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A GENERAL CONTINUOUS-STATE NONLINEAR BRANCHING PROCESS

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In this paper, we consider the unique nonnegative solution to the following generalized version of the stochastic differential equation for a continuous-state branching process:

$$X_t = x + \int_0^t \gamma_0(X_s)\,\mathrm{d}s + \int_0^t \int_0^{\gamma_1(X_{s-})} W(\mathrm{d}s, \mathrm{d}u) + \int_0^t \int_0^\infty \int_0^{\gamma_2(X_{s-})} z\tilde{N}(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u),$$

where $W(\mathrm{d}t, \mathrm{d}u)$ and $\tilde{N}(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u)$ denote a Gaussian white noise and an independent compensated spectrally positive Poisson random measure, respectively, and γ_0, γ_1 and γ_2 are functions on $\mathbb{R}_+$ with both γ_1 and γ_2 taking nonnegative values. Intuitively, this process can be identified as a continuous-state branching process with population-size-dependent branching rates and with competition. Using martingale techniques we find rather sharp conditions on extinction, explosion and coming down from infinity behaviors of the process. Some Foster–Lyapunov-type criteria are also developed for such a process. More explicit results are obtained when γ_i, $i = 0, 1, 2$ are power functions.

1. Introduction.

1.1. *Continuous-state branching processes.* Suppose that $(\Omega, \mathscr{F}, \mathscr{F}_t, \mathbb{P})$ is a filtered probability space satisfying the usual hypotheses. Let $\mathbb{P}_x$ be the law of a process started at x, and denote by $\mathbb{E}_x$ the associated expectation. A continuous-state branching process $X = (X_t)_{t\geq 0}$ is a càdlàg $[0, \infty]$-valued $(\mathscr{F}_t)$-adapted process satisfying the branching property, that is, for any $x, y \geq 0$ and $t, \theta \geq 0$,

$$\mathbb{E}_{x+y}[\mathrm{e}^{-\theta X_t}] = \mathbb{E}_x[\mathrm{e}^{-\theta X_t}]\mathbb{E}_y[\mathrm{e}^{-\theta X_t}]. \tag{1.1}$$

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Consequently, its Laplace transform is determined by

$$\mathbb{E}_x[\mathrm{e}^{-\theta X_t}] = \mathrm{e}^{-x u_t(\theta)},$$

where the nonnegative function $u_t(\theta)$ solves the differential equation

$$\frac{\partial u_t(\theta)}{\partial t} + \psi(u_t(\theta)) = 0$$

with initial value $u_0(\theta) = \theta \geq 0$ and Laplace exponent

$$\psi(\lambda) = b\lambda + \frac{1}{2}\sigma^2\lambda^2 + \int_0^\infty (\mathrm{e}^{-\lambda x} - 1 + \lambda x)\pi(\mathrm{d}x)$$

for $b \in \mathbb{R}$, $\sigma \geq 0$ and for σ-finite measure π on $(0, \infty)$ satisfying $\int_0^\infty (z \wedge z^2)\pi(\mathrm{d}z) < \infty$.

Via the Lamperti random time change, the continuous-state branching process is associated to a spectrally positive Lévy process, which allows many semiexplicit expressions. In particular, extinction and explosion behaviors for continuous-state branching processes were studied by Grey (1974) and Kawazu and Watanabe (1971), respectively, and the conditions for extinction and explosion were expressed in terms of the respective integral tests on the function ψ.

Bertoin and Le Gall (2006) and Dawson and Li (2006, 2012) noticed the following alternative way of characterizing continuous-state branching processes through stochastic differential equations (SDEs in short). Let $\{W(\mathrm{d}t, \mathrm{d}u) : t, u \geq 0\}$ denote an $(\mathscr{F}_t)$-Gaussian white noise with density measure $\mathrm{d}t\,\mathrm{d}u$ on $(0, \infty)^2$. In this paper, we always write $\pi \neq 0$ for the σ-finite measure on $(0, \infty)$. Let $\{N(\mathrm{d}t, \mathrm{d}z, \mathrm{d}u) : t, z, u > 0\}$ denote an independent $(\mathscr{F}_t)$-Poisson random measure with intensity measure $\mathrm{d}t\pi(\mathrm{d}z)\,\mathrm{d}u$ on $(0, \infty)^3$ and let $\{\tilde{N}(\mathrm{d}t, \mathrm{d}z, \mathrm{d}u) : t, z, u > 0\}$ denote the corresponding compensated measure. Then the continuous-state branching process is a pathwise unique nonnegative solution to the following SDE that is called a Dawson–Li SDE in Pardoux (2016):

$$\begin{aligned} X_t = x + b\int_0^t X_s\,\mathrm{d}s + \sigma\int_0^t\int_0^{X_{s-}} W(\mathrm{d}s, \mathrm{d}u) \\ + \int_0^t\int_0^\infty\int_0^{X_{s-}} z\tilde{N}(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u). \end{aligned} \tag{1.2}$$

SDEs similar to (1.2) were studied by Dawson and Li (2006, 2012) and by Fu and Li (2010). See also Bertoin and Le Gall (2003, 2005) for related work.

We refer to Kyprianou (2006), Li (2011, 2012) and Pardoux (2016) for reviews and literature on continuous-state branching processes.

1.2. *Continuous-state branching processes with nonlinear branching.* Models with interactions have gained interests in the study of branching processes. Athreya and Ney (1972) introduced population-size-dependent Galton–Watson branching processes in which the reproduction mechanism depends on the population size; see also Klebaner (1984) and Höpfner (1985) for previous work on population-size-dependent Galton–Watson processes. Another class of interacting Galton–Watson processes is the so-called controlled branching processes. For a controlled branching process, the reproduction law is fixed. But before each branching time, the population is regulated by a control function. Previous work on controlled branching processes can be found in Sevast'yanov and Zubkov (1974) and references therein. A discrete state, continuous time branching process with population dependent branching rate can be found in Chen (1997). When the branching rate function is a power function of the population, the extinction probability for such a branching process was obtained in Chen (2002). When the branching rate is a general positive nonlinear function, such a model called nonlinear Markov branching process was studied in Pakes (2007).

The previous work on discrete-state interacting branching processes motivates the study of their continuous-state counterparts. Some population-size-dependent continuous-state branching processes arising as scaling limits of the corresponding discrete-state branching processes can be found in Li (2006, 2009).

In this paper, we introduce a class of continuous-state branching processes whose branching rates depend on their current population sizes. To this end, we consider a nonnegative solution to the following modification of SDE (1.2):

$$\begin{aligned} X_t = x &+ \int_0^t \gamma_0(X_s)\,\mathrm{d}s + \int_0^t \int_0^{\gamma_1(X_{s-})} W(\mathrm{d}s, \mathrm{d}u) \\ &+ \int_0^t \int_0^\infty \int_0^{\gamma_2(X_{s-})} z\tilde{N}(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u), \end{aligned} \tag{1.3}$$

where γ_0, γ_1 and γ_2 are Borel functions on $\mathbb{R}_+$, and both γ_1 and γ_2 take nonnegative values. The unique nonnegative solution to (1.3) up to the minimum of its first time of hitting 0 and its explosion time can be treated as a continuous-state nonlinear branching process, where $\gamma_i(x)/x, i = 1, 2$ can be interpreted as population-size-dependent branching rates and the drift term involving γ_0 can be related to either competition or population-size-dependent continuous immigration. We refer to Duhalde et al. (2014) for work on continuous-state branching processes with immigration. If $\gamma_i(x) = c_i x$ for $c_1, c_2 \geq 0$, then the solution to (1.3) reduces to the classical continuous-state branching process and satisfies the branching property (1.1). Observe that the solution X to (1.3) can also be treated as a continuous-state controlled branching process.

For $\gamma_2 \equiv 0$, $\gamma_1(z) = z$ and γ_0 satisfying certain conditions, the SDE (1.3) was studied in Pardoux and Wakolbinger (2015) and in Pardoux (2016) where the function γ_0 models an impact of the current population size on the individuals' reproduction dynamics. If the interaction is of the type of competition for rare resources, then increasing the population size results in a reduction of the individuals' birth rate and/or increment of the death rate.

For $\gamma_1(z) = \gamma_2(z) = z$ and $\gamma_0(z) = \theta z - \gamma z^2$ with positive constants θ and γ, the solution to SDE (1.3) can be used to model the density dependence in population dynamics of a large population with competition called logistic branching process, and it was studied in detail by Lambert (2005). The quadratic regulatory term has an ecological interpretation as it describes negative interactions between each pair of individuals in the population. The extinction behavior and the probability distribution of the extinction time were considered in Lambert (2005). A similar model with more general function γ_0 was considered in Le et al. (2013) with its first passage times studied. The total mass for this model was also studied using the Lamperti transform. Berestycki et al. (2018) gave a genealogical description for the process based on interactive pruning of Lévy-trees, and established a Ray–Knight representation result.

For $\gamma_0(z) = \gamma_2(z) \equiv 0$, the extinction/survival behaviors for process X as the total mass process of a superprocess with mean field interaction were discussed in Wang et al. (2017) by a martingale approach. More generally, for $\gamma_2(z) \equiv 0$ the extinction, explosion and coming down from infinity behaviors for the diffusion process X are associated to the classification of its boundaries at 0 and ∞, respectively; see Karlin and Taylor (1981), page 229.

For $\gamma_i(z) = c_i z^r$ with $r > 0$, $c_0 \in \mathbb{R}$ and $c_i \geq 0$ for $i = 1, 2$, the solution to SDE (1.3), called a continuous-state polynomial branching process, was studied by Li (2018), where the parameter r describes the degree of interaction. The polynomial branching process also arises as time-space scaling limit of discrete-state nonlinear branching processes. Intuitively, the functions γ_1 and γ_2 are population-dependent rates for branching events producing small and large amounts of children, respectively. By solving the corresponding Kolmogorov equations, necessary and sufficient conditions in terms of integral tests were obtained for extinction, explosion and coming down from infinity, respectively. Expectations of the extinction time and explosion time were also discussed in Li (2018), which generalizes those results in Chen (2002) for discrete-state processes to the corresponding continuous-state processes. The nonlinear branching processes considered in this paper generalize those in Li (2018) by allowing different rates for different branching events.

Note that if $\tilde{N}$ is the compensated measure of a one-sided α-stable random measure with $\alpha \in (1, 2)$, that is,

$$\pi(\mathrm{d}z) = \frac{\alpha(\alpha-1)}{\Gamma(2-\alpha)} 1_{\{z>0\}} z^{-1-\alpha}\,\mathrm{d}z \tag{1.4}$$

for the Gamma function Γ, then on an enlarged probability space, SDE (1.3) can be transformed into the following SDE:

$$\begin{aligned} X_t = x + \int_0^t \gamma_0(X_s)\,\mathrm{d}s + \int_0^t \sqrt{\gamma_1(X_{s-})}\,\mathrm{d}B_s \\ + \int_0^t \gamma_2(X_{s-})^{1/\alpha} \int_0^\infty u \tilde{M}(\mathrm{d}s, \mathrm{d}u), \end{aligned} \tag{1.5}$$

where $\{B_t : t \geq 0\}$ is a Brownian motion and $\{\tilde{M}(\mathrm{d}t, \mathrm{d}u) : t, u \geq 0\}$ is an independent compensated Poisson random measure with intensity measure $\mathrm{d}t\pi(\mathrm{d}u)$; see Theorem 9.32 in Li (2011) for a similar result. Equation (1.5) has a pathwise unique nonnegative strong solution if $\gamma_0(z) = a_1 z + a_2$, $\gamma_1(z) = z^{r_1}$ and $\gamma_1(z) = z^{r_2}$ for $a_1 \in \mathbb{R}$, $a_2 \geq 0$, $r_1 \in [1/2, 1]$ and $r_2 \in (\alpha - 1, \alpha]$; see Corollary 4.3 in Li and Mytnik (2011). By Theorem 4.1.2 in Li (2012), one can also convert (1.3) to another SDE:

$$X_t = x + \int_0^t \gamma_0(X_s)\,\mathrm{d}s + \int_0^t \sqrt{\gamma_1(X_{s-})}\,\mathrm{d}B_s + \int_0^t \int_0^\infty u\tilde{M}_{\gamma_2}(\mathrm{d}s, \mathrm{d}u),$$

where $\tilde{M}_{\gamma_2}(\mathrm{d}s, \mathrm{d}u)$ is an optional compensated Poisson measure with predictable compensator $\gamma_2(X_{s-})\,\mathrm{d}s\pi(\mathrm{d}u)$.

Using the Lamperti transform for positive self-similar Markov processes, Berestycki et al. (2015) found the extinction condition of solution to (1.5) for

$$\gamma_0(z) = \theta z^\eta f(z), \qquad \gamma_1(z) \equiv 0, \qquad \gamma_2(z) = z^{\alpha\beta} \quad \text{and}$$
$$\pi(\mathrm{d}z) = \frac{\alpha(\alpha-1)}{\Gamma(2-\alpha)} 1_{\{z>0\}} z^{-1-\alpha}\,\mathrm{d}z$$

with $\alpha \in (1, 2)$, $\theta \geq 0$, $\beta \in [1 - 1/\alpha, 1)$, $\eta = 1 - \alpha(1 - \beta) \in [0, 1)$ and for certain nonnegative Lipschitz continuous function f. In particular, for $f \equiv 1$ it is shown that the extinction occurs within finite time with probability one for $0 \leq \theta < \Gamma(\alpha)$ and with probability 0 for $\theta \geq \Gamma(\alpha)$; see Theorems 1.1 and 1.4 of Berestycki et al. (2015).

We refer to Lambert (2005), Berestycki et al. (2010), Bansaye et al. (2016) and Li (2018) for previous studies of coming down from infinity for a branching process with logistic growth, coalescents, birth and death processes and the polynomial branching process, respectively.

Other than the above mentioned results, we are not aware of any previous results on hitting probability and coming down from infinity for solutions to SDEs of type (1.3). There is some literature on nonexplosion of solutions to general SDEs with jumps; see Dong (2018) for a recent result. But we do not find any systematic discussions on the explosion/nonexplosion dichotomy and the coming down from infinity property of the solutions.

The main purpose of this paper is to investigate the extinction, explosion and coming down from infinity behaviors of the continuous-state nonlinear branching process as solution to (1.3) and specify the associated conditions on the functions γ_i, $i = 0, 1, 2$.

For lack of negative jumps, the extinction behaviors depend on the asymptotic behaviors of the function $\gamma_i(x)$ as $x \to 0+$. Intuitively, extinction can either be caused by a large enough negative drift due to γ_0 or large enough fluctuations due to γ_1 or γ_2. Even when the process has a (small) positive drift near 0, it might still die out because of relatively large fluctuations.

We are also interested in the relations between the asymptotics of the functions $\gamma_i(x)$, $i = 0, 1, 2$ as $x \to \infty$ and the explosion and the coming down from infinity behaviors of the nonlinear branching processes as solutions to SDE (1.3).

When γ_i, $i = 0, 1, 2$ are not power functions with the same power, the approach of Li (2018) fails to work. To overcome this difficulty, we adopt an alternative martingale approach that appeared earlier in Wang et al. (2017). Such an approach typically involves understanding how the process exits from consecutive intervals near 0 with the interval lengths decreasing geometrically, or consecutive intervals near ∞ with the interval lengths increasing geometrically. To this end, we construct the corresponding martingale in each situation. These martingales allow to obtain estimates on both the sequential exit probabilities and sequential exit times via optional stopping, where the lack of negative jumps for process X comes in handy. The desired results then follow from Borel–Cantelli-type arguments. Although we focus on SDEs of type (1.3), we expect that this approach could also adapted to study similar properties of solutions to other SDEs with more general jump mechanism, and it remains to be checked how sharp the desired results can be.

In addition, we show that the general nonlinear branching processes considered in this paper are closed under a Lamperti type transform, which allows us to discuss the finiteness of a weighted occupation time until extinction or explosion of the continuous-state nonlinear branching process via considering the extinction or explosion behaviors of the time changed process.

We also find Foster–Lyapunov-type criteria to show the irreducibility of the nonlinear continuous-state branching processes, which is of independent interest. We refer to Chen (2004) and Meyn and Tweedie (1993) for the Foster–Lyapunov-type criteria for explosion and stability of Markov chains.

This paper is structured as follows. After introductions in Sections 1.1 and 1.2 on the continuous-state branching processes, Section 2 summarizes the main results of this paper with an application and examples, where our results are compared with the known results. In Section 3, we show that SDE (1.3) has a unique strong solution up to the first time of reaching 0 or explosion given that the functions γ_i, $i = 0, 1, 2$ are locally Lipschitz on $(0, \infty)$. Section 4 contains Foster–Lyapunov criteria-type results that can be used to show the irreducibility of the solution as a Markov process. Proofs of the main results in Section 2 are included in Section 5.

2. Extinction, explosion and coming down from infinity.

With the convention $\inf \emptyset := \infty$, for $y > 0$ define

$$\tau_y^- \equiv \tau^-(y) := \inf\{t > 0 : X_t < y\}, \qquad \tau_y^+ \equiv \tau^+(y) := \inf\{t > 0 : X_t > y\}$$

and

$$\tau_0^- := \inf\{t > 0 : X_t = 0\}.$$

By a solution to SDE (1.3), we mean a càdlàg process $X = (X_t)_{t\geq 0}$ satisfying (1.3) up to time $\tau_n := \tau_{1/n}^- \wedge \tau_n^+$ for each $n \geq 1$ and $X_t = \limsup_{n\to\infty} X_{\tau_n -}$ for $t \geq \tau := \lim_{n\to\infty} \tau_n$. Then both of the boundary points 0 and ∞ are absorbing for X by definition.

Throughout this subsection, we assume that SDE (1.3) allows a unique weak solution denoted by $X := (X_t)_{t\geq 0}$, and consequently the process X has the strong Markov property. In Theorem 3.1 we are going to show that (1.3) allows a pathwise unique solution if the coefficient functions $\gamma_i, i = 0, 1, 2$ are all locally Lipschitz. We also assume that either $\gamma_1 \not\equiv 0$ or $\gamma_2 \not\equiv 0$ and that the functions γ_0, γ_1 and γ_2 are all locally bounded on $[0, \infty)$.

In the following, we present our main results on extinction, explosion and coming down from infinity properties of process X. Most of the proofs are deferred to Section 5.

For $a > 0$ and $u > 0$, let

$$\begin{aligned} H_a(u) &:= \int_0^\infty \big[(1 + zu^{-1})^{1-a} - 1 - (1 - a)zu^{-1}\big]\pi(\mathrm{d}z) \\ &= a(a - 1)u^{-2} \int_0^\infty z^2\pi(\mathrm{d}z) \int_0^1 (1 + zu^{-1}v)^{-1-a}(1 - v)\,\mathrm{d}v, \end{aligned} \tag{2.1}$$

where we use the following form of Taylor's formula that is often needed in the proofs of this paper; see, for example, Zorich ((2004), page 364) for its proof.

LEMMA 2.1. *If function g has a bounded continuous second derivative on* $[0, \infty)$, *then for any $y, z > 0$ we have*

$$g(y + z) - g(y) - zg'(y) = z^2 \int_0^1 g''(y + zv)(1 - v)\,\mathrm{d}v.$$

Note that for $\pi(\mathrm{d}z) = cz^{-1-\alpha}$ with $\alpha \in (1, 2)$ and $c > 0$,

$$H_a(u) = a(a - 1)u^{-\alpha} \int_0^\infty cy^{1-\alpha}\,\mathrm{d}y \int_0^1 (1 + yv)^{-1-a}(1 - v)\,\mathrm{d}v. \tag{2.2}$$

Put

$$G_a(u) := (a - 1)\gamma_0(u)u^{-1} - 2^{-1}a(a - 1)u^{-2}\gamma_1(u) - \gamma_2(u)H_a(u). \tag{2.3}$$

We choose the function G_a to be of the particular form in (2.3) so that, by Ito's formula, the process constructed in Lemma 5.1 can be shown to be a martingale, which is key for the main proofs in Section 5. The martingale allows to obtain estimates on exits times of the processes X via optional stopping. The conditions for extinction, explosion and coming down from infinity for the process X can be identified from the asymptotic behaviors of $G_a(u)$ for u near 0 or near ∞. An earlier version of G_a can be found in Wang et al. (2017) where it was also used to construct a continuous martingale to study the extinction behavior for the interacting super-Brownian motion.

REMARK 2.2. Suppose that $\pi \neq 0$ and $u \in (0, c)$ for some constant $c > 0$. One can see that:

- If there exists a constant $\alpha \in (1, 2)$ so that

$$\sup_{0<y<c} y^{\alpha-2} \int_0^y z^2 \pi(\mathrm{d}z) \leq b \quad \text{and} \quad \sup_{0<y<c} y^{\alpha-1} \int_y^\infty z\pi(\mathrm{d}z) \leq b,$$

then

$$H_a(u) \leq \frac{(a-1)(a+2)}{2} b u^{-\alpha} \qquad \text{for } a > 1.$$

- If there exists a constant $\alpha \in (1, 2)$ so that

$$\inf_{0<y<c} y^{\alpha-2} \int_0^y z^2 \pi(\mathrm{d}z) \geq b',$$

then

$$-H_a(u) \geq \frac{a(1-a)}{2} b' u^{-\alpha} \qquad \text{for } 0 < a < 1.$$

2.1. *Extinction behaviors*. We first present the two main results on the extinction behaviors for X. Here, we only consider the case that the initial value of X is small. In this way, we only have to impose conditions on function $G(u)$ for small positive values of u. These results, combined with Foster–Lyapunov criteria (Lemmas 4.1 and 4.2), can be used to discuss the extinction behaviors for X with arbitrary initial value.

THEOREM 2.3. (i) *If there exist constants $a > 1$ and $r < 1$ so that $G_a(u) \geq -(\ln u^{-1})^r$ for all small enough $u > 0$, then $\mathbb{P}_x\{\tau_0^- < \infty\} = 0$ for all $x > 0$.*

(ii) *If there exist constants $0 < a < 1$ and $r > 1$ so that $G_a(u) \geq (\ln u^{-1})^r$ for all small enough $u > 0$, then $\mathbb{P}_x\{\tau_0^- < \infty\} > 0$ for all small enough $x > 0$.*

The proof of Theorem 2.3 is deferred to Section 5.

The next results concern the first passage probabilities for which we need the following condition.

CONDITION 2.4. (i) For any x and a with $x > a > 0$,

$$\mathbb{P}_x\{\tau_a^- < \infty\} > 0. \tag{2.4}$$

(ii) For any x and a with $x > a > 0$,

$$\mathbb{P}_x\{\tau_a^- < \infty\} = 1. \tag{2.5}$$

The proof of the next corollary is deferred to Section 5.

COROLLARY 2.5. *Suppose that the assumption of Theorem* 2.3(ii) *holds. Then*:

- $\mathbb{P}_x\{\tau_0^- < \infty\} > 0$ *for all* $x > 0$ *if Condition* 2.4(i) *further holds*;
- $\mathbb{P}_x\{\tau_0^- < \infty\} = 1$ *for all* $x > 0$ *if Condition* 2.4(ii) *further holds*.

For $a \le b$, define

$$\Phi(a,b) := \inf_{y\in[a,b]} \gamma_1(y) + \inf_{y\in[a,b]} \gamma_2(y) 1_{\{\int_0^1 z\pi(\mathrm{d}z)=\infty\}}.$$

We can show that (i) or (ii) of Condition 2.4 hold under certain conditions on $\gamma_i,\ i = 0, 1, 2$.

PROPOSITION 2.6. (i) *Given* $x > a > 0$, *condition* (2.4) *holds if* $\Phi(a,b) > 0$ *and* $\sup_{a\le y\le b}\gamma_0(y) < \infty$ *for all* $b > a$.

(ii) *Given* $x > a > 0$, *suppose that* $\Phi(a,b) > 0$ *for all* $b > a$ *and that* $\gamma_0(y) \le 0$ *for all large enough* y. *Then condition* (2.5) *holds*.

(iii) *If* $\gamma_0(a) \le 0$ *and* $\Phi(a,b) > 0$ *for all* $b \ge a > 0$, *then for each* $x > 0$, $\mathbb{P}_x$*-a.s.* $X_t \to 0$ *as* $t \to \infty$. *Further, by the strong Markov property either*

$$\mathbb{P}_x\{X_t = 0 \textit{ for all } t \textit{ large enough}\} = 1$$

or

$$\mathbb{P}_x\{X_t \to 0,\ \textit{but } X_t > 0 \textit{ for all } t\} = 1,$$

and we say extinguishing occurs in the latter case.

The proof of Proposition 2.6 is deferred to Section 4 after Lemma 4.2.

REMARK 2.7. (i) Combining Proposition 2.6 and Theorem 2.3(ii) we find conditions for extinction with probability one and extinguishing with probability one, respectively.

(ii) If $\gamma_0 = \gamma_2 \equiv 0$, then the process X is the total mass of an interacting super-Brownian motion and Theorem 2.3 generalizes Theorems 3.4 and 3.5 of Wang et al. (2017).

2.2. *Explosion behaviors.* Let $\tau_\infty^+ := \lim_{n\to\infty} \tau_n^+$ be the explosion time. The solution X to SDE (1.3) explodes at a finite time if $\tau_\infty^+ < \infty$. We now present results on the explosion behaviors for X in the following, and again, we only consider the case of large initial values.

THEOREM 2.8. (i) *If there exist constants* $0 < a < 1$ *and* $r < 1$ *so that* $G_a(u) \ge -(\ln u)^r$ *for all* u *large enough, then* $\mathbb{P}_x\{\tau_\infty^+ < \infty\} = 0$ *for all* $x > 0$.

(ii) *If there exist* $a > 1$ *and* $r > 1$ *so that* $G_a(u) \ge (\ln u)^r$ *for all* u *large enough, then* $\mathbb{P}_x\{\tau_\infty^+ < \infty\} > 0$ *for all large* x.

The proof of Theorem 2.8 is deferred to Section 5.

CONDITION 2.9. For any x and b with $b > x > 0$,

$$\mathbb{P}_x\{\tau_b^+ < \infty\} > 0. \tag{2.6}$$

Putting Theorem 2.8(ii) and the above condition together we reach the following remark.

REMARK 2.10. If Condition 2.9 and the assumption in Theorem 2.8(ii) hold, then $\mathbb{P}_x\{\tau_\infty^+ < \infty\} > 0$ for all $x > 0$.

The proof for the next result is deferred to the end of Section 4.

PROPOSITION 2.11. *Given $b > x > 0$, if there exists $a \in (0, x)$ so that*

$$\inf_{y\in[a,b]} \gamma_1(y) + \inf_{y\in[a,b]} \gamma_2(y) > 0,$$

then (2.6) *holds.*

2.3. *Coming down from infinity.* We say that the process X comes down from infinity if

$$\lim_{b\to\infty} \lim_{x\to\infty} \mathbb{P}_x\{\tau_b^- < t\} = 1 \qquad \text{for all } t > 0, \tag{2.7}$$

and it stays infinite if

$$\lim_{x\to\infty} \mathbb{P}_x\{\tau_b^- < t\} = 0 \qquad \text{for all } b, t > 0.$$

We first present equivalent conditions for coming down from infinity. From the proof, one can see that they hold for any real-valued Markov processes with no downward jumps.

PROPOSITION 2.12. *The following statements are equivalent*:

(i) *Process X comes down from infinity.*
(ii) $\lim_{x\to\infty} \mathbb{E}_x[\tau_b^-] < \infty$ *for all large b.*
(iii)

$$\lim_{b\to\infty} \lim_{x\to\infty} \mathbb{E}_x[\tau_b^-] = 0. \tag{2.8}$$

PROOF. For the proof that (i) implies (ii), we refer to the proof of Theorem 1.11 of Li (2018).

Suppose that (ii) holds. Then for any $x' > b$, we have

$$\lim_{x\to\infty} \mathbb{E}_x[\tau_b^-] = \lim_{x\to\infty} (\mathbb{E}_x[\tau_{x'}^-] + \mathbb{E}_{x'}[\tau_b^-]). \tag{2.9}$$

First, letting $x' \to \infty$, and then letting $b \to \infty$ in (2.9), we obtain (2.8); (iii) thus holds.

(i) follows from (iii) by the Markov inequality. □

THEOREM 2.13. (i) *If there exist constants* $a > 1$ *and* $r < 1$ *such that*

$$G_a(u) \geq -(\ln u)^r \tag{2.10}$$

for all u *large, then process* X *stays infinite.*

(ii) *If there exist constants* $0 < a < 1$, $r > 1$ *such that*

$$G_a(u) \geq (\ln u)^r \tag{2.11}$$

for all u *large enough, then process* X *comes down from infinity.*

The proof of Theorem 2.13 is deferred to Section 5.

REMARK 2.14. More recently, for the process X with $\gamma_0 = \gamma_1 = \gamma_2$, the speeds of coming down from infinity are studied in details in Foucart et al. (2019) for the cases that either the function γ_i is regularly varying at infinity or $\gamma_i(x) = g(x)\mathrm{e}^{\theta x}$ for $\theta > 0$ and function g that is regularly varying at infinity.

2.4. *An application: Weighted total population.* Let γ be a strictly positive function defined on $[0, \infty)$ that is bounded on any bounded interval. In the following, we consider the weighted occupation time, or the weighted total population of X before explosion, defined as

$$S := \int_0^{\tau_0^- \wedge \tau_\infty^+} \gamma(X_s)\,\mathrm{d}s.$$

For $t \geq 0$, define

$$U_t := \int_0^{t \wedge \tau_0^- \wedge \tau_\infty^+} \gamma(X_s)\,\mathrm{d}s \quad \text{and} \quad V_t := \inf\{s > 0 : U_s > t\}.$$

Define the process $\bar{X} \equiv \{\bar{X}_t : t \geq 0\}$ by $\bar{X}_t := X_{V_t}$ for $V_t < \infty$ and $\bar{X}_t := X_\infty := \limsup_{t\to\infty} X_t$ for $V_t = \infty$. Define stopping times $\bar{\tau}_0^-$ and $\bar{\tau}_\infty^+$ similarly to τ_0^- and τ_∞^+, respectively, with X replaced by $\bar{X}$.

We first observe that with the above mentioned Lamperti-type transform, a time changed solution to the generalized Dawson–Li equation (1.3) remains a solution to another generalized Dawson–Li equation.

We leave the proof of the next result to the interested readers.

THEOREM 2.15. *For* $i = 0, 1, 2$ *and* $y > 0$ *define* $\bar{\gamma}_i(y) := \gamma_i(y)/\gamma(y)$. *Then there exist, on an extended probability space, a Gaussian white noise* $\{W_0(\mathrm{d}s, \mathrm{d}u) : s \geq 0, u > 0\}$ *with intensity* $\mathrm{d}s\,\mathrm{d}u$ *and an independent compensated Poisson random measure* $\{\tilde{N}_0(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u) : s \geq 0, z > 0, u > 0\}$ *with intensity* $\mathrm{d}s\pi(\mathrm{d}z)\,\mathrm{d}u$ *so* *that* $\{\bar{X}_t : t \geq 0\}$ *solves the following SDE*:

$$\bar{X}_t = x + \int_0^t \bar{\gamma}_0(\bar{X}_s)\,\mathrm{d}s + \int_0^t \int_0^{\bar{\gamma}_1(\bar{X}_s)} W_0(\mathrm{d}s, \mathrm{d}u)$$

$$+ \int_0^t \int_0^\infty \int_0^{\bar{\gamma}_2(\bar{X}_{s-})} z\tilde{N}_0(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u) \tag{2.12}$$

for $0 \leq t < \bar{\tau}_0^- \wedge \bar{\tau}_\infty^+$.

We leave the proof of the next key observation to interested readers.

PROPOSITION 2.16. *We have* $S = \bar{\tau}_0^- \wedge \bar{\tau}_\infty^+$.

REMARK 2.17. By Proposition 2.16 and Theorem 2.15, the finiteness for S is translated into extinction and explosion behaviors for the time changed process $\bar{X}$ for which we can apply Theorems 2.3 and 2.8. More details can be found later in Example 2.23 in Section 2.5. If $\gamma(x) = \gamma_1(x) = \gamma_2(x) \equiv x$ and γ_0 satisfies certain interaction condition, then the behaviors for S have been studied in Theorems 4.3.1 and 4.3.2 of Le (2014).

2.5. *Processes with power branching rate functions*. To obtain more explicit results, in this subsection we only consider processes with power function branching rates, that is,

$$\gamma_i(x) = b_i x^{r_i}, \qquad x > 0, i = 0, 1, 2,$$

for $r_0, r_1, r_2 \geq 0$, $b_0 \in \mathbb{R}$, $b_1, b_2 \geq 0$, $b_1 + b_2 > 0$. In addition, we assume that the measure π is defined in (1.4) with $1 < \alpha < 2$. Then for $a > 0$, by (2.2) and (2.3) we have

$$G_a(u) = (a-1)u^{-1}b_0u^{r_0} - 2^{-1}a(a-1)u^{-2}b_1u^{r_1}$$
$$- a(a-1)u^{-\alpha}b_2u^{r_2}c_{\alpha,a}.$$

The constant $c_{\alpha,a}$ can be computed explicitly:

$$c_{\alpha,a} = \frac{\Gamma(\alpha + a - 1)}{\Gamma(1 + a)}.$$

Then clearly $c_{\alpha,1} = \Gamma(\alpha)$.

EXAMPLE 2.18. In order to apply Theorems 2.3, 2.8 and 2.13, we only need to compare powers and coefficients of the three terms in the polynomial $G_a(u)$ for $0 < a < 1$ or $a > 1$, respectively. To handle the critical case of $r_1 = r_0 + 1$ or (and) $r_2 = r_0 + \alpha - 1$ where some terms have the same power, we further choose the value of a close enough to 1 to obtain the best possible results. For instance, if both $r_1 = r_0 + 1$ and $r_2 = r_0 + \alpha - 1$ hold, for $b_0 > b_1/2 + c_{\alpha,1}b_2$, we choose the constant a satisfying $1 < a < b_0/(b_1/2 + c_{\alpha,a}b_2)$, and for $b_0 < b_1/2 + c_{\alpha,1}b_2$, we choose the constant a satisfying $(b_0/(b_1/2 + c_{\alpha,a}b_2)) \vee 0 < a < 1$.

By Theorem 2.3 and Proposition 2.6, we can obtain explicit and very sharp conditions of extinction/nonextinction for the process X in Example 2.18.

For nonextinction, we have $\mathbb{P}_x\{\tau_0^- = \infty\} = 1$ for all $x > 0$ if one of the following two sets of conditions holds:

(i) $b_0 \leq 0$ and all of the following hold:

- (ia) if $b_0 < 0$, then $r_0 \geq 1$;
- (ib) if $b_1 > 0$, then $r_1 \geq 2$;
- (ic) if $b_2 > 0$, then $r_2 \geq \alpha$;

(ii) $b_0 > 0$ and all of the following hold:

- (iia) if $b_1 > 0$, then $r_1 \geq (r_0 + 1) \wedge 2$;
- (iib) if $b_2 > 0$, then $r_2 \geq (r_0 - 1 + \alpha) \wedge \alpha$;
- (iic)

$$b_0 > \frac{b_1}{2} 1_{\{r_1 = r_0 + 1 < 2\}} + \Gamma(\alpha) b_2 1_{\{r_2 = r_0 + \alpha - 1 < \alpha\}}.$$

In addition, under condition (i), for all $x > 0$

$$\mathbb{P}_x\{\tau_0^- = \infty \text{ and } X_t \to 0 \text{ as } t \to \infty\} = 1,$$

that is, extinguishing occurs.

For extinction with a positive probability, we have $\mathbb{P}_x\{\tau_0^- < \infty\} > 0$ for all $x > 0$ if one of the following two sets of conditions holds:

(i) $b_0 \leq 0$ and at least one of the following hold:

- (ia) $b_0 < 0$ and $r_0 < 1$;
- (ib) $b_1 > 0$ and $r_1 < 2$;
- (ic) $b_2 > 0$ and $r_2 < \alpha$.

(ii) $b_0 > 0$ and at least one of the following hold:

- (iia) $b_1 > 0$ and $r_1 < (r_0 + 1) \wedge 2$;
- (iib) $b_2 > 0$ and $r_2 < (r_0 + \alpha - 1) \wedge \alpha$;
- (iic)

$$b_0 < \frac{b_1}{2} 1_{\{r_1 = r_0 + 1 < 2\}} + \Gamma(\alpha) b_2 1_{\{r_2 = r_0 + \alpha - 1 < \alpha\}}.$$

In addition, $\mathbb{P}_x\{\tau_0^- < \infty\} = 1$ for all $x > 0$ under condition (i).

REMARK 2.19. Note that the above Condition (iic) for $\mathbb{P}_x\{\tau_0^- = \infty\} = 1$ and the above Condition (iic) for $\mathbb{P}_x\{\tau_0^- < \infty\} > 0$ agree with the corresponding results in Berestycki et al. (2015); see the corresponding comments in Section 1.2.

By Theorem 2.8 and Proposition 2.11, we obtain rather sharp conditions of explosion/nonexplosion for the process X in Example 2.18.

For nonexplosion we have $\mathbb{P}_x\{\tau_\infty^+ < \infty\} = 0$ for all $x > 0$ if either $b_0 \le 0$ or at least one of the following is true:

(i) $b_0 > 0$ and $r_0 \le 1$.

(ii) $b_0 > 0$, $r_0 > 1$ and at least one of the following hold:

  (iia) $b_1 > 0$ and $r_1 > r_0 + 1$;

  (iib) $b_2 > 0$ and $r_2 > r_0 + \alpha - 1$;

  (iic)

$$b_0 < \frac{b_1}{2} 1_{\{r_1 = r_0 + 1\}} + \Gamma(\alpha) b_2 1_{\{r_2 = r_0 + \alpha - 1\}}.$$

For explosion with a positive probability, we have $\mathbb{P}_x\{\tau_\infty^+ < \infty\} > 0$ for all $x > 0$ if $b_0 > 0$, $r_0 > 1$ and all of the following hold:

(i) if $b_1 > 0$, then $r_1 \le r_0 + 1$;

(ii) if $b_2 > 0$, then $r_2 \le r_0 + \alpha - 1$;

(iii)

$$b_0 > \frac{b_1}{2} 1_{\{r_1 = r_0 + 1\}} + \Gamma(\alpha) b_2 1_{\{r_2 = r_0 + \alpha - 1\}}.$$

Similarly, by Theorem 2.13 we obtain rather sharp conditions for coming down from infinity.

The process X in Example 2.18 comes down from infinity if one of the following holds:

(i) $b_0 \le 0$ and at least one of the following hold:

  (ia) $b_0 < 0$ and $r_0 > 1$;

  (ib) $b_1 > 0$ and $r_1 > 2$;

  (ic) $b_2 > 0$ and $r_2 > \alpha$.

(ii) $b_0 > 0$ and at least one of the following hold:

  (iia) $b_1 > 0$ and $r_1 > (r_0 + 1) \vee 2$;

  (iib) $b_2 > 0$ and $r_2 > (r_0 + \alpha - 1) \vee \alpha$;

  (iic)

$$b_0 < \frac{b_1}{2} 1_{\{r_1 = r_0 + 1 > 2\}} + \Gamma(\alpha) b_2 1_{\{r_2 = r_0 + \alpha - 1 > \alpha\}}.$$

The process X in Example 2.18 stays infinite if at least one of the following hold:

(i) $b_0 \le 0$ and all of the following hold:

  (ia) if $b_0 < 0$, then $r_0 \le 1$;

(ib) if $b_1 > 0$, then $r_1 \le 2$;
(ic) if $b_2 > 0$, then $r_2 \le \alpha$.

(ii) $b_0 > 0$ and all of the following hold:

(iia) if $b_1 > 0$, then $r_1 \le (r_0 + 1) \vee 2$;
(iib) if $b_2 > 0$, then $r_2 \le (r_0 + \alpha - 1) \vee \alpha$;
(iic)

$$b_0 > \frac{b_1}{2} 1_{\{r_1 = r_0 + 1 > 2\}} + \Gamma(\alpha) b_2 1_{\{r_2 = r_0 + \alpha - 1 > \alpha\}}.$$

From the above example, we make the following observations.

REMARK 2.20. (i) There is no extinction if the process X has a small enough negative drift together with small enough fluctuations near 0. If X has a positive drift, then the requirements on the fluctuations are weaker. Extinction happens with a positive probability if X has either a large enough negative drift or large enough fluctuations near 0. Even if X has a small positive drift near 0, extinction can still happen with a positive probability if the fluctuations are large enough.

(ii) The explosion is caused by a large enough drift associated with the function γ_0. The fluctuations of the process X associated with the functions γ_1 and γ_2 cannot cause explosion. But large enough fluctuations can prevent the explosion from happening.

(iii) A large enough negative drift or large enough fluctuations near infinity can cause coming down from infinity. Even if the process X has a positive drift, large enough fluctuations can still cause coming down from infinity. On the other hand, the process X with a moderate negative drift and moderate fluctuations near infinity stays infinite, and if it allows large fluctuations, with a large enough positive drift it can still stay infinite.

REMARK 2.21. If $b_2 = 0$, then X is a diffusion whose explosion behavior is characterized by Feller's criterion; see, for example, Corollary 4.4 of Cherny and Engelbert (2005). One can check that the explosion/nonexplosion conditions in Example 2.18 are consistent with it.

REMARK 2.22. Example 2.18 recovers, for the case with spectrally positive stable Lévy measure specified in (1.4), the integral tests for extinction, explosion and coming down from infinity in Theorems 1.7, 1.9 and 1.11 of Li (2018), which were proved using a very different approach. Recall that the continuous-state polynomial branching process in Li (2018) is the process X with power branching rate functions satisfying $r_i = r$, $i = 0, 1, 2$. By Example 2.18, we have for the continuous-state polynomial branching process:

- $\mathbb{P}_x\{\tau_0^- < \infty\} > 0$ for all $x > 0$, that is, extinction occurs if and only if

$$2 \cdot 1_{\{b_1 \neq 0\}} + \alpha \cdot 1_{\{b_1 = 0, b_2 \neq 0\}} > r;$$

- $\mathbb{P}_x\{\tau_\infty^+ < \infty\} > 0$ for all $x > 0$, that is, explosion occurs if and only if $b_0 > 0$ and $r > 1$;
- The process X comes down from infinity if and only if $b_0 \leq 0$ and

$$1_{\{b_0 \neq 0\}} + \alpha \cdot 1_{\{b_0=0, b_2 \neq 0\}} + 2 \cdot 1_{\{b_0=0, b_1 \neq 0, b_2=0\}} < r;$$

which agree the integral tests in Li (2018).

The next example is on the finiteness of the weighted total population S of X introduced in Section 2.4. The next results follow from Remark 2.17, Theorem 2.15 and Example 2.18.

EXAMPLE 2.23. Let $\gamma(x) = x^r$ for $0 < r < \min\{r_0, r_1, r_2\}$ in Theorem 2.15. Observe that $\mathbb{P}_x\{S = \bar{\tau}_0^- \wedge \bar{\tau}_\infty^+ = \infty\} = 1$ if and only if $\mathbb{P}_x\{\bar{\tau}_0^- = \infty\} = 1$ and $\mathbb{P}_x\{\bar{\tau}_\infty^+ = \infty\} = 1$. The conditions for $\mathbb{P}_x\{S < \infty\} = \mathbb{P}_x\{\bar{\tau}_0^- \wedge \bar{\tau}_\infty^+ < \infty\} = 0$ for $x > 0$ can be found in Example 2.18.

Similarly, observe that $\mathbb{P}_x\{\bar{\tau}_0^- \wedge \bar{\tau}_\infty^+ < \infty\} > 0$ if and only if $\mathbb{P}_x\{\bar{\tau}_0^- < \infty\} > 0$ or $\mathbb{P}_x\{\bar{\tau}_\infty^+ < \infty\} > 0$. Then the conditions for $\mathbb{P}_x\{S < \infty\} > 0$ can also be found in Example 2.18.

3. Existence and uniqueness of solutions. In this section, we find conditions on the functions γ_i, $i = 0, 1, 2$ under which SDE (1.3) has a pathwise unique solution X, and consequently X is a Markov process. For this purpose, we only need the functions γ_i, $i = 0, 1, 2$ to be locally Lipschitz because we only consider the solutions up to the first time of hitting 0 or explosion.

THEOREM 3.1. *Suppose that the functions γ_i, $i = 0, 1, 2$ are locally Lipschitz; that is, for each closed interval $A \subset (0, \infty)$, there is a constant $c(A) > 0$ so that for any $x, y \in A$,*

$$|\gamma_0(x) - \gamma_0(y)| + |\gamma_1(x) - \gamma_1(y)| + |\gamma_2(x) - \gamma_2(y)| \leq c(A)|x - y|.$$

Then:

(i) *For any initial value $X_0 = x \geq 0$, there exists a pathwise unique solution* (*defined at the beginning of Section* 2) *to SDE* (1.3).

(ii) *If in addition, γ_2 is an increasing function, then for any $y \geq x \in [0, \infty)$ and solutions $X^x := (X_t^x)_{t \geq 0}$ and $X^y := (X_t^y)_{t \geq 0}$ to SDE* (1.3) *with $X_0^x = x$ and $X_0^y = y$, we have*

$$\mathbb{P}\{X_t^y \geq X_t^x \textit{ for all } t \geq 0\} = 1.$$

PROOF. (i) We prove the result by an approximation argument. For each $n \geq 1$ and $i = 0, 1, 2$, define

$$\gamma_i^n(x) := \begin{cases} \gamma_i(n), & n < x < \infty, \\ \gamma_i(x), & 1/n \leq x \leq n, \\ \gamma_i(1/n), & 0 \leq x < 1/n. \end{cases}$$

By pages 245–246 in Ikeda and Watanabe (1989), for each $n \geq 1$, there is a unique strong solution $(\xi_t^n)_{t\geq 0}$ to

$$\begin{aligned}\xi_t^n &= x + \int_0^t \gamma_0^n(\xi_s^n)\,\mathrm{d}s + \int_0^t \int_0^{\gamma_1^n(\xi_s^n)} W(\mathrm{d}s, \mathrm{d}u) \\ &\quad + \int_0^t \int_0^\infty \int_0^{\gamma_2^n(\xi_{s-}^n)} z\tilde{N}(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u).\end{aligned} \tag{3.1}$$

For $m, n \geq 1$, define stopping time

$$\tau_m^n := \inf\{t \geq 0 : \xi_t^n \geq m \quad \text{or} \quad \xi_t^n \leq 1/m\}.$$

Then we have $\xi_t^n = \xi_t^m$ for $t \in [0, \tau_{m\wedge n}^{m\wedge n})$ and $\tau_n^{n+i} = \tau_n^n$, $i = 1, 2, \ldots$. Clearly, the sequence of stopping times $\{\tau_n^n\}$ is increasing in n. Let $\tau := \lim_{n\to\infty} \tau_n^n$. We define the process $X := (X_t)_{t\geq 0}$ by $X_t = \xi_t^n$ for $t \in [0, \tau_n^n)$ and $X_t = \limsup_{n\to\infty} \xi_{\tau_n^n}^n$ for $t \in [\tau, \infty)$. Then

$$\tau_n^n := \inf\{t \geq 0 : X_t \geq n \text{ or } X_t \leq 1/n\}$$

and X is a solution of (1.3). Since the pathwise uniqueness of the solution holds for (3.1) in the time interval $[0, \tau_n^n)$ for each $n \geq 1$, there exists a pathwise unique solution to (1.3).

(ii) Let $(\xi_n^x(t))_{t\geq 0}$ denote the solution of (3.1) to indicate its dependence on the initial state. To apply Theorem 2.2 in Dawson and Li (2012), we identify the notation in Dawson and Li (2012) with that in this paper in the following equations, where the notation on the left-hand sides comes from Dawson and Li (2012) and that on the right-and sides is from the present paper,

$$E = (0, \infty), \qquad U_0 = (0, \infty)^2, \qquad \pi(\mathrm{d}u) = \mathrm{d}u, \qquad g_1(x, z, u) \equiv 0,$$

$$\mu_0(\mathrm{d}z, \mathrm{d}u) = \pi(\mathrm{d}z)\mathrm{d}u, \qquad \tilde{N}_0(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u) = \tilde{N}(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u)$$

and

$$b(x) = b_1(x) = \gamma_0^n(x), \qquad \sigma(x, u) = 1_{\{u\leq\gamma_1^n(x)\}}, \qquad g_0(x, z, u) = z1_{\{u\leq\gamma_2^n(x)\}}.$$

Then conditions (2.a, b, c) in Dawson and Li (2012) are satisfied due to the Lipschitz properties of γ_i^n for $i = 0, 1, 2$. Let

$$l_0(x, y, u) := 1_{\{u\leq\gamma_2(x)\}} - 1_{\{u\leq\gamma_2(y)\}}.$$

Since the function $\gamma_2(x)$ is nondecreasing in x, then for $x < y$ we have

$$\begin{aligned}I(x, y) &:= \int_0^\infty \mathrm{d}u \int_0^1 \frac{l_0(x, y, u)^2(1-t)1_{\{|l_0(x,y,u)|\leq n\}}}{|(x-y) + tl_0(x, y, u)|}\,\mathrm{d}t \\ &= \int_{\gamma_2(x)}^{\gamma_2(y)} \mathrm{d}u \int_0^1 \frac{1-t}{|(x-y)-t|}\,\mathrm{d}t \\ &\leq \big(\gamma_2(y) - \gamma_2(x)\big)\big(\ln(y-x+1) - \ln(y-x)\big) < \infty.\end{aligned}$$

Similarly, $I(x, y) < \infty$ for all $x \geq y$. Then condition (2.d) of Theorem 2.2 in Dawson and Li (2012) holds. Now for any $y \geq x \geq 0$, by Theorem 2.2 in Dawson and Li (2012) we can show that $\xi_n^y(t) \geq \xi_n^x(t)$ a.s. for all n and $t \geq 0$. Consequently, $X_t^y \geq X_t^x$ a.s. for all $t \geq 0$. □

Throughout the rest of this paper, we always assume that SDE (1.3) has a unique weak solution which is a Markov process.

REMARK 3.2. The solution to SDE (1.3) also arises as the weak limit in the Skorokhod space $D([0, \infty), \mathbb{R}_+)$ for a sequence of discrete-state and continuous-time Markov chains that can be interpreted as discrete-state branching processes with population dependent branching rates; see Li et al. (2018) for more details.

4. Foster–Lyapunov criteria for extinction and explosion. In this section, we first present Foster–Lyapunov criteria-type results for the process X which generalize a similar result for Markov chains; see Chen (2004), page 84.

Let $C^2[0, \infty)$ be the space of twice continuously differentiable functions on $[0, \infty)$. Define the operator L on $C^2[0, \infty)$ by

$$Lg(y) := \gamma_0(y)g'(y) + \frac{1}{2}\gamma_1(y)g''(y) + \gamma_2(y)\int_0^\infty \big(g(y+z) - g(y) - zg'(y)\big)\pi(\mathrm{d}z).$$

LEMMA 4.1. *Given $a \geq 0$, let $g \in C^2[0, \infty)$ be a nonnegative function satisfying the following conditions*:

(i) $\sup_{y\in[a,b)} |Lg(y)| < \infty$ *for all $b > a$, that is, Lg is locally bounded on $[a, \infty)$*;

(ii) $\sup_{y\in[a,\infty)} g(y) < \infty$;

(iii) $g(a) > 0$ *and* $\lim_{y\to\infty} g(y) = 0$;

(iv) *For all $b > a$, there is a constant $d_b > 0$ so that $Lg(y) \geq d_b g(y)$ for all $y \in (a, b)$.*

Then for any $x > a$, we have

$$\mathbb{P}_x\{\tau_a^- < \infty\} \geq g(x)/g(a). \tag{4.1}$$

PROOF. For any $b > x > a$, by Itô's formula and conditions (i) and (ii), we have

$$g(X_{t\wedge\tau_a^-\wedge\tau_b^+}) = g(x) + \int_0^{t\wedge\tau_a^-\wedge\tau_b^+} Lg(X_s)\,\mathrm{d}s + \text{mart.}$$

Taking expectations on both sides, we have

$$\mathbb{E}_x\big[g(X_{t\wedge\tau_a^-\wedge\tau_b^+})\big] = g(x) + \int_0^t \mathbb{E}_x\big[Lg(X_s)1_{\{s<\tau_a^-\wedge\tau_b^+\}}\big]\,\mathrm{d}s.$$

By integration by parts,

$$\begin{aligned}\int_0^\infty & \mathrm{e}^{-d_b t}\mathbb{E}_x\big[Lg(X_t)1_{\{t<\tau_a^-\wedge\tau_b^+\}}\big]\,\mathrm{d}t \\ &= \int_0^\infty \mathrm{e}^{-d_b t}\,\mathrm{d}\mathbb{E}_x\big[g(X_{t\wedge\tau_a^-\wedge\tau_b^+})\big] \\ &= d_b\int_0^\infty \mathrm{e}^{-d_b t}\mathbb{E}_x\big[g(X_{t\wedge\tau_a^-\wedge\tau_b^+})\big]\,\mathrm{d}t - g(x).\end{aligned}$$

Then by (iv),

$$\begin{aligned} d_b\int_0^\infty & \mathrm{e}^{-d_b t}\mathbb{E}_x\big[g(X_{t\wedge\tau_a^-\wedge\tau_b^+})\big]\,\mathrm{d}t - g(x) \\ &\geq d_b\int_0^\infty \mathrm{e}^{-d_b t}\mathbb{E}_x\big[g(X_t)1_{\{t<\tau_a^-\wedge\tau_b^+\}}\big]\,\mathrm{d}t.\end{aligned}$$

It follows that

$$\begin{aligned} g(x) &\leq d_b\int_0^\infty \mathrm{e}^{-d_b t}\mathbb{E}_x\big[g(X_{\tau_a^-\wedge\tau_b^+})1_{\{t\geq\tau_a^-\wedge\tau_b^+\}}\big]\,\mathrm{d}t \\ &\leq g(a)\mathbb{P}_x\big\{\tau_a^- < \infty\big\} + \sup_{y\geq b} g(y).\end{aligned}$$

Inequality (4.1) thus follows by letting $b\to\infty$ and (iii). □

The proof for the next lemma is similar to that of Lemma 4.1 and we omit it.

LEMMA 4.2. *Given $0<x<b$, suppose there exist constants $a\in[0,x)$, $d>0$ and a function $g\in C^2[0,\infty)$ satisfying the following conditions*:

(i) $\sup_{y\in[a,b]}|Lg(y)|<\infty$;
(ii) $\sup_{y\in[a,b)}|g(y)|<\infty$;
(iii) $g(a)=0$ *and* $g(x)>0$;
(iv) $Lg(y)\geq dg(y)$ *for all* $y\in[a,b]$.

Then we have $\mathbb{P}_x\{\tau_b^+<\infty\}>0$.

As applications of Lemmas 4.1 and 4.2, we prove Propositions 2.6 and 2.11 in this section.

PROOF OF PROPOSITION 2.6.. (i) Let $g(y)=\mathrm{e}^{-\lambda y}$ with $\lambda>0$ large enough. Note that g satisfies the conditions of Lemma 4.1. Then by Lemma 2.1, we have uniformly for all $a<y<b$,

$$\begin{aligned} Lg(y) \geq \lambda\mathrm{e}^{-\lambda y}\bigg\{&-\sup_{a\leq z\leq b}\big(\gamma_0(z)\vee 0\big)+\frac{\lambda}{2}\inf_{a\leq z\leq b}\gamma_1(z) \\ &+\lambda\inf_{a\leq z\leq b}\gamma_2(z)\int_0^\infty z^2\pi(\mathrm{d}z)\int_0^1 \mathrm{e}^{-\lambda zu}(1-u)\,\mathrm{d}u\bigg\}. \qquad (4.2)\end{aligned}$$

Observe that

$$\begin{aligned}\lambda\int_0^\infty z^2\pi(\mathrm{d}z)\int_0^1 \mathrm{e}^{-\lambda zu}(1-u)\,\mathrm{d}u \\ &\ge 2^{-1}\lambda\int_0^\infty z^2\pi(\mathrm{d}z)\int_0^{1/2}\mathrm{e}^{-\lambda zu}\,\mathrm{d}u \\ &\ge 2^{-1}\int_0^\infty z(1-\mathrm{e}^{-\lambda z/2})\pi(\mathrm{d}z) \\ &\ge 2^{-1}(1-\mathrm{e}^{-1/2})\int_{1/\lambda}^\infty z\pi(\mathrm{d}z)\end{aligned}$$

converges to $\int_0^\infty z\pi(\mathrm{d}z)=\infty$ as $\lambda\to\infty$. It then follows that for each $b>a$ there is a constant $d_b(\lambda)>0$ so that

$$Lg(y)\ge d_b(\lambda)\mathrm{e}^{-\lambda y},\qquad a<y<b \tag{4.3}$$

as λ large enough. Thus by Lemma 4.1, for $x>a$ and large enough λ,

$$\mathbb{P}_x\{\tau_a^-<\infty\}\ge \mathrm{e}^{-\lambda(x-a)}>0, \tag{4.4}$$

which gives (2.4).

(ii) Suppose that there is a constant $c>0$ so that $\gamma_0(y)\le 0$ for all $y\ge c$. Similar to the argument in (4.2) and (4.3), given any $\lambda>0$, uniformly for $c\vee a<y<b$, we have

$$\begin{aligned}Lg(y) &\ge \frac{\lambda^2}{2}\gamma_1(y)\mathrm{e}^{-\lambda y}+\lambda^2\gamma_2(y)\mathrm{e}^{-\lambda y}\int_0^\infty z^2\pi(\mathrm{d}z)\int_0^1\mathrm{e}^{-\lambda zu}(1-u)\,\mathrm{d}u \\ &\ge d_b'(\lambda)\mathrm{e}^{-\lambda y}\end{aligned}$$

for some constant $d_b'(\lambda)>0$. It follows again from Lemma 4.1 that for all $\lambda>0$,

$$\mathbb{P}_x\{\tau_l^-<\infty\}\ge \mathrm{e}^{-\lambda(x-l)}>0,\qquad x>l\ge c\vee a.$$

Letting $\lambda\to 0$, we have

$$\mathbb{P}_x\{\tau_l^-<\infty\}=1,\qquad x>l\ge c\vee a. \tag{4.5}$$

It follows from (4.4) that for large enough λ,

$$\mathbb{P}_x\{\tau_a^-<\infty\}\ge \mathrm{e}^{-\lambda(x-a)},\qquad x>a. \tag{4.6}$$

For any $x>a>0$ and $t>0$, combining (4.5) and (4.6), by the strong Markov property, we have

$$\begin{aligned}&\mathbb{P}_x\{\tau_a^-<\infty\} \\ &\quad=\mathbb{P}_x\{\tau_a^-<t\}+\int_a^{c\vee a}\mathbb{P}_x\{t\le\tau_a^-<\infty,\ X_t\in\mathrm{d}z\}\mathbb{P}_z\{\tau_a^-<\infty\}\end{aligned}$$

$$
\begin{aligned}
&+\int_{c\vee a}^{\infty}\mathbb{P}_x\{t\le\tau_a^-<\infty,\ X_t\in \mathrm{d}z\}\mathbb{P}_z\{\tau_a^-<\infty\}\\
&\ge \mathbb{P}_x\{\tau_a^-<t\}+\int_a^{c\vee a}\mathbb{P}_x\{t\le\tau_a^-<\infty,\ X_t\in \mathrm{d}z\}\mathbb{P}_{c\vee a}\{\tau_a^-<\infty\}\\
&+\int_{c\vee a}^{\infty}\mathbb{P}_x\{t\le\tau_a^-<\infty,\ X_t\in \mathrm{d}z\}\mathbb{P}_{c\vee a}\{\tau_a^-<\infty\}\\
&\ge \mathbb{P}_x\{\tau_a^-<t\}+\mathrm{e}^{-\lambda(c\vee a-a)}\big(1-\mathbb{P}_x\{\tau_a^-<t\}\big). \qquad (4.7)
\end{aligned}
$$

Letting $t\to\infty$ in (4.7), we have $\mathbb{P}_x\{\tau_a^-<\infty\}=1$. The desired result then follows.

(iii) For any small enough $\varepsilon>0$, let

$$A_n:=\{\tau^-(\varepsilon^{n+1})<\infty,\tau_\varepsilon^+\circ\theta_{\tau^-(\varepsilon^{n+1})}<\tau^-(\varepsilon^{n+2})\circ\theta_{\tau^-(\varepsilon^{n+1})}\},\qquad n\ge 1.$$

Since $\gamma_0(y)\le 0$ for all $y\in\mathbb{R}$, then $(X_t)_{t\ge 0}$ is a supermartingale, which implies

$$\varepsilon^{n+1}=X_{\tau^-(\varepsilon^{n+1})}\ge \mathbb{E}_{\varepsilon^{n+1}}[X_{\tau_\varepsilon^+\wedge\tau^-(\varepsilon^{n+2})}]\ge \varepsilon\mathbb{P}_{\varepsilon^{n+1}}\{\tau_\varepsilon^+<\tau_{\varepsilon^{n+2}}^-\}$$

by optional stopping. Thus,

$$\mathbb{P}_x\{A_n\}\le \mathbb{E}_x\big[\mathbb{P}_{\tau^-(\varepsilon^{n+1})}\{\tau_\varepsilon^+<\tau_{\varepsilon^{n+2}}^-\}\big]\le\varepsilon^n.$$

It follows from the Borel–Cantelli lemma that $\mathbb{P}_x\{A_n \text{ i.o.}\}=0$. Therefore, by Proposition 2.6(ii), we have $\mathbb{P}_x$-a.s. $X_t<\varepsilon$ for all t large enough and the desired result follows. □

PROOF OF PROPOSITION 2.11. Observe that there is a constant $b'>0$ so that $\int_0^{b'}z^2\pi(\mathrm{d}z)>0$. Let

$$m_0:=\sup_{y\in[a,b]}|\gamma_0(y)|<\infty,\qquad m_1=\inf_{y\in[a,b]}\gamma_1(y)\quad\text{and}\quad m_2=\inf_{y\in[a,b]}\gamma_2(y).$$

Since $m_1\vee m_2>0$, there exists a large enough constant $c>0$ so that

$$-cm_0+\frac{1}{2}c^2m_1+\frac{1}{2}c^2m_2\int_0^{b'}z^2\pi(\mathrm{d}z)\ge 1.$$

Let g be a convex function, that is, $g''(y)\ge 0$, satisfying $g(y)=\mathrm{e}^{cy}-\mathrm{e}^{ca}$ for $y\in[a,b+b']$ and $g''(y)=0$ for $y>b+b'+1$. Then by Lemma 2.1, it is easy to see that

$$
\begin{aligned}
&\int_0^\infty\big(g(y+z)-g(y)-zg'(y)\big)\pi(\mathrm{d}z)\\
&\quad=\int_0^{b+b'+1}\big(g(y+z)-g(y)-zg'(y)\big)\pi(\mathrm{d}z)\\
&\qquad+\int_{b+b'+1}^\infty\big(g(y+z)-g(y)-zg'(y)\big)\pi(\mathrm{d}z)\\
&\quad\le\frac{1}{2}\sup_{y\in[a,b+b'+1]}g''(y)\int_0^{b+b'+1}z^2\pi(\mathrm{d}z),
\end{aligned}
$$

which implies that condition (i) in Lemma 4.2 is satisfied. Observe that for any $y \in [a, b]$, we have

$$\begin{aligned}\int_0^\infty &\big(g(y+z) - g(y) - zg'(y)\big)\pi(\mathrm{d}z) \\ &\geq \int_0^{b'} \big(g(y+z) - g(y) - zg'(y)\big)\pi(\mathrm{d}z) \\ &= \mathrm{e}^{cy} \int_0^{b'} (\mathrm{e}^{cz} - 1 - cz)\pi(\mathrm{d}z) \geq \frac{1}{2}c^2 \mathrm{e}^{cy} \int_0^{b'} z^2 \pi(\mathrm{d}z).\end{aligned}$$

Therefore, for any $y \in [a, b]$, we have

$$\begin{aligned}Lg(y) &= \gamma_0(y)g'(y) + \frac{1}{2}\gamma_1(y)g''(y) \\ &\quad + \gamma_2(y)\int_0^\infty \big(g(y+z) - g(y) - zg'(y)\big)\pi(\mathrm{d}z) \\ &\geq \gamma_0(y)c\mathrm{e}^{cy} + \frac{c^2}{2}\gamma_1(y)\mathrm{e}^{cy} + \frac{c^2}{2}\gamma_2(y)\mathrm{e}^{cy}\int_0^{b'} z^2\pi(\mathrm{d}z) \\ &\geq \mathrm{e}^{cy}\bigg[-cm_0 + \frac{c^2}{2}m_1 + \frac{c^2}{2}m_2\int_0^{b'} z^2\pi(\mathrm{d}z)\bigg] \geq \mathrm{e}^{Cy} \geq g(y).\end{aligned}$$

Applying Lemma 4.2 yields $\mathbb{P}_x\{\tau_b^+ < \infty\} > 0$. □

5. Proofs of the main results in Section 2. Recall the definitions of H_a and G_a in (2.1) and (2.3), respectively. We now present the martingales we use to show the main results on extinction, explosion and coming down from infinity. It is remarkable that such a martingale is enough to show all the main results in this paper. Some other forms of martingales can only be used to prove partial results.

LEMMA 5.1. *For $b > \varepsilon > c > 0$ let $T := \tau_c^- \wedge \tau_b^+$. Then the process $X_{t\wedge T}^{1-a} \exp\{\int_0^{t\wedge T} G_a(X_s)\,\mathrm{d}s\}$ is an $(\mathscr{F}_t)$-martingale and*

$$\mathbb{E}_\varepsilon\bigg[X_T^{1-a}\exp\bigg\{\int_0^T G_a(X_s)\,\mathrm{d}s\bigg\}\bigg] \leq \varepsilon^{1-a}$$

for $a > 0, a \neq 1$.

PROOF. By Itô's formula, we can see that

$$\begin{aligned}X_t^{1-a} = X_0^{1-a} &- \int_0^t G_a(X_s)X_s^{1-a}\,\mathrm{d}s + (1-a)\int_0^t\int_0^{\gamma_1(X_s)} X_s^{-a}W(\mathrm{d}s, \mathrm{d}u) \\ &+ \int_0^t\int_0^\infty\int_0^{\gamma_2(X_{s-})}\big[(X_{s-}+z)^{1-a} - X_{s-}^{1-a}\big]\tilde{N}(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u),\end{aligned}$$

and it then follows from the integration by parts formula (see, e.g., Protter (2005), page 68) that

$$\begin{aligned}
X_t^{1-a} &\exp\left\{\int_0^t G_a(X_s)\,\mathrm{d}s\right\} \\
&= X_0^{1-a} + \int_0^t X_s^{1-a} \exp\left\{\int_0^s G_a(X_u)\,\mathrm{d}u\right\} G_a(X_s)\,\mathrm{d}s \\
&\quad + \int_0^t \exp\left\{\int_0^s G_a(X_u)\mathrm{d}u\right\} \mathrm{d}\big(X_s^{1-a}\big) \\
&= X_0^{1-a} + \text{local mart.}
\end{aligned}$$

Therefore,

$$t \mapsto X_{t\wedge T}^{1-a} \exp\left\{\int_0^{t\wedge T} G_a(X_s)\,\mathrm{d}s\right\} \tag{5.1}$$

is a local martingale. By Protter ((2005), page 38), (5.1) is a martingale if

$$\mathbb{E}_\varepsilon\left[\sup_{t\in[0,\delta]} X_{t\wedge T}^{1-a} \exp\left\{\int_0^{t\wedge T} G_a(X_s)\,\mathrm{d}s\right\}\right] < \infty \tag{5.2}$$

for each $\delta > 0$. Observe that for $0 \le t \le \delta$

$$\exp\left\{\int_0^{t\wedge T} G_a(X_s)\,\mathrm{d}s\right\}$$

is uniformly bounded from above by a positive constant. Then (5.2) is obvious for $a > 1$. In the following, we consider the case $a < 1$. By the Burkholder–Davis–Gundy inequality, we have

$$\begin{aligned}
&\mathbb{E}_\varepsilon\left[\sup_{t\in[0,\delta]}\left|\int_0^{t\wedge T}\int_0^{\gamma_1(X_{s-})} W(\mathrm{d}s,\mathrm{d}u)\right|^2\right] \\
&\quad \le C\mathbb{E}_\varepsilon\left[\int_0^{\delta\wedge T}\mathrm{d}s\int_0^{\gamma_1(X_{s-})}\mathrm{d}u\right] \le C\delta \sup_{x\in[0,b]}\gamma_1(x)
\end{aligned} \tag{5.3}$$

and

$$\begin{aligned}
&\mathbb{E}_\varepsilon\left[\sup_{t\in[0,\delta]}\left|\int_0^{t\wedge T}\int_0^1\int_0^{\gamma_2(X_{s-})} z\tilde{N}(\mathrm{d}s,\mathrm{d}z,\mathrm{d}u)\right|^2\right] \\
&\quad \le C\mathbb{E}_\varepsilon\left[\int_0^{\delta\wedge T}\mathrm{d}s\int_0^1 z^2\pi(\mathrm{d}z)\int_0^{\gamma_2(X_{s-})}\mathrm{d}u\right] \\
&\quad \le C\delta\int_0^1 z^2\pi(\mathrm{d}z)\sup_{x\in[0,b]}\gamma_2(x).
\end{aligned} \tag{5.4}$$

Observe that

$$\begin{aligned}\mathbb{E}_\varepsilon\Big[\sup_{t\in[0,\delta]}\Big|\int_0^{t\wedge T}\int_1^\infty\int_0^{\gamma_2(X_{s-})} z\tilde{N}(\mathrm{d}s,\mathrm{d}z,\mathrm{d}u)\Big|\Big] &\le \mathbb{E}_\varepsilon\Big[\sup_{t\in[0,\delta]}\Big|\int_0^{t\wedge T}\int_1^\infty\int_0^{\gamma_2(X_{s-})} zN(\mathrm{d}s,\mathrm{d}z,\mathrm{d}u)\Big|\Big]\\ &\quad + \mathbb{E}_\varepsilon\Big[\sup_{t\in[0,\delta]}\Big|\int_0^{t\wedge T}\mathrm{d}s\int_1^\infty z\pi(\mathrm{d}z)\int_0^{\gamma_2(X_{s-})}\mathrm{d}u\Big|\Big]\\ &\le 2\delta\int_1^\infty z\pi(\mathrm{d}z)\sup_{x\in[0,b]}\gamma_2(x). \qquad (5.5)\end{aligned}$$

It then follows from (1.3) and (5.3)–(5.5) that $\mathbb{E}_\varepsilon[\sup_{t\in[0,\delta]} X_{t\wedge T}] < \infty$, which implies (5.2). Now by Fatou's lemma, we get

$$\begin{aligned}\mathbb{E}_\varepsilon\Big[X_T^{1-a}\exp\Big\{\int_0^T G_a(X_s)\,\mathrm{d}s\Big\}\Big] &= \mathbb{E}_\varepsilon\Big[\lim_{t\to\infty} X_{t\wedge T}^{1-a}\exp\Big\{\int_0^{t\wedge T} G_a(X_s)\,\mathrm{d}s\Big\}\Big]\\ &\le \lim_{t\to\infty}\mathbb{E}_\varepsilon\Big[X_{t\wedge T}^{1-a}\exp\Big\{\int_0^{t\wedge T} G_a(X_s)\,\mathrm{d}s\Big\}\Big]\\ &= \varepsilon^{1-a},\end{aligned}$$

which completes the proof. □

PROOF OF THEOREM 2.3. (i) In the present proof for $n = 2, 3, \dots$, let $T_n := \tau^-(\varepsilon^n)\wedge\tau_b^+$ for small enough $0 < \varepsilon < b$. It follows from Lemma 5.1 that

$$\begin{aligned}\varepsilon^{1-a} &\ge \mathbb{E}_\varepsilon\big[X^{1-a}_{\tau^-(\varepsilon^n)\wedge\tau^+(b)}\exp\{-(\ln\varepsilon^{-n})^r(\tau^-(\varepsilon^n)\wedge\tau^+(b)\}\big]\\ &\ge \mathbb{E}_\varepsilon\big[X^{1-a}_{\tau^-(\varepsilon^n)}\exp\{-(\ln\varepsilon^{-n})^r d_n\}1_{\{\tau^-(\varepsilon^n)<\tau_b^+\wedge d_n\}}\big]\\ &= \varepsilon^{(1-a)n}\exp\{\ln\varepsilon^{n(a-1)/2}\}\mathbb{P}_\varepsilon\{\tau^-(\varepsilon^n)<\tau_b^+\wedge d_n\},\end{aligned}$$

where

$$d_n := \frac{\ln\varepsilon^{n(a-1)/2}}{-(\ln\varepsilon^{-n})^r} = \frac{n(a-1)/2\ln\varepsilon^{-1}}{n^r(\ln\varepsilon^{-1})^r}\to\infty$$

as $n\to\infty$. Then

$$\mathbb{P}_\varepsilon\{\tau^-(\varepsilon^n)<\tau_b^+\wedge d_n\}\le\varepsilon^{(a-1)(n-2)/2}.$$

By the Borel–Cantelli lemma, we have

$$\mathbb{P}_\varepsilon\{\tau^-(\varepsilon^n)<\tau_b^+\wedge d_n \text{ i.o.}\} = 0. \qquad (5.6)$$

Then $\mathbb{P}_\varepsilon$-a.s.,

$$\tau^-(\varepsilon^n)\ge\tau_b^+\wedge d_n$$

for all n large enough.

Now if there are infinitely many n so that

$$\tau^-(\varepsilon^n) \geq d_n, \tag{5.7}$$

then we have $\tau_0^- = \infty$; on the other hand, if (5.7) holds for at most finitely many n, then by (5.6) we have $\tau_b^+ < \tau^-(\varepsilon^n)$ for all n large enough. Combining these two cases,

$$\mathbb{P}_\varepsilon\{\tau_0^- = \infty \text{ or } \tau_b^+ < \tau_0^- < \infty\} = 1. \tag{5.8}$$

It follows from the Markov property and lack of negative jumps for X that if $\mathbb{E}_\varepsilon[\mathrm{e}^{-\lambda\tau_0^-}; \tau_0^- < \infty] > 0$ for $\lambda > 0$, then

$$\begin{aligned}\mathbb{E}_\varepsilon[\mathrm{e}^{-\lambda\tau_0^-}; \tau_0^- < \infty] &= \mathbb{E}_\varepsilon[\mathrm{e}^{-\lambda\tau_0^-}; \tau_b^+ < \tau_0^- < \infty] \\ &\leq \mathbb{E}_\varepsilon[\mathrm{e}^{-\lambda\tau_b^+}; \tau_b^+ < \tau_0^-]\mathbb{E}_b[\mathrm{e}^{-\lambda\tau_\varepsilon^-}; \tau_\varepsilon^- < \infty] \\ &\quad \times \mathbb{E}_\varepsilon[\mathrm{e}^{-\lambda\tau_0^-}; \tau_0^- < \infty] \\ &< \mathbb{E}_\varepsilon[\mathrm{e}^{-\lambda\tau_0^-}; \tau_0^- < \infty],\end{aligned}$$

where we need (5.8) for the first equation. Therefore, $\mathbb{E}_\varepsilon[\mathrm{e}^{-\lambda\tau_0^-};\ \tau_0^- < \infty] = 0$ and consequently, $\mathbb{P}_\varepsilon\{\tau_0^- < \infty\} = 0$.

One can also find similar arguments in the proof of Theorem 4.2.2 in Le (2014) and the proof of Theorem 2.8(2) in Le and Pardoux (2015).

(ii) Given $0 < \delta < \frac{1}{3-2a}$, consider the martingale

$$X_{t\wedge T}^{1-a} \exp\left\{\int_0^{t\wedge T} G_a(X_s)\,\mathrm{d}s\right\}$$

for $T = \tau^-(\varepsilon^{1+\delta}) \wedge \tau^+(\varepsilon^{1-\delta})$. By Lemma 5.1,

$$\begin{aligned}\varepsilon^{1-a} &\geq \mathbb{E}_\varepsilon\left[X_{\tau^+(\varepsilon^{1-\delta})}^{1-a} \exp\left\{\int_0^{\tau^+(\varepsilon^{1-\delta})} G_a(X_s)\,\mathrm{d}s\right\} 1_{\{\tau^+(\varepsilon^{1-\delta})<\tau^-(\varepsilon^{1+\delta})\}}\right] \\ &\geq \varepsilon^{(1-a)(1-\delta)}\mathbb{P}_\varepsilon\{\tau^+(\varepsilon^{1-\delta}) < \tau^-(\varepsilon^{1+\delta})\}.\end{aligned}$$

Then

$$\mathbb{P}_\varepsilon\{\tau^+(\varepsilon^{1-\delta}) < \tau^-(\varepsilon^{1+\delta})\} \leq \varepsilon^{(1-a)\delta}. \tag{5.9}$$

Similarly,

$$\varepsilon^{1-a} \geq \mathbb{E}_\varepsilon\left[X_t^{1-a} \exp\left\{\int_0^t G_a(X_s)\,\mathrm{d}s\right\} 1_{\{\tau^+(\varepsilon^{1-\delta})=\tau^-(\varepsilon^{1+\delta})=\infty\}}\right].$$

Letting $t \to \infty$, we have

$$\mathbb{P}_\varepsilon\{\tau^+(\varepsilon^{1-\delta}) = \tau^-(\varepsilon^{1+\delta}) = \infty\} = 0. \tag{5.10}$$

By Lemma 5.1 again, for $t(\varepsilon) := [-(1-\delta)\ln\varepsilon]^{1-r}$ we have

$$\begin{aligned}\varepsilon^{1-a} &\geq \mathbb{E}_\varepsilon\biggl[X^{1-a}_{\tau^-(\varepsilon^{1+\delta})}\exp\biggl\{\int_0^{\tau^-(\varepsilon^{1+\delta})} G_a(X_s)\,\mathrm{d}s\biggr\}1_{\{t(\varepsilon)<\tau^-(\varepsilon^{1+\delta})<\tau^+(\varepsilon^{1-\delta})\}}\biggr]\\ &\geq \varepsilon^{(1-a)(1+\delta)}\mathbb{E}_\varepsilon\bigl[\mathrm{e}^{[-(1-\delta)\ln\varepsilon]^r t(\varepsilon)}1_{\{t(\varepsilon)<\tau^-(\varepsilon^{1+\delta})<\tau^+(\varepsilon^{1-\delta})\}}\bigr]\\ &= \varepsilon^{(1-a)(1+\delta)}\varepsilon^{-(1-\delta)}\mathbb{P}_\varepsilon\bigl\{t(\varepsilon)<\tau^-\bigl(\varepsilon^{1+\delta}\bigr)<\tau^+\bigl(\varepsilon^{1-\delta}\bigr)\bigr\}.\end{aligned}$$

Then

$$\mathbb{P}_\varepsilon\bigl\{t(\varepsilon)<\tau^-\bigl(\varepsilon^{1+\delta}\bigr)<\tau^+\bigl(\varepsilon^{1-\delta}\bigr)\bigr\} \leq \varepsilon^{(a-1)\delta}\varepsilon^{1-\delta} = \varepsilon^{1+(a-2)\delta}. \tag{5.11}$$

Combining (5.9), (5.10) and (5.11), we have

$$\mathbb{P}_\varepsilon\bigl\{\tau^-\bigl(\varepsilon^{1+\delta}\bigr)>t(\varepsilon)\bigr\} \leq \varepsilon^{1+(a-2)\delta}+\varepsilon^{(1-a)\delta} < 2\varepsilon^{(1-a)\delta}.$$

By the strong Markov property and lack of negative jumps for process X,

$$\begin{aligned}&\mathbb{P}_\varepsilon\biggl\{\bigcap_{n=0}^m\bigl\{\tau^-\bigl(\varepsilon^{(1+\delta)^n}\bigr)<\infty,\ \tau^-\bigl(\varepsilon^{(1+\delta)^{n+1}}\bigr)\circ\theta_{\tau^-(\varepsilon^{(1+\delta)^n})}\leq t\bigl(\varepsilon^{(1+\delta)^n}\bigr)\bigr\}\biggr\}\\ &\quad= \prod_{n=0}^m \mathbb{P}_{\varepsilon^{(1+\delta)^n}}\bigl\{\tau^-\bigl(\varepsilon^{(1+\delta)^{n+1}}\bigr)\leq t\bigl(\varepsilon^{(1+\delta)^n}\bigr)\bigr\} \geq \prod_{n=0}^m\bigl[1-2\varepsilon^{(1+\delta)^n(1-a)\delta}\bigr]\\ &\quad\geq \prod_{n=0}^m \mathrm{e}^{-4\varepsilon^{(1+\delta)^n(1-a)\delta}} \geq \mathrm{e}^{-8\varepsilon^{(1-a)\delta}}.\end{aligned}$$

Letting $m\to\infty$, we have

$$\begin{aligned}&\mathbb{P}_\varepsilon\biggl\{\bigcap_{n=0}^\infty\bigl\{\tau^-\bigl(\varepsilon^{(1+\delta)^n}\bigr)<\infty, \tau^-\bigl(\varepsilon^{(1+\delta)^{n+1}}\bigr)\circ\theta_{\tau^-(\varepsilon^{(1+\delta)^n})}\leq t\bigl(\varepsilon^{(1+\delta)^n}\bigr)\bigr\}\biggr\}\\ &\quad\geq \mathrm{e}^{-8\varepsilon^{(1-a)\delta}}.\end{aligned}$$

Since under $\mathbb{P}_\varepsilon$,

$$\tau^-_{0-} = \sum_{n=0}^\infty \tau^-\bigl(\varepsilon^{(1+\delta)^{n+1}}\bigr)\circ\theta_{\tau^-(\varepsilon^{(1+\delta)^n})},$$

then

$$\mathbb{P}_\varepsilon\biggl\{\tau^-_{0-}\leq\sum_{n=0}^\infty t\bigl(\varepsilon^{(1+\delta)^n}\bigr)\biggr\} \geq \mathrm{e}^{-8\varepsilon^{(1-a)\delta}}.$$

Notice that for $\varepsilon_n := \varepsilon^{(1+\delta)^n}$,

$$\sum_{n=1}^\infty t(\varepsilon_n) = \sum_{n=1}^\infty\bigl[(\delta-1)\ln\varepsilon_n\bigr]^{1-r} = \sum_{n=1}^\infty\bigl[(1+\delta)^n(\delta-1)\ln\varepsilon\bigr]^{1-r} < \infty,$$

we thus have

$$\mathbb{P}_\varepsilon\{\tau_{0-}^- < \infty\} \ge \mathrm{e}^{-8\varepsilon^{(1-a)\delta}}.$$

By the definition of solution to SDE (1.3) at the beginning of Section 2, we have

$$\mathbb{P}_\varepsilon\{\tau_0^- = \tau_{0-}^- < \infty\} \ge \mathrm{e}^{-8\varepsilon^{(1-a)\delta}}, \tag{5.12}$$

which completes the proof. □

PROOF OF THEOREM 2.8. (i) In the present proof, for small enough b^{-1} and ε satisfying $0 < b < \varepsilon^{-1}$ and for $n = 2, 3, \dots$, let $T_n := \tau^-(b) \wedge \tau^+(\varepsilon^{-n})$. By Lemma 5.1, we have

$$\begin{aligned}
\varepsilon^{a-1} &\ge \mathbb{E}_{\varepsilon^{-1}}\big[X^{1-a}_{\tau^+(\varepsilon^{-n})\wedge\tau^-(b)} \exp\{-(\ln \varepsilon^{-n})^r (\tau^+(\varepsilon^{-n}) \wedge \tau^-(b))\}\big] \\
&\ge \mathbb{E}_{\varepsilon^{-1}}\big[X^{1-a}_{\tau^+(\varepsilon^{-n})} \exp\{-(\ln \varepsilon^{-n})^r d_n\} 1_{\{\tau^+(\varepsilon^{-n}) < \tau^-(b)\wedge d_n\}}\big] \\
&\ge \varepsilon^{(a-1)n} \mathbb{E}_{\varepsilon^{-1}}\big[\exp\{\ln \varepsilon^{(1-a)n/2}\} 1_{\{\tau^+(\varepsilon^{-n}) < \tau^-(b)\wedge d_n\}}\big]
\end{aligned}$$

for b and ε^{-1} large enough, where

$$d_n := \frac{(1-a)n \ln \varepsilon^{-1}}{2(\ln \varepsilon^{-n})^r} = \frac{(1-a)n^{1-r}}{2}(\ln \varepsilon^{-1})^{1-r} \to \infty$$

as $n \to \infty$. Then

$$\mathbb{P}_{\varepsilon^{-1}}\{\tau^+_{\varepsilon^{-n}} < \tau^-(b) \wedge d_n\} \le \varepsilon^{(1-a)(n-2)/2}$$

for large enough b and ε^{-1}. The desired result of part (i) then follows from an argument similar to that in the proof for Theorem 2.3(i).

(ii) Taking $T := \tau^-(\varepsilon^{-1+\delta}) \wedge \tau^+(\varepsilon^{-1-\delta})$ in Lemma 5.1, we get

$$\begin{aligned}
\varepsilon^{a-1} &\ge \mathbb{E}_{\varepsilon^{-1}}\bigg[X^{1-a}_{\tau^-(\varepsilon^{-1+\delta})} \exp\bigg\{\int_0^{\tau^-(\varepsilon^{-1+\delta})} G_a(X_s)\,\mathrm{d}s\bigg\} \\
&\qquad\qquad \times 1_{\{\tau^-(\varepsilon^{-1+\delta}) < \tau^+(\varepsilon^{-1-\delta})\}}\bigg] \\
&\ge \varepsilon^{(a-1)(1-\delta)} \mathbb{P}_{\varepsilon^{-1}}\{\tau^-(\varepsilon^{-1+\delta}) < \tau^+(\varepsilon^{-1-\delta})\}.
\end{aligned}$$

Then

$$\mathbb{P}_{\varepsilon^{-1}}\{\tau^-(\varepsilon^{-(1-\delta)}) < \tau^+(\varepsilon^{-(1+\delta)})\} \le \varepsilon^{(a-1)\delta}. \tag{5.13}$$

Similarly,

$$\begin{aligned}
\varepsilon^{a-1} &\ge \mathbb{E}_{\varepsilon^{-1}}\bigg[X_n^{1-a} \exp\bigg\{\int_0^n G_a(X_s)\,\mathrm{d}s\bigg\} 1_{\{\tau^-(\varepsilon^{-1+\delta})\wedge\tau^+(\varepsilon^{-1-\delta}) > n\}}\bigg] \\
&\ge \varepsilon^{(1+\delta)(a-1)} \mathrm{e}^{n\varepsilon^{-1+\delta}} \mathbb{P}_{\varepsilon^{-1}}\{\tau^-(\varepsilon^{-1+\delta}) \wedge \tau^+(\varepsilon^{-1-\delta}) > n\}.
\end{aligned}$$

Letting $n \to \infty$, we have

$$\mathbb{P}_{\varepsilon^{-1}}\{\tau^-(\varepsilon^{-(1-\delta)}) = \tau^+(\varepsilon^{-(1+\delta)}) = \infty\} = 0. \tag{5.14}$$

Let $t(y) := (\ln y^{1-\delta})^{1-r}$ for $y > 1$ and small enough δ. With T replaced by $t(\varepsilon^{-1}) \wedge \tau^-(\varepsilon^{-1+\delta}) \wedge \tau^+(\varepsilon^{-1-\delta})$, similar to the above argument we get

$$\begin{aligned}
\varepsilon^{a-1} &\geq \mathbb{E}_{\varepsilon^{-1}}\bigg[X^{1-a}_{t(\varepsilon^{-1})} \exp\bigg\{\int_0^{t(\varepsilon^{-1})} G_a(X_s)\,\mathrm{d}s\bigg\} \\
&\qquad \times 1_{\{t(\varepsilon^{-1})<\tau^+(\varepsilon^{-(1+\delta)})<\tau^-(\varepsilon^{-(1-\delta)})\}}\bigg] \\
&\geq \varepsilon^{(a-1)(1+\delta)}\mathbb{E}_{\varepsilon^{-1}}\big[\mathrm{e}^{((\delta-1)\ln\varepsilon)^r t(\varepsilon^{-1})} 1_{\{t(\varepsilon^{-1})<\tau^+(\varepsilon^{-(1+\delta)})<\tau^-(\varepsilon^{-(1-\delta)})\}}\big] \\
&= \varepsilon^{(a-1)(1+\delta)}\mathbb{E}_{\varepsilon^{-1}}\big[\mathrm{e}^{(\delta-1)\ln\varepsilon} 1_{\{t(\varepsilon^{-1})<\tau^+(\varepsilon^{-(1+\delta)})<\tau^-(\varepsilon^{-(1-\delta)})\}}\big].
\end{aligned}$$

Then

$$\begin{aligned}
&\mathbb{P}_{\varepsilon^{-1}}\{t(\varepsilon^{-1}) < \tau^+(\varepsilon^{-(1+\delta)}) < \tau^-(\varepsilon^{-(1-\delta)})\} \\
&\quad \leq \varepsilon^{(1-a)\delta}\mathrm{e}^{-(\delta-1)\ln\varepsilon} = \varepsilon^{1-a\delta}.
\end{aligned} \tag{5.15}$$

Combining (5.13), (5.14) and (5.15), we have

$$\mathbb{P}_{\varepsilon^{-1}}\{\tau^+(\varepsilon^{-(1+\delta)}) > t(\varepsilon^{-1})\} \leq 2\varepsilon^{(a-1)\delta}. \tag{5.16}$$

Write $\tilde{\tau}_0 := 0$ and $\tilde{\tau}_{n+1} := \tau^+((X_{\tilde{\tau}_n} \vee 1)^{1+\delta}) \circ \tilde{\tau}_n + \tilde{\tau}_n$ for $n = 0, 1, 2, \dots$ with the convention $X_\infty = 0$. Notice that X allows possible positive jumps, and under $\mathbb{P}_{\varepsilon^{-1}}$ for $n \geq 1$, $X_{\tilde{\tau}_n} \geq \varepsilon^{-(1+\delta)^n}$ if $\tilde{\tau}_n < \infty$.

Observe that under $\mathbb{P}_{\varepsilon^{-1}}$, if $\tilde{\tau}_n < \infty$ for all $n \geq 1$, then

$$\sum_{n=1}^\infty t(X_{\tilde{\tau}_n}) \leq \sum_{n=1}^\infty \big[\ln \varepsilon^{-(1+\delta)^n(1-\delta)}\big]^{1-r} = \sum_{n=1}^\infty \big[(1+\delta)^n(\delta-1)\ln\varepsilon\big]^{1-r} < \infty.$$

By the strong Markov property and estimate (5.16), we can show that

$$\begin{aligned}
\mathbb{P}_{\varepsilon^{-1}}\{\tau^+_\infty < \infty\} &\geq \mathbb{P}_{\varepsilon^{-1}}\Big\{\lim_{n\to\infty} \tilde{\tau}_n < \infty\Big\} \\
&\geq \mathbb{P}_{\varepsilon^{-1}}\{\tilde{\tau}_{n+1} - \tilde{\tau}_n < t(X_{\tilde{\tau}_n}) \text{ for all } n \geq 1\} \\
&\geq \prod_{n=1}^\infty \big[1 - 2\varepsilon^{(a-1)\delta(1+\delta)^n}\big] > 0.
\end{aligned}$$

This completes the proof. □

PROOF OF THEOREM 2.13. To show part (i), for any constants $d > 0$ and $b > 0$ such that (2.10) holds for all $u > b$, for any $0 < \varepsilon < b^{-1}$, we have

$$\begin{aligned}
&\mathbb{P}_{\varepsilon^{-1}}\{\tau_b^- < d\} \\
&\quad\le \mathbb{P}_{\varepsilon^{-1}}\{\tau_b^- < d \wedge \tau^+(\varepsilon^{-2})\} \\
&\qquad + \sum_{n=1}^{\infty} \mathbb{P}_{\varepsilon^{-1}}\Big\{\tau^+(\varepsilon^{-2^n}) < \tau_b^- < d, \sup_{0\le s\le \tau_b^-} X_s \in [\varepsilon^{-2^n}, \varepsilon^{-2^{n+1}})\Big\} \\
&\quad\le \mathbb{P}_{\varepsilon^{-1}}\{\tau_b^- < d \wedge \tau^+(\varepsilon^{-2})\} \\
&\qquad + \sum_{n=1}^{\infty} \mathbb{P}_{\varepsilon^{-1}}\{\tau^+(\varepsilon^{-2^n}) < \tau_b^-, \\
&\qquad \tau_b^- \circ \theta(\tau^+(\varepsilon^{-2^n})) < d \wedge \tau^+(\varepsilon^{-2^{n+1}}) \circ \theta(\tau^+(\varepsilon^{-2^n}))\} \\
&\quad\le \mathbb{P}_{\varepsilon^{-1}}\{\tau_b^- < d \wedge \tau^+(\varepsilon^{-2})\} \\
&\qquad + \sum_{n=1}^{\infty} \mathbb{E}_{\varepsilon^{-1}}\big[1_{\{\tau^+(\varepsilon^{-2^n})<\tau_b^-\}} \mathbb{P}_{X_{\tau^+(\varepsilon^{-2^n})}}\{\tau_b^- < d \wedge \tau^+(\varepsilon^{-2^{n+1}})\}\big].
\end{aligned} \tag{5.17}$$

By Lemma 5.1, for $\varepsilon^{-2^n} \le x < \varepsilon^{-2^{n+1}}$, $T := \tau_b^- \wedge \tau^+(\varepsilon^{-2^{n+1}}) \wedge d$ and $n = 0, 1, 2, \dots,$

$$\begin{aligned}
x^{1-a} &\ge \mathbb{E}_x\Big[X_T^{1-a} \exp\Big\{\int_0^T G_a(X_s)\,\mathrm{d}s\Big\}\Big] \\
&\ge \mathbb{E}_x\Big[X_T^{1-a} \exp\Big\{-\int_0^T (\ln(X_s))^r\,\mathrm{d}s\Big\}\Big] \\
&\ge b^{1-a}\mathbb{E}_x\big[\exp\{-d(\ln\varepsilon^{-2^{n+1}})^r\};\ \tau_b^- < \tau^+(\varepsilon^{-2^{n+1}}) \wedge d\big] \\
&= b^{1-a}\mathbb{E}_x\big[\exp\{-d(2^{n+1}\ln\varepsilon^{-1})^r\};\ \tau_b^- < \tau^+(\varepsilon^{-2^{n+1}}) \wedge d\big].
\end{aligned}$$

Then

$$\begin{aligned}
\mathbb{P}_x\{\tau_b^- < \tau^+(\varepsilon^{-2^{n+1}}) \wedge d\} &\le b^{a-1}\exp\{(1-a)2^n\ln\varepsilon^{-1} + d(2^{n+1}\ln\varepsilon^{-1})^r\} \\
&\le b^{a-1}\mathrm{e}^{(1-a)2^{n-1}\ln\varepsilon^{-1}} = b^{a-1}\varepsilon^{(a-1)2^{n-1}}
\end{aligned}$$

for all small enough $\varepsilon > 0$. It follows from (5.17) and the strong Markov property that

$$\mathbb{P}_{\varepsilon^{-1}}\{\tau_b^- < d\} \le b^{a-1}\sum_{n=0}^{\infty}\varepsilon^{(a-1)2^{n-1}},$$

which goes to 0 as $\varepsilon \to 0+$. Since $b > 0$ and $d > 0$ are arbitrary, the process X thus stays at infinity.

We now proceed to show the part (ii). Write $t(x) := (1+\delta)^r(\ln x)^{1-r}$ for $x > 1$. Then by Lemma 5.1, for large x,

$$\begin{aligned} x^{1-a} &\ge \mathbb{E}_x\biggl[X^{1-a}_{\tau^+(x^{(1+\delta)})}\exp\biggl\{\int_0^{\tau^+(x^{(1+\delta)})}G_a(X_s)\,\mathrm{d}s\biggr\}1_{\{\tau^-(x^{(1+\delta)^{-1}})>\tau^+(x^{1+\delta})\}}\biggr] \\ &\ge x^{(1-a)(1+\delta)}\mathbb{P}_x\bigl\{\tau^-\bigl(x^{(1+\delta)^{-1}}\bigr) > \tau^+\bigl(x^{1+\delta}\bigr)\bigr\}. \end{aligned}$$

Then

$$\mathbb{P}_x\bigl\{\tau^-\bigl(x^{(1+\delta)^{-1}}\bigr) > \tau^+\bigl(x^{1+\delta}\bigr)\bigr\} \le x^{-\delta(1-a)}. \tag{5.18}$$

By condition (2.11), we also have

$$\begin{aligned} x^{1-a} &\ge \mathbb{E}_x\biggl[X^{1-a}_{\tau^-(x^{(1+\delta)^{-1}})}\exp\biggl\{\int_0^{\tau^-(x^{(1+\delta)^{-1}})}G_a(X_s)\,\mathrm{d}s\biggr\} \\ &\qquad\times 1_{\{t(x)<\tau^-(x^{(1+\delta)^{-1}})<\tau^+(x^{1+\delta})\}}\biggr] \\ &\ge x^{(1-a)/(1+\delta)}\mathbb{E}_x\biggl[\exp\biggl\{\int_0^{t(x)}(\ln X_s)^r\,\mathrm{d}s\biggr\}1_{\{t(x)<\tau^-(x^{(1+\delta)^{-1}})<\tau^+(x^{1+\delta})\}}\biggr] \\ &\ge x^{(1-a)/(1+\delta)}\mathrm{e}^{(1+\delta)^{-r}(\ln x)^r t(x)}\mathbb{P}_x\bigl\{t(x) < \tau^-\bigl(x^{(1+\delta)^{-1}}\bigr) < \tau^+\bigl(x^{1+\delta}\bigr)\bigr\} \\ &= x^{(1-a)/(1+\delta)}x\mathbb{P}_x\bigl\{t(x) < \tau^-\bigl(x^{(1+\delta)^{-1}}\bigr) < \tau^+\bigl(x^{1+\delta}\bigr)\bigr\}. \end{aligned}$$

It follows that

$$\mathbb{P}_x\bigl\{t(x) < \tau^-\bigl(x^{(1+\delta)^{-1}}\bigr) < \tau^+\bigl(x^{1+\delta}\bigr)\bigr\} \le x^{\frac{\delta(1-a)}{1+\delta}-1} = x^{-(1+\delta a)/(1+\delta)}. \tag{5.19}$$

Combining (5.18) and (5.19), we have

$$\mathbb{P}_x\bigl\{t(x) < \tau^-\bigl(x^{(1+\delta)^{-1}}\bigr)\bigr\} \le x^{-(1+\delta a)/(1+\delta)} + x^{-\delta(1-a)} \le 2x^{-\delta(1-a)}$$

for small enough $\delta > 0$. Then for $b \equiv b(\delta)$ large enough, by the strong Markov property

$$\begin{aligned} &\mathbb{P}_{b^{(1+\delta)^m}}\biggl\{\bigcap_{n=1}^m\bigl\{\tau^-\bigl(b^{(1+\delta)^n}\bigr) < \infty, \tau^-\bigl(b^{(1+\delta)^{n-1}}\bigr)\circ\theta_{\tau^-(b^{(1+\delta)^n})} \le t\bigl(b^{(1+\delta)^n}\bigr)\bigr\}\biggr\} \\ &\quad= \prod_{n=1}^m \mathbb{P}_{b^{(1+\delta)^n}}\bigl\{\tau^-\bigl(b^{(1+\delta)^{n-1}}\bigr) \le t\bigl(b^{(1+\delta)^n}\bigr)\bigr\} \\ &\quad\ge \prod_{n=1}^m\bigl(1 - 2b^{-\delta(1-a)(1+\delta)^{n-1}}\bigr) \ge \prod_{n=1}^m \mathrm{e}^{-4b^{-\delta(1-a)(1+\delta)^{n-1}}} \\ &\quad= \mathrm{e}^{-4\sum_{n=1}^m b^{-\delta(1-a)(1+\delta)^{n-1}}} \ge \mathrm{e}^{-8b^{-\delta(1-a)}}. \end{aligned}$$

Let $m \to \infty$. Then

$$\lim_{x\to\infty} \mathbb{P}_x\left\{\tau_b^- \le \sum_{n=1}^{\infty} t\big(b^{(1+\delta)^n}\big) = (1+\delta)^r (\ln b)^{1-r} \sum_{n=1}^{\infty} (1+\delta)^{(1-r)n} < \infty\right\}$$

$$\ge \mathrm{e}^{-8b^{-\delta(1-a)}} \tag{5.20}$$

for $r > 1$. Letting $b \to \infty$ in (5.20), we obtain the limit (2.7) and the process X comes down from infinity. □

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