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ENTROPY-CONTROLLED LAST-PASSAGE PERCOLATION[1]

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We introduce a natural generalization of Hammersley's Last-Passage Percolation (LPP) called *Entropy-controlled* Last-Passage Percolation (E-LPP), where points can be collected by paths with a global (path-entropy) constraint which takes into account the whole structure of the path, instead of a local (1-Lipschitz) constraint as in Hammersley's LPP. Our main result is to prove quantitative tail estimates on the maximal number of points that can be collected by a path with entropy bounded by a prescribed constant. The E-LPP turns out to be a key ingredient in the context of the directed polymer model when the environment is heavy-tailed, which we consider in (Berger and Torri (2018)). We give applications in this context, which are essentials tools in (Berger and Torri (2018)): we show that the limiting variational problem conjectured in (*Ann. Probab.* **44** (2016) 4006–4048), Conjecture 1.7, is finite, and we prove that the discrete variational problem converges to the continuous one, generalizing techniques used in (*Comm. Pure Appl. Math.* **64** (2011) 183–204; *Probab. Theory Related Fields* **137** (2007) 227–275).

1. Introduction: Hammersley's LPP and beyond. Let us recall the original Hammersley's Last-Passage Percolation (LPP) problem of the maximal number of points that can be collected by up/right paths, also known as Ulam's problem [16] of the maximal increasing subsequence of a random permutation. Let $m \in \mathbb{N}$, and $(Z_i)_{1\le i\le m}$ be m points independently drawn uniformly on the square $[0,1]^2$. We denote the coordinates of these points $Z_i := (x_i, y_i)$ for $1 \le i \le m$. A sequence $(z_{i_\ell})_{1\le \ell\le k}$ is said to be *increasing* if $x_{i_\ell} > x_{i_{\ell-1}}$ and $y_{i_\ell} > y_{i_{\ell-1}}$ for any $1 \le \ell \le k$ (by convention $i_0 = 0$ and $z_0 = (0,0)$). The question is to find the length of the longest increasing sequence among the m points, which is equivalent to finding the length of the longest increasing subsequence of a random (uniform) permutation of length m: we let

$$\mathcal{L}_m = \max\{k : \exists (i_1,\dots,i_k) \text{ s.t. } (Z_{i_\ell})_{1\le \ell\le k} \text{ is increasing}\}. \tag{1.1}$$

Hammersley [12] first proved that $m^{-1/2}\mathcal{L}_m$ converges a.s. and in L^1 to some constant, that was believed to be 2. Later, the constant has been proven to be indeed 2 (see [13, 17]), and estimates related to $\mathcal{L}_m$ were improved by a series of papers,

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culminating in a seminal paper by Baik, Deift and Johansson [3], showing that $m^{-1/6}(\mathcal{L}_m - 2\sqrt{m})$ converges in distribution to the Tracy–Widom distribution.

The main goal of the present article is to define the *Entropy-controlled Last Passage Percolation* (E-LPP), a natural extension of Hammersley's LPP. We introduce the concept of entropy constraint, which depends on the structure of the whole path, and is related to the moderate deviation rate function of the simple symmetric random walk.

The E-LPP turns out to be crucial in the analysis of the directed polymer model in a heavy-tailed environment in $(1+1)$-dimension. We refer to [7–9] for the definition and a general overview of the directed polymer model. This model has attracted much attention in recent years, in particular because it is in the KPZ universality class: in particular, it is conjectured that at any fixed inverse temperature β, the transversal fluctuation exponent ξ is equal to $2/3$. Alberts, Khanin and Quastel [1] recently introduced the concept of *intermediate disorder regime* in which β scales with n, the size of the system. In the setting of a heavy-tailed environment, this was considered first by Auffinger–Louidor [2], who showed that rescaling suitably β, the model has transversal fluctuations exponent $\xi = 1$. Dey and Zygouras [10] then proved that with a different (stronger) rescaling of β, the model has Brownian fluctuations, that is $\xi = 1/2$. Additionally, Dey and Zygouras proposed a phase-diagram that connects the exponent of the transversal fluctuation of the polymer ξ and the decay rate of β with the tail exponent α of the heavy-tailed distribution of the environment. In [4], we start to complete this picture by giving a complete description in the case of $\alpha \in (0,2)$: one of the main results is a proof of Conjecture 1.7 of [10], describing explicitly the scaling-limit of the model. One crucial tool needed in [4] is the E-LPP defined in the present article, which allows to go beyond the Lipschitz setting of [2, 11], and treat intermediate transversal fluctuations $1/2 < \xi < 1$.

1.1. *Organization of the article.* All our results are stated in Section 2: in Section 2.1 we give the definition and results for the E-LPP in continuous and in discrete settings; in Section 2.3, we consider the problem of E-LPP with heavy-tail weights that appears in [4], and we show that the continuous limit in [4], Theorem 2.4, is well defined, completing the proof of [10], Conjecture 1.7; in Section 2.4 we state the convergence of the discrete energy-entropy variational problem to its continuous counterpart. This result is crucial to prove the convergence in Theorems 2.2–2.7 of [4]. The proofs of the all results are presented in Sections 3 to 5.

2. Main results. Operating a rotation by 45° clockwise, we may map Hammersley's LPP problem (cf. Section 1) to that of the maximal number of points that can be collected by 1-Lipschitz paths $s : [0,1] \to \mathbb{R}$. We now introduce a new (natural) model where the Lipschitz constraint is replaced by a path entropy constraint.

2.1. *Entropy-controlled LPP.* For $t > 0$, and a finite set $\Delta = \{(t_i, x_i); 1 \le i \le j\} \subset [0,t] \times \mathbb{R}$ with $|\Delta| = j \in \mathbb{N}$ and with $0 \le t_1 \le t_2 \le \cdots \le t_j \le t$, we can define the entropy of Δ as

$$\text{Ent}(\Delta) := \frac{1}{2} \sum_{i=1}^{j} \frac{(x_i - x_{i-1})^2}{t_i - t_{i-1}}, \tag{2.1}$$

where we used the convention that $(t_0, x_0) = (0,0)$. If there exists some $1 \le i \le j$ such that $t_i = t_{i-1}$, then we set $\text{Ent}(\Delta) = +\infty$. This corresponds to the definition (2.7) of the entropy of a continuous path $s : [0,t] \to \mathbb{R}$, applied to the linear interpolation of the points of Δ: to any set Δ we can therefore canonically associate a (continuous) path with the same entropy. The set Δ is seen as a set of points a path has to go through. For $S = (S_i)_{i \ge 0}$, a simple symmetric random walk on $\mathbb{Z}$, and if $\Delta \subset \mathbb{N} \times \mathbb{Z}$, we have that $\mathbf{P}(\Delta \subset S) \le e^{-\text{Ent}(\Delta)}$ ($\Delta \subset S$ means that $S_{t_i} = x_i$ for all $i \le |\Delta|$)—we used that for the simple random walk $\mathbf{P}(S_i = x) \le e^{-x^2/2i}$ by a standard Chernoff bound argument.

Then, for any fixed $B > 0$, we will consider the maximal number of points that can be collected by paths with entropy smaller than B, among a random set Υ_m of m points, whose law is denoted $\mathbb{P}$. We now consider two types of problems, depending on how this set Υ_m is constructed:

(i) *continuous* setting: for $t, x > 0$, we consider a domain $\boldsymbol{\Lambda}_{t,x} := [0,t] \times [-x,x]$, and $\boldsymbol{\Upsilon}_m = \boldsymbol{\Upsilon}_m(t,x) = \{Y_1, \dots, Y_m\}$ where $(Y_i)_{1 \le i \le m}$ is a collection of independent r.v. chosen uniformly in $\boldsymbol{\Lambda}_{t,x}$;

(ii) *discrete* setting: for $n, h \in \mathbb{N}$, we consider a domain $\Lambda_{n,h} := [\![0,n]\!] \times [\![-h,h]\!]$, and $\Upsilon_m = \Upsilon_m(n,h) = \{Y_1, \dots, Y_m\}$ is a set of m distinct points taken uniformly in $\Lambda_{n,h}$.

We are then able to define the entropy-controlled LPP by

$$\mathcal{L}_m^{(B)}(t,x) = \max_{\substack{\Delta \subset \boldsymbol{\Upsilon}_m(t,x) \\ \text{Ent}(\Delta) \le B}} |\Delta|, \qquad L_m^{(B)}(n,h) = \max_{\substack{\Delta \subset \Upsilon_m(n,h) \\ \text{Ent}(\Delta) \le B}} |\Delta|, \tag{2.2}$$

the maximal number of points than can be included in a set Δ that has entropy smaller than B. In other words, it is the maximal number of points in $\boldsymbol{\Upsilon}_m$ or Υ_m that can be collected by a path of entropy smaller than B. Note that we use the different font to be able to differentiate the setting: $\mathcal{L}$, $\boldsymbol{\Lambda}$, $\boldsymbol{\Upsilon}$ for the continuous case and L, Λ, Υ for the discrete one. We refer to Figure 1 for a picture of an optimizing path for the continuous E-LPP with entropy constraint $B = 1$, which is compared to an optimizing path for Hammersley's (1-Lipschitz) LPP.

We show the following result—the lower bound is not needed for our applications, but can be found in [5].

THEOREM 2.1. *There are constants* $C_0, c_0, c_0' > 0$ *such that*: *for any* $t, x, B > 0, n, h \ge 1$

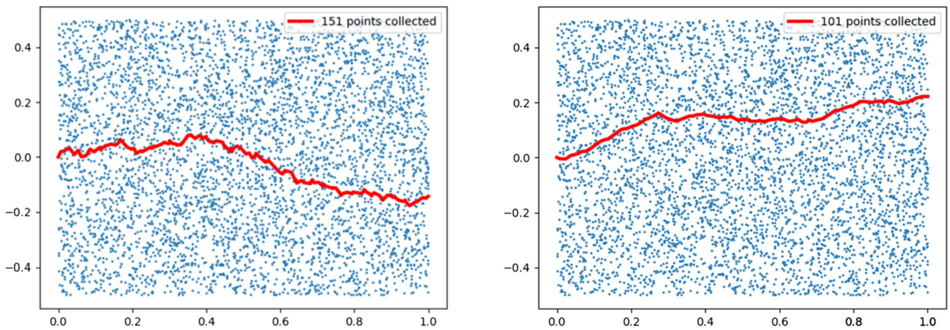

FIG. 1. *We sample $m = 5000$ points uniformly and independently in the box* $[0, 1] \times [-0.5, 0.5]$ (*note that* $\sqrt{m} \approx 70.71$). *On the left*: *a* (*near-*)*optimal path for the* continuous *E-LPP* (*with constraint* $\mathrm{Ent}(s) \le B = 1$), *which collects* 151 *points* (*we used a simulated annealing procedure*, *hence the* near-*optimality of the path*). *On the right*: *an optimal path for Hammersley's* (1-*Lipschitz*) *LPP*, *which collects* 101 *points*.

(i) continuous setting: *for all $m \ge 1$ and all $k \le m$*

$$\mathbb{P}\big(\mathcal{L}_m^{(B)}(t, x) \ge k\big) \le \left(\frac{C_0 (Bt/x^2)^{1/2} m}{k^2}\right)^k. \tag{2.3}$$

(ii) discrete setting: *for all $1 \le m \le nh$ and all $k \le m$*

$$\mathbb{P}\big(L_m^{(B)}(n, h) \ge k\big) \le \left(\frac{C_0 (Bn/h^2)^{1/2} m}{k^2}\right)^k. \tag{2.4}$$

The proof of Theorem 2.1 is not difficult but a bit technical, and we give it in Section 3.

This result shows in particular that $\mathcal{L}_m^{(B)}(t, x)$ is of order $((Bt/x^2)^{1/4}\sqrt{m}) \wedge m$, resp. $L_m^{(B)}(n, h)$ is of order $((Bn/h^2)^{1/4}\sqrt{m}) \wedge m$, as stressed by the following corollary. We stress that keeping track of the dependence on B is essential for the applications we have in mind.

COROLLARY 2.2. *For any $b > 0$, there is a constant $c_b > 0$ such that, for any $m \ge 1$, and any positive B, and any t, x, respectively, n, h,*

$$\mathbb{E}\left[\left(\frac{\mathcal{L}_m^{(B)}(t, x)}{((Bt/x^2)^{1/4}\sqrt{m}) \wedge m}\right)^b\right] \le c_b; \qquad \mathbb{E}\left[\left(\frac{L_m^{(B)}(n, h)}{((Bn/h^2)^{1/4}\sqrt{m}) \wedge m}\right)^b\right] \le c_b.$$

REMARK 2.3. On may view Theorem 2.1 as a generalization of [11], Proposition 3.3. More precisely, we recover [11], Proposition 3.3, by considering $\Lambda_{n,n} = [\![n, n]\!]^2$ and replacing the entropy condition $\mathrm{Ent}(\Delta) \le B$ by a *Lipschitz* condition, that is considering only the sets Δ whose points can be interpolated using a Lipschitz path. Let us denote $L_m^{(\mathrm{Lip})}(n)$ the LPP obtained. Now observe that if Δ satisfies the Lipschitz condition we have that $\mathrm{Ent}(\Delta) \le n/2$ (recall the definition (2.1)):

as a consequence it holds that $L_m^{(n/2)}(n,n) \geq L_m^{(\mathrm{Lip})}(n)$. We also stress that our definition of E-LPP opens the way to many extensions: in particular as soon as one is able to properly define the entropy of a path (i.e., of a set Δ), one could extend the results to the case of paths with unbounded jumps or even nondirected paths: this is the object of [5], where a general notion of path-constrained LPP is developed and studied.

Let us stress here that one might want to reverse the point of view, and estimate the minimal entropy needed for a path to visit at least k points. This turns out to be essential in Section 4 of [4]. One realizes that

$$\inf_{\substack{\Delta \subset \Upsilon_m \\ |\Delta| \geq k}} \mathrm{Ent}(\Delta) \leq B \quad \Longleftrightarrow \quad \sup_{\substack{\Delta \subset \Upsilon_m \\ \mathrm{Ent}(\Delta) \leq B}} |\Delta| \geq k.$$

Hence, an easy consequence of Theorem 2.1 is that for any $k \leq n$ (we state it only in the discrete setting)

$$\mathbb{P}\Big(\inf_{\Delta \subset \Upsilon_m, |\Delta| \geq k} \mathrm{Ent}(\Delta) \leq B\Big) \leq \left(\frac{C_0 (Bn/h^2)^{1/2} m}{k^2}\right)^k. \tag{2.5}$$

It therefore says that, with high probability, a path that collects k points in $\Upsilon_m \subset \Lambda_{n,h}$ has an entropy larger than a constant times $k^4/m^2 \times h^2/n$.

2.2. *Open questions and directions.* Our main goal has been to introduce a generalized last-passage percolation, and Theorem 2.1 give the first estimates on the model. Below, we explain how these estimates are already extremely useful in the context of the directed polymer model; see Sections 2.3 and 2.4. However, many questions are raised, and we provide here a few important open problems and future questions of investigation.

(a) As a consequence of Theorem 2.1 (and Theorem 3.1 in [5] for the lower bound), we have that $\mathcal{L}_m^{(B)}(t,x)$ is of order $((Bt/x^2)^{1/4}\sqrt{m}) \wedge m$. The next step would then be to show that for fixed B, t, x, $\mathcal{L}_m^{(B)}(t,x)/((Bt/x^2)^{1/4}\sqrt{m})$ converges a.s. to a constant $\mathtt{C}$ as $m \to \infty$, and identify the constant $\mathtt{C}$. In this direction, we consider in [5] the E-LPP in a Poissonian setting: the convergence then follows from a subadditivity argument, and we expect that a “de-Poissonization” argument would allow to transfer the convergence to the setting presented here.

(b) Once the constant $\mathtt{C}$ has been determined, the next natural step is to identify the fluctuations of $\mathcal{L}_m^{(B)}(t,x) - \mathtt{C}(Bt/x^2)^{1/4}\sqrt{m}$, and also to determine the transversal fluctuations of optimal paths. In [5], Appendix A, numerical simulations are presented, and they suggest that $m^{-1/6}(\mathcal{L}_m^{(B)}(t,x) - \mathtt{C}(Bt/x^2)^{1/4}\sqrt{m})$ converges in distribution, in the spirit of the result of Baik, Deift and Johansson [3] (see Section 1).

2.3. *Application I*: *Continuous E-LPP with heavy-tail weights*. In [4], we consider the *directed polymer model in heavy-tail environment*, in weak-coupling regimes (i.e., when the temperature diverges as the size of the system goes to $+\infty$). We prove the convergence of the log-partition function (suitably centered and rescaled) to a continuous energy-entropy variational problem $\mathcal{T}_\beta$, defined below in (2.9) (or in [4], Section 2.2).

Our E-LPP appears crucial to achieve this. In particular, it enables us to show that the variational problem $\mathcal{T}_\beta$ is well defined when the tail decay exponent α is in $(1/2, 2)$: this is Theorem 2.4, which proves the first part of [10], Conjecture 1.7. The second part of this conjecture, that is, that $\mathcal{T}_\beta$ is indeed the scaling limit of the log-partition function of the directed polymer in heavy-tail environment, is proved in [4], Theorem 2.4.

Let us recall some notation from Section 2.2 in [4]. The set of allowed paths (scaling limits of random walk trajectories) is

$$\mathscr{D} := \{s : [0, 1] \to \mathbb{R};\, s \text{ continuous and a.e. differentiable}\}, \tag{2.6}$$

and the (continuum) *entropy* of a path $s \in \mathscr{D}$ is defined by

$$\mathrm{Ent}(s) = \frac{1}{2}\int_0^1 \big(s'(t)\big)^2\, dt. \tag{2.7}$$

This last definition derives from the rate function of the moderate deviation of the simple random walk (see [15] or [4], equation (2.14)).

We let $\mathcal{P} := \{(w_i, t_i, x_i)\}_{i\geq 1}$ be a Poisson point process on $[0, \infty) \times [0, 1] \times \mathbb{R}$ of intensity $\mu(\mathrm{d}w\,\mathrm{d}t\,\mathrm{d}x) = \frac{\alpha}{2} w^{-\alpha-1}\mathbb{1}_{\{w>0\}}\,\mathrm{d}w\,\mathrm{d}t\,\mathrm{d}x$, where $\alpha \in (0, 2)$. For a quenched realization of $\mathcal{P}$, (minus) the *energy* of a continuous path $s \in \mathscr{D}$ is then defined by

$$\pi(s) = \pi_{\mathcal{P}}(s) := \sum_{(w,t,x)\in\mathcal{P}} w\mathbb{1}_{\{(t,x)\in s\}}, \tag{2.8}$$

where $(t, x) \in s$ means that (t, x) is visited by the path s, that is, $s_t = x$.

Using (2.7) and (2.8), we define the *energy–entropy competition variational problem*: for any $\beta \geq 0$, we let

$$\mathcal{T}_\beta := \sup_{s\in\mathscr{D},\mathrm{Ent}(s)<+\infty} \{\beta\pi(s) - \mathrm{Ent}(s)\}. \tag{2.9}$$

The next result shows that it is well defined, and gives some of its properties.

THEOREM 2.4. *For $\alpha \in (1/2, 2)$, we have the scaling relation*

$$\mathcal{T}_\beta \overset{(\mathrm{d})}{=} \beta^{\frac{2\alpha}{2\alpha-1}}\mathcal{T}_1, \tag{2.10}$$

and $\mathcal{T}_\beta \in (0, +\infty)$ for all $\beta > 0$ a.s. Moreover, $\mathbb{E}[(\mathcal{T}_\beta)^\upsilon] < \infty$ for any $\upsilon < \alpha - 1/2$. We also have that a.s. the map $\beta \mapsto \mathcal{T}_\beta$ is continuous, and that the supremum in (2.9) is attained by some unique continuous path s^*_β with $\mathrm{Ent}(s^*_\beta) < \infty$.

On the other hand, for $\alpha \in (0, 1/2]$ we have $\mathcal{T}_\beta = +\infty$ for all $\beta > 0$ a.s.

REMARK 2.5. As we discuss in Section 2.5 of [4], the fact that the maximizer of $\mathcal{T}_\beta$ is unique could be used to show the concentration of the paths around s^*_β under the polymer measure $\mathbf{P}^\omega_{n,\beta_n}$, in analogy with the result obtained by Auffinger and Louidor in Theorem 2.1 of [2].

2.4. *Application II*: *Discrete E-LPP with heavy-tail weights*. In this section, we discuss the convergence of a discrete energy-entropy variational problem $T^{\beta_{n,h}}_{n,h}$ defined below in (2.15), to its continuous counterpart $\mathcal{T}_\beta$ (2.9). This is a crucial result that we need in [4] to prove Theorems 2.4–2.7.

We introduce the discrete field $\{\omega_{i,x}; (i,x) \in \mathbb{N} \times \mathbb{Z}\}$, which are i.i.d. nonnegative random variables of law $\mathbb{P}$: there is some slowly varying function $L(\cdot)$ and some $\alpha > 0$ such that

$$\mathbb{P}(\omega > x) = L(x) x^{-\alpha}. \tag{2.11}$$

This random field is the discrete counterpart of the Poisson point process $\mathcal{P}$ introduced in Section 2.3. We refer to Section 5.1 for further details.

Let us consider $F(x) = \mathbb{P}(\omega \le x)$ be the disorder distribution (cf. (2.11)), and define the function $m(x)$ by

$$m(x) := F^{-1}\left(1 - \frac{1}{x}\right) \qquad \text{so } \mathbb{P}\big(\omega > m(x)\big) = \frac{1}{x}. \tag{2.12}$$

The second identity characterizes $m(x)$ up to asymptotic equivalence: we have that $m(\cdot)$ is a regularly varying function with exponent $1/\alpha$.

For any given box $\Lambda_{n,h} = [\![1, n]\!] \times [\![-h, h]\!]$, we can rewrite the discrete field in this region $(\omega_{i,x})_{(i,x)\in\Lambda_{n,h}}$ using the *order statistics*: we let $M_r^{(n,h)}$ be the rth largest value of $(\omega_{i,x})_{(i,x)\in\Lambda_{n,h}}$ and $Y_r^{(n,h)} \in \Lambda_{n,h}$ its position—note that $(Y_r^{(n,h)})_{r=1}^{|\Lambda_{n,h}|}$ is simply a random permutation of the points of $\Lambda_{n,h}$. In such a way,

$$(\omega_{i,j})_{(i,j)\in\Lambda_n} = \big(M_r^{(n,h)}, Y_r^{(n,h)}\big)_{r=1}^{|\Lambda_{n,h}|}. \tag{2.13}$$

In the following, we refer to $(M_r^{(n,h)})_{r=1}^{|\Lambda_{n,h}|}$ as the *weight* sequence. We now define (minus) the energy collected by a set $\Delta \subset \Lambda_{n,h}$ and its contribution by the first ℓ weights (with $1 \le \ell \le |\Lambda_{n,h}|$) as follows:

$$\begin{aligned} \Omega_{n,h}(\Delta) &:= \sum_{r=1}^{|\Lambda_{n,h}|} M_r^{(n,h)} \mathbb{1}_{\{Y_r^{(n,h)} \in \Delta\}}; \\ \Omega^{(\ell)}_{n,h}(\Delta) &:= \sum_{r=1}^{\ell} M_r^{(n,h)} \mathbb{1}_{\{Y_r^{(n,h)} \in \Delta\}}. \end{aligned} \tag{2.14}$$

We also set $\Omega^{(>\ell)}_{n,h}(\Delta) := \Omega_{n,h}(\Delta) - \Omega^{(\ell)}_{n,h}(\Delta)$.

In such a way, we can define the (discrete) variational problem

$$T_{n,h}^{\beta_{n,h}} := \max_{\Delta \subset \Lambda_{n,h}} \{\beta_{n,h}\Omega_{n,h}(\Delta) - \mathrm{Ent}(\Delta)\}, \tag{2.15}$$

with $\beta_{n,h}$ some function of n, h (soon to be specified), and $\mathrm{Ent}(\Delta)$ as defined in (2.1). We also define analogues of (2.15) with a restriction to the ℓ largest weights, or beyond the ℓth weight

$$\begin{aligned} T_{n,h}^{\beta_{n,h},(\ell)} &:= \max_{\Delta \subset \Lambda_{n,h}} \{\beta_{n,h}\Omega_{n,h}^{(\ell)}(\Delta) - \mathrm{Ent}(\Delta)\}, \\ T_{n,h}^{\beta_{n,h},(>\ell)} &:= \max_{\Delta \subset \Lambda_{n,h}} \{\beta_{n,h}\Omega_{n,h}^{(>\ell)}(\Delta) - \mathrm{Ent}(\Delta)\}. \end{aligned} \tag{2.16}$$

The following proposition is crucial for the proof of Theorem 2.7 below, and is also a central tool in [4], Section 4.

PROPOSITION 2.6. *The following hold true*:

- *For any $a < \alpha$, there is a constant $c_a > 0$ such that for any $1 \le \ell \le nh$, for any $b > 1$*,

$$\mathbb{P}\left(T_{n,h}^{\beta_{n,h},(\ell)} \ge b \times (\beta_{n,h} m(nh))^{4/3} \left(\frac{n}{h^2}\right)^{1/3}\right) \le c_a b^{-3a/4}. \tag{2.17}$$

- *We also have that there is a constant $c > 0$ such that for any $b > 1$*,

$$\mathbb{P}\left(T_{n,h}^{\beta_{n,h},(>\ell)} \ge b \times (\beta_{n,h} m(nh/\ell))^{4/3} \left(\frac{\ell^2 n}{h^2}\right)^{1/3}\right) \le c b^{-\alpha\ell/4} + e^{-cb^{1/4}}. \tag{2.18}$$

The proof is postponed to Section 5.2. Observe that we need here to keep track of the dependence on n, h: to that end, estimates obtained in Section 2.1 will be crucial. Note already that if $\frac{n}{h^2}\beta_{n,h} m(nh) \to \beta \in (0,\infty)$, as $n, h \to \infty$, it gives that $T_{n,h}^{\beta_{n,h},(\ell)}$ is of order $\beta^4 h^2/n$.

In the next result, we prove the convergence in distribution for (2.15), which generalizes the convergence of related variational problems considered in [2, 11].

THEOREM 2.7. *Suppose that $\frac{n}{h^2}\beta_{n,h} m(nh) \to \nu \in [0,\infty)$ as $n, h \to \infty$. For every $\alpha \in (1/2, 2)$ and for any $q > 0$, we have the following convergence in distribution*:

$$\frac{n}{h^2} T_{n,qh}^{\beta_{n,h}} \xrightarrow[n\to\infty]{(\mathrm{d})} \mathcal{T}_{\nu,q} := \sup_{s \in \mathscr{M}_q} \{\nu\pi(s) - \mathrm{Ent}(s)\}, \tag{2.19}$$

with $\mathscr{M}_q := \{s \in \mathscr{D}, \mathrm{Ent}(s) < \infty, \max_{t\in[0,1]} |s(t)| \le q\}$. We also have

$$\frac{n}{h^2} T_{n,qh}^{\beta_{n,h},(\ell)} \xrightarrow[n\to\infty]{(\mathrm{d})} \mathcal{T}_{\nu,q}^{(\ell)} := \sup_{s \in \mathscr{M}_q} \{\nu\pi^{(\ell)}(s) - \mathrm{Ent}(s)\}, \tag{2.20}$$

where $\pi^{(\ell)} := \sum_{r=1}^{\ell} M_r \mathbb{1}_{\{Y_r \in s\}}$ *with* $\{(M_r, Y_r)\}_{r \geq 1}$ *the order statistics of* $\mathcal{P}$ *restricted to* $[0,1] \times [-q, q]$; *see Section* 5.1. *Finally, we have*

$$\mathcal{T}_{\nu,q}^{(\ell)} \xrightarrow[\ell \to \infty]{\text{a.s.}} \mathcal{T}_{\nu,q}, \quad \text{and} \quad \mathcal{T}_{\nu,q} \xrightarrow[q \to \infty]{\text{a.s.}} \mathcal{T}_{\nu}. \tag{2.21}$$

3. Proof of Theorem 2.1 and Corollary 2.2.

3.1. *Proof of Theorem* 2.1. We start with the proof in the continuous setting. The discrete setting follows the same lines and details will be skipped.

Continuous setting. Let us consider $\mathcal{E}_k^{(t,B)}$ the set of k-tuples in $[0,t] \times \mathbb{R}$ (i.e., up to time t) that have entropy smaller than B:

$$\mathcal{E}_k^{(t,B)} = \left\{ \big((t_\ell, x_\ell)\big)_{1 \leq \ell \leq k} \subset [0,t] \times \mathbb{R}; \begin{array}{c} 0 < t_1 < \cdots < t_k < t; \\ \mathrm{Ent}\big((t_\ell, x_\ell)_{1 \leq \ell \leq k}\big) \leq B \end{array} \right\}.$$

We can compute exactly the volume of $\mathcal{E}_k^{(t,B)}$.

LEMMA 3.1. *We have, for any* $t > 0$ *and* $B > 0$

$$\mathrm{Vol}\big(\mathcal{E}_k^{(t,B)}\big) = C_k \times B^{k/2} t^{3k/2}, \qquad \text{with } C_k = \frac{\pi^k/\sqrt{2}}{\Gamma(k/2+1)\Gamma(3k/2+1)}.$$

In particular, it gives that there exists some constant C *such that*

$$\mathrm{Vol}\big(\mathcal{E}_k^{(t,B)}\big) \leq \left(\frac{C B^{1/2} t^{3/2}}{k^2} \right)^k.$$

PROOF. The key to the computation is the induction formula below, based on the decomposition over the left-most point in $\mathcal{E}_k^{(t,B)}$ at position (u, y) (by symmetry we can assume $y \geq 0$): it leaves $k-1$ points with remaining time $t-u$ and entropy smaller than $B - \frac{y^2}{2u}$,

$$\mathrm{Vol}\big(\mathcal{E}_k^{(t,B)}\big) = 2 \int_{u=0}^{t} \int_{y=0}^{\sqrt{2Bu}} \mathrm{Vol}\big(\mathcal{E}_{k-1}^{(t-u, B-y^2/2u)}\big)\, dy\, du. \tag{3.1}$$

The induction is only calculations. For $k = 1$, we have

$$\mathrm{Vol}\big(\mathcal{E}_1^{(t,B)}\big) = 2 \int_{u=0}^{t} \int_{y=0}^{\sqrt{2Bu}} du\, dy = 2\sqrt{2B} \int_0^t u^{1/2}\, du = \frac{4\sqrt{2}}{3} B^{1/2} t^{3/2},$$

so that we indeed have that $C_1 = \pi(\sqrt{2}\Gamma(3/2)\Gamma(5/2))^{-1}$.

For $k \geq 2$, by induction, we have

$$\mathrm{Vol}\big(\mathcal{E}_k^{(t,B)}\big) = 2C_{k-1} \int_{u=0}^{t} \int_{y=0}^{\sqrt{2Bu}} (t-u)^{3(k-1)/2} \left(B - \frac{y^2}{2u} \right)^{(k-1)/2} dy\, du.$$

Then, by a change of variable $w = y^2/(2Bu)$, we get that

$$\int_{y=0}^{\sqrt{2Bu}} \left(B - \frac{y^2}{2u}\right)^{(k-1)/2} dy = B^{(k-1)/2} \int_0^1 (1-w)^{(k-1)/2} \sqrt{\frac{Bu}{2}} w^{-1/2}\, dw$$
$$= \frac{1}{\sqrt{2}} B^{k/2} u^{1/2} \frac{\Gamma((k-1)/2+1)\Gamma(1/2)}{\Gamma(k/2+1)}.$$

Moreover, we also have

$$\int_{u=0}^{t} u^{1/2}(t-u)^{3(k-1)/2}\, du = t^{3(k-1)/2+1/2+1} \int_0^1 v^{1/2}(1-v)^{3(k-1)/2}\, dv$$
$$= t^{3k/2} \frac{\Gamma(3/2)\Gamma(3(k-1)/2+1)}{\Gamma(3k/2+1)}.$$

Hence, the constant C_k verifies

$$C_k = 2C_{k-1} \times \sqrt{\pi} \frac{\Gamma((k-1)/2+1)}{\Gamma(k/2+1)} \times \frac{\sqrt{\pi}}{2} \frac{\Gamma(3(k-1)/2+1)}{\Gamma(3k/2+1)},$$

which completes the induction, in view of the formula for C_{k-1}.

For the inequality in the second part of the lemma, we simply use Stirling's formula to get that there is a constant $c > 0$ such that

$$\Gamma(k/2+1) \geq (ck)^{k/2} \quad \text{and} \quad \Gamma(3k/2+1) \geq (ck)^{3k/2}. \qquad \square$$

Let us denote $\mathcal{N}_k$ the number of sets $\Delta \subset \Upsilon_m(t,x)$ with $|\Delta| = k$, that have entropy at most B. We write

$$\mathbb{P}\big(\mathcal{L}_m^{(B)}(t,x) \geq k\big) = \mathbb{P}(\mathcal{N}_k \geq 1) \leq \mathbb{E}[\mathcal{N}_k].$$

Since all the points are exchangeable, we get

$$\mathbb{E}[\mathcal{N}_k] = \binom{m}{k} \mathbb{P}\big(\exists \sigma \in \mathfrak{S}_k \text{ s.t. } (Z_{\sigma(1)}, \dots, Z_{\sigma(k)}) \in \mathcal{E}_k^{(t,B)}\big),$$

where $Z_1 = (t_1, x_1), \dots, Z_k = (t_k, x_k)$ are independent uniform r.v. on the domain $\mathbf{\Lambda}_{t,x}$ (with volume $2tx$). We then have that

$$\mathbb{P}\big(\exists \sigma \in \mathfrak{S}_k \text{ s.t. } (Z_{\sigma(1)}, \dots, Z_{\sigma(k)}) \in \mathcal{E}_k^{(t,B)}\big) = k! \frac{\mathrm{Vol}(\mathcal{E}_k^{(t,B)})}{(2tx)^k}.$$

We therefore obtain, using that $\binom{m}{k} \leq m^k/k!$, together with Lemma 3.1

$$\mathbb{P}\big(\mathcal{L}_m^{(B)}(t,x) \geq k\big) \leq \left(\frac{CB^{1/2}t^{1/2}m}{2xk^2}\right)^k. \tag{3.2}$$

This gives the upper bound of Theorem 2.1(i).

Discrete setting: *Upper bound*. The proof follows the same idea as above: we skip most of the details. Define $E_k^{(n,B)}$ the set of k-tuples in $[\![1,n]\!] \times \mathbb{Z}$ that have entropy smaller than B:

$$E_k^{(n,B)} := \left\{ \big((t_\ell, x_\ell)\big)_{1\le \ell\le k} \subset [\![1,n]\!] \times \mathbb{Z};\ \begin{array}{c} 0 < t_1 < \cdots < t_k \le n; \\ \mathrm{Ent}\big((t_\ell, x_\ell)_{1\le\ell\le k}\big) \le B \end{array} \right\}.$$

We can estimate the cardinality of $E_k^{(n,B)}$; however, not in an exact manner as in the continuous case.

LEMMA 3.2. *For any $n \in \mathbb{N}$, it holds true that*

$$\mathrm{Vol}\big(E_k^{(n,B)}\big) \le 2^k C_k \times B^{k/2} n^{3k/2}, \qquad \textit{with } C_k = \frac{\pi^k/\sqrt{2}}{\Gamma(k/2+1)\Gamma(3k/2+1)}.$$

PROOF. The analogous of (3.1) is here

$$\mathrm{Vol}\big(E_k^{(n,B)}\big) = 2 \sum_{i=1}^{n} \sum_{y=0}^{\sqrt{2Bi}} \mathrm{Vol}\big(E_{k-1}^{(n-i, B-x^2/2i)}\big). \tag{3.3}$$

The induction is again straightforward calculations: we can use the computations made in the continuous setting, together with the comparison between finite sums and Riemann integrals, that is,

$$\begin{aligned} &\sum_{i=0}^{n} g(i) \le \int_0^{n+1} g(z)\,\mathrm{d}z \qquad \text{if } g \text{ is increasing,} \\ &\sum_{i=0}^{n} g(i) \le g(0) + \int_0^{n} g(z)\,\mathrm{d}z \qquad \text{if } g \text{ is decreasing.} \end{aligned} \tag{3.4}$$

Details are left to the reader. □

Again, we have $\mathbb{P}(L_m^{(B)}(n,h) \ge k) \le \mathbb{E}[N_k]$, where N_k is the number of sets $\Delta \subset \Upsilon_m \subset \Lambda_{n,h}$ with $|\Delta| = k$, that have entropy at most B. Then

$$\mathbb{E}[N_k] = \binom{m}{k} \mathbb{P}\big(\exists \sigma \in \mathfrak{S}_k \text{ s.t. } (Z_{\sigma(1)}^{(n,h)}, \dots, Z_{\sigma(k)}^{(n,h)}) \in \mathcal{E}_k^{(n,B)}\big),$$

where $(Z_1^{(n,h)}, \dots, Z_k^{(n,h)})$ are a uniform random choice of k distinct points from $\Lambda_{n,h}$ (which contains $n(2h+1)$ points)—the main difference with the continuous setting comes from the fact that the Z_i's are not independent. We therefore have that, using Lemma 3.2,

$$\mathbb{E}[n_k] = \binom{m}{k} \frac{\mathrm{Vol}(E_k^{(n,B)})}{\binom{2nh+n}{k}} \le \frac{m^k}{(2nh)^k} \left(\frac{CB^{1/2}}{k^2}\right)^k.$$

We also used that $\binom{m}{k} \le m^k/k!$ and that $\binom{2nh+n}{k} \ge (2nh+n-k)^k/k!$ with $k \le n$. This concludes the proof of the upper bound in Theorem 2.1(ii).

3.2. *Proof of Corollary* 2.2. We prove it in the continuous setting, the discrete one being similar. From Theorem 2.1, we deduce that for any $u \geq (eC_0)^{1/2}$, we have

$$\mathbb{P}\big(\mathcal{L}_m^{(B)}(t,x) \geq u\big(Bt/x^2\big)^{1/4}\sqrt{m}\big) \leq \exp\big(-u\big(Bt/x^2\big)^{1/4}\sqrt{m}\big). \tag{3.5}$$

Applying this inequality with $u = (eC_0)^{1/2}$, and using also the a priori bound $\mathcal{L}_m^{(B)}(n,h) \leq m$, we get that for any $b > 0$

$$\begin{aligned}
&\mathbb{E}\bigg[\bigg(\frac{\mathcal{L}_m^{(B)}(t,x)}{((Bt/x^2)^{1/4}m^{1/2}) \wedge m}\bigg)^b\bigg] \\
&\quad \leq (eC_0)^{b/2} + \int_{(eC_0)^{b/2}}^{+\infty} \mathbb{P}\bigg(\frac{\mathcal{L}_m^{(B)}(t,x)}{((Bt/x^2)^{1/4}m^{1/2}) \wedge m} > u^{1/b}\bigg)\,du \\
&\quad \leq (eC_0)^{b/2} + cst.
\end{aligned}$$

4. Proof of Theorem 2.4. Let us recall that $\mathcal{P} := \{(w_i, t_i, x_i) : i \geq 1\}$ is a Poisson point process on $[0,\infty) \times [0,1] \times \mathbb{R}$ of intensity $\mu(\mathrm{d}w\,\mathrm{d}t\,\mathrm{d}x) = \frac{\alpha}{2} w^{-\alpha-1} \mathbb{1}_{\{w>0\}}\,\mathrm{d}w\,\mathrm{d}t\,\mathrm{d}x$, as introduced in Section 2.3.

4.1. *Ideas of the proof.* First, we prove that $\mathcal{T}_\beta = +\infty$ when $\alpha \leq 1/2$. Then we prove the scaling relation (2.10), and finally we show the finiteness of the υth moment ($\upsilon < \alpha - 1/2$). We stress that the core of the proof is based on an application of the continuous E-LPP: roughly, the idea of the proof is to decompose the variational problem (2.9) according to the value of the entropy:

$$\mathcal{T}_\beta = \sup_{B \geq 0}\Big\{\beta \sup_{s \in \mathscr{D}, \mathrm{Ent}(s) = B} \pi(s) - B\Big\}. \tag{4.1}$$

Then a simple scaling argument gives that

$$\sup_{s : \mathrm{Ent}(s) \leq B} \pi(s) \overset{(d)}{=} B^{\frac{1}{2\alpha}} \sup_{s : \mathrm{Ent}(s) \leq 1} \pi(s).$$

The E-LPP appears essential to show that the last supremum is finite; see, in particular, Lemma 4.1 below. Then, at a heuristic level, we get that $\mathcal{T}_\beta$ is finite because in (4.1) we have $B^{\frac{1}{2\alpha}} \ll B$ as $B \to \infty$ (remember that $\alpha > 1/2$). In the last part of the proof, we prove the continuity of $\beta \mapsto \mathcal{T}_\beta$ and of the existence and uniqueness of the maximizer in (2.9).

4.2. *Case* $\alpha \leq 1/2$. Let us prove here that $\mathcal{T}_\beta = +\infty$ when $\alpha \in (0, 1/2]$. For any k in $\mathbb{Z}$, we define the event

$$\mathcal{G}_k := \Big\{\mathcal{P} \cap [\beta^{-1}2^{2k+1}, +\infty) \times \Big[\frac{1}{2}, 1\Big] \times [2^{k-1}, 2^k] \neq \varnothing\Big\}.$$

On the event $\mathcal{G}_k$, we denote (w_k, t_k, x_k) a point of $\mathcal{P}$ such that $w_k \geq \beta^{-1} 2^{2k+1}$ and $(t_k, x_k) \in [\frac{1}{2}, 1] \times [2^{k-1}, 2^k]$: considering the path going straight to (t_k, x_k) we get that

$$\mathcal{T}_\beta \geq \beta w_k - \frac{x_k^2}{2t_k} \geq 2^{2k} \qquad \text{on the event } \mathcal{G}_k.$$

Then it is just a matter of estimating $\mathbb{P}(\mathcal{G}_k)$. We stress that considering $\mathcal{M}_k$ the maximal weight in $[\frac{1}{2}, 1] \times [2^{k-1}, 2^k]$, we find that $\mathcal{M}_k$ is of order $(2^k)^{1/\alpha}$ (as a maximum of a field of independent heavy-tail random variables, or using the scaling relations below), so that we get that: if $\alpha < 1/2$, $\mathbb{P}(\mathcal{G}_k) \to 1$ as $k \to +\infty$; if $\alpha = 1/2$, there is a constant $c > 0$ such that $\mathbb{P}(\mathcal{G}_k) \geq c$ for all $k \in \mathbb{Z}$; if $\alpha > 1/2$, $\mathbb{P}(\mathcal{G}_k) \to 1$ as $k \to -\infty$. Note that the events $\mathcal{G}_k$ are independent, so an application of Borel–Cantelli lemma gives that for $\alpha \leq 1/2$, a.s. $\mathcal{G}_k$ occurs for infinitely many $k \in \mathbb{N}$: since $\mathcal{T}_\beta \geq 2^{2k}$ on $\mathcal{G}_k$, it implies that $\mathcal{T}_\beta = +\infty$ a.s. for $\alpha \leq 1/2$.

On the other hand, let us remark that this argument also proves that when $\alpha > 1/2$, a.s. there exists some $k_0 \leq -1$ such that $\mathcal{G}_{k_0}$ occurs, and thus $\mathcal{T}_\beta \geq 2^{2k_0} > 0$.

4.3. *Scaling relations.* For any $\alpha \in (0, 2)$ and $a > 0$, we consider two functions $\varphi(w, t, x) := (w, t, ax)$ and $\psi(w, t, x) := (a^{-1/\alpha} w, t, x)$ which scale space by a (hence the entropy by a^2) and weights by $a^{-1/\alpha}$, respectively. The random sets $\varphi(\mathcal{P})$ and $\psi(\mathcal{P})$ are still two Poisson point processes with the same law, that is, $\varphi(\mathcal{P}) \overset{(\mathrm{d})}{=} \psi(\mathcal{P})$. This implies that (recall the definition (2.8))

$$\pi(as) \overset{(\mathrm{d})}{=} a^{1/\alpha} \pi(s).$$

Therefore,

$$\sup_{s \in \mathscr{D}, \mathrm{Ent}(s) < \infty} \{\beta \pi(s) - a^2 \,\mathrm{Ent}(s)\} \overset{(\mathrm{d})}{=} \sup_{s \in \mathscr{D}, \mathrm{Ent}(s) < \infty} \{\beta a^{-1/\alpha} \pi(s) - \mathrm{Ent}(s)\}. \tag{4.2}$$

Consequently, for any $\alpha \in (0, 2)$, $a^2 \mathcal{T}_{\beta/a^2} \overset{(\mathrm{d})}{=} \mathcal{T}_{\beta a^{-1/\alpha}}$. In particular, for any $\beta > 0$ it holds true that for $\alpha > 1/2$

$$\mathcal{T}_\beta \overset{(\mathrm{d})}{=} \beta^{\frac{2\alpha}{2\alpha - 1}} \mathcal{T}_1. \tag{4.3}$$

Let us note that $\mathcal{T}_1 > 0$ a.s., as we already observed in Section 4.2.

4.4. *Finite moments of $\mathcal{T}_\beta$.* We show that for $\alpha \in (1/2, 2)$ $\mathbb{E}[(\mathcal{T}_\beta)^\upsilon] < \infty$ for any $\upsilon < \alpha - 1/2$, which readily implies that $\mathcal{T}_\beta < \infty$ a.s. For any interval $[c, d)$ with $0 \leq c < d$, we let

$$\mathcal{T}_\beta([c, d)) := \sup_{s \in \mathscr{D}, \mathrm{Ent}(s) \in [c, d)} \{\beta \pi(s) - \mathrm{Ent}(s)\}, \tag{4.4}$$

and we observe that $\mathcal{T}_\beta = \mathcal{T}_\beta([0,1)) \vee \sup_{k\geq 0} \mathcal{T}_\beta([2^k, 2^{k+1}))$. Moreover, as in (4.2) we have

$$\begin{aligned}\mathcal{T}_\beta\big([2^k, 2^{k+1})\big) &\overset{(\mathrm{d})}{=} \sup_{s:\mathrm{Ent}(s)\in[1,2)} \big\{2^{\frac{k}{2\alpha}}\beta\pi(s) - 2^k \,\mathrm{Ent}(s)\big\} \\ &\leq 2^{\frac{k}{2\alpha}}\beta \sup_{s:\mathrm{Ent}(s)\leq 2} \pi(s) - 2^k. \end{aligned} \tag{4.5}$$

We show the following lemma.

LEMMA 4.1. *For any $a < \alpha$, we have that there is a constant $c_a > 0$ such that for any $t > 1$ we get*

$$\mathbb{P}\Big(\sup_{s\in\mathscr{D},\mathrm{Ent}(s)\leq 2} \pi(s) > t\Big) \leq c_a t^{-a}.$$

Note that this bound is sharp: let $\mathcal{M}$ be the largest weight in $[1/2, 1] \times [-1, 1]$, then we have that for $t > 1$, the probability in Lemma 4.1 is bounded below by $\mathbb{P}(\mathcal{M} > t) \geq c_\alpha t^{-\alpha}$. (The last inequality comes from the form of the intensity of the Poisson point process.)

From this lemma and (4.5), we get that for any $t \geq -1$ and any k large enough so that $\beta^{-1}2^{-\frac{k}{2\alpha}}2^{-k} > 2$, we get

$$\begin{aligned}\mathbb{P}\big(\mathcal{T}_\beta\big([2^k, 2^{k+1})\big) > t\big) &\leq \mathbb{P}\Big(\sup_{s\in\mathscr{D},\mathrm{Ent}(s)\leq 2} \pi(s) > \beta^{-1}2^{-\frac{k}{2\alpha}}\big(t + 2^k\big)\Big) \\ &\leq c_a \beta^a 2^{k\frac{a}{2\alpha}}\big(t+2^k\big)^{-a}. \end{aligned} \tag{4.6}$$

Then, for any $t \geq 1$ and $a < \alpha$, we get by a union bound that

$$\begin{aligned}\mathbb{P}(\mathcal{T}_\beta > t) &\leq \sum_{k=0}^{\infty} \mathbb{P}\big(\mathcal{T}_\beta\big([2^k, 2^{k+1})\big) > t\big) \\ &\leq c'_a 2^a \beta^a t^{-a} \sum_{k=0}^{\log_2 t} 2^{k\frac{a}{2\alpha}} + c'_a 2^a \beta^a \sum_{k>\log_2 t} 2^{-ak(1-\frac{1}{2\alpha})} \\ &\leq c''_a \beta^a t^{-a} t^{\frac{a}{2\alpha}} + c''_a t^{-a(1-\frac{1}{2\alpha})} \leq 2c''_a \beta^a t^{-a(1-\frac{1}{2\alpha})}, \end{aligned}$$

where we used that $t + 2^k \geq t/2$ if $k \leq \log_2 t$, and $t + 2^k \geq 2^k/2$ if $k > \log_2 t$. For the second sum, we also used that $1 - \frac{1}{2\alpha} > 0$ when $\alpha > 1/2$. In particular, this shows that for any $\delta > 0$, there is some constant $c_{\delta,\beta} > 0$ such that for any $t \geq 1$

$$\mathbb{P}(\mathcal{T}_\beta > t) \leq c_{\delta,\beta} t^{-(\alpha-\frac{1}{2})+\delta}, \tag{4.7}$$

which proves that $\mathbb{E}[(\mathcal{T}_\beta)^\upsilon] < \infty$ for any $\upsilon < \alpha - 1/2$.

PROOF OF LEMMA 4.1. Recall that $\mathrm{Ent}(s) \le 2$ implies that $\max|s| \le 2$. Therefore we can restrict our Poisson point process to $\mathbb{R}_+ \times [0,1] \times [-2,2]$. In this case (cf. Section 5.1 below), we rewrite a realization of the Poisson point process by using its order statistics. We introduce $(Y_i)_{i\in\mathbb{N}}$ an i.i.d. sequence of uniform random variables on $[0,1]\times[-2,2]$ and $(M_i)_{i\in\mathbb{N}}$ be a random sequence independent of $(Y_i)_{i\in\mathbb{N}}$ defined by $M_i = 4^{1/\alpha}(\mathcal{E}_1 + \cdots + \mathcal{E}_i)^{-1/\alpha}$ with $(\mathcal{E}_j)_{j\ge1}$ an i.i.d. sequence of Exp(1) random variables. In such a way, $\mathcal{P} \stackrel{(d)}{=} (M_i, Y_i)_{i\in\mathbb{N}}$ and $\pi(s) = \sum_{i=1}^\infty M_i \mathbb{1}_{\{Y_i\in s\}}$.

The proof is then a consequence of Theorem 2.1 (with $B=1$), which allows to use the same ideas as in [11], Proposition 3.3. We develop the argument used in [11] in a more robust way, which makes it easier to adapt to the discrete setting. Using the notation introduced in Section 1, for any $i\ge 0$, we denote $\boldsymbol{\Upsilon}_i = \{Y_1,\dots,Y_i\}$ ($\boldsymbol{\Upsilon}_0 = \varnothing$), and let $\Delta_i = \Delta_i(s) = s\cap \boldsymbol{\Upsilon}_i$ be the set of the i largest weights collected by s. The E-LPP can be written here as $\mathcal{L}_i^{(2)} := \max_{s:\mathrm{Ent}(s)\le2}|\Delta_i(s)|$—we drop here the dependence on t, x.

Using that M_i is a nonincreasing sequence, we write

$$\pi(s) = \sum_{j=0}^\infty \sum_{i=2^j}^{2^{j+1}-1} M_i \mathbb{1}_{\{Y_i\in s\}} \le \sum_{j=0}^\infty M_{2^j}\mathcal{L}^{(2)}_{2^{j+1}}. \tag{4.8}$$

Then we fix some $\delta > 0$ such that $1/\alpha - 1/2 > 2\delta$, and we let $C = \sum_{j=0}^\infty 2^{j(1/2-1/\alpha+2\delta)}$: we obtain via a union bound that

$$\begin{aligned}\mathbb{P}\Big(\sup_{\mathrm{Ent}(s)\le2}\pi(s) > t\Big) &\le \sum_{j=0}^\infty \mathbb{P}\Big(M_{2^j}\mathcal{L}^{(2)}_{2^{j+1}} > \frac{1}{C}t2^{j(1/2-1/\alpha+2\delta)}\Big)\\ &\le \sum_{j=0}^\infty \Big[\mathbb{P}\big(\mathcal{L}^{(2)}_{2^{j+1}} > C'\log t\,(2^{j+1})^{1/2+\delta}\big)\\ &\qquad + \mathbb{P}\Big(M_{2^j} > C''\frac{t}{\log t}(2^j)^{-1/\alpha+\delta}\Big)\Big].\end{aligned} \tag{4.9}$$

Here, C' is a constant that we choose large in a moment, and C'' is a constant depending on C, C'; we also work with $t\ge2$ for simplicity.

For the first probability in the sum, we obtain from Theorem 2.1(i) (with $m = 2^{j+1}$ and $k = C'\log t\,(2^{j+1})^{\frac12+\delta}$) that provided $C'(\log t)2^{j\delta} \ge 2C_0^{1/2}$,

$$\mathbb{P}\big(\mathcal{L}^{(2)}_{2^{j+1}} > C'\log t\,(2^{j+1})^{1/2+\delta}\big) \le \Big(\frac12\Big)^{C'(\log t)2^{j\delta}} \le t^{-\log 2C'2^{j\delta}}.$$

Hence, for t sufficiently large we get that

$$\sum_{j=0}^\infty \mathbb{P}\big(\mathcal{L}^{(2)}_{2^{j+1}} > C'\log t\,(2^{j+1})^{1/2+\delta}\big) \le ct^{-C'\log 2} \le ct^{-a} \tag{4.10}$$

provided that we fixed C' large.

For the second probability in the sum, recall that $M_i \stackrel{(d)}{=} 4^{1/\alpha}\mathrm{Gamma}(i)^{-1/\alpha}$, so that for any $a < \alpha$, $\mathbb{E}[(i^{1/\alpha}M_i)^a]$ is bounded by a constant independent of i. Therefore, Markov's inequality gives that

$$\mathbb{P}\bigg(M_{2^j} > C''\frac{t}{\log t}(2^j)^{-1/\alpha+\delta}\bigg) \le c(\log t)^a t^{-a}(2^j)^{-a\delta},$$

so that

$$\sum_{j=0}^{\infty}\mathbb{P}\bigg(M_{2^j} > C''\frac{t}{\log t}(2^j)^{-1/\alpha+\delta}\bigg) \le c(\log t)^a t^{-a}. \tag{4.11}$$

Plugging (4.10) and (4.11) into (4.9), we obtain that for any $a' < a < \alpha$ there are constants $c > 0$ such that for any $t \ge 2$

$$\mathbb{P}\Big(\sup_{\mathrm{Ent}(s)\le 2}\pi(s) > t\Big) \le 2c(\log t)^a t^{-a} \le c' t^{-a'},$$

which completes the proof. □

4.5. *Continuity of $\beta \mapsto \mathcal{T}_\beta$.* An obvious and crucial fact that we use along the way is that for any realization of $\mathcal{P}$, $\beta \mapsto \mathcal{T}_\beta$ is nondecreasing.

4.5.1. *Left continuity.* Let us first show that $\beta \mapsto \mathcal{T}_\beta$ is left continuous, since it is less technical. Fix $\varepsilon > 0$. For any β, there exists a path $s_\beta^{(\varepsilon)}$ with $\pi(s_\beta^{(\varepsilon)}) < \infty$ such that $\mathcal{T}_\beta \le \beta\pi(s_\beta^{(\varepsilon)}) - \mathrm{Ent}(s_\beta^{(\varepsilon)}) + \varepsilon$. Using this path $s_\beta^{(\varepsilon)}$, we then simply write that for any $\delta > 0$

$$\mathcal{T}_\beta \ge \mathcal{T}_{\beta-\delta} \ge (\beta-\delta)\pi\big(s_\beta^{(\varepsilon)}\big) - \mathrm{Ent}\big(s_\beta^{(\varepsilon)}\big).$$

Letting $\delta \downarrow 0$, we get that the right-hand side converges to $\beta\pi(s_\beta^{(\varepsilon)}) - \mathrm{Ent}(s_\beta^{(\varepsilon)}) \ge \mathcal{T}_\beta - \varepsilon$. Since ε is arbitrary, one concludes that $\lim_{\delta\uparrow 0}\mathcal{T}_{\beta-\delta} = \mathcal{T}_\beta$, that is $\beta \mapsto \mathcal{T}_\beta$ is left continuous.

4.5.2. *Right continuity.* It remains to prove that a.s. $\beta \mapsto \mathcal{T}_\beta$ is right continuous. We prove a preliminary result.

LEMMA 4.2. *For any $K > 0$, $\mathbb{P}$-a.s. there exists $B_0 > 0$ such that for any $0 \le \beta \le K$,*

$$\mathcal{T}_\beta = \mathcal{T}_\beta\big([0, B_0]\big), \tag{4.12}$$

where $\mathcal{T}_\beta([0, B_0])$ is defined in (4.4).

PROOF. Let us recall that $\mathcal{T}_\beta = \mathcal{T}_\beta([0,1)) \vee \sup_{k\geq 0} \mathcal{T}_\beta([2^k, 2^{k+1}))$. Using (4.6) with $t=-1$, for any $a<\alpha$ we have that

$$\mathbb{P}\big(\mathcal{T}_\beta\big([2^k,2^{k+1})\big) > -1\big) \leq c_a \beta^a 2^{k\frac{a}{2\alpha}} \big(2^k-1\big)^{-a} \leq c_{a,K} 2^{ka(\frac{1}{2\alpha}-1)}.$$

Since $\frac{1}{2\alpha}-1<0$, by Borel–Cantelli lemma we obtain that $\mathbb{P}$-a.s. there exists $k_0>0$ such that $\mathcal{T}_\beta([2^k,2^{k+1})) \leq -1$ for all $k \geq k_0$. This concludes the proof. □

Then, since we now consider paths with entropy bounded by B_0, we can restrict the Poisson point process $\mathcal{P}$ to $\mathbb{R}_+ \times [0,1] \times [-\sqrt{2B_0}, \sqrt{2B_0}]$. In this case, we write a realization of the Poisson point process by using its order statistics. More precisely we introduce $M_i := (8B_0)^{1/2\alpha}(\mathcal{E}_1 + \cdots + \mathcal{E}_i)^{-1/\alpha}$, where $(\mathcal{E}_i)_{i\in\mathbb{N}}$ is an i.i.d. sequence of exponential of mean 1 and $(Y_i)_{i\in\mathbb{N}}$ is an i.i.d. sequence of uniform random variables on $[0,1]\times[-\sqrt{2B_0}, \sqrt{2B_0}]$, independent of $(\mathcal{E}_i)_{i\in\mathbb{N}}$. Then $\mathcal{P} \overset{(d)}{=} (M_i, Y_i)_{i\in\mathbb{N}}$ and $\pi(s) = \sum_{i=1}^{\infty} M_i \mathbb{1}_{\{Y_i \in s\}}$.

For any $\ell \in \mathbb{N}$, we let $\pi^{(\ell)} := \sum_{i=1}^{\ell} M_i \mathbb{1}_{Y_i\in s}$ be the "truncated" energy of a path: we can write for any $\beta<K$, and any $\delta>0$ such that $\beta+\delta \leq K$

$$\begin{aligned}\mathcal{T}_{\beta+\delta} &= \mathcal{T}_{\beta+\delta}\big([0,B_0]\big)\\ &\leq \sup_{s\in\mathscr{D}, \mathrm{Ent}(s)\leq B_0} \big\{(\beta+\delta)\pi^{(\ell)}(s) - \mathrm{Ent}(s)\big\}\\ &\quad + (\beta+\delta) \sup_{s\in\mathscr{D}, \mathrm{Ent}(s)\leq B_0} \big|\pi(s) - \pi^{(\ell)}(s)\big|.\end{aligned}$$

We control the last term with the following lemma.

LEMMA 4.3. *It holds that*

$$\max_{s\in\mathscr{D}, \mathrm{Ent}(s)\leq B_0} \big|\pi(s)-\pi^{(\ell)}(s)\big| \xrightarrow[\ell\to\infty]{\text{a.s.}} 0.$$

Hence, for any fixed ε, we can a.s. choose some ℓ_ε such that for any $\beta<K$ and any $\delta>0$ with $\beta+\delta\leq K$,

$$\mathcal{T}_\beta \leq \mathcal{T}_{\beta+\delta} \leq \sup_{s\in\mathscr{D}, \mathrm{Ent}(s)\leq B_0} \big\{(\beta+\delta)\pi^{(\ell)}(s) - \mathrm{Ent}(s)\big\} + K\varepsilon.$$

Then, letting $\delta \downarrow 0$, and since the supremum can now be reduced to a finite set (we consider only ℓ points), we get that for any $\beta<K$,

$$\mathcal{T}_\beta \leq \lim_{\delta\downarrow 0} \mathcal{T}_{\beta+\delta} \leq \sup_{s\in\mathscr{D}, \mathrm{Ent}(s)\leq B_0} \big\{\beta\pi^{(\ell)}(s) - \mathrm{Ent}(s)\big\} + \varepsilon \leq \mathcal{T}_\beta + \varepsilon.$$

Since ε is arbitrary, this shows that $\lim_{\delta\downarrow 0}\mathcal{T}_{\beta+\delta} = \mathcal{T}_\beta$ a.s., that is, $\beta \mapsto \mathcal{T}_\beta$ is right continuous.

PROOF OF LEMMA 4.3. For any $i\in\mathbb{N}$, we consider $\boldsymbol{\Upsilon}_i = \{Y_1,\dots,Y_i\}$ and for any given path s we define $\Delta_i = \Delta_i(s) = s \cap \boldsymbol{\Upsilon}_i$ the set of the i largest weights collected by s. Then let $\mathcal{L}_i^{(B_0)} = \sup_{s\in\mathscr{D}_{B_0}} |\Delta_i(s)|$, as introduced in (2.2). Realizing that $\mathbb{1}_{\{Y_i\in s\}} = |\Delta_i(s)| - |\Delta_{i-1}(s)|$, and integrating by parts (as done in [11]), we obtain for any $s \in \mathscr{D}_{B_0}$,

$$\begin{aligned}\pi(s) - \pi^{(\ell)}(s) &= \sum_{i>\ell} M_i \mathbb{1}_{\{Y_i\in s\}} \\ &= \lim_{n\to\infty} \sum_{i=\ell+1}^{n} M_i\big(|\Delta_i| - |\Delta_{i-1}|\big) \\ &= \lim_{n\to\infty}\left(\sum_{i=\ell+1}^{n-1} |\Delta_i|(M_i - M_{i+1}) + M_n|\Delta_n| - M_\ell|\Delta_\ell|\right) \\ &\le \sum_{i=\ell+1}^{\infty} \mathcal{L}_i^{(B_0)}(M_i - M_{i+1}) + \limsup_{n\to\infty} M_n \mathcal{L}_n^{(B_0)}.\end{aligned} \tag{4.13}$$

At this stage, the law of large numbers gives that $\lim_{n\to\infty} n^{1/\alpha} M_n = (8B_0)^{1/2\alpha}$ a.s., and Corollary 2.2 gives that $\limsup_{n\to\infty} n^{-1/2}\mathcal{L}_n^{(B_0)} < +\infty$ a.s. Since $\alpha < 2$, we therefore conclude that $\limsup_{n\to\infty} M_n\mathcal{L}_n^{(B_0)} = 0$ a.s.

We let $U_\ell := \sum_{i>\ell} \mathcal{L}_i^{(B_0)}(M_i - M_{i-1})$. We are going to show that there exists some ℓ_0 such that $\sum_{i>\ell_0} \mathcal{L}_i^{(B)}(M_i - M_{i-1}) < \infty$ a.s., and thus $\lim_{\ell\to\infty} U_\ell = 0$ a.s. We show that $\mathbb{E}[U_{\ell_0}^2]$ is finite for ℓ_0 large enough. For any $\varepsilon > 0$, by Cauchy–Schwarz inequality we have that

$$U_{\ell_0} \le \left(\sum_{i>\ell_0} (i^{-\frac{1}{2}-\varepsilon})^2\right)^{1/2}\left(\sum_{i>\ell_0} \big(i^{-\frac{1}{2}+\varepsilon}\mathcal{L}_i^{(B_0)}(M_i - M_{i+1})\big)^2\right)^{1/2}.$$

Then we get that for ℓ_0 large enough

$$\begin{aligned}\mathbb{E}[U_{\ell_0}^2] &\le C\sum_{i>\ell_0} i^{1+2\varepsilon}\mathbb{E}[(\mathcal{L}_i^{(B)})^2]\mathbb{E}[(M_i - M_{i-1})^2] \\ &\le C'_{B_0}\sum_{i>\ell_0} i^{1+2\varepsilon} \times i \times i^{-2-2/\alpha} < +\infty.\end{aligned}$$

Here, we used Corollary 2.2 and a straightforward calculation that gives $\mathbb{E}[(M_i - M_{i-1})^2] \le ci^{-2-2/\alpha}$ for i large enough (see, for instance, equation (7.2) in [11]). Provided ε is small enough so that $2\varepsilon - 2/\alpha < -1$, we obtain that $\mathbb{E}[U_{\ell_0}^2] < \infty$. □

4.6. *Existence and uniqueness of the maximizer.* As a consequence of Lemma 4.2, to show that the supremum is attained and is unique in (2.9), it is enough to prove the following result.

LEMMA 4.4. *For a.e. realization of $\mathcal{P}$ and for any $B > 0$, we have that*

$$s^*_\beta(B) = \arg\max_{s \in \mathscr{D}_B} \{\beta\pi(s) - \mathrm{Ent}(s)\}$$

exists, and it is unique. Here, we defined $\mathscr{D}_B := \{s \in \mathscr{D} :\ \mathrm{Ent}(s) \le B\}$.

PROOF. Our first step is to show that $\mathscr{D}_B$ is compact for the uniform norm $\|\cdot\|_\infty$. Let us observe that for any $s : [0, 1] \to \mathbb{R}$, the condition $\mathrm{Ent}(s) \le B$ implies that

$$|s(x) - s(y)| \le \int_y^x |s'(t)|\, dt \le (2B)^{1/2}|x - y|^{1/2}, \qquad \forall x, y \in [0, 1],$$

so that s belongs to the Hölder Space $C^{1/2}([0, 1])$. Hence, $\mathscr{D}_B$ is included in $C^{1/2}([0, 1])$ which is compact for the uniform norm $\|\cdot\|_\infty$ by the Ascoli–Arzelà theorem. We therefore only need to show that $\mathscr{D}_B$ is closed for the uniform norm $\|\cdot\|_\infty$.

For this purpose, we consider a convergent sequence s_n and we denote by $\mathbf{s}$ its limit. Since $\mathrm{Ent}(s_n) = \frac{1}{2}\|s'_n\|^2_{L^2}$ for all n, we have that $(s'_n)_{n\in\mathbb{N}}$ belongs to the (closed) ball of radius $(2B)^{1/2}$ of $L^2([0, 1])$. By Banach–Alaoglu theorem, the sequence $(s'_n)_{n\in\mathbb{N}}$ contains a weakly convergent subsequence. This means that there exist n_k and $s^\star$ such that

$$\int_0^1 \varphi(x)s'_{n_k}(x)\, \mathrm{d}x \xrightarrow[k\to\infty]{} \int_0^1 \varphi(s)s^\star(x)\, dx \qquad \forall\varphi \in L^2([0, 1]).$$

By uniqueness of the limit (and taking $\varphi(x) = \mathbb{1}_{\{[0,y]\}}(x)$), this relation implies that $\mathbf{s}(y) = \int_0^y s^\star(x)\, dx$, that is $\mathbf{s}' = s^\star$ almost everywhere. Since the L^2 norm is weakly lower semicontinuous by the Hahn–Banach theorem that is, $\|s^\star\|_{L^2} \le \liminf_{k\to\infty} \|s'_{n_k}\|_{L^2}$, we obtain that $\mathbf{s} \in \mathscr{D}_B$, so $\mathscr{D}_B$ is closed. As a by-product of this argument, we also have that the entropy function $s \mapsto \mathrm{Ent}(s)$ is lower semicontinuous on $(\mathscr{D}_B, \|\cdot\|_\infty)$.

Existence of the maximizer. Since $\mathscr{D}_B$ is compact, the existence of the maximizer comes from the fact that the function

$$t_\beta(s) := \beta\pi(s) - \mathrm{Ent}(s) \tag{4.14}$$

is upper semicontinuous, thanks to the extreme value theorem tells. Since we have already shown that $s \mapsto \mathrm{Ent}(s)$ is lower semicontinuous, we only need to prove the following.

LEMMA 4.5. *For a.e. realization of $\mathcal{P}$ and for any $B > 0$, the function $s \mapsto \pi(s)$ is upper semicontinuous on $(\mathscr{D}_B, \|\cdot\|_\infty)$.*

PROOF. We recall that if $s \in \mathscr{D}_B$ then $\max_{t\in[0,1]} |s(t)| \le \sqrt{2B}$. Therefore, using the same notation as above, we can write a realization of the Poisson point process $\mathcal{P}$ by using its order statistics: $\mathcal{P} = (M_i, Y_i)_{i\in\mathbb{N}}$, $\pi(s) = \sum_{i=1}^{\infty} M_i \mathbb{1}_{\{Y_i \in s\}}$, and recall that for any $\ell \in \mathbb{N}$ we let $\pi^{(\ell)} := \sum_{i=1}^{\ell} M_i \mathbb{1}_{\{Y_i \in s\}}$. Thanks to (4.3), we only need to prove that for any fixed $\ell \in \mathbb{N}$ the function $s \mapsto \pi^{(\ell)}(s)$ is upper semicontinuous: then $\pi(s)$, as the uniform limit of $\pi^{(\ell)}$, is still upper semicontinuous.

For any $s \in \mathscr{D}_B$, we let $\iota_s := \boldsymbol{\Upsilon}_\ell \setminus \{s \cap \boldsymbol{\Upsilon}_\ell\}$ be the set of all points of $\boldsymbol{\Upsilon}_\ell = \{Y_1, \dots, Y_\ell\}$ that are not in s. Since there are finitely many points, we realize that there exists $\eta = \eta(s, \ell) > 0$ such that $d_H(\iota_s, \mathrm{graph}(s)) > \eta$, with d_H is the Hausdorff distance.

Given $s \in \mathscr{D}_B$, we consider a sequence $(s_n)_n, s_n \in \mathscr{D}_B$ that converges to s, $\lim_{n\to\infty} \|s_n - s\|_\infty = 0$. We observe that whenever $\|s_n - s\|_\infty \le \eta/2$, we have that $d_H(\iota_s, \mathrm{graph}(s_n)) > \eta/2$. This means that for n large enough

$$\{s_n \cap \boldsymbol{\Upsilon}_\ell\} \subset \{s \cap \boldsymbol{\Upsilon}_\ell\},$$

which implies that $\pi^{(\ell)}(s) \ge \limsup_{n\to\infty} \pi^{(\ell)}(s_n)$. □

Uniquenes of the maximizer. The strategy is very similar to the one used in [2], Lemma 4.1 or [11], Lemma 4.2. For any $s \in \mathscr{D}_B$, we let $I(s) := \{s \cap \boldsymbol{\Upsilon}_\infty\}$, where we $\boldsymbol{\Upsilon}_\infty = \{Y_i, i \in \mathbb{N}\}$.

Let us assume that we have two maximizers $s_1 \ne s_2$. Since $\boldsymbol{\Upsilon}_\infty$ is dense in $[0, 1] \times [-\sqrt{2B}, \sqrt{2B}]$, we have that $I(s_1) \ne I(s_2)$. In particular, there exists i_0 such that $Y_{i_0} \in I(s_1)$ and $Y_{i_0} \notin I(s_2)$, and since s_1 and s_2 are two maximizers of (4.14) it means

$$\max_{s\,:\,Y_{i_0}\in I(s)} t_\beta(s) = \max_{s\,:\,Y_{i_0}\notin I(s)} t_\beta(s).$$

This implies that

$$\beta M_{i_0} = \max_{s\,:\,Y_{i_0}\notin I(s)} t_\beta(s) - \max_{s\,:\,Y_{i_0}\in I(s)} \left\{ \beta \sum_{j, j\ne i_0} M_j \mathbb{1}_{\{Y_j\in s\}} - \mathrm{Ent}(s) \right\}. \tag{4.15}$$

Conditioning on $(Y_j)_{j\in\mathbb{N}}$ and $(M_j)_{j\in\mathbb{N}, j\ne i_0}$ we have that the left-hand side has a continuous distribution—the distribution of $M_{i_0}^{-\alpha}$ conditional on $(Y_j)_{j\in\mathbb{N}}$ and $(M_j)_{j\in\mathbb{N}, j\ne i_0}$ is uniform on the interval $[M_{i_0-1}^{-\alpha}, M_{i_0+1}^{-\alpha}]$ –, while the right-hand side is a constant—it is independent of M_{i_0}. Therefore, the event (4.15) has zero probability, and by sigma subadditivity we get that $\mathbb{P}(I(s_1) \ne I(s_2)) = 0$, which contradicts the existence of two distinct maximizers. □

5. Proof of Proposition 2.6 and Theorem 2.7. Let us state right away a lemma that will prove to be useful in the rest of the paper.

LEMMA 5.1. *For any $\eta > 0$, there exists a constant c such that, for any $t > 1$ and any $\ell \le nh$, we have*

$$\mathbb{P}\Big(M_\ell^{(n,h)} > tm\Big(\frac{nh}{\ell}\Big)\Big) \le (ct)^{-(1-\eta)\alpha\ell}.$$

PROOF. We simply write that by a union bound

$$\mathbb{P}\Big(M_\ell^{(n,h)} > tm\Big(\frac{nh}{\ell}\Big)\Big) \le \binom{nh}{\ell}\mathbb{P}\Big(\omega_1 > tm\Big(\frac{nh}{\ell}\Big)\Big)^\ell$$
$$\le \Big(c\frac{nh}{\ell}\mathbb{P}\Big(\omega_1 > tm\Big(\frac{nh}{\ell}\Big)\Big)\Big)^\ell.$$

Then, since $\mathbb{P}(\omega_1 > x)$ is regularly varying with exponent $-\alpha$, Potter's bound (cf. [6]) gives that there is a constant c_η such that for any $t \ge 1$

$$\mathbb{P}\Big(\omega_1 > tm\Big(\frac{nh}{\ell}\Big)\Big) \le c_\eta t^{-(1-\eta)\alpha}\mathbb{P}\Big(\omega_1 > m\Big(\frac{nh}{\ell}\Big)\Big) = c_\eta t^{-(1-\eta)\alpha}\frac{nh}{\ell},$$

where we used the definition of $m(\cdot)$ in the last identity. This completes the proof. □

5.1. *Continuum limit of the order statistics.* For any $q > 0$, let $\Lambda_{n,qh} = [\![1, n]\!] \times [\![-qh, qh]\!]$ and let $(M_r^{(n,qh)}, Y_r^{(n,qh)})_{r=1}^{|\Lambda_{n,qh}|}$ be the order statistics in that box; cf. (2.13). If we rescale $\Lambda_{n,qh}$ by $n \times h$, and we let $(\widetilde{Y}_r^{(n,qh)})_{r=1}^{|\Lambda_{n,h}|}$ be the rescaled permutation, that is, a random permutation of the points of the set $([0, 1] \times [-q, q]) \cap (\frac{\mathbb{N}}{n} \times \frac{\mathbb{Z}}{h})$. Then for any fixed $\ell \in \mathbb{N}$,

$$\big(\widetilde{Y}_1^{(n,qh)}, \dots, \widetilde{Y}_\ell^{(n,qh)}\big) \stackrel{(d)}{\to} (Y_1, \dots, Y_\ell), \qquad \text{as } n, h \to \infty, \tag{5.1}$$

where $(Y_i)_{i\in\mathbb{N}}$ is an i.i.d. sequence of uniform random variables on $[0, 1]\times[-q, q]$. For the continuum limit for the weight sequence $(M_r^{(n,qh)})_{r=1}^{|\Lambda_{n,qh}|}$, we use some basic facts of the classical extreme value theory (see, e.g., [14]), that is for all $\ell \in \mathbb{N}$,

$$\Big(\widetilde{M}_i^{(n,qh)} := \frac{M_i^{(n,qh)}}{m(nh)}, i = 1, \dots, \ell\Big) \stackrel{(d)}{\to} (M_i, i = 1, \dots, \ell), \tag{5.2}$$

where $(M_i)_{i\in\mathbb{N}}$ is the *continuum weight sequence*. The sequence $(M_i)_{i\ge1}$ can be defined as $M_i := (2q)^{1/\alpha}(\mathcal{E}_1 + \cdots + \mathcal{E}_i)^{-\frac{1}{\alpha}}$, where $(\mathcal{E}_i)_{i\in\mathbb{N}}$ is an i.i.d. sequence of exponential random variables of mean 1, independent of the Y_i's.

In such a way $(M_i, Y_i)_{i\in\mathbb{N}}$ is the order statistics associated with a realization of a Poisson point process on $[0, \infty) \times [0, 1] \times [-q, q]$ of intensity $\mu(\mathrm{d}w, \mathrm{d}t, \mathrm{d}x) = \frac{\alpha}{2}w^{-\alpha-1}\mathbb{1}_{\{w>0\}}\,\mathrm{d}w\,\mathrm{d}t\,\mathrm{d}x$.

PROOF OF (2.21). This is a simple consequence of the monotonicity of $\ell \mapsto \mathcal{T}_{\nu,q}^{(\ell)}$ and of $q \mapsto \mathcal{T}_{\nu,q}$ (together with the fact that $\mathcal{T}_\nu$ is well defined). □

5.2. *Proof of Proposition* 2.6. Let us first focus on $T_{n,h}^{\beta_{n,h},(\ell)}$. As in (4.4) in the continuous setting, we introduce, for any interval $[c,d)$,

$$T_{n,h}^{\beta_{n,h},(\ell)}([c,d)) := \max_{\Delta\subset\Lambda_{n,h},\mathrm{Ent}(\Delta)\in[c,d)} \{\beta_{n,h}\Omega_{n,h}^{(\ell)}(\Delta) - \mathrm{Ent}(\Delta)\}. \tag{5.3}$$

Then we realize that for any $d > 0$,

$$T_{n,h}^{\beta_{n,h},(\ell)} = T_{n,h}^{\beta_{n,h},(\ell)}([0,d)) \vee \sup_{k\geq 1} T_{n,h}^{\beta_{n,h},(\ell)}([2^{k-1}d, 2^k d)).$$

Using that

$$T_{n,h}^{\beta_{n,h},(\ell)}([2^{k-1}d, 2^k d)) \leq \beta_{n,h} \sup_{\Delta:\mathrm{Ent}(\Delta)\leq 2^k d} \Omega_{n,h}^{(\ell)}(\Delta) - 2^{k-1}d, \qquad \text{for } k \geq 1,$$

$$T_{n,h}^{\beta_{n,h},(\ell)}([0,d)) \leq \beta_{n,h} \sup_{\Delta:\mathrm{Ent}(\Delta)\leq d} \Omega_{n,h}^{(\ell)}(\Delta),$$

with the choice $d = b\widehat{\beta}$ and $\widehat{\beta} := (\beta_{n,h}m(nh))^{4/3}(n/h^2)^{1/3}$, a union bound gives that

$$\begin{aligned}\mathbb{P}(T_{n,h}^{\beta_{n,h},(\ell)} \geq b\widehat{\beta}) &\leq \sum_{k\geq 0} \mathbb{P}\Big(\beta_{n,h} \sup_{\Delta:\mathrm{Ent}(\Delta)\leq 2^k b\widehat{\beta}} \Omega_{n,h}^{(\ell)}(\Delta) \geq 2^{k-1}b\widehat{\beta}\Big) \\ &\leq \sum_{k\geq 0} \mathbb{P}\Big(\sup_{\Delta:\mathrm{Ent}(\Delta)\leq 2^k b\widehat{\beta}} \Omega_{n,h}^{(\ell)}(\Delta) \geq 2^{k-1}bm(nh)\big(\widehat{\beta}n/h^2\big)^{1/4}\Big),\end{aligned} \tag{5.4}$$

where we use that $\widehat{\beta}$ satisfies the equation $\widehat{\beta} = \beta_{n,h}m(nh)(\widehat{\beta}n/h^2)^{1/4}$.

We then need the following lemma, analogous to Lemma 4.1.

LEMMA 5.2. *For any $a < \alpha$, there exists a constant c such that for any $B \geq 1$, $n, h \geq 1$ and any $t > 1$,*

$$\mathbb{P}\Big(\sup_{\Delta:\mathrm{Ent}(\Delta)\leq B} \Omega_{n,h}^{(\ell)}(\Delta) \geq t \times m(nh)\big(Bn/h^2\big)^{1/4}\Big) \leq ct^{-a}.$$

Applying this lemma in (5.4) (with $B = 2^k b\widehat{\beta}$, $t = 2^{3k/4-1}b^{3/4}$), we get that for any $k \geq 0$,

$$\mathbb{P}\Big(\sup_{\Delta:\mathrm{Ent}(\Delta)\leq 2^k b\widehat{\beta}} \Omega_{n,h}^{(\ell)}(\Delta) \geq 2^{k-1}bm(nh)\big(\widehat{\beta}n/h^2\big)^{1/4}\Big) \leq c\big(2^k b\big)^{-3a/4},$$

so that summing over k in (5.4), we get Proposition 2.6.

PROOF OF LEMMA 5.2. We mimic here the proof of Lemma 4.1, but we need to keep the dependence on the parameters n, h, B. For $i \geq 0$, we denote $\Upsilon_i := \{Y_1^{(n,h)}, \dots, Y_i^{(n,h)}\}$ with the $Y_j^{(n,h)}$ introduced in Section 2.4 ($\Upsilon_0 = \varnothing$), and for any Δ we let $\Delta_i := \Delta \cap \Upsilon_i$ be the restriction of Δ to the i largest weights. As in (4.8), we can write

$$(5.5)\qquad \frac{1}{m(nh)(Bn/h^2)^{1/4}} \times \sup_{\Delta:\mathrm{Ent}(\Delta)\le B} \Omega_{n,h}^{(\ell)}(\Delta) \le \sum_{j=0}^{\log_2 \ell} \widetilde{M}_{2^j} \widetilde{L}_{2^{j+1}},$$

where $\widetilde{M}_i = M_i^{(n,h)}/m(nh)$ and $\widetilde{L}_i = L_i^{(B)}(n,h)/(Bn/h^2)^{1/4}$ are the rescaled weights and E-LPP (we drop the dependence on n, h, B for notational convenience).

As in the proof of Lemma 4.1, we fix some $\delta > 0$ such that $1/\alpha - 1/2 > 2\delta$, and as for (4.9), the probability in Lemma 5.2 is bounded by

$$(5.6)\quad \sum_{j=0}^{\log_2 \ell} \Big[\mathbb{P}\big(\widetilde{L}_{2^{j+1}} > C' \log t \big(2^{j+1}\big)^{1/2+\delta}\big) + \mathbb{P}\Big(\widetilde{M}_{2^j} > C'' \frac{t}{\log t} \big(2^j\big)^{-1/\alpha+\delta}\Big)\Big].$$

For the first probability in the sum, we obtain from Theorem 2.1(ii) that provided that $C'(\log t) 2^{j\delta} \ge 2C_0^{1/2}$,

$$(5.7)\qquad \mathbb{P}\big(\widetilde{L}_{2^{j+1}} > C' \log t \big(2^{j+1}\big)^{1/2+\delta}\big) \le \Big(\frac{1}{2}\Big)^{C'(\log t) 2^{j\delta}} \le t^{-(\log 2) C' 2^{j\delta}}.$$

Then the first sum in (5.6) is bounded by t^{-a} provided that C' had been fixed large enough.

For the second probability in (5.6), we use Lemma 5.1 above to get that for any $a < \alpha$,

$$\begin{aligned} \mathbb{P}\Big(\widetilde{M}_{2^j} > C'' \frac{t}{\log t} \big(2^j\big)^{-1/\alpha+\delta}\Big) &\le \mathbb{P}\Big(M_{2^j}^{(n,h)} > C''' \frac{t}{\log t} \big(2^j\big)^{\delta/2} m\big(nh2^{-j}\big)\Big) \\ &\le c(\log t)^a t^{-a} \big(2^j\big)^{-a\delta}. \end{aligned} \qquad (5.8)$$

For the first inequality, we used Potter's bound to get that $m(nh2^{-j}) \le cm(nh)(2^j)^{-1/\alpha+\delta/2}$. We conclude that the second sum in (5.6) is bounded by a constant times $(\log t)^a t^{-a}$.

All together, and possibly decreasing the value a a (by an arbitrarily small anount), this yields Lemma 5.2. $\square$

Let us now turn to the case of $T_{n,h}^{\beta_{n,h},(>\ell)}$. We first need an analogue of Lemma 5.2.

LEMMA 5.3. *There exists a constant c such that for any $B \ge 1$, $n, h \in \mathbb{N}$ and $0 \le \ell \le nh$, for any $t > 1$,*

$$\mathbb{P}\Big(\sup_{\Delta:\mathrm{Ent}(\Delta)\le B} \Omega_{n,h}^{(>\ell)}(\Delta) \ge t \times m(nh/\ell)\ell^{1/2}\big(Bn/h^2\big)^{1/4}\Big) \le ct^{-\alpha\ell/3} + e^{-c\sqrt{t}}.$$

PROOF. Analogously to (5.5), we get that

$$\frac{1}{m(nh/\ell)\ell^{1/2}(Bn/h^2)^{1/4}} \times \sup_{\Delta:\mathrm{Ent}(\Delta)\le B} \Omega_{n,h}^{(>\ell)}(\Delta)$$
$$\le \sum_{j=0}^{\log_2(nh/\ell)} \frac{M_{2^j\ell}^{(n,h)}}{m(nh/\ell)} \frac{L_{2^{j+1}\ell}^{(n,h)}}{\ell^{1/2}(Bn/h^2)^{1/4}}. \tag{5.9}$$

Then we get similar to (5.7)–(5.8) that for any $\delta > 0$: (a) thanks to Theorem 2.1(ii), we have

$$\mathbb{P}\left(\frac{L_{2^{j+1}\ell}^{(n,h)}}{\ell^{1/2}(Bn/h^2)^{1/4}} \ge C'\sqrt{t}(2^{j+1})^{1/2+\delta}\right) \le \left(\frac{1}{2}\right)^{C'\sqrt{t}2^{j\delta}} \le e^{-c\sqrt{t}2^{\delta j}}; \tag{5.10}$$

(b) thanks to Lemma 5.1, we have

$$\mathbb{P}\left(\frac{M_{2^j\ell}^{(n,h)}}{m(nh/\ell)} \ge C''\sqrt{t}(2^j)^{-1/\alpha+\delta}\right) \le ct^{-\alpha\ell/3}(2^j)^{-\alpha\delta\ell/2}. \tag{5.11}$$

Lemma 5.3 follows from a bound analogous to (5.6). □

Then, setting $\widehat{\beta}_\ell = (\beta_{nh}m(nh/\ell))^{4/3}(\ell^2 n/h^2))^{\frac{1}{3}}$ so that we have $\widehat{\beta}_\ell = \beta_{n,h}m(nh/\ell)\ell^{1/2}(\widehat{\beta}_\ell n/h^2)^{1/4}$, we obtain similar to (5.4) that

$$\begin{aligned}
&\mathbb{P}\big(T_{n,h}^{\beta_{n,h},(>\ell)} \ge b \times \widehat{\beta}_\ell\big) \\
&\quad \le \sum_{k\ge 0} \mathbb{P}\Big(\beta_{n,h} \sup_{\Delta:\mathrm{Ent}(\Delta)\le 2^k b\widehat{\beta}_\ell} \Omega_{n,h}^{(>\ell)}(\Delta) \ge 2^{k-1} b\widehat{\beta}_\ell\Big) \\
&\quad \le \sum_{k\ge 0} \mathbb{P}\Big(\sup_{\Delta:\mathrm{Ent}(\Delta)\le 2^k b\widehat{\beta}_\ell} \Omega_{n,h}^{(>\ell)}(\Delta) \ge 2^{k-1} bm(nh/\ell)\big(\ell^2\widehat{\beta}_\ell n/h^2\big)^{1/4}\Big) \\
&\quad \le \sum_{k\ge 0} \big(c(2^k b)^{-\alpha\ell/4} + e^{-c2^{3k/8}b^{3/8}}\big) \le c' b^{-\alpha\ell/4} + e^{-c' b^{1/4}}.
\end{aligned}$$

This concludes the proof of Proposition 2.6.

5.3. *Proof of Theorem* 2.7. For any $q > 0$, we consider the Poisson point process restricted to $[0, 1] \times [-q, q]$, and we label its elements according to its order statistics $(M_i, Y_i)_{i\in\mathbb{N}}$. For any $\Delta \subset [0, 1] \times [-q, q]$, we define $\pi^{(\ell)}(\Delta) = \sum_{i=1}^{\ell} M_i \mathbb{1}_{\{Y_i\in\Delta\}}$ and $\pi^{(>\ell)}(\Delta) := \pi(\Delta) - \pi^{(\ell)}(\Delta)$. In analogy with the discrete setting (cf. (2.16)), we define

$$\begin{aligned}
\mathcal{T}_{\nu,q}^{(>\ell)} &= \sup_{s\in\mathscr{M}_q} \big\{\nu\pi^{(>\ell)}(s) - \mathrm{Ent}(s)\big\}, \\
\mathcal{T}_{\nu,q}^{(\ell)} &= \sup_{s\in\mathscr{M}_q} \big\{\nu\pi^{(\ell)}(s) - \mathrm{Ent}(s)\big\}.
\end{aligned} \tag{5.12}$$

We first show the convergence (2.20) of the large-weights variational problem, before we prove (2.19).

Convergence of the large weights. Note that the maximum of $T_{n,qh}^{\beta_{n,h},(\ell)}$ and $\mathcal{T}_{\nu,q}^{(\ell)}$ are achieved on $\Upsilon_\ell = \Upsilon_\ell(q)$ and $\boldsymbol{\Upsilon}_\ell = \boldsymbol{\Upsilon}_\ell(q)$, respectively, that is,

$$(5.13)\qquad \begin{aligned} T_{n,qh}^{\beta_{n,h},(\ell)} &= \max_{\Delta\subset\Upsilon_\ell}\{\beta_{n,h}\Omega_{n,h}^{(\ell)}(\Delta) - \mathrm{Ent}(\Delta)\},\\ \mathcal{T}_{\nu,q}^{(\ell)} &= \sup_{\Delta\subset\boldsymbol{\Upsilon}_\ell}\{\nu\pi^{(\ell)}(\Delta) - \mathrm{Ent}(\Delta)\},\end{aligned}$$

where $\Upsilon_\ell(q)$ (resp., $\boldsymbol{\Upsilon}_\ell(q)$) is the set of the locations of the ℓ largest weights inside $\Lambda_{n,qh}$ (resp., $\boldsymbol{\Lambda}_{1,q}$). Since we have only a finite number of points, the convergence (2.20) is a consequence of (5.1) and (5.2) and the Skorokhod representation theorem.

Restriction to the large weights. To show the convergence (2.19), it is therefore enough to control the contribution of the large weights. Let $\delta > 0$ such that $\frac{1}{\alpha} - \frac{1}{2} > \delta$. Using Potter's bound (cf. [6]), we have that

$$\big(\beta_{n,h} m(nh/\ell)\big)^{\frac{4}{3}}\left(\frac{\ell^2 n}{h^2}\right)^{\frac{1}{3}} \le c\frac{h^2}{n}\ell^{-\frac{4}{3}(\frac{1}{\alpha}-\frac{1}{2}-\delta)}.$$

Plugging it into (2.18) and taking $b = b_{\ell,\varepsilon} := \varepsilon\ell^{\frac{4}{3}(\frac{1}{\alpha}-\frac{1}{2}+\delta)}$, we obtain that

$$(5.14)\qquad \mathbb{P}\left(\frac{n}{h^2}T_{n,qh}^{\beta_{n,h},(>\ell)} \ge \varepsilon\right) \le c' b_{\ell,\varepsilon}^{-\alpha\ell/4} + e^{-c' b_{\ell,\varepsilon}^{1/4}} \stackrel{\ell\to\infty}{\longrightarrow} 0,$$

uniformly on n, h. Combined with (2.20) and the first part of (2.21), this gives the convergence (2.19).

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