2) by $(\varepsilon^{-\frac{1}{2H}} x, \varepsilon^{-\frac{1}{2H}} u_1, \varepsilon^{-\frac{1}{2H}} u_2)$, and using the fact that $\Psi_\varepsilon(\varepsilon^{\frac{1}{2H}} x, \varepsilon^{\frac{1}{2H}} u_1, \varepsilon^{\frac{1}{2H}} u_2) = \varepsilon^{-d} \Psi_1(x, u_1, u_2)$, we get

$$\varepsilon^{d-\frac{3}{H}+2} \int_{\mathcal{S}_i^T} \Psi_\varepsilon(x, u_1, u_2)\, dx\, d\vec{u} \le \varepsilon^{2-\frac{3}{2H}} \int_{\mathcal{S}_i} \Psi_1(x, u_1, u_2)\, dx\, d\vec{u},$$

where the sets $\mathcal{S}_i$ are defined by (3.18). Therefore, using $\mu(x, u_1, u_2)^2 \le (u_1 u_2)^{2H}$, we obtain

$$\varepsilon^{d-\frac{3}{H}+2} \int_{\mathcal{S}_i^T} \Psi_\varepsilon(x, u_1, u_2)\, dx\, d\vec{u} \le \varepsilon^{2-\frac{3}{2H}} \int_{\mathcal{S}_i} \frac{\mu(x, u_1, u_2)^2}{(u_1 u_2)^{2H}} \Theta_1(x, u_1, u_2)^{-\frac{d}{2}}\, dx\, d\vec{u}. \tag{3.20}$$

The integral appearing in the right-hand side of the previous inequality is finite by Lemma 5.3 (see equation (5.7) for $p = 2$ and $i = 1, 2$). Relation (3.19) for $i = 1, 2$ is then obtained by taking $\varepsilon \to 0$ in (3.20).

It then remains to prove (3.19) for $i = 3$. Changing the coordinates (x, u_1, u_2) by $(a := u_1, b := x - u_1, c := u_2)$, we get

$$\int_{\mathcal{S}_3^T} \Psi_\varepsilon(x, u_1, u_2)\, dx\, d\vec{u} \le \int_{[0,T]^3} \Psi_\varepsilon(a + b, a, c)\, da\, db\, dc. \tag{3.21}$$

We bound the right-hand side of the previous inequality as follows. First we write

$$\begin{aligned} \mu(a + b, a, c) &= \frac{1}{2}\big((a + b + c)^{2H} + b^{2H} - (b + c)^{2H} - (a + b)^{2H}\big) \\ &= H(2H - 1)ac \int_{[0,1]^2} (b + av_1 + cv_2)^{2H-2}\, dv_1\, dv_2. \end{aligned} \tag{3.22}$$

Notice that if $a > c$, then $b + av_1 + cv_2 \ge v_1(b + a) \ge v_1(b + \frac{a}{2} + \frac{c}{2})$, and if $c > a$, then $b + av_1 + cv_2 \ge v_2(b + c) \ge v_2(b + \frac{a}{2} + \frac{c}{2})$. Therefore, since $H > \frac{3}{4}$, by (3.22) we deduce that there exists a constant $K > 0$, such that

$$\mu(a + b, a, c) \le Kac(a + b + c)^{2H-2}. \tag{3.23}$$

On the other hand, if Σ denotes the covariance matrix of $(B_a, B_{a+b+c} - B_{a+b})$, we can write

$$\Theta_\varepsilon(a + b, a, c) = \varepsilon^2 + \varepsilon\big(a^{2H} + c^{2H}\big) + |\Sigma|.$$

As a consequence, by part (3) of Lemma 5.1, we deduce that $\Theta_\varepsilon(a + b, a, c) \ge \varepsilon^2 + \delta(ac)^{2H}$ for some constant $\delta \in (0, 1)$. Hence, by (3.17) and (3.23), that there exists a constant $C > 0$, such that

$$\Psi_\varepsilon(a + b, a, c) \le C(ac)^{4-4H}(a + b + c)^{8H-8}\big(\varepsilon^2 + (ac)^{2H}\big)^{-\frac{d}{2}}. \tag{3.24}$$

Next we bound the right-hand side of (3.24) by using Young's inequality. Since $H > \frac{3}{4}$ and $Hd > \frac{3}{2}$, then

$$0 < \frac{3 - 2H}{Hd} < \frac{3}{2Hd} < 1. \tag{3.25}$$

Using the relation (3.25), as well as the fact that $\frac{3}{4} < H < 1$, we deduce that there exists a constant $y > 0$, such that

$$4H - 4 + 4Hdy < 0, \tag{3.26}$$

$$4H - 3 - 4Hdy > 0, \tag{3.27}$$

$$(Hd)^{-1}(3 - 2H) + y < 1. \tag{3.28}$$

By (3.28), the constant $\gamma := \frac{3-2H}{Hd} + y$ belongs to (0, 1), and hence, by Young's inequality, we have

$$(1-\gamma)\varepsilon^2 + \gamma(ac)^{2H} \geq \varepsilon^{2(1-\gamma)}(ac)^{2H\gamma}. \tag{3.29}$$

In addition, by (3.26), we have

$$\begin{aligned}(a+b+c)^{8H-8} &= (a+b+c)^{4H-4-4Hdy}(a+b+c)^{4H-4+4Hdy} \\ &\leq b^{4H-4-4Hdy}(a+c)^{4H-4+4Hdy} \leq b^{4H-4-4Hdy}(2\sqrt{ac})^{4H-4+4Hdy},\end{aligned} \tag{3.30}$$

where the last inequality follows from the arithmetic mean-geometric mean inequality. Hence, by (3.24), (3.29) and (3.30), we obtain

$$\varepsilon^{d-\frac{3}{H}+2}\int_{[0,T]^3}\Psi_\varepsilon(a+b,a,c)\,da\,db\,dc \leq \varepsilon^{dy}C\int_{[0,T]^3} b^{4H-4-4Hdy}(ac)^{-1+Hdy}\,da\,db\,dc. \tag{3.31}$$

The integral in the right-hand side is finite by (3.27). Relation (3.19) for $i=3$ then follows from (3.21) and (3.31). □

The next result extends Lemma 3.2 to the case $H=\frac{3}{4}$.

Lemma 3.3. *Let $d \geq 3$ be fixed. Then, if $H=\frac{3}{4}$,*

$$\lim_{\varepsilon\to 0}\frac{\varepsilon^{\frac{d}{2}-1}}{\sqrt{\log(1/\varepsilon)}}\big\|I_T^\varepsilon - \mathbb{E}\big[I_T^\varepsilon\big] - I_2\big(h_{2,T}^\varepsilon\big)\big\|_{L^2(\Omega)} = 0. \tag{3.32}$$

Proof. For $T>0$ fixed, define the quantity

$$Q_\varepsilon := \big\|I_T^\varepsilon - \mathbb{E}\big[I_T^\varepsilon\big] - I_2\big(h_{2,T}^\varepsilon\big)\big\|_{L^2(\Omega)}^2.$$

As in the proof of equation (3.15) in Lemma 3.2, we can show that there exists a constant $C>0$ such that

$$Q_\varepsilon \leq C\int_{[0,T]^3}\Psi_\varepsilon(x,u_1,u_2)\,dx\,d\vec{u}, \tag{3.33}$$

where

$$\Psi_\varepsilon(x,u_1,u_2) := \frac{\mu(x,u_1,u_2)^4}{(\varepsilon+u_1^{\frac{3}{2}})^2(\varepsilon+u_2^{\frac{3}{2}})^2}\Theta_\varepsilon(x,u_1,u_2)^{-\frac{d}{2}}. \tag{3.34}$$

Hence, by splitting the domain of integration in (3.33) as $[0,T]^3=\bigcup_{i=1}^3\mathcal{S}_i^T$, we deduce that relation (3.32) holds, provided that

$$\lim_{\varepsilon\to 0}\frac{\varepsilon^{d-2}}{\log(1/\varepsilon)}\int_{\mathcal{S}_i^T}\Psi_\varepsilon(x,u_1,u_2)\,d\vec{u}=0, \tag{3.35}$$

for $i=1,2,3$. To prove (3.35) for $i=1,2$, we change the coordinates (x,u_1,u_2) by $(\varepsilon^{-\frac{2}{3}}x,\varepsilon^{-\frac{2}{3}}u_1,\varepsilon^{-\frac{2}{3}}u_2)$ and use the fact that $\Psi_\varepsilon(\varepsilon^{\frac{2}{3}}x,\varepsilon^{\frac{2}{3}}u_1,\varepsilon^{\frac{2}{3}}u_2)=\varepsilon^{-d}\Psi_1(x,u_2,u_2)$, in order to get

$$\frac{\varepsilon^{d-2}}{\log(1/\varepsilon)}\int_{\mathcal{S}_i^T}\Psi_\varepsilon(x,u_1,u_2)\,dx\,d\vec{u} \leq \frac{1}{\log(1/\varepsilon)}\int_{\mathcal{S}_i}\Psi_1(x,u_1,u_2)\,dx\,d\vec{u}. \tag{3.36}$$

As a consequence, by applying the inequality $\mu(x,u_1,u_2)^2 \leq (u_1u_2)^{\frac{3}{2}}$, we get

$$\frac{\varepsilon^{d-2}}{\log(1/\varepsilon)}\int_{\mathcal{S}_i^T}\Psi_\varepsilon(x,u_1,u_2)\,dx\,d\vec{u} \leq \frac{1}{\log(1/\varepsilon)}\int_{\mathcal{S}_i}\frac{\mu(x,u_1,u_2)^2}{(u_1u_2)^{\frac{3}{2}}}\Theta_1(x,u_1,u_2)^{-\frac{d}{2}}\,dx\,d\vec{u}. \tag{3.37}$$

The integral appearing the right-hand side of the previous inequality is finite for $i=1,2$ by Lemma 5.3 (see equation (5.7) for $p=2$). Relation (3.35) for $i=1,2$ is then obtained by taking $\varepsilon\to 0$ in (3.37).

It then suffices to handle the case $i=3$. Define the function $K(x,u_1,u_2)$ by

$$K(x,u_1,u_2) := \frac{\mu(x,u_1,u_2)^4}{(u_1u_2)^3}\Theta_1(x,u_1,u_2)^{-\frac{d}{2}}. \tag{3.38}$$

Notice that

$$\frac{1}{\log(1/\varepsilon)}\int_{\mathcal{S}_3}\Psi_1(x,u_1,u_2)\,dx\,d\vec{u} \leq \frac{1}{\log(1/\varepsilon)}\int_{\mathcal{S}_3}K(x,u_1,u_2)\,dx\,d\vec{u}. \tag{3.39}$$

From (3.22), it easily follows that $\mu(a+b,a,c)\leq \frac{3ac}{2}(a+b+c)^{-\frac{1}{2}}$, and thus,

$$K(a+b,a,c) \leq \frac{3^4}{2^4}ac(a+b+c)^{-2}\Theta_1(a+b,a,c)^{-\frac{d}{2}}.$$

Notice that $\Theta_1(a+b,a,c) = 1+a^{\frac{3}{2}}+c^{\frac{3}{2}}+|\Sigma|$, where Σ denotes the covariance matrix of $(B_a, B_{a+b+c}-B_{a+b})$. Therefore, by part (3) of Lemma 5.1, we deduce that

$$\Theta_1(a+b,a,c) \geq 1+a^{\frac{3}{2}}+c^{\frac{3}{2}}+\delta(ac)^{\frac{3}{2}},$$

for some constant $\delta\in(0,1)$. From here it follows that there exists a constant $C>0$, such that

$$K(a+b,a,c) \leq Cac(a+b+c)^{-2}\big(1+a^{\frac{3}{2}}+c^{\frac{3}{2}}+a^{\frac{3}{2}}c^{\frac{3}{2}}\big)^{-\frac{d}{2}}.$$

Therefore, using the fact that $1+m+n+mn\geq(1\vee m)(1\vee n)$ for all $m,n\geq 0$, and defining $\varrho_1 := a\vee c$, $\varrho_2 := a\wedge c$, we get

$$K(a+b,a,c) \leq C(\varrho_1\varrho_2)(b\vee\varrho_1)^{-2}\big((1\vee\varrho_1)^{\frac{3}{2}}(1\vee\varrho_2)^{\frac{3}{2}}\big)^{-\frac{d}{2}}.$$

Using the previous inequality, as well as the condition $d\geq 3$, we can easily check that $K(a+b,a,c)$ is integrable in $\mathbb{R}_+^3$, which in turn implies that $K(x,u_1,u_2)$ is integrable in $\mathcal{S}_3$. Using this observation, as well as relations (3.36) and (3.39), we obtain

$$\lim_{\varepsilon\to 0}\frac{\varepsilon^{d-2}}{\log(1/\varepsilon)}\int_{\mathcal{S}_3^T}\Psi_\varepsilon(x,u_1,u_2)\,dx\,d\vec{u} = 0,$$

as required. The proof is now complete. □

The next result provides a useful approximation for $I_2(h_{2,T}^\varepsilon)$.

Lemma 3.4. *Assume that $H=\frac{3}{4}$ and $d\geq 3$. Let $h_{2,T}^\varepsilon$ be defined as in (2.16) and consider the following approximation of $I_2(h_{2,T}^\varepsilon)$*

$$\widetilde{J}_T^\varepsilon := -\frac{(2\pi)^{-\frac{d}{2}}\varepsilon^{-\frac{d}{2}+1}}{2}\sum_{j=1}^d\int_0^T\int_0^\infty \frac{u^{\frac{3}{2}}}{\varepsilon^{\frac{1}{3}}(1+u^{\frac{3}{2}})^{\frac{d}{2}+1}}H_2\bigg(\frac{B_{s+\varepsilon^{\frac{2}{3}}u}^{(j)}-B_s^{(j)}}{\sqrt{\varepsilon}u^{\frac{3}{2}}}\bigg)\,du\,ds. \tag{3.40}$$

Then we have that

$$\lim_{\varepsilon\to 0}\frac{\varepsilon^{\frac{d}{2}-1}}{\sqrt{\log(1/\varepsilon)}}\big\|I_2\big(h_{2,T}^{\varepsilon}\big)-\widetilde{J}_T^{\varepsilon}\big\|_{L^2(\Omega)}=0.$$

Proof. Using (2.13), we can easily check that

$$I_2\big(h_{2,T}^{\varepsilon}\big)=-\frac{(2\pi)^{-\frac{d}{2}}}{2}\sum_{j=1}^{d}\int_0^T\int_0^{T-u}\frac{u^{\frac{3}{2}}}{(\varepsilon+u^{\frac{3}{2}})^{\frac{d}{2}+1}}H_2\bigg(\frac{B_{s+u}^{(j)}-B_s^{(j)}}{u^{\frac{3}{4}}}\bigg)\,ds\,du.$$

Making the change of variables $v:=\varepsilon^{-\frac{2}{3}}u$, we can easily deduce that

$$\widetilde{J}_T^{\varepsilon}-I_2\big(h_{2,T}^{\varepsilon}\big)=-\frac{(2\pi)^{-\frac{d}{2}}\varepsilon^{-\frac{d}{2}+\frac{2}{3}}}{2}\sum_{j=1}^{d}\int_0^T\int_{\varepsilon^{-\frac{2}{3}}(T-s)}^{\infty}\frac{v^{\frac{3}{2}}}{(1+v^{\frac{3}{2}})^{\frac{d}{2}+1}}H_2\bigg(\frac{B_{s+\varepsilon^{\frac{2}{3}}u}^{(j)}-B_s^{(j)}}{\sqrt{\varepsilon}u^{\frac{3}{4}}}\bigg)\,du\,ds. \tag{3.41}$$

Set

$$\Phi^{\varepsilon}=\varepsilon^{d-2}\big\|\widetilde{J}_T^{\varepsilon}-I_2\big(h_{2,T}^{\varepsilon}\big)\big\|_{L^2(\Omega)}^2.$$

Using (3.41), as well as the fact that

$$\mathbb{E}\big[H_2\big(v_1^{-H}(B_{s_1+v_1}-B_{s_1})\big)H_2\big(v_2^{-H}(B_{s_2+v_2}-B_{s_2})\big)\big]=2(v_1v_2)^{-2H}\mu(s_2-s_1,v_1,v_2)^2, \tag{3.42}$$

for all $s_1,s_2,v_1,v_2\geq 0$, we can easily check that

$$\Phi^{\varepsilon}=\frac{d(2\pi)^{-d}}{2}\int_{[0,T]^2}\int_{\mathbb{R}_+^2}\mathbb{1}_{[T,\infty)}\big(s_1+\varepsilon^{\frac{2}{3}}u_1\big)\mathbb{1}_{[T,\infty)}\big(s_2+\varepsilon^{\frac{2}{3}}u_2\big)V_{\varepsilon,s_2-s_1}(u_1,u_2)\,d\vec{u}\,d\vec{s},$$

where

$$V_{\varepsilon,x}(u_1,u_2):=\varepsilon^{-\frac{8}{3}}\psi(u_1,u_2)\mu\big(x,\varepsilon^{\frac{2}{3}}u_1,\varepsilon^{\frac{2}{3}}u_2\big)^2,$$

and

$$\psi(u_1,u_2):=\big(1+u_1^{\frac{3}{2}}\big)^{-\frac{d}{2}-1}\big(1+u_2^{\frac{3}{2}}\big)^{-\frac{d}{2}-1}. \tag{3.43}$$

Hence, using the fact that $\mu(x,v_1,v_2)=\mu(-x,v_2,v_1)$, we can write

$$\Phi^{\varepsilon}=d(2\pi)^{-d}\int_0^T\int_0^{s_2}\int_{\mathbb{R}_+^2}\mathbb{1}_{[T,\infty)}\big(s_1+\varepsilon^{\frac{2}{3}}u_1\big)\mathbb{1}_{[T,\infty)}\big(s_2+\varepsilon^{\frac{2}{3}}u_2\big)V_{\varepsilon,s_2-s_1}(u_1,u_2)\,d\vec{u}\,d\vec{s}. \tag{3.44}$$

Changing the coordinates (s_1,s_2,u_1,u_2) by $(s:=s_1,x:=s_2-s_1,u_1,u_2)$ in the expression (3.44), and then integrating the variable s, we obtain

$$\big|\Phi^{\varepsilon}\big|=d(2\pi)^{-d}\int_0^T\int_{\mathbb{R}_+^2}\big(T-\big(T-\varepsilon^{\frac{2}{3}}u_1\big)_+\vee\big(T-x-\varepsilon^{\frac{2}{3}}u_2\big)_+\big)V_{\varepsilon,x}(u_1,u_2)\,d\vec{u}\,dx,$$

and consequently, there exists a constant $C>0$ such that

$$\big|\Phi^{\varepsilon}\big|\leq C\int_0^T\int_{\mathbb{R}_+^2}r_{\varepsilon^{\frac{2}{3}}}(u_1)V_{\varepsilon,x}(u_1,u_2)\,d\vec{u}\,dx, \tag{3.45}$$

where $r_\delta(u_1) := T - (T - \delta u_1)_+$. Making the change of variable $v := \varepsilon^{-\frac{2}{3}} x$ in (3.45) and using the fact that $V_{\varepsilon, \varepsilon^{\frac{2}{3}} v}(u_1, u_2) = \varepsilon^{-\frac{2}{3}} G_{1,v}^{(1)}(u_1, u_2)$, we get

$$\left|\Phi^\varepsilon\right| \leq C \int_0^{\varepsilon^{-\frac{2}{3}} T} \int_{\mathbb{R}_+^2} r_{\varepsilon^{\frac{2}{3}}}(u_1) G_{1,v}^{(1)}(u_1, u_2)\, d\vec{u}\, dv.$$

Therefore, defining $N := \varepsilon^{-\frac{2}{3}}$, so that $\log(1/\varepsilon) = \frac{3 \log N}{2}$, we obtain

$$\frac{|\Phi^\varepsilon|}{\log(1/\varepsilon)} \leq \frac{2C}{3 \log N} \int_0^{NT} \int_{\mathbb{R}_+^2} r_{\frac{1}{N}}(u_1) G_{1,x}^{(1)}(u_1, u_2)\, d\vec{u}\, dx.$$

To bound the right-hand side of the previous relation we split the domain of integration as follows. Define the sets $\mathcal{S}_i$, for $i = 1, 2, 3$, by (3.18). Then

$$\limsup_{\varepsilon \to 0} \frac{|\Phi^\varepsilon|}{\log(1/\varepsilon)} \leq \frac{2C}{3} \sum_{i=1}^3 \limsup_{N \to \infty} \frac{1}{\log N} \int_0^{NT} \int_{\mathbb{R}_+^2} \mathbb{1}_{\mathcal{S}_i}(x, u_1, u_2) r_{\frac{1}{N}}(u_1) G_{1,x}^{(1)}(u_1, u_2)\, d\vec{u}\, dx. \tag{3.46}$$

By relations (2.29) and (2.38), there exists a constant $C > 0$, such that

$$G_{1,x}^{(1)}(u_1, u_2) \leq C (u_1 u_2)^{-\frac{3}{2}} \mu^2(x, u_1, u_2) \Theta_1(x, u_1, u_2)^{-\frac{d}{2}}. \tag{3.47}$$

Hence, by Lemma 5.3, the terms with $i = 1$ and $i = 2$ in the sum in the right-hand side of (3.46) converge to zero. From this observation, we conclude that there exists a constant $C > 0$, such that

$$\limsup_{\varepsilon \to 0} \frac{|\Phi^\varepsilon|}{\log(1/\varepsilon)} \leq \limsup_{N \to \infty} \frac{C}{\log N} \int_0^{NT} \int_{\mathbb{R}_+^2} \mathbb{1}_{\mathcal{S}_3}(x, u_1, u_2) r_{\frac{1}{N}}(u_1) G_{1,x}^{(1)}(u_1, u_2)\, d\vec{u}\, dx. \tag{3.48}$$

Using Lemma 5.2, we can easily show that there exists a constant $C > 0$, such for every $(x, u_1, u_2) \in \mathcal{S}_3$, the following inequality holds

$$G_{1,x}^{(1)}(u_1, u_2) = \psi(u_1, u_2) \mu(x, u_1, u_2)^2 \leq C \psi(u_1, u_2)(x + u_1 + u_2)^{-1} (u_1 u_2)^2, \tag{3.49}$$

where $\psi(u_1, u_2)$ is defined in (3.43). From (3.48) and (3.49), it follows that

$$\limsup_{\varepsilon \to 0} \frac{|\Phi^\varepsilon|}{\log(1/\varepsilon)} \leq \limsup_{N \to \infty} \frac{C}{\log N} \int_0^{NT} \int_{\mathbb{R}_+^2} r_{\frac{1}{N}}(u_1)(x + u_1 + u_2)^{-1} (u_1 u_2)^2 \psi(u_1, u_2)\, d\vec{u}\, dx.$$

In addition, we have that

$$\begin{aligned} &\limsup_{N \to \infty} \frac{1}{\log N} \int_0^1 \int_{\mathbb{R}_+^2} r_{\frac{1}{N}}(u_1)(x + u_1 + u_2)^{-1} (u_1 u_2)^2 \psi(u_1, u_2)\, d\vec{u}\, dx \\ &\quad \leq \limsup_{N \to \infty} \frac{T}{\log N} \int_0^1 \int_{\mathbb{R}_+^2} (u_1 + u_2)^{-1} (u_1 u_2)^2 \psi(u_1, u_2)\, d\vec{u} = 0, \end{aligned}$$

and consequently,

$$\limsup_{\varepsilon \to 0} \frac{|\Phi^\varepsilon|}{\log(1/\varepsilon)} \leq \limsup_{N \to \infty} \frac{C}{\log N} \int_1^{NT} \int_{\mathbb{R}_+^2} r_{\frac{1}{N}}(u_1) x^{-1} (u_1 u_2)^2 \psi(u_1, u_2)\, d\vec{u}\, dx.$$

For $\delta > 0$ fixed, let $M > 1$ be such that

$$\int_M^\infty \int_0^\infty (u_1 u_2)^2 \psi(u_1, u_2)\, d\vec{u} < \delta. \tag{3.50}$$

Using (3.50), as well as the fact that $r_{\frac{1}{N}}(u)$ is increasing on u, we obtain

$$\frac{1}{\log N} \int_1^{NT} \int_M^\infty \int_0^\infty x^{-1} (u_1 u_2)^2 \psi(u_1, u_2)\, d\vec{u}\, dx \le \delta \left(1 + \frac{\log(T)}{\log N}\right),$$

and

$$\begin{aligned}
&\limsup_{N\to\infty} \frac{1}{\log N} \int_1^{NT} \int_0^M \int_0^\infty r_{\frac{1}{N}}(u_1) x^{-1} (u_1 u_2)^2 \psi(u_1, u_2)\, d\vec{u}\, dx \\
&\quad \le \limsup_{N\to\infty} \left(1 + \frac{\log(T)}{\log N}\right) \int_{\mathbb{R}_+^2} r_{\frac{1}{N}}(M) (u_1 u_2)^2 \psi(u_1, u_2)\, d\vec{u} = 0.
\end{aligned}$$

As a consequence,

$$\limsup_{\varepsilon\to 0} \frac{|\Phi^\varepsilon|}{\log(1/\varepsilon)} \le C\delta.$$

Hence, taking $\delta \to 0$, we get

$$\lim_{\varepsilon\to 0} \frac{\Phi^\varepsilon}{\log(1/\varepsilon)} = 0,$$

as required. □

Finally, we describe the behavior of the covariance function of $I_2(h_{2,T}^\varepsilon)$ for the case $H = \frac{3}{4}$.

Theorem 3.5. *Let $T_1, T_2 \ge 0$ be fixed. Then, if $d \ge 3$ and $H = \frac{3}{4}$,*

$$\lim_{\varepsilon\to 0} \frac{\varepsilon^{d-2}}{\log(1/\varepsilon)} \mathbb{E}\big[I_2\big(h_{2,T_1}^\varepsilon\big) I_2\big(h_{2,T_2}^\varepsilon\big)\big] = \rho^2 (T_1 \wedge T_2),$$

where ρ is a finite constant defined by

$$\rho := \frac{\sqrt{3d}}{2^{\frac{d+5}{2}} \pi^{\frac{d}{2}}} \int_0^\infty \big(1 + u^{\frac{3}{2}}\big)^{-\frac{d}{2}-1} u^2\, du. \tag{3.51}$$

Proof. Consider the approximation $\widetilde{J}_T^\varepsilon$ of $I_2(h_{2,T}^\varepsilon)$, introduced in (3.40). By Lemma 3.4,

$$\lim_{\varepsilon\to 0} \frac{\varepsilon^{d-2}}{\log(1/\varepsilon)} \big\| \widetilde{J}_T^\varepsilon - I_2\big(h_{2,T}^\varepsilon\big)\big\|_{L^2(\Omega)}^2 \to 0.$$

Therefore, it suffices to show that

$$\lim_{\varepsilon\to 0} \frac{\varepsilon^{d-2}}{\log(1/\varepsilon)} \mathbb{E}\big[\widetilde{J}_{T_1}^\varepsilon \widetilde{J}_{T_2}^\varepsilon\big] = \rho^2 (T_1 \wedge T_2). \tag{3.52}$$

As in Lemma 3, to prove (3.52), it suffices to show that for each $a < b < \alpha < \beta$,

$$\lim_{\varepsilon\to 0} \frac{\varepsilon^{d-2}}{\log(1/\varepsilon)} \mathbb{E}\big[\big(\widetilde{J}_b^\varepsilon - \widetilde{J}_a^\varepsilon\big)\big(\widetilde{J}_\beta^\varepsilon - \widetilde{J}_\alpha^\varepsilon\big)\big] = 0, \tag{3.53}$$

and

$$\lim_{\varepsilon\to 0}\frac{\varepsilon^{d-2}}{\log(1/\varepsilon)}\mathbb{E}\big[\big(\widetilde{J}^{\varepsilon}_b-\widetilde{J}^{\varepsilon}_a\big)^2\big]=\rho^2(b-a). \tag{3.54}$$

First we prove (3.53). Set

$$\Phi^{\varepsilon}=\varepsilon^{d-2}\mathbb{E}\big[\big(\widetilde{J}^{\varepsilon}_b-\widetilde{J}^{\varepsilon}_a\big)\big(\widetilde{J}^{\varepsilon}_\beta-\widetilde{J}^{\varepsilon}_\alpha\big)\big].$$

Using (3.42) and (3.40), we can easily check that

$$\Phi^{\varepsilon}=\frac{d(2\pi)^{-d}}{2}\int_{\alpha}^{\beta}\int_a^b\int_{\mathbb{R}_+^2}V_{\varepsilon,s_2-s_1}(u_1,u_2)\,d\vec{u}\,d\vec{s}, \tag{3.55}$$

where

$$V_{\varepsilon,x}(u_1,u_2):=\varepsilon^{-\frac{8}{3}}\psi(u_1,u_2)\mu\big(x,\varepsilon^{\frac{2}{3}}u_1,\varepsilon^{\frac{2}{3}}u_2\big)^2,$$

and $\psi(u_1,u_2)$ is defined by (3.38). Changing the coordinates (s_1,s_2,u_1,u_2) by $(s:=s_1,x:=s_2-s_1,u_1,u_2)$ in (3.55), and then integrating the variable s, we can show that

$$\big|\Phi^{\varepsilon}\big|\le d(2\pi)^{-d}\beta\int_{\gamma}^{\beta}\int_{\mathbb{R}_+^2}V_{\varepsilon,x}(u_1,u_2)\,d\vec{u}\,dx, \tag{3.56}$$

where the constant γ is defined by $\gamma:=\alpha-b$. Making the change of variable $v:=\varepsilon^{-\frac{2}{3}}x$ and using the fact that

$$V_{\varepsilon,\varepsilon^{\frac{2}{3}}v}(u_1,u_2)=\varepsilon^{-\frac{2}{3}}G^{(1)}_{1,v}(u_1,u_2),$$

we get

$$\big|\Phi^{\varepsilon}\big|\le d(2\pi)^{-d}\beta\int_{\varepsilon^{-\frac{2}{3}}\gamma}^{\varepsilon^{-\frac{2}{3}}\beta}\int_{\mathbb{R}_+^2}G^{(1)}_{1,v}(u_1,u_2)\,d\vec{u}\,dv.$$

Therefore, defining $N:=\varepsilon^{-\frac{2}{3}}$, so that $\log(1/\varepsilon)=\frac{3\log N}{2}$, we obtain

$$\frac{|\Phi^{\varepsilon}|}{\log(1/\varepsilon)}\le\frac{2d(2\pi)^{-d}\beta}{3\log N}\int_{N\gamma}^{N\beta}\int_{\mathbb{R}_+^2}G^{(1)}_{1,x}(u_1,u_2)\,d\vec{u}\,dx.$$

To bound the right-hand side of the previous relation we split the domain of integration as follows. Define the sets $\mathcal{S}_i$, for $i=1,2,3$, by (3.18). Then, there exists $C>0$, such that

$$\limsup_{\varepsilon\to 0}\frac{|\Phi^{\varepsilon}|}{\log(1/\varepsilon)}\le\sum_{i=1}^{3}\limsup_{N\to\infty}\frac{C}{\log N}\int_{N\gamma}^{N\beta}\int_{\mathbb{R}_+^2}\mathbb{1}_{\mathcal{S}_i}(x,u_1,u_2)G^{(1)}_{1,x}(u_1,u_2)\,d\vec{u}\,dx. \tag{3.57}$$

Taking into account (3.47), by Lemma 5.3, the terms with $i=1$ and $i=2$ in the sum in the right-hand side of (3.57) converge to zero. From this observation, we conclude that

$$\limsup_{\varepsilon\to 0}\frac{|\Phi^{\varepsilon}|}{\log(1/\varepsilon)}\le\limsup_{N\to\infty}\frac{C}{\log N}\int_{N\gamma}^{N\beta}\int_{\mathbb{R}_+^2}\mathbb{1}_{\mathcal{S}_3}(x,u_1,u_2)G^{(1)}_{1,x}(u_1,u_2)\,d\vec{u}\,dx. \tag{3.58}$$

By Lemma 5.2, there exists $C>0$, such for every $(x,u_1,u_2)\in\mathcal{S}_3$,

$$G^{(1)}_{1,x}(u_1,u_2)=\psi(u_1,u_2)\mu(x,u_1,u_2)^2\le C\psi(u_1,u_2)x^{-1}(u_1u_2)^2. \tag{3.59}$$

From (3.58) and (3.59), we obtain

$$\limsup_{\varepsilon\to 0}\frac{|\Phi^{\varepsilon}|}{\log(1/\varepsilon)}\leq C\limsup_{N\to\infty}\frac{\log(N\beta)-\log(N\gamma)}{\log N}\int_{\mathbb{R}_+^2}\psi(u_1,u_2)(u_1u_2)^2\,d\vec{u},$$

for some constant $C>0$. The function $(1+u^{\frac{3}{2}})^{-\frac{d}{2}-1}u^2$ is integrable for u in $\mathbb{R}_+$ due to the condition $d\geq 3$, and hence, from the previous inequality we conclude that

$$\limsup_{\varepsilon\to 0}\big(\log(1/\varepsilon)\big)^{-1}\big|\Phi^{\varepsilon}\big|=0. \tag{3.60}$$

Relation (3.53) then follows from (3.60).

Next we prove (3.54). By taking $\alpha=a$ and $\beta=b$ in relation (3.55), we obtain

$$\varepsilon^{d-2}\mathbb{E}\big[\big(\widetilde{J}_b^{\varepsilon}-\widetilde{J}_a^{\varepsilon}\big)^2\big]=d(2\pi)^{-d}\int_a^b\int_a^{s_2}\int_{\mathbb{R}_+^2}V_{\varepsilon,s_2-s_1}(u_1,u_2)\,d\vec{u}\,d\vec{s}.$$

Changing the coordinates (s_1,s_2,t_1,t_2) by $(s_1,x:=\varepsilon^{-\frac{2}{3}}(s_2-s_1),u_1:=t_1-s_1,u_2:=t_2-s_2)$, integrating the variable s_1 and using the fact that $V_{\varepsilon,\varepsilon^{\frac{2}{3}}\hat{x}}(u_1,u_2)=\varepsilon^{-\frac{2}{3}}G_{1,\hat{x}}^{(1)}(u_1,u_2)$, we get

$$\varepsilon^{d-2}\mathbb{E}\big[\big(\widetilde{J}_b^{\varepsilon}-\widetilde{J}_a^{\varepsilon}\big)^2\big]=d(2\pi)^{-d}\int_0^{\varepsilon^{-\frac{2}{3}}(b-a)}\int_{\mathbb{R}_+^2}\big(b-\varepsilon^{\frac{2}{3}}x-a\big)G_{1,x}^{(1)}(u_1,u_2)\,d\vec{u}\,dx.$$

Therefore, defining $N:=\varepsilon^{-\frac{2}{3}}$, so that $\log(1/\varepsilon)=\frac{3\log N}{2}$, we obtain

$$\begin{aligned}&\frac{\varepsilon^{d-2}}{\log(1/\varepsilon)}\mathbb{E}\big[\big(\widetilde{J}_b^{\varepsilon}-\widetilde{J}_a^{\varepsilon}\big)^2\big]\\&\quad=\frac{2d(2\pi)^{-d}}{3\log N}\sum_{i=1}^{3}\int_0^{N(b-a)}\int_{\mathbb{R}_+^2}\left(b-\frac{x}{N}-a\right)\mathbb{1}_{\mathcal{S}_i}(x,u_1,u_2)G_{1,x}^{(1)}(u_1,u_2)\,d\vec{u}\,dx.\end{aligned} \tag{3.61}$$

By inequality (3.47) and Lemma 5.3, the terms with $i=1$ and $i=2$ in the sum in the right-hand side of (3.61) converge to zero. From this observation, it follows that

$$\begin{aligned}&\lim_{\varepsilon\to 0}\frac{\varepsilon^{d-2}}{\log(1/\varepsilon)}\mathbb{E}\big[\big(\widetilde{J}_b^{\varepsilon}-\widetilde{J}_a^{\varepsilon}\big)^2\big]\\&\quad=\lim_{N\to\infty}\frac{2d(2\pi)^{-d}}{3\log N}\int_0^{N(b-a)}\int_{\mathbb{R}_+^2}(b-a)\mathbb{1}_{\mathcal{S}_3}(x,u_1,u_2)G_{1,x}^{(1)}(u_1,u_2)\,d\vec{u}\,dx\\&\qquad-\lim_{N\to\infty}\frac{2d(2\pi)^{-d}}{3N\log N}\int_0^{N(b-a)}\int_{\mathbb{R}_+^2}\mathbb{1}_{\mathcal{S}_3}(x,u_1,u_2)xG_{1,x}^{(1)}(u_1,u_2)\,d\vec{u}\,dx,\end{aligned} \tag{3.62}$$

provided that the limits in the right-hand side exist. By (3.59), there exists a constant $C>0$ such that

$$\begin{aligned}&\frac{1}{N\log N}\int_0^{N(b-a)}\int_{\mathbb{R}_+^2}\mathbb{1}_{\mathcal{S}_3}(x,u_1,u_2)xG_{1,x}^{(1)}(u_1,u_2)\,d\vec{u}\,dx\\&\quad\leq\frac{C}{N\log N}\int_0^{N(b-a)}\int_{\mathbb{R}_+^2}\psi(u_1,u_2)(u_1u_2)^2\,d\vec{u}\,dx=\frac{C(b-a)}{\log N}\int_{\mathbb{R}_+^2}\psi(u_1,u_2)(u_1u_2)^2\,d\vec{u}.\end{aligned}$$

Since $d \geq 3$, the integral in the right-hand side is finite, and hence

$$\lim_{N\to\infty} \frac{1}{N\log N}\int_0^{N(b-a)}\int_{\mathbb{R}_+^2} \mathbb{1}_{\mathcal{S}_3}(x,u_1,u_2)xG_{1,x}^{(1)}(u_1,u_2)\,d\vec{u}\,dx = 0.$$

Therefore, by equation (3.62) and L'Hôpital rule,

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\frac{\varepsilon^{d-2}}{\log(1/\varepsilon)}\mathbb{E}\big[\big(\widetilde{J}_b^\varepsilon - \widetilde{J}_a^\varepsilon\big)^2\big] \\
&\quad = \lim_{N\to\infty}\frac{2d(2\pi)^{-d}}{3\log N}\int_0^{N(b-a)}\int_{\mathbb{R}_+^2}(b-a)\mathbb{1}_{\mathcal{S}_3}(x,u_1,u_2)G_{1,x}^{(1)}(u_1,u_2)\,d\vec{u}\,dx \\
&\quad = \lim_{N\to\infty}\frac{2d(2\pi)^{-d}}{3}\int_{\mathbb{R}_+^2} N(b-a)^2\mathbb{1}_{\mathcal{S}_3}\big(N(b-a),u_1,u_2\big)G_{1,N(b-a)}^{(1)}(u_1,u_2)\,d\vec{u}\,dx. \qquad (3.63)
\end{aligned}$$

By (3.59), the integrand in the right-hand side is bounded by $C\psi(u_1,u_2)(u_1u_2)^2$, for some constant $C > 0$. On the other hand, using (2.5), we can easily check that

$$\begin{aligned}
\mu(x,v_1,v_2) = \big\langle \mathbb{1}_{[0,v_1]}, \mathbb{1}_{[x,x+v_2]}\big\rangle_{\mathfrak{H}} &= H(2H-1)v_1v_2\int_{[0,1]^2}|x+v_2w_2-v_1w_1|^{2H-2}\,d\vec{w} \\
&= \frac{3v_1v_2}{8}\int_{[0,1]^2}|x+v_2w_2-v_1w_1|^{-\frac{1}{2}}\,d\vec{w}, \qquad (3.64)
\end{aligned}$$

so that

$$\lim_{N\to\infty} N(b-a)\mu\big(N(b-a),u_1,u_2\big)^2 = \frac{3^2(u_1u_2)^2}{2^6}, \qquad (3.65)$$

and hence,

$$\lim_{N\to\infty} N(b-a)\mathbb{1}_{\mathcal{S}_3}\big(N(b-a),u_1,u_2\big)G_{1,N(b-a)}^{(1)}(u_1,u_2) = \frac{3^2}{2^6}\psi(u_1,u_2)(u_1u_2)^2.$$

Therefore, by applying the dominated convergence theorem to (3.63), we get

$$\lim_{\varepsilon\to 0}\frac{\varepsilon^{d-2}}{\log(1/\varepsilon)}\mathbb{E}\big[\big(\widetilde{J}_b^\varepsilon - \widetilde{J}_a^\varepsilon\big)^2\big] = (b-a)\frac{3d}{2^{d+5}\pi^d}\bigg(\int_{\mathbb{R}_+}\big(1+u^{\frac{3}{2}}\big)^{-\frac{d}{2}-1}u^2\,du\bigg)^2. \qquad \square$$

Relation (3.54) follows from the previous inequality. The proof is now complete.

4. Proof of Theorems 1.2, 1.3 and 1.4

In the sequel, $W = \{W_t\}_{t\geq 0}$ will denote a standard one-dimensional Brownian motion independent of B, and $X^j = \{X_t^j\}_{t\geq 0}$ will denote the second order Hermite process introduced in Section 2.

Proof of Theorem 1.2. We start with the proof of Theorem 1.2, which will be done in two steps.

Step 1. First we prove the convergence of the finite dimensional distributions, namely, we will show that for every $r \in \mathbb{N}$, and $T_1,\dots,T_r \geq 0$ fixed, it holds

$$\varepsilon^{\frac{d}{2}-\frac{3}{4H}}\big(\big(I_{T_1}^\varepsilon,\dots,I_{T_r}^\varepsilon\big) - \mathbb{E}\big[\big(I_{T_1}^\varepsilon,\dots,I_{T_r}^\varepsilon\big)\big]\big) \stackrel{\text{Law}}{\to} \sigma(W_{T_1},\dots,W_{T_r}), \qquad (4.1)$$

as $\varepsilon \to 0$, where σ is the finite constant defined by (3.3). To this end, define the kernels h_{2q,T_i}^ε by (2.16), and the constants σ_q^2 by (3.1), for $q \in \mathbb{N}$. Notice that the constants σ_q^2 are well defined due to the condition $\frac{3}{2d} < H < \frac{3}{4}$.

Define as well the matrices $C_q = \{C_q^{i,j} \mid 1 \le i, j \le r\}$ and $C = \{C^{i,j} \mid 1 \le i, j \le r\}$, by $C_q^{i,j} := \sigma_q^2(T_i \wedge T_j)$, and $C^{i,j} := \sigma^2(T_i \wedge T_j)$. Since $I_{T_i}^{\varepsilon}$ has chaos decomposition (2.15), by Theorem 2.1, we deduce that in order to prove the convergence (4.1), it suffices to show the following properties:

(i) For every fixed $q \ge 1$, and $1 \le i, j \le r$, we have

$$\varepsilon^{d-\frac{3}{2H}}(2q)!\langle h_{2q,T_i}^{\varepsilon}, h_{2q,T_j}^{\varepsilon}\rangle_{(\mathfrak{H}^d)^{\otimes 2q}} \to \sigma_q^2(T_i \wedge T_j), \quad \text{as } \varepsilon \to 0.$$

(ii) The constants σ_q^2 satisfy $\sum_{q=1}^{\infty} \sigma_q^2 = \sigma^2$. In particular, $C^{i,j} = \lim_{Q\to\infty} \sum_{q=1}^{Q} C_q^{i,j}$,

(iii) For all $q \ge 1$ and $i = 1, \dots, r$, the random variables $\varepsilon^{\frac{d}{2}-\frac{3}{4H}} I_{2q}(h_{2q,T_i}^{\varepsilon})$ converge in law to a centered Gaussian distribution as $\varepsilon \to 0$,

(iv) $\lim_{Q\to\infty} \sup_{\varepsilon\in(0,1)} \varepsilon^{d-\frac{3}{2H}} \sum_{q=Q}^{\infty} (2q)!\|h_{2q,T_i}^{\varepsilon}\|_{(\mathfrak{H}^d)^{\otimes 2q}}^2 = 0$, for every $i = 1, \dots, r$.

Part (i) follows from Theorem 3.1. Condition (ii) follows from equation (3.2). In [5, Theorem 2], it was proved that for $T > 0$ fixed, $\varepsilon^{\frac{d}{2}-\frac{3}{4H}} I_{2q}(h_{2q,T}^{\varepsilon})$ converges in law to a centered Gaussian random variable when $\varepsilon \to 0$, and

$$\lim_{Q\to\infty} \sup_{\varepsilon\in(0,1)} \varepsilon^{d-\frac{3}{2H}} \sum_{q=Q}^{\infty} (2q)! \left\| h_{2q,T}^{\varepsilon} \right\|_{(\mathfrak{H}^d)^{\otimes 2q}}^2 = 0,$$

which proves conditions (iii) and (iv). This finishes the proof of (4.1).

Step 2. We are going to show the tightness of the sequence of processes $\{\varepsilon^{\frac{d}{2}-\frac{3}{4H}}(I_T^{\varepsilon} - \mathbb{E}[I_T^{\varepsilon}])\}_{T\ge 0}$. To this end, we will prove that there exists a sufficiently small $p > 2$, depending only on d and H, such that for every $0 \le T_1 \le T_2$, it holds

$$\sup_{\varepsilon\in(0,1)} \mathbb{E}\left[\left|\varepsilon^{\frac{d}{2}-\frac{3}{4H}}\left(I_{T_2}^{\varepsilon} - \mathbb{E}\left[I_{T_2}^{\varepsilon}\right] - \left(I_{T_1}^{\varepsilon} - \mathbb{E}\left[I_{T_1}^{\varepsilon}\right]\right)\right)\right|^p\right] \le C|T_2 - T_1|^{\frac{p}{2}}, \tag{4.2}$$

for some constant $C > 0$ only depending on d, p and H. The tightness property for $\{\varepsilon^{\frac{d}{2}-\frac{3}{4H}}(I_T^{\varepsilon} - \mathbb{E}[I_T^{\varepsilon}])\}_{T\ge 0}$ then follows from the Billingsley criterion (see [2, Theorem 12.3]).

In order to prove (4.2) we proceed as follows. Define, for $0 \le T_1 \le T_2$ fixed, the random variable $Z_{\varepsilon} = Z_{\varepsilon}(T_1, T_2)$, by

$$Z_{\varepsilon} := I_{T_2}^{\varepsilon} - \mathbb{E}\left[I_{T_2}^{\varepsilon}\right] - \left(I_{T_1}^{\varepsilon} - \mathbb{E}\left[I_{T_1}^{\varepsilon}\right]\right). \tag{4.3}$$

From the chaos decomposition (2.15), we can easily check that $\mathbb{E}[DL^{-1}Z_{\varepsilon}]$ coincides with the derivative of the first chaotic component of $L^{-1}Z_{\varepsilon}$, which is identically zero 0. Hence, by (2.2), there exists a constant $c_p > 0$ such that

$$\|Z_{\varepsilon}\|_{L^p(\Omega)} \le c_p \left\| D^2 L^{-1} Z_{\varepsilon} \right\|_{L^p(\Omega;(\mathfrak{H}^d)^{\otimes 2})}. \tag{4.4}$$

The right-hand side of the previous inequality can be estimated as follows. From (2.4), we can easily check that

$$D^2 L^{-1} Z_{\varepsilon} = \int_0^{\infty} \int_{\mathcal{K}_{T_1,T_2}} D^2 P_{\theta}\left[p_{\varepsilon}(B_t - B_s)\right] ds\, dt\, d\theta, \tag{4.5}$$

where $\mathcal{K}_{T_1,T_2}$ is defined by (2.31). Let $\widetilde{B}$ be an independent copy of B. Using Mehler's formula (2.3) and the semigroup property of the heat kernel, we obtain

$$\begin{aligned} P_{\theta}\left[p_{\varepsilon}(B_t - B_s)\right] &= \widetilde{\mathbb{E}}\left[p_{\varepsilon}\left(e^{-\theta}(B_t - B_s) + \sqrt{1 - e^{-2\theta}}(\widetilde{B}_t - \widetilde{B}_s)\right)\right] \\ &= p_{\lambda_{\varepsilon}(\theta,s,t)}\left(e^{-\theta}(B_t - B_s)\right), \end{aligned} \tag{4.6}$$

where the function $\lambda_{\varepsilon} = \lambda_{\varepsilon}(\theta, s, t)$ is defined by

$$\lambda_{\varepsilon}(\theta, s, t) := \varepsilon + \left(1 - e^{-2\theta}\right)(t - s)^{2H}. \tag{4.7}$$

This implies that for every multi-index $\mathbf{i}=(i_1,i_2)$, with $1\le i_1,i_2\le d$, we have

$$D^2P_\theta\big[p_\varepsilon(B_t-B_s)\big](\mathbf{i},x_1,x_2)=e^{-2\theta}\mathbb{1}_{[s,t]}(x_1)\mathbb{1}_{[s,t]}(x_2)\times\lambda_\varepsilon(\theta,s,t)^{-1}p_{\lambda_\varepsilon(\theta,s,t)}\big(e^{-\theta}(B_t-B_s)\big)g_{\mathbf{i},\lambda_\varepsilon(\theta,s,t)}\big(e^{-\theta}(B_t-B_s)\big), \tag{4.8}$$

where the function $g_{\mathbf{i},\lambda}$, for $\lambda>0$, is defined by

$$g_{\mathbf{i},\lambda}(x_1,\dots,x_d)=\begin{cases}\lambda^{-1}x_{i_1}^2-1 & \text{if } i_1=i_2,\\ \lambda^{-1}x_{i_1}x_{i_2} & \text{if } i_1\ne i_2.\end{cases}$$

From (4.5) and (4.8), we deduce that

$$\begin{aligned}\big\|D^2L^{-1}Z_\varepsilon\big\|^2_{(\mathfrak{H}^d)^{\otimes 2}}=\int_{\mathbb{R}_+^2}\int_{\mathcal{K}^2_{T_1,T_2}}&e^{-2\theta-2\beta}\mu(s_2-s_1,t_1-s_1,t_2-s_2)^2\\&\times\big(\lambda_\varepsilon(\theta,s_1,t_1)\lambda_\varepsilon(\beta,s_2,t_2)\big)^{-1}p_{\lambda_\varepsilon(\theta,s_1,t_1)}\big(e^{-\theta}(B_{t_1}-B_{s_1})\big)\\&\times p_{\lambda_\varepsilon(\beta,s_2,t_2)}\big(e^{-\beta}(B_{t_2}-B_{s_2})\big)\sum_{\mathbf{i}}g_{\mathbf{i},\lambda_\varepsilon(\theta,s_1,t_1)}\big(e^{-\theta}(B_{t_1}-B_{s_1})\big)\\&\times g_{\mathbf{i},\lambda_\varepsilon(\beta,s_2,t_2)}\big(e^{-\beta}(B_{t_2}-B_{s_2})\big)\,d\vec{s}\,d\vec{t}\,d\theta\,d\beta,\end{aligned}\tag{4.9}$$

where the sum runs over all the possible multi-indices $\mathbf{i}=(i_1,i_2)$, with $1\le i_1,i_2\le d$. Using Minkowski inequality, as well as (4.4) and (4.9), we deduce that

$$\begin{aligned}\|Z_\varepsilon\|^2_{L^p(\Omega)}&\le c_p^2\big\|D^2L^{-1}Z_\varepsilon\big\|^2_{L^p(\Omega;(\mathfrak{H}^d)^{\otimes 2})}=c_p^2\Big\|\big\|D^2L^{-1}Z_\varepsilon\big\|^2_{(\mathfrak{H}^d)^{\otimes 2}}\Big\|_{L^{\frac{p}{2}}(\Omega)}\\&\le c_p^2\int_{\mathbb{R}_+^2}\int_{\mathcal{K}^2_{T_1,T_2}}e^{-2\theta-2\beta}\mu(s_2-s_1,t_1-s_1,t_2-s_2)^2\\&\quad\times\big(\lambda_\varepsilon(\theta,s_1,t_1)\lambda_\varepsilon(\beta,s_2,t_2)\big)^{-1}\Big\|p_{\lambda_\varepsilon(\theta,s_1,t_1)}\big(e^{-\theta}(B_{t_1}-B_{s_1})\big)\\&\quad\times p_{\lambda_\varepsilon(\beta,s_2,t_2)}\big(e^{-\beta}(B_{t_2}-B_{s_2})\big)\sum_{\mathbf{i}}g_{\mathbf{i},\lambda_\varepsilon(\theta,s_1,t_1)}\big(e^{-\theta}(B_{t_1}-B_{s_1})\big)\\&\quad\times g_{\mathbf{i},\lambda_\varepsilon(\beta,s_2,t_2)}\big(e^{-\beta}(B_{t_2}-B_{s_2})\big)\Big\|_{L^{\frac{p}{2}}(\Omega)}\,d\vec{s}\,d\vec{t}\,d\theta\,d\beta.\end{aligned}\tag{4.10}$$

Next we bound the $L^{\frac{p}{2}}(\Omega)$-norm in the right-hand side of the previous inequality. Let $y\in(0,1)$ be fixed. We can easily check that there exists a constant $C>0$ only depending on y, such that for every $\lambda_1,\lambda_2>0$ and $\eta,\xi\in\mathbb{R}^d$, and every multi-index $\mathbf{i}=(i_1,i_2)$, with $1\le i_1,i_2\le d$,

$$\big|g_{\mathbf{i},\lambda_1}(\eta)g_{\mathbf{i},\lambda_2}(\xi)\big|\le\big(1+\lambda_1^{-1}\|\eta\|^2\big)\big(1+\lambda_2^{-1}\|\xi\|^2\big)\le Ce^{\frac{y}{2}(\lambda_1^{-1}\|\eta\|^2+\lambda_2^{-1}\|\xi\|^2)}.\tag{4.11}$$

From (4.10) and (4.11), it follows that there exists a constant $C>0$, not depending on ε,T_1,T_2, such that

$$\begin{aligned}\|Z_\varepsilon\|^2_{L^p(\Omega)}\le C\int_{\mathbb{R}_+^2}\int_{\mathcal{K}^2_{T_1,T_2}}&e^{-2\theta-2\beta}\mu(s_2-s_1,t_1-s_1,t_2-s_2)^2\\&\times\big(\lambda_\varepsilon(\theta,s_1,t_1)\lambda_\varepsilon(\beta,s_2,t_2)\big)^{-1}\\&\times\Big\|p_{\frac{\lambda_\varepsilon(\theta,s_1,t_1)}{1-y}}\big(e^{-\theta}(B_{t_1}-B_{s_1})\big)p_{\frac{\lambda_\varepsilon(\beta,s_2,t_2)}{1-y}}\big(e^{-\beta}(B_{t_2}-B_{s_2})\big)\Big\|_{L^{\frac{p}{2}}(\Omega)}\,d\vec{s}\,d\vec{t}\,d\theta\,d\beta.\end{aligned}\tag{4.12}$$

Proceeding as in the proof of (2.34), we can easily check that the quantity

$$S := \mathbb{E}\big[p_{\frac{\lambda_\varepsilon(\theta,s_1,t_1)}{1-y}}\big(e^{-\theta}(B_{t_1}-B_{s_1})\big)^{\frac{p}{2}} p_{\frac{\lambda_\varepsilon(\beta,s_2,t_2)}{1-y}}\big(e^{-\beta}(B_{t_2}-B_{s_2})\big)^{\frac{p}{2}}\big],$$

satisfies

$$S = (2\pi)^{-\frac{d(p-2)}{2}}\left(\frac{\lambda_\varepsilon(\theta,s_1,t_1)\lambda_\varepsilon(\beta,s_2,t_2)}{(1-y)^2}\right)^{-\frac{dp}{4}+\frac{d}{2}}\frac{2^d}{p^d}e^{d(\theta+\beta)} \\ \times\left|\frac{2}{p(1-y)}\begin{pmatrix}\lambda_\varepsilon(\theta,s_1,t_1)e^{2\theta} & 0\\ 0 & \lambda_\varepsilon(\beta,s_2,t_2)e^{2\beta}\end{pmatrix}+\Sigma\right|^{-\frac{d}{2}},$$

where $\Sigma=\{\Sigma_{i,j}\}_{1\le i,j\le 2}$, denotes the covariance matrix of $(B^{(1)}_{t_1}-B^{(1)}_{s_1}, B^{(1)}_{t_2}-B^{(1)}_{s_2})$, whose components are given by $\Sigma_{1,1}=(t_1-s_1)^{2H}$, $\Sigma_{1,2}=\Sigma_{2,1}=\mu(s_2-s_1,t_1-s_1,t_2-s_2)$, and $\Sigma_{2,2}=(t_2-s_2)^{2H}$. Therefore, there exists a constant $C>0$ only depending on p and d, such that

$$S\le C\big(\lambda_\varepsilon(\theta,s_1,t_1)\lambda_\varepsilon(\beta,s_2,t_2)\big)^{-\frac{dp}{4}+\frac{d}{2}} \\ \times e^{d(\theta+\beta)}\left|\frac{2}{p(1-y)}\begin{pmatrix}\lambda_\varepsilon(\theta,s_1,t_1)e^{2\theta} & 0\\ 0 & \lambda_\varepsilon(\beta,s_2,t_2)e^{2\beta}\end{pmatrix}+\Sigma\right|^{-\frac{d}{2}}.$$

Choosing $y<1-\frac{2}{p}$, so that $\frac{p(1-y)}{2}\Sigma\ge\Sigma$, we deduce that there exists a constant $C>0$ only depending on p, y and d, such that

$$S\le C\big(\lambda_\varepsilon(\theta,s_1,t_1)\lambda_\varepsilon(\beta,s_2,t_2)\big)^{-\frac{dp}{4}+\frac{d}{2}} \\ \times e^{d(\theta+\beta)}\left|\begin{pmatrix}\lambda_\varepsilon(\theta,s_1,t_1)e^{2\theta}+(t_1-s_1)^{2H} & \mu(s_2-s_1,t_1-s_2,t_2-s_2)\\ \mu(s_2-s_1,t_1-s_2,t_2-s_2) & \lambda_\varepsilon(\beta,s_2,t_2)e^{2\beta}+(t_2-s_2)^{2H}\end{pmatrix}\right|^{-\frac{d}{2}}.$$

Hence, by the multilinearity of the determinant function,

$$S\le C\big(\lambda_\varepsilon(\theta,s_1,t_1)\lambda_\varepsilon(\beta,s_2,t_2)\big)^{-\frac{dp}{4}+\frac{d}{2}} \\ \times\left|\begin{pmatrix}\lambda_\varepsilon(\theta,s_1,t_1)+e^{-2\theta}(t_1-s_1)^{2H} & e^{-2\beta}\mu(s_2-s_1,t_1-s_2,t_2-s_2)\\ e^{-2\theta}\mu(s_2-s_1,t_1-s_2,t_2-s_2) & \lambda_\varepsilon(\beta,s_2,t_2)+e^{-2\beta}(t_2-s_2)^{2H}\end{pmatrix}\right|^{-\frac{d}{2}}. \tag{4.13}$$

By relation (4.7), we have that $\lambda_\varepsilon(\theta,s,t)+e^{-2\theta}(t-s)^{2H}=\varepsilon+(t-s)^{2H}$ for every $\theta,s,t>0$. As a consequence, relation (4.13) can be written as

$$\begin{aligned}S&\le C\big(\lambda_\varepsilon(\theta,s_1,t_1)\lambda_\varepsilon(\beta,s_2,t_2)\big)^{-\frac{dp}{4}+\frac{d}{2}} \\ &\quad\times\big(\varepsilon^2+\varepsilon\big((t_1-s_1)^{2H}+(t_2-s_2)^{2H}\big)+(t_1-s_1)^{2H}(t_2-s_2)^{2H}-e^{-2\beta-2\theta}\mu^2\big)^{-\frac{d}{2}} \\ &\le C\big(\lambda_\varepsilon(\theta,s_1,t_1)\lambda_\varepsilon(\beta,s_2,t_2)\big)^{-\frac{dp}{4}+\frac{d}{2}}\Theta_\varepsilon(s_2-s_1,t_1-s_1,t_2-s_2)^{-\frac{d}{2}},\end{aligned} \tag{4.14}$$

where $\Theta_\varepsilon(x,u_1,u_2)$ is defined by (2.35). From (4.7), (4.12) and (4.14), it follows that

$$\begin{aligned}\|Z_\varepsilon\|^2_{L^p(\Omega)}&\le C\int_{\mathbb{R}^2_+}\int_{\mathcal{S}^2_{T_1,T_2}}e^{-2\theta-2\beta}\mu(s_2-s_1,t_1-s_1,t_2-s_2)^2 \\ &\quad\times\Big(\big(\varepsilon+\big(1-e^{-2\theta}\big)(t_1-s_1)^{2H}\big)\big(\varepsilon+\big(1-e^{-2\beta}\big)(t_2-s_2)^{2H}\big)\Big)^{-1-\frac{d}{2}+\frac{d}{p}} \\ &\quad\times\Theta_\varepsilon(s_2-s_1,t_1-s_1,t_2-s_2)^{-\frac{d}{p}}\,d\vec{s}\,d\vec{t}\,d\theta\,d\beta.\end{aligned} \tag{4.15}$$

Changing the coordinates (s_1, t_1, s_2, t_2) by $(s_1, x := s_2 - s_1, u_1 := t_1 - s_1, u_2 := t_2 - s_2)$ in (4.15), we get

$$\|Z_\varepsilon\|^2_{L^p(\Omega)} \le 2C \int_{\mathbb{R}^2_+} e^{-2\theta-2\beta} \int_{[0,T_2]^3} \int_{(T_1-u_1)_+ \vee (T_1-x-u_2)_+}^{(T_2-u_1)_+ \wedge (T_2-x-u_2)_+} ds_1$$
$$\times \mu(x, u_1, u_2)^2 \big(\big(\varepsilon + \big(1 - e^{-2\theta}\big)u_1^{2H}\big)\big(\varepsilon + \big(1 - e^{-2\beta}\big)u_2^{2H}\big)\big)^{-1-\frac{d}{2}+\frac{d}{p}}$$
$$\times \Theta_\varepsilon(x, u_1, u_2)^{-\frac{d}{p}}\, dx\, d\vec{u}\, d\theta\, d\beta.$$

Integrating the variable s_1, and making the change of variables $\eta := 1 - e^{-2\theta}$, and $\xi := 1 - e^{-2\beta}$, we deduce that there exists a constant $C > 0$, such that

$$\|Z_\varepsilon\|^2_{L^p(\Omega)} \le C(T_2 - T_1) \int_{[0,T_2]^3} \mu(x, u_1, u_2)^2 \Theta_\varepsilon(x, u_1, u_2)^{-\frac{d}{p}}$$
$$\times \int_{[0,1]^2} \big(\big(\varepsilon + \eta u_1^{2H}\big)\big(\varepsilon + \xi u_2^{2H}\big)\big)^{-1-\frac{d}{2}+\frac{d}{p}}\, d\eta\, d\xi\, dx\, d\vec{u}. \tag{4.16}$$

Changing the coordinates (x, u_1, u_2) by $(\varepsilon^{-\frac{1}{2H}}x, \varepsilon^{-\frac{1}{2H}}u_1, \varepsilon^{-\frac{1}{2H}}u_2)$ in (4.16), and using the fact that $\Theta_\varepsilon(\varepsilon^{-\frac{1}{2H}}x, \varepsilon^{-\frac{1}{2H}}u_1, \varepsilon^{-\frac{1}{2H}}u_2) = \varepsilon^2\Theta_1(x, u_1, u_2)$, we get

$$\big\|\varepsilon^{\frac{d}{2}-\frac{3}{4H}} Z_\varepsilon\big\|^2_{L^p(\Omega)} \le C(T_2 - T_1) \int_{\mathbb{R}^3_+} \mu(x, u_1, u_2)^2 \Theta_1(x, u_1, u_2)^{-\frac{d}{p}}$$
$$\times \int_{[0,1]^2} \big(\big(1 + \eta u_1^{2H}\big)\big(1 + \xi u_2^{2H}\big)\big)^{-1-\frac{d}{2}+\frac{d}{p}}\, d\eta\, d\xi\, dx\, d\vec{u}.$$

Integrating the variables η and ξ, we obtain

$$\big\|\varepsilon^{\frac{d}{2}-\frac{3}{4H}} Z_\varepsilon\big\|^2_{L^p(\Omega)} \le C\left(\frac{d}{2} - \frac{d}{p}\right)^{-2} (T_2 - T_1) \int_{\mathbb{R}^3_+} (u_1u_2)^{-2H} \mu(x, u_1, u_2)^2 \Theta_1(x, u_1, u_2)^{-\frac{d}{p}}$$
$$\times \big(1 - \big(1 + u_1^{2H}\big)^{-\frac{d}{2}+\frac{d}{p}}\big)\big(1 - \big(1 + u_2^{2H}\big)\big)^{-\frac{d}{2}+\frac{d}{p}}\, dx\, d\vec{u}.$$

Hence, choosing $p > 2$, we deduce that there exists a constant C only depending on H, d and p, such that

$$\big\|\varepsilon^{\frac{d}{2}-\frac{3}{4H}} Z_\varepsilon\big\|^2_{L^p(\Omega)} \le C(T_2 - T_1) \int_{\mathbb{R}^3_+} \frac{\mu(x, u_1, u_2)^2}{u_1^{2H}u_2^{2H}} \Theta_1(x, u_1, u_2)^{-\frac{d}{p}}\, dx\, d\vec{u}. \tag{4.17}$$

Since $Hd > \frac{3}{2}$, we can choose p so that $2 < p < \frac{4Hd}{3}$. For this choice of p, the integral in the right-hand side of (4.17) is finite by Lemma 5.3. Therefore, from (4.17), it follows that there exists a constant $C > 0$, independent of T_1, T_2 and ε, such that $\|\varepsilon^{\frac{d}{2}-\frac{3}{4H}} Z_\varepsilon\|^2_{L^p(\Omega)} \le C(T_2 - T_1)$, which in turn implies that

$$\mathbb{E}\big[\big|\varepsilon^{\frac{d}{2}-\frac{3}{4H}} Z_\varepsilon\big|^p\big] \le C(T_2 - T_1)^{\frac{p}{2}}. \tag{4.18}$$

Relation (4.2) then follows from (4.18). This finishes the proof of Theorem 1.2. □

Proof of Theorem 1.3. Now we proceed with the proof of Theorem 1.3, in which we will prove (1.5) and (1.7) in the case $H > \frac{3}{4}$. In order to prove (1.5), it suffices to show that for every $T > 0$,

$$\varepsilon^{\frac{d}{2}-\frac{3}{2H}+1}\big(I_T^\varepsilon - \mathbb{E}\big[I_T^\varepsilon\big] - I_2\big(h_{2,T}^\varepsilon\big)\big) \stackrel{L^2(\Omega)}{\to} 0, \tag{4.19}$$

and

$$\varepsilon^{\frac{d}{2}-\frac{3}{2H}+1} I_2\big(h_{2,T}^{\varepsilon}\big) \overset{L^2(\Omega)}{\to} -\Lambda \sum_{j=1}^{d} X_T^j, \tag{4.20}$$

as $\varepsilon \to 0$. Relation (4.19) follows from Lemma 3.2. In order to prove the convergence (4.20) we proceed as follows. Using (2.13), we can easily check that

$$I_2\big(h_{2,T}^{\varepsilon}\big) = -\frac{(2\pi)^{-\frac{d}{2}}}{2} \sum_{j=1}^{d} \int_0^T \int_0^{T-u} \big(\varepsilon + u^{2H}\big)^{-\frac{d}{2}-1} u^{2H} H_2\big(u^{-H}\big(B_{s+u}^{(j)} - B_s^{(j)}\big)\big)\, ds\, du.$$

Making the change of variable $v := \varepsilon^{-\frac{1}{2H}} u$, we get

$$\begin{aligned}
&\varepsilon^{\frac{d}{2}-\frac{3}{2H}+1} I_2\big(h_{2,T}^{\varepsilon}\big) \\
&\quad = -\frac{(2\pi)^{-\frac{d}{2}}}{2} \sum_{j=1}^{d} \int_0^{\varepsilon^{-\frac{1}{2H}} T} \int_0^{T-\varepsilon^{\frac{1}{2H}} v} \big(1 + v^{2H}\big)^{-\frac{d}{2}-1} v^{2H} \varepsilon^{1-\frac{1}{H}} H_2\bigg(\frac{B^{(j)}_{s+\varepsilon^{\frac{1}{2H}} v} - B_s^{(j)}}{\sqrt{\varepsilon} v^H}\bigg)\, ds\, dv \\
&\quad = -\frac{(2\pi)^{-\frac{d}{2}}}{2} \sum_{j=1}^{d} \int_0^{\varepsilon^{-\frac{1}{2H}} T} \big(1 + u^{2H}\big)^{-\frac{d}{2}-1} u^2 I_2\big(\varphi^{\varepsilon^{\frac{1}{2H}} u}_{j,T-\varepsilon^{\frac{1}{2H}} u}\big)\, du,
\end{aligned} \tag{4.21}$$

where the kernel $\varphi^{\varepsilon^{\frac{1}{2H}} u}_{j,T-\varepsilon^{\frac{1}{2H}} u}$ is defined by (2.6). From (4.21), it follows that for every $\varepsilon, \eta > 0$,

$$\begin{aligned}
&\mathbb{E}\big[\varepsilon^{\frac{d}{2}-\frac{3}{2H}+1} I_2\big(h_{2,T}^{\varepsilon}\big) \eta^{\frac{d}{2}-\frac{3}{2H}+1} I_2\big(h_{2,T}^{\eta}\big)\big] \\
&\quad = \frac{(2\pi)^{-d}}{2} \sum_{j=1}^{d} \int_0^{\varepsilon^{-\frac{1}{2H}} T} \int_0^{\eta^{-\frac{1}{2H}} T} \big(1 + u_1^{2H}\big)^{-\frac{d}{2}-1} \big(1 + u_2^{2H}\big)^{-\frac{d}{2}-1} \\
&\qquad \times (u_1 u_2)^2 \big\langle \varphi^{\varepsilon^{\frac{1}{2H}} u_1}_{j,T-\varepsilon^{\frac{1}{2H}} u_1}, \varphi^{\eta^{\frac{1}{2H}} u_2}_{j,T-\eta^{\frac{1}{2H}} u_1} \big\rangle_{(\mathfrak{H}^d)^{\otimes 2}}\, d\vec{u}.
\end{aligned} \tag{4.22}$$

By (2.7),

$$\begin{aligned}
\lim_{\varepsilon \to 0} \big\langle \varphi^{\varepsilon^{\frac{1}{2H}} u_1}_{j,T-\varepsilon^{\frac{1}{2H}} u_1}, \varphi^{\eta^{\frac{1}{2H}} u_2}_{j,T-\eta^{\frac{1}{2H}} u_1} \big\rangle_{(\mathfrak{H}^d)^{\otimes 2}} &= H^2(2H-1)^2 \int_{[0,T]^2} |s_1 - s_2|^{4H-4}\, d\vec{s} \\
&= \frac{H^2(2H-1)}{4H-3} T^{4H-2}.
\end{aligned} \tag{4.23}$$

On the other hand, by (2.8), there exists a constant $C_{H,T} > 0$, only depending on H and T, such that

$$0 \le \big\langle \varphi^{\varepsilon^{\frac{1}{2H}} u_1}_{j,T-\varepsilon^{\frac{1}{2H}} u_1}, \varphi^{\eta^{\frac{1}{2H}} u_2}_{j,T-\eta^{\frac{1}{2H}} u_1} \big\rangle_{(\mathfrak{H}^d)^{\otimes 2}} \le C_{H,K}.$$

Hence, using the pointwise convergence (4.23), we can apply the dominated convergence theorem to (4.22), in order to obtain

$$\lim_{\varepsilon, \nu \to 0} \mathbb{E}\big[\varepsilon^{\frac{d}{2}-\frac{3}{2H}+1} I_2\big(h_{2,T}^{\varepsilon}\big) \eta^{\frac{d}{2}-\frac{3}{2H}+1} I_2\big(h_{2,T}^{\eta}\big)\big] = \frac{d(2\pi)^{-d} \Lambda^2 H^2 (2H-1) T^{4H-2}}{2(4H-3)},$$

where the constant Λ is defined by (1.6). From the previous identity, it follows that $\varepsilon^{\frac{d}{2}-\frac{3}{2H}-1}I_2(h_{2,T}^{\varepsilon})$ converges to $I_2(\widetilde{h}_T)$ as $\varepsilon \to 0$, for some $\widetilde{h}_T \in (\mathfrak{H}^d)^{\otimes 2}$.

Recall that the element $\pi_T^j \in (\mathfrak{H}^d)^{\otimes d}$, is defined as the limit in $(\mathfrak{H}^d)^{\otimes 2}$, as $\varepsilon \to 0$, of $\varphi_{j,T}^{\varepsilon}$, and is characterized by relation (2.9). In order to prove (4.20), it suffices to show that $\widetilde{h}_T = \Lambda \sum_{j=1}^d \pi_T^j$, or equivalently, that

$$\langle \widetilde{h}_T, f_1 \otimes f_2 \rangle_{(\mathfrak{H}^d)^{\otimes 2}} = -\Lambda \sum_{j=1}^d \langle \pi_T^j, f_1 \otimes f_2 \rangle_{(\mathfrak{H}^d)^{\otimes 2}},$$

for vectors of step functions with compact support $f_i = (f_i^{(1)}, \dots, f_i^{(d)}) \in \mathfrak{H}^d$, $i = 1, 2$. By (4.21),

$$\lim_{\varepsilon\to 0} \langle \widetilde{h}_T, f_1 \otimes f_2 \rangle_{(\mathfrak{H}^d)^{\otimes 2}} = \lim_{\varepsilon\to 0} -\frac{(2\pi)^{-\frac{d}{2}}}{2} \int_0^{\varepsilon^{-\frac{1}{2H}}T} \left(1+u^{2H}\right)^{-\frac{d}{2}} u^2 \langle \varphi_{j,T-\varepsilon^{\frac{1}{2H}}u}^{\varepsilon^{\frac{1}{2H}}u}, f_1 \otimes f_2 \rangle_{(\mathfrak{H}^d)^{\otimes 2}}\, du. \tag{4.24}$$

Proceeding as in the proof of (4.23), we can easily check that

$$\lim_{\varepsilon\to 0} \langle \varphi_{j,T-\varepsilon^{\frac{1}{2H}}u}^{\varepsilon^{\frac{1}{2H}}u}, f_1 \otimes f_2 \rangle_{(\mathfrak{H}^d)^{\otimes 2}} = -H^2(2H-1)^2 \sum_{j=1}^d \int_0^T \prod_{i=1,2} \int_0^T |s-\eta|^{2H-2} f_i^{(j)}(\eta)\, d\eta\, ds.$$

Moreover, by (2.8),

$$\left| \langle \varphi_{j,T-\varepsilon^{\frac{1}{2H}}u}^{\varepsilon^{\frac{1}{2H}}u}, f_1 \otimes f_2 \rangle_{(\mathfrak{H}^d)^{\otimes 2}} \right| \le \left\| \varphi_{j,T-\varepsilon^{\frac{1}{2H}}u}^{\varepsilon^{\frac{1}{2H}}u} \right\|_{(\mathfrak{H}^d)^{\otimes 2}} \|f_1\|_{\mathfrak{H}^d} \|f_2\|_{\mathfrak{H}^d} \le C_{H,T} \|f_1\|_{\mathfrak{H}^d} \|f_2\|_{\mathfrak{H}^d},$$

for some constant $C_{H,T} > 0$ only depending on T and H. Therefore, applying the dominated convergence theorem in (4.24), we get

$$\lim_{\varepsilon\to 0} \langle \widetilde{h}_T, f_1 \otimes f_2 \rangle_{(\mathfrak{H}^d)^{\otimes 2}} = -\Lambda H^2(2H-1)^2 \sum_{j=1}^d \int_0^T \prod_{i=1,2} \int_0^T |s-\eta|^{2H-2} f_i^{(j)}(\eta)\, d\eta\, ds, \tag{4.25}$$

and from the characterization (2.9), we conclude that $\widetilde{h}_T = -\Lambda \sum_{j=1}^d \pi_T^j$, as required. This finishes the proof of (4.20), which, by (4.19), implies that the convergence (1.5).

It only remains to prove (1.7). By (1.5), it suffices to show the tightness property for $\varepsilon^{\frac{d}{2}-\frac{3}{2H}+1}(I_T^{\varepsilon} - \mathbb{E}[I_T^{\varepsilon}])$, which, as in the proof of (1.2), can be reduced to proving that there exists $p > 2$, such that for every $0 \le T_1 \le T_2 \le K$,

$$\mathbb{E}\left[\left|\varepsilon^{\frac{d}{2}-\frac{3}{2H}+1} Z_{\varepsilon}\right|^p\right] \le C(T_2-T_1)^{\frac{p}{2}}, \tag{4.26}$$

where Z_{ε} is defined by (4.3), and C is some constant only depending on d, H, K and p. Changing the coordinates (x, u_1, u_2) by $(x, \varepsilon^{-\frac{1}{2H}}u_1, \varepsilon^{-\frac{1}{2H}}u_2)$ in (4.16), and using the fact that

$$\Theta_{\varepsilon}\left(x, \varepsilon^{\frac{1}{2H}}u_1, \varepsilon^{\frac{1}{2H}}u_2\right) = \varepsilon^2 \Theta_1\left(\varepsilon^{-\frac{1}{2H}}x, u_1, u_2\right),$$

we can easily check that

$$\begin{aligned} \left\| \varepsilon^{\frac{d}{2}-\frac{3}{2H}+1} Z_{\varepsilon} \right\|_{L^p(\Omega)}^2 \le C(T_2-T_1) &\int_{\mathbb{R}_+^2} \int_0^{T_2} \varepsilon^{-\frac{2}{H}} \mu\left(x, \varepsilon^{\frac{1}{2H}}u_1, \varepsilon^{\frac{1}{2H}}u_2\right)^2 \\ &\times \Theta_1\left(\varepsilon^{-\frac{1}{2H}}x, u_1, u_2\right)^{-\frac{d}{p}} \int_{[0,1]^2} \left(\left(1+\eta u_1^{2H}\right)\left(1+\xi u_2^{2H}\right)\right)^{-1-\frac{d}{2}+\frac{d}{p}} d\eta\, d\xi\, dx\, d\vec{u}, \end{aligned}$$

and hence, if $p>2$, we obtain

$$\left\|\varepsilon^{\frac{d}{2}-\frac{3}{2H}+1}Z_{\varepsilon}\right\|_{L^{p}(\Omega)}^{2}\leq C(T_2-T_1)\int_{\mathbb{R}_{+}^{2}}\int_{0}^{T_2}\varepsilon^{-\frac{2}{H}}\mu\big(x,\varepsilon^{\frac{1}{2H}}u_1,\varepsilon^{\frac{1}{2H}}u_2\big)^{2}(u_1u_2)^{-2H} \times\Theta_1\big(\varepsilon^{-\frac{1}{2H}}x,u_1,u_2\big)^{-\frac{d}{p}}\,dx\,d\vec{u}. \tag{4.27}$$

By Lemma 5.4, if $T_1,T_2\in[0,K]$, for some $K>0$, the integral in the right-hand side of the previous inequality is bounded by a constant only depending on H,d,p and K. Relation (4.26) then follows from (4.27). This finishes the proof of the tightness property for $\varepsilon^{\frac{d}{2}-\frac{3}{2H}+1}(I_T^{\varepsilon}-\mathbb{E}[I_T^{\varepsilon}])$ in the case $H>\frac{3}{4}$. □

Proof of Theorem 1.4. Finally we prove Theorem 1.4. First we show the convergence of the finite dimensional distributions, namely, that for every $r\in\mathbb{N}$ and $T_1,\dots,T_r\geq 0$ fixed, it holds

$$\frac{\varepsilon^{\frac{d}{2}-1}}{\sqrt{\log(1/\varepsilon)}}\big(\big(I_{T_1}^{\varepsilon},\dots,I_{T_r}^{\varepsilon}\big)-\mathbb{E}\big[\big(I_{T_1}^{\varepsilon},\dots,I_{T_r}^{\varepsilon}\big)\big]\big)\stackrel{\text{Law}}{\to}\rho(W_{T_1},\dots,W_{T_r}), \tag{4.28}$$

where ρ is defined by (3.51). Consider the random variable $\widetilde{J}_T^{\varepsilon}$ introduced in (3.40). By Lemma 3.3, we have

$$\lim_{\varepsilon\to 0}\frac{\varepsilon^{\frac{d}{2}-1}}{\sqrt{\log(1/\varepsilon)}}\big\|I_T^{\varepsilon}-\mathbb{E}\big[I_T^{\varepsilon}\big]-I_2\big(h_{2,T}^{\varepsilon}\big)\big\|_{L^2(\Omega)}=0, \tag{4.29}$$

and by Lemma 3.4

$$\lim_{\varepsilon\to 0}\frac{\varepsilon^{\frac{d}{2}-1}}{\sqrt{\log(1/\varepsilon)}}\big\|I_2\big(h_{2,T}^{\varepsilon}\big)-\widetilde{J}_T^{\varepsilon}\big\|_{L^2(\Omega)}=0. \tag{4.30}$$

Consequently,

$$\lim_{\varepsilon\to 0}\frac{\varepsilon^{\frac{d}{2}-1}}{\sqrt{\log(1/\varepsilon)}}\big\|I_T^{\varepsilon}-\mathbb{E}\big[I_T^{\varepsilon}\big]-\widetilde{J}_T^{\varepsilon}\big\|_{L^2(\Omega)}=0,$$

and hence, relation (4.28) is equivalent to

$$\frac{\varepsilon^{\frac{d}{2}-1}}{\sqrt{\log(1/\varepsilon)}}\big(\widetilde{J}_{T_1}^{\varepsilon},\dots,\widetilde{J}_{T_r}^{\varepsilon}\big)\stackrel{\text{Law}}{\to}\rho(W_{T_1},\dots,W_{T_r}). \tag{4.31}$$

By the Peccati–Tudor criterion, the convergence (4.31) holds provided that $\widetilde{J}_t^{\varepsilon}$ satisfies the following conditions:

(i) For every $1\leq i,j\leq r$,

$$\frac{\varepsilon^{d-2}}{\log(1/\varepsilon)}\mathbb{E}\big[\widetilde{J}_{T_i}^{\varepsilon}\widetilde{J}_{T_j}^{\varepsilon}\big]\to\rho^2(T_i\wedge T_j),\quad\text{as }\varepsilon\to 0.$$

(ii) For all $i=1,\dots,r$, the random variables $\frac{\varepsilon^{\frac{d}{2}-1}}{\sqrt{\log(1/\varepsilon)}}\widetilde{J}_{T_i}^{\varepsilon}$ converge in law to a centered Gaussian distribution as $\varepsilon\to 0$.

Relation (i) follows from relation (4.30), as well as Theorem 3.5. Hence, it suffices to check (ii). To this end, consider the following Riemann sum approximation for $\widetilde{J}_T^{\varepsilon}$

$$R_{T,M}^{\varepsilon}:=-\frac{c_{\log}\varepsilon^{\frac{2}{3}-\frac{d}{2}}}{2^M}\sum_{k=2}^{M2^M}\int_0^T\sum_{j=1}^{d}\frac{u(k)^{\frac{3}{2}}}{(1+u(k))^{\frac{3}{2}}}H_2\left(\frac{B_{s+\varepsilon^{\frac{2}{3}}u(k)}^{(j)}-B_s^{(j)}}{\sqrt{\varepsilon}u(k)^{\frac{3}{4}}}\right)ds, \tag{4.32}$$

where $c_{\log} := \frac{(2\pi)^{-\frac{d}{2}}}{2}$ and $u(k) := \frac{k}{2^M}$, for $k = 2, \dots, M2^M$. We will prove that $\frac{\varepsilon^{\frac{d}{2}-1}}{\sqrt{\log(1/\varepsilon)}}(R^{\varepsilon}_{T,M} - \widetilde{J}^{\varepsilon}_T)$ converges to zero, uniformly in $\varepsilon \in (0, 1/e)$, and $\frac{\varepsilon^{\frac{d}{2}-1}}{\sqrt{\log(1/\varepsilon)}} R^{\varepsilon}_{T,M} \overset{\text{Law}}{\to} T\mathcal{N}(0, \widetilde{\rho}^2_M)$ as $\varepsilon \to 0$ for some constant $\widetilde{\rho}^2_M$ satisfying $\widetilde{\rho}^2_M \to \rho^2$ as $M \to \infty$. The result will then follow by a standard approximation argument. We will separate the argument in the following steps.

Step I. We prove that $\frac{\varepsilon^{\frac{d}{2}-1}}{\sqrt{\log(1/\varepsilon)}}(R^{\varepsilon}_{T,M} - \widetilde{J}^{\varepsilon}_T) \to 0$ in $L^2(\Omega)$ as $M \to \infty$ uniformly in $\varepsilon \in (0, 1/e)$, namely,

$$\lim_{M\to\infty} \sup_{\varepsilon\in(0,1/e)} \frac{\varepsilon^{\frac{d}{2}-1}}{\sqrt{\log(1/\varepsilon)}} \big\| R^{\varepsilon}_{T,M} - \widetilde{J}^{\varepsilon}_T \big\|_{L^2(\Omega)} = 0. \tag{4.33}$$

For $\varepsilon \in (0, 1/e)$ fixed, we decompose the term $\widetilde{J}^{\varepsilon}_T$ as

$$\widetilde{J}^{\varepsilon}_T = \widetilde{J}^{\varepsilon,M}_{T,1} + \widetilde{J}^{\varepsilon,M}_{T,2}, \tag{4.34}$$

where

$$\widetilde{J}^{\varepsilon,M}_{T,1} := -c_{\log}\varepsilon^{\frac{3}{2}-\frac{d}{2}} \int_0^T \int_{2^{-M}}^M \sum_{j=1}^d \frac{u^{\frac{3}{2}}}{(1+u^{\frac{3}{2}})^{\frac{d}{2}+1}} H_2\bigg(\frac{B^{(j)}_{s+\varepsilon^{\frac{2}{3}}u} - B^{(j)}_s}{\sqrt{\varepsilon}u^{\frac{3}{4}}}\bigg)\,du\,ds$$

and

$$\widetilde{J}^{\varepsilon,M}_{T,2} := -c_{\log}\varepsilon^{\frac{3}{2}-\frac{d}{2}} \int_0^T \int_0^\infty \mathbb{1}_{(0,2^{-M})\cup(M,\infty)}(u) \sum_{j=1}^d \frac{u^{\frac{3}{2}}}{(1+u^{\frac{3}{2}})^{\frac{d}{2}+1}} H_2\bigg(\frac{B^{(j)}_{s+\varepsilon^{\frac{2}{3}}u} - B^{(j)}_s}{\sqrt{\varepsilon}u^{\frac{3}{4}}}\bigg)\,du\,ds.$$

From (4.34), we deduce that the relation (4.33) is equivalent to

$$\lim_{M\to\infty} \sup_{\varepsilon\in(0,1/e)} \frac{\varepsilon^{\frac{d}{2}-1}}{\sqrt{\log(1/\varepsilon)}} \big\| R^{\varepsilon}_{T,M} - \widetilde{J}^{\varepsilon,M}_{T,1} \big\|_{L^2(\Omega)} = 0, \tag{4.35}$$

provided that

$$\lim_{M\to\infty} \sup_{\varepsilon\in(0,1/e)} \frac{\varepsilon^{\frac{d}{2}-1}}{\sqrt{\log(1/\varepsilon)}} \big\| \widetilde{J}^{\varepsilon,M}_{T,2} \big\|_{L^2(\Omega)} = 0. \tag{4.36}$$

To prove (4.36) we proceed as follows. First we use the relation (3.42) to write

$$\begin{aligned}\frac{\varepsilon^{d-2}}{\log(1/\varepsilon)} \big\| \widetilde{J}^{\varepsilon,M}_{T,2} \big\|^2_{L^2(\Omega)} &= \frac{2dc^2_{\log}}{\log(1/\varepsilon)} \int_{[0,T]^2} \int_{[0,\varepsilon^{-\frac{2}{3}}T]} \prod_{i=1,2} \mathbb{1}_{(0,2^{-M})\cup(M,\infty)}(u_i) \\ &\quad \times \psi(u_1,u_2)\varepsilon^{-8/3}\mu\big(s_2-s_1, \varepsilon^{\frac{2}{3}}u_1, \varepsilon^{\frac{2}{3}}u_1\big)^2\,ds_1\,d\vec{s}\,d\vec{u},\end{aligned}$$

where $\psi(u_1, u_2)$ is defined by (3.38). Changing the coordinates (s_1, s_2, u_1, u_2) by $(s := s_1, x := \varepsilon^{-\frac{2}{3}}(s_2 - s_1), u_1, u_2)$ when $s_1 \le s_2$, and by $(s := s_2, x := \varepsilon^{-\frac{2}{3}}(s_1 - s_2), u_1, u_2)$ when $s_1 \ge s_2$, integrating the variable s, and using the identity $\mu(\varepsilon^{\frac{2}{3}}x, \varepsilon^{\frac{2}{3}}u_1, \varepsilon^{\frac{2}{3}}u_2)^2 = \varepsilon^2\mu(x, u_1, u_2)$, we get

$$\frac{\varepsilon^{d-2}}{\log(1/\varepsilon)} \big\| \widetilde{J}^{\varepsilon,M}_{T,2} \big\|^2_{L^2(\Omega)} \le \frac{4Tdc^2_{\log}}{\log(1/\varepsilon)} \int_{[0,\varepsilon^{-\frac{2}{3}}T]^3} \prod_{i=1,2} \mathbb{1}_{(0,2^{-M})\cup(M,\infty)}(u_i) G^{(1)}_{1,x}(u_1,u_2)\,dx\,d\vec{u}, \tag{4.37}$$

where the function $G^{(1)}_{1,x}(u_1,u_2)$ is defined by (2.23). Define the regions $\mathcal{S}_i$ by (3.18). Splitting the domain of integration of the right-hand side of (4.37) into $[0,\varepsilon^{-\frac{2}{3}}T]^3=\bigcup_{i=1}^3([0,\varepsilon^{-\frac{2}{3}}T]^3\cap\mathcal{S}_i)$, we obtain

$$\big(\log(1/\varepsilon)\big)^{-1}\varepsilon^{d-2}\big\|\widetilde{J}^{\varepsilon,M}_{T,2}\big\|^2_{L^2(\Omega)}\le 4T\big(\log(1/\varepsilon)\big)^{-1}dc^2_{\log}\sum_{i=1}^3\int_{[0,\varepsilon^{-\frac{2}{3}}T]^3}\mathbb{1}_{\mathcal{S}_i}(x,u_1,u_2)\times\prod_{i=1,2}\mathbb{1}_{(0,2^{-M})\cup(M,\infty)}(u_i)G^{(1)}_{1,x}(u_1,u_2)\,dx\,d\vec{u},$$

and hence, dropping the normalization term $\frac{1}{\log(1/\varepsilon)}$ in the regions $\mathcal{S}_1,\mathcal{S}_2$, we obtain

$$\big(\log(1/\varepsilon)\big)^{-1}\varepsilon^{d-2}\big\|\widetilde{J}^{\varepsilon,M}_{T,2}\big\|^2_{L^2(\Omega)}\le 4Tdc^2_{\log}\int_{[0,\varepsilon^{-\frac{2}{3}}T]^3}\left(\frac{\mathbb{1}_{\mathcal{S}_3}(x,u_1,u_2)}{\log(1/\varepsilon)}+\sum_{i=1}^2\mathbb{1}_{\mathcal{S}_i}(x,u_1,u_2)\right)\times\prod_{j=1,2}\mathbb{1}_{(0,2^{-M})\cup(M,\infty)}(u_j)G^{(1)}_{1,x}(u_1,u_2)\,dx\,d\vec{u}.$$

The integrands corresponding to $i=1,2$ converge pointwise to zero as $M\to\infty$, and are bounded by the functions $\mathbb{1}_{\mathcal{S}_i}(x,u_1,u_2)G^{(1)}_{1,x}(u_1,u_2)$, which, by relations (2.29) and (2.38), are in turn bounded by

$$\mathbb{1}_{\mathcal{S}_i}(x,u_1,u_2)C\frac{\mu(x,u_1,u_2)^2}{(u_1u_2)^{2H}}\Theta_1(x,u_1,u_2)^{-\frac{d}{2}},\tag{4.38}$$

for some constant $C>0$. In addition, by Lemma 5.3, the function (4.38) is integrable for $i=1,2$, and hence, by the dominated convergence theorem,

$$\limsup_{M\to\infty}\sup_{\varepsilon\in(0,1/e)}\frac{\varepsilon^{d-2}}{\log(1/\varepsilon)}\big\|\widetilde{J}^{\varepsilon,M}_{T,2}\big\|^2_{L^2(\Omega)}\le\limsup_{M\to\infty}\sup_{\varepsilon\in(0,1/e)}\frac{4Tdc^2_{\log}}{\log(1/\varepsilon)}\int_{[0,\varepsilon^{-\frac{2}{3}}T]^3}\mathbb{1}_{\mathcal{S}_3}(x,u_1,u_2)\times\prod_{i=1,2}\mathbb{1}_{(0,2^{-M})\cup(M,\infty)}(u_i)G^{(1)}_{1,x}(u_1,u_2)\,dx\,du_1\,du_2.\tag{4.39}$$

On the other hand, by equation (5.5) in Lemma 5.2, we deduce that there exists a constant $C>0$, such that for every $(x,u_1,u_2)\in\mathcal{S}_3$,

$$G^{(1)}_{1,x}(u_1,u_2)\le C(x+u_1+u_2)^{-1}(u_1u_2)^2\psi(u_1,u_2).\tag{4.40}$$

Therefore, from (4.39) we deduce that

$$\limsup_{M\to\infty}\sup_{\varepsilon\in(0,1/e)}\frac{\varepsilon^{d-2}}{\log(1/\varepsilon)}\big\|\widetilde{J}^{\varepsilon,M}_{T,2}\big\|^2_{L^2(\Omega)}\le\limsup_{M\to\infty}\sup_{\varepsilon\in(0,1/e)}\frac{4Cdc^2_{\log}T}{\log(1/\varepsilon)}\int_0^{\varepsilon^{-\frac{2}{3}}T}\int_{\mathbb{R}^2_+}\prod_{i=1,2}\mathbb{1}_{(0,2^{-M})\cup(M,\infty)}(u_i)\times(x+u_1+u_2)^{-1}(u_1u_2)^2\psi(u_1,u_2)\,d\vec{u}\,dx,$$

so that there exists a constant $C > 0$ such that

$$\limsup_{M\to\infty} \sup_{\varepsilon\in(0,1/e)} \frac{\varepsilon^{d-2}}{\log(1/\varepsilon)} \| \widetilde{J}^{\varepsilon,M}_{T,2} \|^2_{L^2(\Omega)} \\ \leq \limsup_{M\to\infty} \sup_{\varepsilon\in(0,1/e)} CT \int_{\mathbb{R}^2_+} \prod_{i=1,2} \mathbb{1}_{(0,2^{-M})\cup(M,\infty)}(u_i)\psi(u_1,u_2) \\ \times \left(\frac{\log(\varepsilon^{-\frac{2}{3}}T + u_1 + u_2) - \log(u_1+u_2)}{\log(1/\varepsilon)} \right)(u_1 u_2)^2 \, d\vec{u} = 0,$$

where the last equality easily follows from the dominated convergence theorem. This finishes the proof of (4.36).

To prove (4.35) we proceed as follows. Define the intervals $I_k := (\frac{k-1}{2^M}, \frac{k}{2^M}]$. Then, we can write $R^{\varepsilon}_{T,M}$ and $\widetilde{J}^{M}_{T,1}$, as

$$R^{\varepsilon}_{T,M} = -\sum_{k=2}^{M2^M} c_{\log}\varepsilon^{\frac{3}{2}-\frac{d}{2}} \int_0^T \int_{\mathbb{R}_+} \sum_{j=1}^{d} \mathbb{1}_{I_k}(u) \frac{u(k)^{\frac{3}{2}}}{(1+u(k)^{\frac{3}{2}})^{\frac{d}{2}+1}} H_2\left(\frac{B^{(j)}_{s+\varepsilon^{\frac{2}{3}}u(k)} - B^{(j)}_s}{\sqrt{\varepsilon}u(k)^{\frac{3}{4}}} \right) du\, ds,$$

and

$$\widetilde{J}^{\varepsilon,M}_{T,1} = -\sum_{k=2}^{M2^M} c_{\log}\varepsilon^{\frac{3}{2}-\frac{d}{2}} \int_0^T \int_{\mathbb{R}_+} \sum_{j=1}^{d} \mathbb{1}_{I_k}(u) \frac{u(k)^{\frac{3}{2}}}{(1+u(k)^{\frac{3}{2}})^{\frac{d}{2}+1}} H_2\left(\frac{B^{(j)}_{s+\varepsilon^{\frac{2}{3}}u} - B^{(j)}_s}{\sqrt{\varepsilon}u^{\frac{3}{4}}} \right) du\, ds.$$

Notice that by (3.42),

$$\mathbb{E}\left[H_2\left(\frac{B_{s_1+\varepsilon^{\frac{2}{3}}v_1} - B_{s_1}}{\sqrt{\varepsilon}v_1^{\frac{3}{4}}} \right) H_2\left(\frac{B_{s_1+\varepsilon^{\frac{2}{3}}v_2} - B_{s_2}}{\sqrt{\varepsilon}v_2^{\frac{3}{4}}} \right) \right] = 2(v_1 v_2)^{-\frac{3}{2}} \mu\big(\varepsilon^{-\frac{2}{3}}(s_2-s_1), v_1, v_2\big)^2,$$

and hence,

$$\frac{\varepsilon^{d-2}}{\log(1/\varepsilon)} \| \widetilde{J}^{\varepsilon,M}_{T,1} - R^{\varepsilon}_{T,M} \|^2_{L^2(\Omega)} = \frac{2dc^2_{\log}}{\log(1/\varepsilon)} \int_{[0,T]^2} \int_{\mathbb{R}^2_+} \sum_{k_1,k_2=2}^{M2^M} \mathbb{1}_{I_{k_1}}(u_1)\mathbb{1}_{I_{k_2}}(u_2) \\ \times \varepsilon^{-\frac{2}{3}} A^M_{k_1,k_2}\big(\varepsilon^{-\frac{2}{3}}(s_2-s_1), u_1, u_2\big) \, d\vec{s}\, d\vec{u},$$

where the function $A^M_{k_1,k_2}(x,u_1,u_2)$ is defined by

$$A^M_{k_1,k_2}(x,u_1,u_2) := \big(G^{(1)}_{1,x}(u_1,u_2) - G^{(1)}_{1,x}\big(u(k_1),u_2\big) \\ - G^{(1)}_{1,x}\big(u_1,u(k_2)\big) + G^{(1)}_{1,x}\big(u(k_1),u(k_2)\big)\big).$$

Changing the coordinates (s_1,s_2,u_1,u_2) by $(s := s_1, x := \varepsilon^{-\frac{2}{3}}(s_2-s_1), u_1, u_1)$ in the case $s_2 \geq s_1$ and by $(s := s_2, x := \varepsilon^{-\frac{2}{3}}(s_1-s_2), u_1, u_1)$ in the case $s_1 \geq s_2$, and integrating the variable s, we deduce that there exists a constant $C > 0$, such that

$$\frac{\varepsilon^{d-2}}{\log(1/\varepsilon)} \| \widetilde{J}^{\varepsilon,M}_{T,1} - R^{\varepsilon}_{T,M} \|^2_{L^2(\Omega)} \leq \frac{CT}{\log(1/\varepsilon)} \int_0^{\varepsilon^{-\frac{2}{3}}T} \int_{\mathbb{R}^2_+} \sum_{k_1,k_2=2}^{M2^M} \mathbb{1}_{I_{k_1}}(u_1)\mathbb{1}_{I_{k_2}}(u_2) \\ \times \big| A^M_{k_1,k_2}(x,u_1,u_2) \big| \, d\vec{u}\, dx. \tag{4.41}$$

In order to bound the term $|A^M_{k_1,k_2}(x,u_1,u_2)|$ we proceed as follows. Consider the function

$$D^M_x(u_1,u_2) := \psi\big(u_1-2^{-M},u_2-2^{-M}\big)\mu\big(x,u_1+2^{-M}\big)^2 - \psi\big(u_1+2^{-M},u_2+2^{-M}\big)\mu\big(x,u_1-2^{-M}\big)^2,$$

where $\psi(u_1,u_2)$ is defined by (3.43). By relation (2.5), we have that

$$\mu(x,u_1,u_2) = \frac{3}{8}\int_0^{u_1}\int_x^{x+u_2}|v_1-v_2|^{-\frac{1}{2}}\,d\vec{v} = \frac{3u_1u_2}{8}\int_{[0,1]^2}|x+v_2u_2-v_1u_1|^{-\frac{1}{2}}\,d\vec{v}, \tag{4.42}$$

and consequently, $\mu(x,u_1,u_2)\le\mu(x,v_1,v_2)$ for every $u_1\le v_1$ and $u_2\le v_2$. Using this observation, we can easily show that for every $v_1\in[u_1-2^{-M},u_1+2^{-M}]$ and $v_2\in[u_2-2^{-M},u_2+2^{-M}]$, the following inequality holds

$$\psi\big(u_1+2^{-M},u_2+2^{-M}\big)^{-\frac{d}{2}}\mu\big(x,u_1-2^{-M}\big)^2 \le G^{(1)}_{1,x}(v_1,v_2) \le \psi\big(u_1-2^{-M},u_2-2^{-M}\big)^{-\frac{d}{2}}\mu\big(x,u_1+2^{-M}\big)^2.$$

Hence, for every $u_1\in I_{k_1}$ and $u_2\in I_{k_2}$,

$$\big|A^M_{k_1,k_2}(u_1,u_2)\big| \le 2D^M_x(u_1,u_2). \tag{4.43}$$

Using relations (4.41) and (4.43), as well as the fact that

$$\sum_{k_1,k_2=2}^{M2^M}\mathbb{1}_{I_{k_1}}(u_1)\mathbb{1}_{I_{k_2}}(u_2) = \mathbb{1}_{[2^{-M},M]^2}(u_1,u_2),$$

we obtain

$$\frac{\varepsilon^{d-2}}{\log(1/\varepsilon)}\big\|\widetilde{J}^{\varepsilon,M}_{T,1}-R^{\varepsilon}_{T,M}\big\|^2_{L^2(\Omega)} \le \frac{CT}{\log(1/\varepsilon)}\int_0^{\varepsilon^{-\frac{2}{3}}T}\int_{\mathbb{R}^2_+}\mathbb{1}_{[2^{-M},M]^2}(u_1,u_2)D^M_x(u_1,u_2)\,d\vec{u}\,dx. \tag{4.44}$$

To bound the integral in the right-hand side we proceed as follows. Define $N:=\varepsilon^{-\frac{2}{3}}$, so that $\log(1/\varepsilon)=\frac{3\log N}{2}$. Then, applying L'Hôpital's rule in (4.44), we deduce that there is a constant $C>0$, such that

$$\begin{aligned}
&\limsup_{\varepsilon\to 0}\frac{\varepsilon^{d-2}}{\log(1/\varepsilon)}\big\|\widetilde{J}^{\varepsilon,M}_{T,1}-R^{\varepsilon}_{T,M}\big\|^2_{L^2(\Omega)} \\
&\quad\le \limsup_{N\to\infty}\frac{CT}{\log N}\int_0^{NT}\int_{\mathbb{R}^2_+}\mathbb{1}_{[2^{-M},M]^2}(u_1,u_2)D^M_x(u_1,u_2)\,d\vec{u}\,dx \\
&\quad= \limsup_{N\to\infty}CT\int_{\mathbb{R}^2_+}\mathbb{1}_{[2^{-M},M]^2}(u_1,u_2)NT\,D^M_{NT}(u_1,u_2)\,d\vec{u}.
\end{aligned} \tag{4.45}$$

On the other hand, using equation (5.5) in Lemma 5.2, we have that for every $(x,u_1,u_2)\in\mathcal{S}_3$,

$$x\mu(x,u_1,u_2)^2 \le x(x+u_1+u_2)^{-1}(u_1u_2)^2 \le (u_1u_2)^2.$$

Hence, using (3.65) and the dominated convergence theorem in (4.45), we deduce that there is a constant $C > 0$, such that

$$\begin{aligned}\limsup_{\varepsilon\to 0} \frac{\varepsilon^{d-2}}{\log(1/\varepsilon)} \big\| \widetilde{J}_{T,1}^{\varepsilon,M} - R_{T,M}^{\varepsilon} \big\|_{L^2(\Omega)}^2 \\ \le CT \int_{\mathbb{R}_+^2} \mathbb{1}_{[2^{-M},M]^2}(u_1,u_2)\big(\psi\big(u_1-2^{-M},u_2-2^{-M}\big)\big(\big(u_1+2^{-M}\big)\big(u_1+2^{-M}\big)\big)^2 \\ - \psi\big(u_1+2^{-M},u_2+2^{-M}\big)\big(\big(u_1-2^{-M}\big)\big(u_1-2^{-M}\big)\big)^2\big)\, d\vec{u} \end{aligned} \tag{4.46}$$

Let $M_0 \in \mathbb{N}$ and $\delta > 0$ be fixed. Using the fact that integrands in (4.46) are decreasing on M and

$$\sum_{k_1,k_2=2}^{M_0 2^{M_0}} \mathbb{1}_{I_{k_1}}(x_1)\mathbb{1}_{I_{k_2}}(x_2) = \mathbb{1}_{[2^{-M_0},M_0]}(x_1)\mathbb{1}_{[2^{-M_0},M_0]}(x_2) \le 1,$$

we can easily check from the definition of the convergence (4.46), that there exists $\gamma = \gamma(M_0,\delta) > 0$ such that for every $M > M_0$, the following inequality holds

$$\begin{aligned}\sup_{\varepsilon\in(0,\gamma)} \frac{\varepsilon^{d-2}}{\log(1/\varepsilon)} \big\| \widetilde{J}_{T,1}^{\varepsilon,M} - R_{T,M}^{\varepsilon} \big\|_{L^2(\Omega)}^2 \\ \le \delta + CT \int_{\mathbb{R}_+^2} \big(\psi\big(u_1-2^{-M_0},u_2-2^{-M_0}\big)\big(\big(u_1+2^{-M_0}\big)\big(u_1+2^{-M_0}\big)\big)^2 \\ - \psi\big(u_1+2^{-M_0},u_2+2^{-M_0}\big)\big(\big(u_1-2^{-M_0}\big)\big(u_1-2^{-M_0}\big)\big)^2\big)\, d\vec{u}. \end{aligned} \tag{4.47}$$

To handle the term $\sup_{\varepsilon\in(\gamma,1/e)} \frac{\varepsilon^{d-2}}{\log(1/\varepsilon)} \| \widetilde{J}_{T,1}^{\varepsilon,M} - R_{T,M}^{\varepsilon} \|_{L^2(\Omega)}^2$, we use (4.44) to get

$$\sup_{\varepsilon\in(\gamma,1/e)} \frac{\varepsilon^{d-2}}{\log(1/\varepsilon)} \big\| \widetilde{J}_{T,1}^{\varepsilon,M} - R_{T,M}^{\varepsilon} \big\|_{L^2(\Omega)}^2 \le CT \int_0^{\gamma^{-\frac{2}{3}}T} \int_{\mathbb{R}_+^2} \mathbb{1}_{[2^{-M},M]^2}(u_1,u_2) D_x^M(u_1,u_2)\, d\vec{u}\, dx. \tag{4.48}$$

From (4.47) and (4.48), we conclude that there exists a constant $C > 0$, only depending on T, such that for every $M > M_0$,

$$\begin{aligned}\sup_{\varepsilon\in(0,1/e)} \frac{\varepsilon^{d-2}}{\log(1/\varepsilon)} \big\| \widetilde{J}_{T,1}^{\varepsilon,M} - R_{T,M}^{\varepsilon} \big\|_{L^2(\Omega)}^2 \\ \le \delta + CT \int_{\mathbb{R}_+^2} \sum_{k_1,k_2=2}^{M_0 2^{M_0}} \big(\psi\big(u_1-2^{-M_0},u_2-2^{-M_0}\big)\big(\big(u_1+2^{-M_0}\big)\big(u_1+2^{-M_0}\big)\big)^2 \\ - \psi\big(u_1+2^{-M_0},u_2+2^{-M_0}\big)\big(\big(u_1-2^{-M_0}\big)\big(u_1-2^{-M_0}\big)\big)^2\big)\, d\vec{u} \\ + CT \int_0^{\gamma^{-\frac{2}{3}}T} \int_{\mathbb{R}_+^2} \mathbb{1}_{[2^{-M},M]^2}(u_1,u_2) D_x^M(u_1,u_2)\, d\vec{u}\, dx. \end{aligned} \tag{4.49}$$

Taking first the limit as $M \to \infty$ and then as $M_0 \to \infty$ in (4.49), and applying the dominated convergence theorem, we get

$$\limsup_{M\to\infty} \sup_{\varepsilon\in(0,1)} \frac{\varepsilon^{d-2}}{\log(1/\varepsilon)} \big\| \widetilde{J}_{T,1}^{\varepsilon,M} - R_{T,M}^{\varepsilon} \big\|_{L^2(\Omega)}^2 \le \delta.$$

Relation (4.35) is then obtained by taking $\delta \to 0$ in the previous inequality.

Step II. Next we prove that

$$\lim_{\varepsilon\to 0}\frac{\varepsilon^{d-2}}{\log(1/\varepsilon)}\mathbb{E}\big[\big(R^{\varepsilon}_{T,M}\big)^2\big]=T\widetilde{\rho}^2_M, \tag{4.50}$$

where $\widetilde{\rho}_M$ is given by

$$\widetilde{\rho}_M=\frac{\sqrt{3d}}{2^{\frac{d+5}{2}}\pi^{\frac{d}{2}}2^M}\sum_{k=2}^{M2^M}\big(1+u(k)^{\frac{3}{2}}\big)^{-\frac{d}{2}-1}u(k)^2, \tag{4.51}$$

and $u(k)=\frac{k}{2^M}$. Notice that in particular, $\widetilde{\rho}^2_M$ satisfies

$$\lim_{M\to\infty}\widetilde{\rho}^2_M=\rho^2,$$

where ρ^2 is defined by (3.51). To prove (4.51) we proceed as follows. Recall that the constant $c_{\log}$ is defined by $c_{\log}=\frac{(2\pi)^{-\frac{d}{2}}}{2}$. Then, from the definition of $R^{\varepsilon}_{T,M}$ (see equation (4.32)), it easily follows that

$$\frac{\varepsilon^{d-2}}{\log(1/\varepsilon)}\mathbb{E}\big[\big(R^{\varepsilon}_{T,M}\big)^2\big]=\frac{2dc^2_{\log}}{\log(1/\varepsilon)2^{2M}}\int_{[0,T]^2}\sum_{k_1,k_2=2}^{M2^M}\varepsilon^{-\frac{2}{3}}G^{(1)}_{1,\varepsilon^{-\frac{2}{3}}(s_2-s_1)}\big(u(k_1),u(k_2)\big)\,d\vec{s}.$$

Changing the coordinates (s_1,s_2) by $(s_1,x:=s_2-s_1)$, and then integrating the variable s_1, we get

$$\begin{aligned}\frac{\varepsilon^{d-2}}{\log(1/\varepsilon)}\mathbb{E}\big[\big(R^{\varepsilon}_{T,M}\big)^2\big]&=\frac{4dc^2_{\log}}{\log(1/\varepsilon)2^{2M}}\int_0^T T\sum_{k_1,k_2=2}^{M2^M}\varepsilon^{-\frac{2}{3}}G^{(1)}_{1,\varepsilon^{-\frac{2}{3}}x}\big(\varepsilon^{\frac{2}{3}}u(k_1),\varepsilon^{\frac{2}{3}}u(k_2)\big)\,dx\\&\quad-\frac{4dc^2_{\log}}{\log(1/\varepsilon)2^{2M}}\int_0^T x\sum_{k_1,k_2=2}^{M2^M}\varepsilon^{-\frac{2}{3}}G^{(1)}_{1,\varepsilon^{-\frac{2}{3}}x}\big(\varepsilon^{\frac{2}{3}}u(k_1),\varepsilon^{\frac{2}{3}}u(k_2)\big)\,dx.\end{aligned}$$

Using relation (3.59) as well as the Cauchy–Schwarz inequality $\mu(x,u_1,u_2)\le(u_1u_2)^{\frac{3}{4}}$, we can easily deduce that there exists a constant $C>0$, depending on $u_1,\dots,u_{M2^M}$, but not on x or ε, such that

$$G^{(1)}_{1,\varepsilon^{-\frac{2}{3}}x}\big(u(k_1),u(k_2)\big)\le C\varepsilon^{\frac{2}{3}}x^{-1},$$

and hence,

$$\lim_{\varepsilon\to 0}\frac{1}{\log(1/\varepsilon)}\int_0^T x\sum_{k_1,k_2=2}^{M2^M}\varepsilon^{-\frac{2}{3}}G^{(1)}_{1,\varepsilon^{-\frac{2}{3}}x}\big(u(k_1),u(k_2)\big)\,dx=0,$$

which implies that

$$\begin{aligned}\lim_{\varepsilon\to 0}\frac{\varepsilon^{d-2}}{\log(1/\varepsilon)}\mathbb{E}\big[\big(R^{\varepsilon}_{T,M}\big)^2\big]&=\lim_{\varepsilon\to 0}\frac{4dc^2_{\log}T}{\log(1/\varepsilon)2^{2M}}\int_0^T\sum_{k_1,k_2=2}^{M2^M}\varepsilon^{-\frac{2}{3}}G_{1,\varepsilon^{-\frac{2}{3}}x}\big(u(k_1),u(k_2)\big)\,dx\\&=\lim_{\varepsilon\to 0}\frac{4dc^2_{\log}T}{\log(1/\varepsilon)2^{2M}}\int_0^{\varepsilon^{-\frac{2}{3}}T}\sum_{k_1,k_2=2}^{M2^M}G_{1,x}\big(u(k_1),u(k_2)\big)\,dx,\end{aligned}$$

where the last equality follows by making the change of variables $\widetilde{x} := \varepsilon^{-\frac{2}{3}}x$. Hence, writing $N := \varepsilon^{-\frac{2}{3}}$, so that $\log(1/\varepsilon) = \frac{2\log N}{3}$, and using L'Hôpital's rule, we get

$$\begin{aligned}\lim_{\varepsilon\to 0}\frac{\varepsilon^{d-2}}{\log(1/\varepsilon)}\mathbb{E}\big[\big(R^{\varepsilon}_{T,M}\big)^2\big] &= \lim_{N\to\infty}\frac{8dc^2_{\log}T}{3\log N 2^{2M}}\int_0^{NT}\sum_{k_1,k_2=2}^{M2^M} G^{(1)}_{1,x}\big(u(k_1),u(k_2)\big)\,dx\\ &= \lim_{N\to\infty}\frac{8dc^2_{\log}T}{3\cdot 2^{2M}}\sum_{k_1,k_2=2}^{M2^M} NTG^{(1)}_{1,NT}\big(u(k_1),u(k_2)\big) = \widetilde{\rho}^2_M,\end{aligned} \tag{4.52}$$

where the last identity follows from (2.23) and (3.65). This finishes the proof of (4.50).

Step III. Next we prove the convergence in law of $\frac{\varepsilon^{\frac{d}{2}-1}}{\sqrt{\log(1/\varepsilon)}}\widetilde{J}^{\varepsilon}_T$ to a Gaussian random variable with variance ρ^2. From Steps I and II, it suffices to show that

$$R^{\varepsilon}_{T,M}\overset{\text{Law}}{\to}\mathcal{N}\big(0,\widetilde{\rho}^2_M\big),\quad \text{as } \varepsilon\to 0, \tag{4.53}$$

In order to prove (4.53) we proceed as follows. Define the random vector

$$D^{\varepsilon} = \big(D^{\varepsilon}_k\big)_{k=2}^{M2^M},$$

where

$$D^{\varepsilon}_k := -\frac{c_{\log}u(k)^{\frac{3}{2}}}{2^M(1+u(k)^{\frac{3}{2}})^{\frac{d}{2}+1}}\sum_{j=1}^{d}\frac{1}{\varepsilon^{\frac{1}{3}}\sqrt{\log(1/\varepsilon)}}\int_0^T H_2\bigg(\frac{B^{(j)}_{s+\varepsilon^{\frac{2}{3}}u(k)} - B^{(j)}_s}{\sqrt{\varepsilon}u(k)^{\frac{3}{4}}}\bigg)\,ds,$$

and $c_{\log} = \frac{(2\pi)^{-\frac{d}{2}}}{2}$. Notice that

$$\frac{\varepsilon^{\frac{d}{2}-1}}{\sqrt{\log(\varepsilon)}}R^{\varepsilon}_{T,M} = \sum_{k=2}^{M2^M}D^{\varepsilon}_k.$$

We will prove that D^{ε} converges to a centered Gaussian vector. By the Peccati–Tudor criterion (see [12]), it suffices to prove that the components of the vector D^{ε} converge to a Gaussian distribution, and the covariance matrix of D^{ε} is convergent. To prove the former statement, define

$$\Psi^j_{k_1,k_2}(\varepsilon) := \mathbb{E}\bigg[\int_0^T H_2\bigg(\frac{B^{(j)}_{s_1+\varepsilon^{\frac{2}{3}}u(k_1)} - B^{(j)}_{s_1}}{\sqrt{\varepsilon}u(k_1)^{\frac{3}{4}}}\bigg)\,ds_1\int_0^T H_2\bigg(\frac{B^{(j)}_{s_2+\varepsilon^{\frac{2}{3}}u(k_2)} - B^{(j)}_{s_2}}{\sqrt{\varepsilon}u(k_2)^{\frac{3}{4}}}\bigg)\,ds_2\bigg].$$

Proceeding as in the proof of (4.52), we can show that for $2\le k_1,k_2\le M2^M$,

$$\begin{aligned}\Psi^j_{k_1,k_2}(\varepsilon) &= \frac{2(u(k_1)u(k_2))^{-\frac{3}{2}}}{\varepsilon^{\frac{8}{3}}\log(1/\varepsilon)}\int_{[0,T]^2}\mu\big(s_2-s_1,\varepsilon^{\frac{2}{3}}u(k_1),\varepsilon^{\frac{2}{3}}u(k_2)\big)^2\,d\vec{s}\\ &= \frac{8(u(k_1)u(k_2))^{-\frac{3}{2}}}{3\log(\varepsilon^{-\frac{2}{3}})}\int_0^{\varepsilon^{-\frac{2}{3}}T}\int_0^{T-\varepsilon^{\frac{2}{3}}x}\mu\big(x,u(k_1),u(k_2)\big)^2\,ds\,dx.\end{aligned}$$

As in the proof of (4.52), we can use L'Hôpital's rule, (3.65) and the previous identity, to get

$$\lim_{\varepsilon\to 0}\Psi^{i,j}_n = \lim_{\varepsilon\to 0}\frac{8(u(k_1)u(k_2))^{-\frac{3}{2}}T}{3\log(\varepsilon^{-\frac{2}{3}})}\int_0^{\varepsilon^{-\frac{2}{3}}T}\mu\big(x,u(k_1),u(k_2)\big)^2\,dx = \frac{3}{2^3}T\sqrt{u(k)u(j)}.$$

From here, it follows that

$$\lim_{\varepsilon\to 0}\mathbb{E}\big[D^{\varepsilon}_{k_1}D^{\varepsilon}_{k_2}\big]=\Sigma_{i,j}:=\frac{3dT}{2^{d+5}\pi^d 2^{2M}}\psi\big(u(k_1),u(k_2)\big)\big(u(k_1)u(k_2)\big)^2,$$

namely, the covariance matrix of D^{ε} converges to the matrix $\Sigma=(\Sigma_{k,j})_{2\le k,j\le M2^M}$. In addition, by [3, Equation(1.4)], for $2\le k\le M2^M$ fixed, the sequence of random variables D^{ε}_k converges to a Gaussian random variable as $\varepsilon\to 0$. Therefore, by the Peccati–Tudor criterion, the random vector D converges to a jointly Gaussian vector $Z=(Z_k)_{k=2}^{M2^M}$, with mean zero and covariance Σ. In particular, we have

$$\frac{\varepsilon^{\frac{d}{2}-1}}{\sqrt{\log(\varepsilon)}}R^{\varepsilon}_{T,M}=\sum_{k=2}^{M2^M}D^{\varepsilon}_k\overset{\text{Law}}{\to}\mathcal{N}\Bigg(0,\sum_{j,k=2}^{M2^M}\Sigma_{k,j}\Bigg)\quad\text{as } \varepsilon\to 0.$$

Relation (4.53) easily follows from the previous identity.

Since (4.28) holds, in order to finish the proof of Theorem 1.4 it suffices to prove tightness. As before, we define, for $T_1\le T_2$ belonging to a compact interval $[0,K]$, the random variable Z_{ε} by the formula (4.3). Then, by the Billingsley criterion, it suffices to prove that there exist constants $C>0$ and $p>2$, only depending on K, such that

$$\mathbb{E}\Bigg[\bigg|\frac{\varepsilon^{\frac{d}{2}-1}}{\sqrt{\log(1/\varepsilon)}}Z_{\varepsilon}\bigg|^p\Bigg]\le C(T_2-T_1)^{\frac{p}{2}}. \tag{4.54}$$

Using relation (4.27) with $H=\frac{3}{4}$, we can easily check that

$$\begin{aligned}\frac{\varepsilon^{d-2}}{\log(1/\varepsilon)}\|Z_{\varepsilon}\|^2_{L^p(\Omega)}&\le\frac{C(T_2-T_1)}{\log(1/\varepsilon)}\int_{\mathbb{R}^2_+}\int_0^{T_2}\varepsilon^{-\frac{8}{3}}\mu\big(x,\varepsilon^{\frac{2}{3}}u_1,\varepsilon^{\frac{2}{3}}u_2\big)^2(u_1u_2)^{-2H}\\&\quad\times\Theta_1\big(\varepsilon^{-\frac{2}{3}}x,u_1,u_2\big)^{-\frac{d}{p}}\,dx\,d\vec{u}\\&\le\sup_{\varepsilon\in(0,1/e)}\frac{C(T_2-T_1)}{\log(1/\varepsilon)}\int_{\mathbb{R}^2_+}\int_0^{T_2}\varepsilon^{-\frac{8}{3}}\mu\big(x,\varepsilon^{\frac{2}{3}}u_1,\varepsilon^{\frac{2}{3}}u_2\big)^2(u_1u_2)^{-2H}\\&\quad\times\Theta_1\big(\varepsilon^{-\frac{2}{3}}x,u_1,u_2\big)^{-\frac{d}{p}}\,dx\,d\vec{u}.\end{aligned} \tag{4.55}$$

The right-hand side in the previous identity is finite for $p>2$ sufficiently small by Lemma 5.5, and hence, there exists a constant $p>2$ such that

$$\frac{\varepsilon^{d-2}}{\log(1/\varepsilon)}\mathbb{E}\big[|Z_{\varepsilon}|^p\big]\le C(T_2-T_1)^{\frac{p}{2}}.$$

This finishes the proof of the tightness property for $\frac{\varepsilon^{\frac{d}{2}-1}}{\sqrt{\log(1/\varepsilon)}}(I^{\varepsilon}_T-\mathbb{E}[I^{\varepsilon}_{T_1}])$. The proof of Theorem 1.4 is now complete. □

5. Technical lemmas

In this section we prove some technical lemmas, which where used in the proof of Theorems 1.2, 1.3 and 1.4.

Lemma 5.1. *Let $s_1,s_2,t_1,t_2\in\mathbb{R}_+$ be such that $s_1\le s_2$, and $s_i\le t_i$ for $i=1,2$. Denote by Σ the covariance matrix of $(B_{t_1}-B_{s_1},B_{t_2}-B_{s_2})$. Then, there exists a constants $0<\delta<1$ and $k>0$, such that the following inequalities hold*

(1) *If $s_1<s_2<t_1<t_2$,*

$$|\Sigma|\ge\delta\big((a+b)^{2H}c^{2H}+(b+c)^{2H}a^{2H}\big), \tag{5.1}$$

where $a:=s_2-s_1$, $b:=t_1-s_2$ and $c:=t_2-t_1$.

(2) *If* $s_1 < s_2 < t_2 < t_1$,

$$|\Sigma| \geq \delta b^{2H}\big(a^{2H} + c^{2H}\big), \tag{5.2}$$

where $a := s_2 - s_1$, $b := t_2 - s_2$ *and* $c := t_1 - t_2$.

(3) *If* $s_1 < t_1 < s_2 < t_2$,

$$|\Sigma| \geq \delta a^{2H} c^{2H}, \tag{5.3}$$

where $a := t_1 - s_1$ *and* $c := t_2 - s_2$.

Proof. Relations (5.1)–(5.3) follow from Lemma B.1 in [7]. The inequalities (5.1) and (5.3) where also proved in [5, Lemma 9], but the lower bound given in this lemma for the case $s_1 < s_2 < t_2 < t_1$ is not correct. □

Lemma 5.2. *There exists a constant* $k > 0$, *such that for every* $s_1 < t_1 < s_2 < t_2$,

$$\mu(a + b, a, c) \leq k b^{2H-2} ac, \tag{5.4}$$

where $a := t_1 - s_1$, $b := s_2 - t_1$ *and* $c := t_2 - s_2$. *In addition, if* $H > \frac{1}{2}$,

$$\mu(x, u_1, u_2) \leq k(x + u_1 + u_2)^{2H-2} u_1 u_2, \tag{5.5}$$

where $x := s_2 - s_1$, $u_1 := t_1 - s_1$ *and* $u_2 := t_2 - s_2$.

Proof. We can easily check that

$$\mu(a + b, a, c) = \frac{1}{2}\big((a + b + c)^{2H} + b^{2H} - (b + c)^{2H} - (a + b)^{2H}\big),$$

and hence,

$$\mu(a + b, a, c) = H(2H - 1)ac \int_{[0,1]^2} |b + av_1 + cv_2|^{2H-2} \, d\vec{v},$$

Relation (5.4) follows by dropping the term $av_1 + cv_2$ in the previous integral, while (5.5) follows from the following computation, which is valid for every $H > \frac{1}{2}$,

$$\begin{aligned}
\mu(a + b, a, c) &= H(2H - 1)ac \int_{[0,1]^2} |b + av_1 + cv_2|^{2H-2} \, dv_1 \, dv_2 \\
&\leq H(2H - 1)ac \int_0^1 \big|(a \vee b \vee c)v\big|^{2H-2} \, dv \\
&= Hac|a \vee b \vee c|^{2H-2} \leq H4^{2H-2} ac|2a + b + c|^{2H-2} \\
&= 4^{2H-2} H(x + u_1 + u_2)^{2H-2} u_1 u_2.
\end{aligned} \tag{5.6}$$

□

Lemma 5.3. *Define the functions* μ *and* Θ_1 *by* (2.19) *and* (2.35) *respectively. Let* $\frac{3}{2d} < H < 1$, *and* $0 < p < \frac{4Hd}{3}$ *be fixed. Then, the following integral is convergent*

$$\int_{\mathcal{S}_i} \frac{\mu(x, u_1, u_2)^2}{u_1^{2H} u_2^{2H}} \Theta_1(x, u_1, u_2)^{-\frac{d}{p}} \, dx \, d\vec{u} < \infty, \tag{5.7}$$

for $i = 1, 2$, *where the sets* $\mathcal{S}_i$ *are defined by* (3.18). *Moreover, if* $H < \frac{3}{4}$, *then*

$$\int_{\mathbb{R}_+^3} \frac{\mu(x, u_1, u_2)^2}{u_1^{2H} u_2^{2H}} \Theta_1(x, u_1, u_2)^{-\frac{d}{p}} \, dx \, d\vec{u} < \infty. \tag{5.8}$$

Proof. Denote the integrand in (5.8) and (5.7) by $\Psi(x, u_1, u_2)$, namely,

$$\Psi(x, u_1, u_2) = \mu(x, u_1, u_2)^2 (u_1 u_2)^{-2H} \Theta_1(x, u_1, u_2)^{-\frac{d}{p}}. \tag{5.9}$$

We can decompose the domain of integration of (5.8), as $\mathbb{R}_+^3 = \mathcal{S}_1 \cup \mathcal{S}_2 \cup \mathcal{S}_3$, where $\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3$ are defined by (3.18). Then, it suffices to show that

$$\int_{\mathcal{S}_i} \Psi(x, u_1, u_2) \, dx \, d\vec{u} < \infty, \tag{5.10}$$

for $i = 1, 2$ provided that $0 < p < \frac{4Hd}{3}$, and for $i = 3$, provided that $0 < p < \frac{4Hd}{3}$ and $H < \frac{3}{4}$. First consider the case $i = 1$. Changing the coordinates (x, u_1, u_2) by $(a := x, b := u_1 - x, c := x + u_2 - u_1)$ in (5.10) for $i = 1$, we get

$$\int_{\mathcal{S}_1} \Psi(x, u_1, u_2) \, dx \, d\vec{u} = \int_{\mathbb{R}_+^3} \Psi(a, a+b, b+c) \, da \, db \, dc.$$

To bound the integral in the right-hand side we proceed as follows. First we notice that the term $\mu(a, a+b, b+c)$ is given by

$$\mu(a, a+b, b+c) = \frac{1}{2}\big((a+b+c)^{2H} + b^{2H} - c^{2H} - a^{2H}\big).$$

By the Cauchy–Schwarz inequality, $|\mu(a, a+b, b+c)| \leq (a+b)^H (b+c)^H$. In addition, by (5.1) there exists a constant $\delta > 0$ such that

$$(a+b)^{2H}(b+c)^{2H} - \mu(a, a+b, b+c)^2 \geq \delta\big((a+b)^{2H} c^{2H} + (b+c)^{2H} a^{2H}\big). \tag{5.11}$$

As a consequence,

$$\Psi(a, a+b, b+c) \leq \big(1 + (a+b)^{2H} + (b+c)^{2H} + \delta\big((a+b)^{2H} c^{2H} + (b+c)^{2H} a^{2H}\big)\big)^{-\frac{d}{p}}.$$

Define $\varrho_1 = \varrho_1(a, b, c)$ and $\varrho_2 = \varrho_2(a, b, c)$ as the first and second largest element of $\{a, b, c\}$. Hence, we deduce that there exists a constant $K > 0$ such that

$$\Psi(a, a+b, b+c) \leq K\big(1 + \varrho_1^{2H} + \varrho_1^{2H} \varrho_2^{2H}\big)^{-\frac{d}{p}} \leq K(1 \vee \varrho_1)^{-\frac{2Hd}{p}} (1 \vee \varrho_1)^{-\frac{2Hd}{p}}. \tag{5.12}$$

Using the condition $p < \frac{4Hd}{3}$, as well as the previous inequality, we can easily check that $\Psi(a, a+b, b+c)$ is integrable in $\mathbb{R}_+^3$, which in turn implies that $\Psi(x, u_1, u_2)$ is integrable in $\mathcal{S}_1$, as required.

Next we consider the case $i = 2$. Changing the coordinates (x, u_1, u_2) by $(a := x, b := u_2, c := u_1 - x - u_2)$ in (5.10) for $i = 2$, we get

$$\int_{\mathcal{S}_2} \Psi(x, u_1, u_2) \, dx \, du_1 \, du_2 = \int_{\mathbb{R}_+^3} \Psi(a, a+b+c, b) \, da \, db \, dc.$$

To bound the integral in the right-hand side we proceed as follows. First notice that the term $\mu(a, a+b+c, b)$ is given by

$$\mu(a, a+b+c, b) = \frac{1}{2}\big((b+c)^{2H} + (a+b)^{2H} - c^{2H} - a^{2H}\big). \tag{5.13}$$

By the Cauchy–Schwarz inequality, $|\mu(a,a+b+c,b)| \le b^H(a+b+c)^H$. In addition, by (5.2), there exists a constant $\delta > 0$ such that

$$b^{2H}(a+b+c)^{2H} - \mu(a,a+b+c,b)^2 \ge \delta b^{2H}\big(a^{2H}+c^{2H}\big).$$

As a consequence,

$$\Psi(a,a+b+c,b) \le \big(1+b^{2H}+(a+b+c)^{2H}+\delta b^{2H}\big(a^{2H}+c^{2H}\big)\big)^{-\frac{d}{p}}.$$

From here it follows that there exists a constant $K > 0$ such that (5.12) holds. Using the condition $p < \frac{4Hd}{3}$, as well as the previous inequalities, we can easily check that $\Psi(a,a+b+c,b)$ is integrable in the region $\{(a,b,c) \in \mathbb{R}_+^3 \mid b \ge a \wedge c\}$.

Next we check the integrability of $\Psi(a,a+b+c,b)$ in $\{(a,b,c) \in \mathbb{R}_+^3 \mid b \le a \wedge c\}$. Applying the mean value theorem in (5.13), we can easily check that

$$\mu(a,a+b+c,b) = \frac{1}{2}\big(2H(a+\xi_1)^{2H-1}b + 2H(c+\xi_2)^{2H-1}b\big), \tag{5.14}$$

for some ξ_1, ξ_2 between 0 and b. Therefore, if $H < \frac{1}{2}$, we obtain

$$\mu(a,a+b+c,b) \le H\big(a^{2H-1}+c^{2H-1}\big)b, \tag{5.15}$$

which in turn implies that

$$\begin{aligned}\Psi(a,a+b+c,b) \le{}& H^2\big(a^{2H-1}+c^{2H-1}\big)^2 b^{2-2H}(a+b+c)^{-2H}\\ &\times\big(1+b^{2H}+(a+b+c)^{2H}+\delta b^{2H}\big(a^{2H}+c^{2H}\big)\big)^{-\frac{d}{p}}.\end{aligned} \tag{5.16}$$

For the case $H \ge \frac{1}{2}$, we use (5.14), in order to obtain

$$\mu(a,a+b+c,b) \le H\big((a+b)^{2H-1}+(c+b)^{2H-1}\big)b,$$

which in turn implies that

$$\begin{aligned}\Psi(a,a+b+c,b) \le{}& H^2\big((a+b)^{2H-1}+(c+b)^{2H-1}\big)^2 b^{2-2H}(a+b+c)^{-2H}\\ &\times\big(1+b^{2H}+(a+b+c)^{2H}+\delta b^{2H}\big(a^{2H}+c^{2H}\big)\big)^{-\frac{d}{p}}.\end{aligned} \tag{5.17}$$

From (5.16), we deduce that, if $H < \frac{1}{2}$, there exists a constant $K > 0$ such that

$$\Psi(a,a+b+c,b) \le K(a\wedge c)^{4H-2}b^{2-2H}(a\vee c)^{-2H}\big(1+(a\vee c)^{2H}+b(a\vee c)^{2H}\big)^{-\frac{d}{p}}. \tag{5.18}$$

In turn, from (5.17), it follows that if $H \ge \frac{1}{2}$, there exists a constant $K > 0$, such that

$$\Psi(a,a+b+c,b) \le K(a\vee c)^{4H-2}b^{2-2H}\big(1+(a\vee c)^{2H}+b^{2H}(a\vee c)^{2H}\big)^{-\frac{d}{p}}. \tag{5.19}$$

Using the conditions $H < \frac{3}{4}$ and $p < \frac{4Hd}{3}$, we can easily check that $2H < \frac{Hd}{2p}$, which, by (5.18) and (5.19), implies that $\Psi(a,a+b+c,b)$ is integrable in $\{(a,b,c) \in \mathbb{R}_+^3 \mid b \le a \wedge c\}$. From here it follows that $\Psi(a,a+b+c,b)$ is integrable in $\mathbb{R}_+^3$, and hence $\Psi(x,u_1,u_2)$ is integrable in $\mathcal{S}_2$, as required.

Finally we consider the case $i = 3$ for $H < \frac{3}{4}$. Changing the coordinates (x,u_1,u_2) by $(a := u_1, b := x-u_1, c := u_2)$ in (5.10) for $i = 3$, we get

$$\int_{\mathcal{S}_3}\Psi(x,u_1,u_2)\,dx\,d\vec{u} = \int_{\mathbb{R}^3}\Psi(a+b,a,c)\,da\,db\,dc.$$

To bound the integral in the right-hand side we proceed as follows. First we notice that the term $\mu(a+b,a,c)$ is given by

$$\mu(a+b,a,c)=\frac{1}{2}\big((a+b+c)^{2H}+b^{2H}-(b+c)^{2H}-(a+b)^{2H}\big). \tag{5.20}$$

By the Cauchy–Schwarz inequality, $\mu(a+b,a,c)\leq a^H c^H$. In addition, by (5.3), there exist constants $k,\delta>0$ such that

$$a^{2H}c^{2H}-\mu(a+b,a,c)^2\geq \delta a^{2H}c^{2H}, \tag{5.21}$$

and

$$\mu(a+b,a,c)\leq kb^{2H-2}ac. \tag{5.22}$$

From (5.21) and (5.22), we deduce the following bounds for Ψ

$$\Psi(a+b,a,c)\leq \big(1+a^{2H}+c^{2H}+\delta a^{2H}c^{2H}\big)^{-\frac{d}{p}}, \tag{5.23}$$

$$\Psi(a+b,a,c)\leq 2Hb^{4H-4}(ac)^{-2H+2}\big(1+a^{2H}+c^{2H}+\delta a^{2H}c^{2H}\big)^{-\frac{d}{p}}. \tag{5.24}$$

Using (5.23), as well as the condition $p<\frac{4Hd}{3}$, we can easily check that $\Psi(a+b,a,c)$ is integrable in the region $\{(a,b,c)\in\mathbb{R}^3_+ \mid b\leq a\wedge c\}$.

Next we check the integrability of $\Psi(a+b,a,c)$ in the region $\{(a,b,c)\in\mathbb{R}^3_+ \mid b\geq a\vee c\}$. Since $H<\frac{3}{4}$, from (5.24) it follows that there exists a constant $C>0$ such that

$$\begin{aligned}\int_{(a\vee c)}^{\infty}\Psi(a+b,a,c)\,db&\leq C(ac)^{-2H+2}(a\vee c)^{4H-3}\big(1+a^{2H}+c^{2H}+a^{2H}c^{2H}\big)^{-\frac{d}{p}}\\&\leq C(ac)^{\frac{1}{2}}\big(1+a^{2H}+c^{2H}+a^{2H}c^{2H}\big)^{-\frac{d}{p}}.\end{aligned}$$

The integrability of $\Psi(a+b,a,c)$ in the region $\{(a,b,c)\in\mathbb{R}^3_+ \mid b\geq a\vee c\}$ then follows from condition the $p<\frac{4Hd}{3}$.

Finally, we prove the integrability of $\Psi(a+b,a,c)$ in the regions $\{(a,b,c)\in\mathbb{R}^3_+ \mid a\leq b\leq c\}$ and $\{(a,b,c)\in\mathbb{R}^3_+ \mid c\leq b\leq a\}$. Let $a,b,c\geq 0$ be such that $a\leq b\leq c$. Applying the mean value theorem to (5.20), we can easily show that

$$\mu(a+b,a,c)=\frac{1}{2}\big(\xi_1^{2H-1}a-\xi_2^{2H-1}a\big),$$

for some ξ_1 between $c+b$ and $a+b+c$, and ξ_2 between b and $a+b$. Hence, if $H\leq\frac{1}{2}$, it follows that

$$\big|\mu(a+b,a,c)\big|\leq\frac{1}{2}\big(|\xi_1|^{2H-1}a+|\xi_2|^{2H-1}a\big)\leq\frac{1}{2}\big((c+b)^{2H-1}a+b^{2H-1}a\big).$$

From here it follows that there exists a constant $C>0$, only depending on H such that

$$\big|\mu(a+b,a,c)\big|\leq Cb^{2H-1}a. \tag{5.25}$$

Using inequalities (5.21) and (5.25), we deduce that there exists a constant $K>0$ such that

$$\Psi(a+b,a,c)\leq Kb^{4H-2}a^{2-2H}c^{-2H}\big(1+a^{2H}+c^{2H}+a^{2H}c^{2H}\big)^{-\frac{d}{p}}.$$

From here, it follows that

$$\Psi(a+b,a,c)\leq Kb^{4H-2}a^{2-2H}c^{-2H}\big(1+a^{2H}+c^{2H}+a^{2H}c^{2H}\big)^{-\frac{d}{p}}. \tag{5.26}$$

Using the condition $H \leq \frac{3}{4}$, we can easily show that $2H - \frac{2Hd}{p} \leq \frac{3}{2} - \frac{2Hd}{p} < 0$. Hence, from (5.26), we deduce that $\Psi(a+b,a,c)$ is integrable in $\{(a,b,c) \in \mathbb{R}_+^3 \mid a \leq b \leq c\}$. The integrability of $\Psi(a+b,a,c)$ over the region $\{(a,b,c) \in \mathbb{R}_+^3 \mid c \leq b \leq a\}$ in the case $H \leq \frac{1}{2}$, follows from a similar argument. To handle the case $H > \frac{1}{2}$, we proceed as follows. From (5.20), we can easily show that for every $a,b,c \geq 0$ such that $a \leq b \leq c$,

$$\mu(a+b,a,c) = H(2H-1)ac \int_{[0,1]^2} (b + a\xi + c\eta)^{2H-2}\, d\xi\, d\eta \leq H(2H-1)ac \int_0^1 (c\eta)^{2H-2}\, d\eta,$$

and hence

$$\mu(a+b,a,c) \leq Hac^{2H-1}.$$

From here it follows that

$$\Psi(a+b,a,c) \leq a^{2-2H} c^{2H-2} \left(1 + a^{2H} + c^{2H} + a^{2H} c^{2H}\right)^{-\frac{d}{p}}.$$

Using the condition $p < \frac{4Hd}{3}$, we deduce that $\Psi(a+b,a,c)$ is integrable in $\{(a,b,c) \in \mathbb{R}_+^3 \mid a \leq b \leq c\}$. The integrability of $\Psi(a+b,a,c)$ over the region $\{(a,b,c) \in \mathbb{R}_+^3 \mid c \leq b \leq a\}$ in the case $H > \frac{1}{2}$, follows from a similar argument. From the previous analysis it follows that $\Psi(a+b,a,c)$ is integrable in $\mathbb{R}_+^3$, and hence $\Psi(x,u_1,u_2)$ is integrable in $\mathcal{S}_3$, as required. The proof is now complete. □

Following similar arguments to those presented in the proof of Lemma 5.3, we can prove the following result

Lemma 5.4. *Let the functions μ and Θ_1 be defined by* (2.19) *and* (2.35) *respectively. Then, for every $\frac{3}{4} < H < 1$ and $0 < p < \frac{4Hd}{3}$,*

$$\sup_{\varepsilon \in (0,1)} \int_{\mathbb{R}_+^2} \int_0^T \varepsilon^{-\frac{2}{H}} \frac{\mu(x, \varepsilon^{\frac{1}{2H}} u_1, \varepsilon^{\frac{1}{2H}} u_2)^2}{u_1^{2H} u_2^{2H}} \Theta_1\left(\varepsilon^{-\frac{1}{2H}} x, u_1, u_2\right)^{-\frac{d}{p}} dx\, d\vec{u} < \infty. \tag{5.27}$$

Proof. Denote by $\kappa_\varepsilon(x,u_1,u_2)$ the function

$$\kappa_\varepsilon(x,u_1,u_2) := \varepsilon^{-\frac{2}{H}} \mu\left(x, \varepsilon^{\frac{1}{2H}} u_1, \varepsilon^{\frac{1}{2H}} u_2\right)^2 (u_1 u_2)^{-2H} \Theta_1\left(\varepsilon^{-\frac{1}{2H}} x, u_1, u_2\right)^{-\frac{d}{p}}.$$

To prove (5.27), it suffices to show that

$$\sup_{\varepsilon \in (0,1)} \int_{\mathbb{R}_+^2} \int_0^T \mathbb{1}_{\mathcal{S}_i}\left(x, \varepsilon^{\frac{1}{2H}} u_1, \varepsilon^{\frac{1}{2H}} u_2\right) \kappa_\varepsilon(x,u_1,u_2)\, dx\, d\vec{u} < \infty, \tag{5.28}$$

for $i = 1,2,3$. To prove (5.28) in the case $i = 1,2$, we make the change of variable $\widehat{x} := \varepsilon^{-\frac{1}{2H}} x$, in order to get

$$\begin{aligned} &\int_{\mathbb{R}_+^2} \int_0^T \mathbb{1}_{\mathcal{S}_i}\left(x, \varepsilon^{\frac{1}{2H}} u_1, \varepsilon^{\frac{1}{2H}} u_2\right) \kappa_\varepsilon(x,u_1,u_2)\, dx\, d\vec{u} \\ &\quad = \varepsilon^{-\frac{3}{2H}+2} \int_{\mathbb{R}_+^2} \int_0^{\varepsilon^{-\frac{1}{2H}} T} \mathbb{1}_{\mathcal{S}_i}(\widehat{x}, u_1, u_2) \Psi(\widehat{x}, u_1, u_2)\, d\widehat{x}\, d\vec{u}, \end{aligned}$$

where Ψ is defined by (5.9). Hence,

$$\int_{\mathbb{R}_+^2} \int_0^T \mathbb{1}_{\mathcal{S}_i}\left(x, \varepsilon^{\frac{1}{2H}} u_1, \varepsilon^{\frac{1}{2H}} u_2\right) \kappa_\varepsilon(x,u_1,u_2)\, dx\, d\vec{u} \leq \int_{\mathcal{S}_i} \Psi(x,u_1,u_2)\, dx\, d\vec{u}. \tag{5.29}$$

In Lemma 5.3, we proved that $\int_{\mathcal{S}_1}\Psi(x,u_1,u_2)\,dx\,d\vec{u}<\infty$, provided that $p<\frac{4Hd}{3}$. To handle the case $i=2$, we change the coordinates (x,u_1,u_2) by $(a:=x, b:=u_2, c:=u_1-x-u_2)$, in order to get

$$\int_{\mathcal{S}_2}\Psi(x,u_1,u_2)\,dx\,d\vec{u}=\int_{\mathbb{R}_+^3}\Psi(a,a+b+c,b)\,da\,db\,dc.$$

By (5.12), $\Psi(a,a+b+c,b)$ is integrable in $\{(a,b,c)\in\mathbb{R}_+^3 \mid b\geq a\wedge c\}$. In addition, since $2H-\frac{1}{2}\leq\frac{3}{2}<Hd$, by (5.19), $\Psi(a,a+b+c,b)$ is integrable in $\{(a,b,c)\in\mathbb{R}_+^3 \mid b\leq a\wedge c\}$, and hence, $\Psi(x,u_1,u_2)$ is integrable in $\mathcal{S}_2$, as required. It then remains to prove (5.28) in the case $i=3$. By (5.6), for every $(x,v_1,v_2)\in\mathcal{S}_3$,

$$\left|\mu(x,v_1,v_2)\right|\leq Cv_1v_2x^{2H-2}. \tag{5.30}$$

On the other hand, for every $(x,\varepsilon^{\frac{1}{2H}}u_1,\varepsilon^{\frac{1}{2H}}u_2)\in\mathcal{S}_3$, it holds $(\varepsilon^{-\frac{1}{2H}}x,u_1,u_2)\in\mathcal{S}_3$, and hence, by (5.21),

$$\Theta_1\big(\varepsilon^{-\frac{1}{2H}}x,u_1,u_2\big)\geq\delta u_1^{2H}u_2^{2H}. \tag{5.31}$$

By (5.30) and (5.31), we obtain

$$\kappa_\varepsilon(x,u_1,u_2)\leq C(u_1u_2)^{2-2H}x^{4H-4}\big(1+u_1^{2H}+u_2^{2H}+u_1^{2H}u_2^{2H}\big)^{-\frac{d}{p}}, \tag{5.32}$$

for some constant $C>0$, and hence,

$$\begin{aligned}&\int_{\mathbb{R}_+^2}\int_0^T \mathbb{1}_{\mathcal{S}_3}\big(x,\varepsilon^{\frac{1}{2H}}u_1,\varepsilon^{\frac{1}{2H}}u_2\big)\kappa_\varepsilon(x,u_1,u_2)\,dx\,d\vec{u}\\ &\quad\leq\int_{\mathbb{R}_+^2}\int_0^T (u_1u_2)^{2-2H}x^{4H-4}\big(1+u_1^{2H}+u_2^{2H}+u_1^{2H}u_2^{2H}\big)^{-\frac{d}{p}}\,dx\,d\vec{u}.\end{aligned}$$

Since $H>\frac{3}{4}$, then $3-2H<\frac{3}{2}<Hd$, and hence, the integral in the right-hand side of the previous identity is finite, which implies that (5.28) holds for $i=3$, as required. The proof is now complete. □

Lemma 5.5. *Let $d\geq 3$, and $T>0$ be fixed. Let the functions μ and Θ_ε be defined by* (2.19) *and* (2.35) *respectively and and assume that $H=\frac{3}{4}$. Then, for every $0<p<d$,*

$$\sup_{\varepsilon\in(0,1/e)}\frac{\varepsilon^{-8/3}}{\log(1/\varepsilon)}\int_{\mathbb{R}_+^2}\int_0^T (u_1u_2)^{-\frac{3}{2}}\mu\big(x,\varepsilon^{\frac{2}{3}}u_1,\varepsilon^{\frac{2}{3}}u_2\big)^2\Theta_1\big(\varepsilon^{-\frac{2}{3}}x,u_1,u_2\big)^{-\frac{d}{p}}\,dx\,d\vec{u}<\infty.$$

Proof. Denote by $\kappa_\varepsilon(x,u_1,u_2)$ the function

$$\kappa_\varepsilon(x,u_1,u_2):=\frac{\varepsilon^{-8/3}}{\log(1/\varepsilon)}\mu\big(x,\varepsilon^{\frac{2}{3}}u_1,\varepsilon^{\frac{2}{3}}u_2\big)^2(u_1u_2)^{-\frac{3}{2}}\Theta_1\big(\varepsilon^{-\frac{2}{3}}x,u_1,u_2\big)^{-\frac{d}{p}}.$$

As in Lemma 5.4, it suffices to show that

$$\sup_{\varepsilon\in(0,1)}\int_{\mathbb{R}_+^2}\int_0^T \mathbb{1}_{\mathcal{S}_i}\big(x,\varepsilon^{\frac{2}{3}}u_1,\varepsilon^{\frac{2}{3}}u_2\big)\kappa_\varepsilon(x,u_1,u_2)\,dx\,d\vec{u}<\infty, \tag{5.33}$$

for $i=1,2,3$, where the regions $\mathcal{S}_i$ are defined by (3.18). The cases $i=1,2$ are handled similarly to Lemma 5.4, so it suffices to prove (5.28) in the case $i=3$. Suppose $(x,\varepsilon^{\frac{2}{3}}u_1,\varepsilon^{\frac{2}{3}}u_2)\in\mathcal{S}_3$. Then, by Lemma 5.2, there exists a constant $C>0$, such that

$$\big|\mu\big(x,\varepsilon^{\frac{2}{3}}u_1,\varepsilon^{\frac{2}{3}}u_2\big)\big|\leq C\varepsilon^{4/3}\big(x+\varepsilon^{\frac{2}{3}}u_1+\varepsilon^{\frac{2}{3}}u_2\big)^{-\frac{1}{2}}u_1u_2=C\varepsilon\big(\varepsilon^{-\frac{2}{3}}x+u_1+u_2\big)^{-\frac{1}{2}}u_1u_2.$$

In addition, by Lemma 5.1 we have that $u_1^{\frac{3}{2}}u_2^{\frac{3}{2}} - \mu(\varepsilon^{-\frac{2}{3}}x, u_1, u_2)^2 \geq \delta(u_1u_2)^{\frac{3}{2}}$, for some $\delta > 0$. Therefore, we conclude that there exists a constant $C > 0$, such that

$$\kappa_\varepsilon(x,u_1,u_2) \leq \frac{\varepsilon^{-\frac{2}{3}}C^2}{\log(1/\varepsilon)}\big(\varepsilon^{-\frac{2}{3}}x+u_1+u_2\big)^{-1}\sqrt{u_1u_2}\big(1+u_1^{\frac{3}{2}}+u_2^{\frac{3}{2}}+u_1^{\frac{3}{2}}u_2^{\frac{3}{2}}-\mu(x,u_1,u_2)^2\big)^{-\frac{d}{p}}$$
$$\leq \frac{\varepsilon^{-\frac{2}{3}}C^2\delta^{-\frac{d}{p}}}{\log(1/\varepsilon)}\big(\varepsilon^{-\frac{2}{3}}x+u_1+u_2\big)^{-1}\sqrt{u_1u_2}\big(1+u_1^{\frac{3}{2}}+u_2^{\frac{3}{2}}+u_1^{\frac{3}{2}}u_2^{\frac{3}{2}}\big)^{-\frac{d}{p}}.$$

Consequently, there exists a constant $C > 0$, such that

$$\int_{\mathbb{R}_+^2}\int_0^T \mathbb{1}_{\mathcal{S}_i}(x,u_1,u_2)\kappa_\varepsilon(x,u_1,u_2)\,dx\,d\vec{u}$$
$$\leq \frac{C\varepsilon^{-\frac{2}{3}}}{\log(1/\varepsilon)}\int_0^T\int_{\mathbb{R}_+^2}\big(\varepsilon^{-\frac{2}{3}}x+u_1+u_2\big)^{-1}\sqrt{u_1u_2}\big(1+u_1^{\frac{3}{2}}+u_2^{\frac{3}{2}}+u_1^{\frac{3}{2}}u_2^{\frac{3}{2}}\big)^{-\frac{d}{p}}\,d\vec{u}\,dx.$$

Hence, making the change of variable $\widetilde{x} := \varepsilon^{-\frac{2}{3}}x$, we obtain

$$\int_{\mathbb{R}_+^2}\int_0^T \mathbb{1}_{\mathcal{S}_i}(x,u_1,u_2)\kappa_\varepsilon(x,u_1,u_2)\,dx\,d\vec{u}$$
$$= \frac{C}{\log(1/\varepsilon)}\int_{\mathbb{R}_+^2}\int_0^1 (x+u_1+u_2)^{-1}\sqrt{u_1u_2}\big(1+u_1^{\frac{3}{2}}+u_2^{\frac{3}{2}}+u_1^{\frac{3}{2}}u_2^{\frac{3}{2}}\big)^{-\frac{d}{p}}\,dx\,d\vec{u}$$
$$+ \frac{C}{\log(1/\varepsilon)}\int_{\mathbb{R}_+^2}\int_1^{\varepsilon^{-\frac{2}{3}}T} (x+u_1+u_2)^{-1}\sqrt{u_1u_2}\big(1+u_1^{\frac{3}{2}}+u_2^{\frac{3}{2}}+u_1^{\frac{3}{2}}u_2^{\frac{3}{2}}\big)^{-\frac{d}{p}}\,dx\,d\vec{u}. \tag{5.34}$$

Applying the inequalities $(x+u_1+u_2)^{-1} \leq (u_1+u_2)^{-1} \leq \frac{1}{2}(u_1u_2)^{-\frac{1}{2}}$ for $x \in [0,1]$, and $(x+u_1+u_2)^{-1} \leq x^{-1}$ for $x \geq 1$, in the first and second terms in the right-hand side of (5.34), and then integrating the variable x, we can show that

$$\int_{\mathbb{R}_+^2}\int_0^T \mathbb{1}_{\mathcal{S}_i}(x,u_1,u_2)\kappa_\varepsilon(x,u_1,u_2)\,dx\,d\vec{u} \leq C\int_{\mathbb{R}_+^2}\left(\frac{(u_1u_2)^{-\frac{1}{2}}+\frac{2}{3}\log(1/\varepsilon)+\log(T)}{\log(1/\varepsilon)}\right)$$
$$\times\sqrt{u_1u_2}\big(1+u_1^{\frac{3}{2}}+u_2^{\frac{3}{2}}+u_1^{\frac{3}{2}}u_2^{\frac{3}{2}}\big)^{-\frac{d}{p}}\,dx\,d\vec{u},$$

and consequently, for every $\varepsilon < 1/e$,

$$\int_{\mathbb{R}_+^2}\int_0^T \mathbb{1}_{\mathcal{S}_i}(x,u_1,u_2)\kappa_\varepsilon(x,u_1,u_2)\,dx\,d\vec{u}$$
$$\leq C\int_{\mathbb{R}_+^2}\big((u_1u_2)^{-\frac{1}{2}}+\log(T)\big)\sqrt{u_1u_2}\big(1+u_1^{\frac{3}{2}}+u_2^{\frac{3}{2}}+u_1^{\frac{3}{2}}u_2^{\frac{3}{2}}\big)^{-\frac{d}{p}}\,dx\,d\vec{u}.$$

The right-hand side of the previous inequality is finite due to the condition $0 < p < d$. This finishes the proof of (5.33). □

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