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SECOND-ORDER BSDE UNDER MONOTONICITY CONDITION AND LIQUIDATION PROBLEM UNDER UNCERTAINTY

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In this work, we investigate an optimal liquidation problem under Knightian uncertainty. We obtain the value function and an optimal control characterised by the solution of a second-order BSDE with monotone generator and with a singular terminal condition.

CONTENTS

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1. Introduction. This paper is devoted to the study of an optimal liquidation problem under uncertainty. Roughly speaking, for some $\vartheta > 1$, we want to minimize the functional cost

$$J(\mathcal{X}) = \sup_{\mathbb{P}\in\mathcal{P}} \mathbb{E}^{\mathbb{P}}\left[\int_0^T (\eta_s|\dot{\mathcal{X}}_s|^\vartheta + \gamma_s|\mathcal{X}_s|^\vartheta)\, ds + \xi|\mathcal{X}_T|^\vartheta\right] \tag{1.1}$$

over all progressively measurable processes X that satisfy the dynamics

$$\mathcal{X}_s = x + \int_0^s \dot{\mathcal{X}}_u\, du.$$

The nonnegative quantity ξ is a penalty on the remaining value $\mathcal{X}_T$ of the state process $\mathcal{X}$. In particular when $\xi = +\infty$, $J(\mathcal{X})$ is finite only if the terminal constraint $\mathcal{X}_T\mathbf{1}_{\xi=+\infty} = 0$ is satisfied. If the set of probability measures $\mathcal{P}$ is a singleton, then the problem is solved in [5] and [29] using a backward stochastic differential equation (BSDE for short) with singular terminal condition. Our goal is to extend these results to the case where there is model uncertainty, that is when the probability measure $\mathbb{P}$ is not unique. Minimizing (1.1) corresponds for an agent to compute the worst case scenario for the liquidation of her portfolio.

The analysis of optimal control problems with state constraint on the terminal value is motivated by models of optimal portfolio liquidation under stochastic price impact (see, e.g., [3, 4, 15, 16, 19] or [27], among many others). For a fixed probability $\mathbb{P}$ (i.e., without the supremum in (1.1)):

$$J(\mathcal{X},\mathbb{P}) = \mathbb{E}^{\mathbb{P}}\left[\int_0^T (\eta_s|\dot{\mathcal{X}}_s|^\vartheta + \gamma_s|\mathcal{X}_s|^\vartheta)\, ds + \xi|\mathcal{X}_T|^\vartheta\right] \tag{1.2}$$

this position targeting problem (1.2) and some variants have been studied in [5, 17, 18, 29] or [38]. In this framework, the state process $\mathcal{X}$ denotes the agent's position in the financial market. At each point in time t, she can trade in the primary venue at a rate $\dot{\mathcal{X}}_t$ which generates costs $\eta_t|\dot{\mathcal{X}}_t|^\vartheta$ incurred by the stochastic price impact parameter η_t. The term $\gamma_t|\mathcal{X}_t|^\vartheta$ can be understood as a measure of risk associated to the open position. $J(\mathcal{X},\mathbb{P})$ thus represents the overall expected costs for closing an initial position x over the time period $[0, T]$ using strategy $\mathcal{X}$, with a terminal cost $\xi|\mathcal{X}_T|^\vartheta$. The penalization ξ is $\mathbb{F}_T$-measurable and takes value in $[0,\infty]$. The total cost $J(\mathcal{X},\mathbb{P})$ is finite if and only if $\mathcal{X}_T\mathbf{1}_{\xi=+\infty} = 0$ a.s. The case $\xi = +\infty$ a.s. corresponds to the liquidation constraint: $X_T = 0$ a.s., that is, the position has to be closed imperatively. The optimal strategies and the value function of this control problem (1.2) are characterized in [5] and [29] (see also [17] for the use of BSPDEs) by the minimal supersolution (y, z, m) of the BSDE

$$dy_t = \frac{y_t^{\mathfrak{q}}}{(\mathfrak{q}-1)\eta_t^{\mathfrak{q}-1}}\, dt - \gamma_t\, dt + z_t\, dW_t + dm_t \tag{1.3}$$

with $\liminf_{t\to T} y_t \geq \xi$. Here, $\mathfrak{q} > 1$ is the Hölder conjugate of ϑ and m is a martingale orthogonal to W. Since ξ can be equal to $+\infty$, such a BSDE is called *singular*. In [5] and [29], sufficient conditions on the coefficient processes η and γ are provided such that there exists a minimal supersolution to (1.3) and then by a verification theorem based on a penalization argument, it is proved that $\inf_{\mathcal{X}} J(\mathcal{X}, \mathbb{P}) = y_0|x|^p$ (see the details in Section 3.1).

When $\mathbb{P}$ is not unique, we need to solve

$$J(\mathcal{X}) = \sup_{\mathbb{P}\in\mathcal{P}} J(\mathcal{X}, \mathbb{P}) = \Big(\sup_{\mathbb{P}\in\mathcal{P}} y_0^{\mathbb{P}}\Big)|x|^p,$$

where $y^{\mathbb{P}}$ is the minimal supersolution of (1.3) under the probability measure $\mathbb{P}$. From the theory of second-order BSDE (2BSDE for short) introduced by [40] and [41], our problem can be solved with this useful tool. Nevertheless, the generator f of our BSDE (1.3) is not Lipschitz continuous but only monotone w.r.t. y:

$$f(t, y) = -\frac{y^{\mathfrak{q}}}{(\mathfrak{q}-1)\eta_t^{\mathfrak{q}-1}} + \gamma_t.$$

This condition has been already considered in [36], but under the additional assumption that the generator is of linear growth. The possibility to extend the existing results to a general monotone driver is mentioned in the paper [37] (see in particular Section 2.4.5). Thereby following the ideas of [37], we want to show that a 2BSDE with monotone generator w.r.t. y still has a unique solution.

Let us precise the main contributions of the paper. Roughly speaking, the paper [37] shows that if nice properties are known for BSDEs and reflected BSDEs, then it is possible to construct a solution for the 2BSDE. Hence to follow the scheme of [37], several properties of classical (reflected or not) BSDEs are needed. Several general results can be found in [28] and [30] for BSDEs in a general filtration and [26, 31] or [9] for reflected BSDEs (see also the references therein). But some technical results were missing in the general setting of the 2BSDEs: *Lipschitz approximation of BSDEs* (Lemma A.4) or *existence and uniqueness of the solution for a reflected BSDEs in a general filtration* (without quasi-left continuity) *when the driver is monotone* (Proposition A.2). These results, even though useful for applications to 2BSDE, are expectable and the techniques employed are rather standard. This is the reason why they are postponed in the Appendix.

The first part is devoted to 2BSDEs with a monotone driver (condition (**H**)). The probabilistic setting is the same as [37]. But to overcome this difficulty induced by monotonicity, we will impose some stronger integrability conditions[3] **C1** and **C2** on the terminal value ξ (and on the process f^0). Our main results are Theorem 1 (uniqueness) and Theorem 2 (existence). Although the sketch is almost the same as in [37], several technical issues have to be taken into account in our setting. Moreover, the monotonicity of the driver forces us to change the minimality condition on the nondecreasing process $K^{\mathbb{P}}$. Instead of the classical assumption

$$\operatorname*{essinf}^{\mathbb{P}}_{\mathbb{P}'\in\mathcal{P}(t,\mathbb{P},\mathbb{F}_+)} \mathbb{E}^{\mathbb{P}'}\big[K_T^{\mathbb{P}'} - K_t^{\mathbb{P}'}|\mathcal{F}_t^+\big] = 0, \qquad 0 \le t \le T, \mathbb{P}\text{-a.s.}, \forall \mathbb{P} \in \mathcal{P}$$

we have here

$$\operatorname*{essinf}^{\mathbb{P}}_{\mathbb{P}'\in\mathcal{P}(t,\mathbb{P},\mathbb{F}_+)} \mathbb{E}^{\mathbb{P}'}\left[\int_t^T \exp\left(\int_t^s \lambda_u^{\mathbb{P}'}\,du\right) dK_s^{\mathbb{P}'}\middle|\mathcal{F}_t^+\right] = 0,$$
$$0 \le t \le T, \mathbb{P}\text{-a.s.}, \forall \mathbb{P} \in \mathcal{P},$$

where $\lambda_s^{\mathbb{P}'}$ is the increment of the generator evaluated at the solution Y of the 2BSDE and at the solution $y^{\mathbb{P}'}$ of the classical BSDE under $\mathbb{P}'$. In the Lipschitz setting, $\lambda^{\mathbb{P}'}$ is bounded and thus can be removed, whereas under the monotone assumption, it is only bounded from above (see Definition 1, Conditions (2.6) and (2.16) and the discussion in Section 2.5).

Then we come back to our initial goal: the resolution of the optimal control problem (1.1). From our results on 2BSDEs, we can now obtain directly the value function and an optimal control for the unconstrained problem (Proposition 1). For the constrained problem, a known difficulty concerns the filtration. Indeed to avoid the possibility of a uncontrolled jump for the orthogonal martingale part at the terminal time T, some additional hypothesis on the filtration is needed (see [35], Section 2.2). Under this technical condition on the filtration, we prove that the 2BSDE with singular terminal condition has a minimal super-solution (Theorem 3 and Remark 9) and that we can solve (1.1) using this supersolution (Theorem 4).

The paper is decomposed as follows. In Section 2, we use the scheme developed in [37] to obtain existence and uniqueness for the 2BSDE. In Section 3, we solve the control problem (1.1) using 2BSDE with monotone driver and singular terminal condition. In the Appendix, we recall and develop some results concerning BSDE and reflected BSDE with monotone driver.

2. Second-order BSDE with monotone generator. This section is devoted to the extension of the results in [37] to the case where the generator f is only monotone w.r.t. y (see **H2**). Compared with [36], we do not assume that f is of linear growth w.r.t. y (see **H4**). Nevertheless, let us immediately emphasize that we will assume stronger integrability assumptions of ξ and f^0 to overcome this difficulty and we change the minimality condition on the nondecreasing process K (see Condition (2.6)).

[3]Sufficient to solve our control problem. Weaker integrability assumptions are left for future research.

2.1. *The probabilistic setting.* Our framework is exactly the same as in [37]. Let us recall the notation and assumptions. Let $\mathbb{N}^* := \mathbb{N} \setminus \{0\}$ and let $\mathbb{R}_+^*$ be the set of real positive numbers. For every d-dimensional vector b with $d \in \mathbb{N}^*$, we denote by $b^1, \dots, b^d$ its coordinates and for $\alpha, \beta \in \mathbb{R}^d$ we denote by $\alpha \cdot \beta$ the usual inner product, with associated norm $\|\cdot\|$, which we simplify to $|\cdot|$ when d is equal to 1. We also let $\mathbf{1}_d$ be the vector whose coordinates are all equal to 1. For any $(l, c) \in \mathbb{N}^* \times \mathbb{N}^*$, $\mathcal{M}_{l,c}(\mathbb{R})$ will denote the space of $l \times c$ matrices with real entries. Elements of the matrix $M \in \mathcal{M}_{l,c}$ will be denoted by $(M^{i,j})_{1\le i\le l, 1\le j\le c}$, and the transpose of M will be denoted by $M^\top$. When $l = c$, we let $\mathcal{M}_l(\mathbb{R}) := \mathcal{M}_{l,l}(\mathbb{R})$. We also identify $\mathcal{M}_{l,1}(\mathbb{R})$ and $\mathbb{R}^l$. Let $\mathbb{S}_d^{\ge 0}$ denote the set of all symmetric positive semidefinite $d \times d$ matrices. We fix a map $\psi : \mathbb{S}_d^{\ge 0} \longrightarrow \mathcal{M}_d(\mathbb{R})$ which is (Borel) measurable and satisfies $\psi(a)(\psi(a))^\top = a$ for all $a \in \mathbb{S}_d^{\ge 0}$, and denote $a^{\frac{1}{2}} := \psi(a)$.

2.1.1. *Canonical space.* Let $d \in \mathbb{N}^*$, we denote by $\Omega := C([0,T], \mathbb{R}^d)$ the canonical space of all $\mathbb{R}^d$-valued continuous paths ω on $[0,T]$ such that $\omega_0 = 0$, equipped with the canonical process X, *that is*, $X_t(\omega) := \omega_t$, for all $\omega \in \Omega$. Denote by $\mathbb{F} = (\mathcal{F}_t)_{0\le t\le T}$ the canonical filtration generated by X, and by $\mathbb{F}_+ = (\mathcal{F}_t^+)_{0\le t\le T}$ the right limit of $\mathbb{F}$ with $\mathcal{F}_t^+ := \bigcap_{s>t} \mathcal{F}_s$ for all $t \in [0,T)$ and $\mathcal{F}_T^+ := \mathcal{F}_T$. We equip Ω with the uniform convergence norm $\|\omega\|_\infty := \sup_{0\le t\le T} \|\omega_t\|$, so that the Borel σ-field of Ω coincides with $\mathcal{F}_T$. Let $\mathbb{P}_0$ denote the Wiener measure on Ω under which X is a Brownian motion.

Let $\mathbb{M}_1$ denote the collection of all probability measures on $(\Omega, \mathcal{F}_T)$. Notice that $\mathbb{M}_1$ is a Polish space equipped with the weak convergence topology. We denote by $\mathfrak{B}$ its Borel σ-field. Then for any $\mathbb{P} \in \mathbb{M}_1$, denote by $\mathcal{F}_t^{\mathbb{P}}$ the completed σ-field of $\mathcal{F}_t$ under $\mathbb{P}$. Denote also the completed filtration by $\mathbb{F}^{\mathbb{P}} = (\mathcal{F}_t^{\mathbb{P}})_{t\in[0,T]}$ and $\mathbb{F}_+^{\mathbb{P}}$ the right limit of $\mathbb{F}^{\mathbb{P}}$, so that $\mathbb{F}_+^{\mathbb{P}}$ satisfies the usual conditions. Moreover, for $\mathcal{P} \subset \mathbb{M}_1$, we introduce the universally completed filtration $\mathbb{F}^U := (\mathcal{F}_t^U)_{0\le t\le T}$, $\mathbb{F}^{\mathcal{P}} := (\mathcal{F}_t^{\mathcal{P}})_{0\le t\le T}$, and $\mathbb{F}^{\mathcal{P}+} := (\mathcal{F}_t^{\mathcal{P}+})_{0\le t\le T}$, defined as follows:

$$\mathcal{F}_t^U := \bigcap_{\mathbb{P}\in\mathbb{M}_1} \mathcal{F}_t^{\mathbb{P}}, \qquad \mathcal{F}_t^{\mathcal{P}} := \bigcap_{\mathbb{P}\in\mathcal{P}} \mathcal{F}_t^{\mathbb{P}}, \quad t \in [0,T],$$

$$\mathcal{F}_t^{\mathcal{P}+} := \mathcal{F}_{t+}^{\mathcal{P}}, \quad t \in [0,T), \quad \text{and} \quad \mathcal{F}_T^{\mathcal{P}+} := \mathcal{F}_T^{\mathcal{P}}.$$

We also introduce an enlarged canonical space $\overline{\Omega} := \Omega \times \Omega'$, where Ω' is identical to Ω. By abuse of notation, we denote by (X, B) its canonical process, *that is*, $X_t(\bar\omega) := \omega_t$, $B_t(\bar\omega) := \omega'_t$ for all $\bar\omega := (\omega, \omega') \in \overline{\Omega}$, by $\overline{\mathbb{F}} = (\overline{\mathcal{F}}_t)_{0\le t\le T}$ the canonical filtration generated by (X, B), and by $\overline{\mathbb{F}}^X = (\overline{\mathcal{F}}_t^X)_{0\le t\le T}$ the filtration generated by X. Similarly, we denote the corresponding right-continuous filtrations by $\overline{\mathbb{F}}_+^X$ and $\overline{\mathbb{F}}_+$, and the augmented filtration by $\overline{\mathbb{F}}_+^{X,\overline{\mathbb{P}}}$ and $\overline{\mathbb{F}}_+^{\overline{\mathbb{P}}}$, given a probability measure $\overline{\mathbb{P}}$ on $\overline{\Omega}$.

2.1.2. *Semimartingale measures.* We say that a probability measure $\mathbb{P}$ on $(\Omega, \mathcal{F}_T)$ is a semimartingale measure if X is a semimartingale under $\mathbb{P}$. Then on the canonical space Ω, there is some $\mathbb{F}$-progressively measurable nondecreasing process (see, e.g., Karandikar [22]), denoted by $\langle X\rangle = (\langle X\rangle_t)_{0\le t\le T}$, which coincides with the quadratic variation of X under each semimartingale measure $\mathbb{P}$. Denote further

$$\widehat{a}_t := \limsup_{\varepsilon\searrow 0} \frac{\langle X\rangle_t - \langle X\rangle_{t-\varepsilon}}{\varepsilon}.$$

For every $t \in [0, T]$, let $\mathcal{P}_t^W$ denote the collection of all probability measures $\mathbb{P}$ on $(\Omega, \mathcal{F}_T)$ such that:

- $(X_s)_{s\in[t,T]}$ is a $(\mathbb{P}, \mathbb{F})$-semimartingale admitting the canonical decomposition (see, e.g., [20], Theorem I.4.18)

  $$X_s = \int_t^s b_r^{\mathbb{P}}\, dr + X_s^{c,\mathbb{P}}, \qquad s \in [t, T], \mathbb{P}\text{-a.s.},$$

  where $b^{\mathbb{P}}$ is an $\mathbb{F}^{\mathbb{P}}$-predictable $\mathbb{R}^d$-valued process, and $X^{c,\mathbb{P}}$ is the continuous local martingale part of X under $\mathbb{P}$.
- $(\langle X\rangle_s)_{s\in[t,T]}$ is absolutely continuous in s with respect to the Lebesgue measure, and $\widehat{a}_t$ takes values in $\mathbb{S}_d^{\ge 0}$, $\mathbb{P}$-a.s. for all $t \in [0, T]$.

Given a random variable or process λ defined on Ω, we can naturally define its extension on $\overline{\Omega}$ (which, abusing notation slightly, we still denote by λ) by

$$\lambda(\bar{\omega}) := \lambda(\omega) \qquad \forall \bar{\omega} = (\omega, \omega') \in \overline{\Omega}. \tag{2.1}$$

In particular, the process $\widehat{a}$ can be extended on $\overline{\Omega}$. Given a probability measure $\mathbb{P} \in \mathcal{P}_t^W$, we define a probability measure $\overline{\mathbb{P}}$ on the enlarged canonical space $\overline{\Omega}$ by $\overline{\mathbb{P}} := \mathbb{P} \otimes \mathbb{P}_0$, so that X in $(\overline{\Omega}, \overline{\mathcal{F}}_T, \overline{\mathbb{P}}, \overline{\mathbb{F}})$ is a semimartingale with the same triplet of characteristics as X in $(\Omega, \mathcal{F}_T, \mathbb{P}, \mathbb{F})$, B is a $\overline{\mathbb{F}}$-Brownian motion, and X is independent of B. Then for every $\mathbb{P} \in \mathcal{P}_t^W$, there is some $\mathbb{R}^d$-valued, $\overline{\mathbb{F}}$-Brownian motion $W^{\mathbb{P}} = (W_r^{\mathbb{P}})_{t\le r\le s}$ such that (see, e.g., Theorem 4.5.2 of [42])

$$X_s = \int_t^s b_r^{\mathbb{P}}\, dr + \int_t^s \widehat{a}_r^{\frac{1}{2}}\, dW_r^{\mathbb{P}}, \qquad s \in [t, T], \overline{\mathbb{P}}\text{-a.s.}, \tag{2.2}$$

where we extend the definition of $b^{\mathbb{P}}$ and $\widehat{a}$ on $\overline{\Omega}$ as in (2.1), and where we recall that $\widehat{a}^{\frac{1}{2}}$ has been defined in the notation above.

Notice that when $\widehat{a}_r$ is nondegenerate $\mathbb{P}$-a.s., for all $r \in [t, T]$, then we can construct the Brownian motion $W^{\mathbb{P}}$ on Ω by

$$W_t^{\mathbb{P}} := \int_0^t \widehat{a}_s^{-\frac{1}{2}}\, dX_s^{c,\mathbb{P}}, \qquad t \in [0, T], \mathbb{P}\text{-a.s.},$$

and do not need to consider the above enlarged space equipped with an independent Brownian motion to construct $W^{\mathbb{P}}$.

REMARK 1 (On the choice of $\widehat{a}^{\frac{1}{2}}$). The measurable map $\psi : a \longmapsto a^{\frac{1}{2}}$ is fixed throughout the paper. A first choice is to take $a^{\frac{1}{2}}$ as the unique nonnegative symmetric square root of a (see, e.g., Lemma 5.2.1 of [42]). One can also use the Cholesky decomposition to obtain $a^{\frac{1}{2}}$ as a lower triangular matrix. Finally, the reader can read [37], Remark 2.2, where the sets $\mathcal{P}(t,\omega)$ are given by the collections of probability measures induced by a family of controlled diffusion processes. In this case, one can take $\widehat{a}^{\frac{1}{2}}$ in the following way:

$$a = \begin{pmatrix} \sigma\sigma^T & \sigma \\ \sigma^T & I_n \end{pmatrix} \quad \text{and} \quad a^{\frac{1}{2}} = \begin{pmatrix} \sigma & 0 \\ I_n & 0 \end{pmatrix} \qquad \text{for some } \sigma \in \mathcal{M}_{m,n}. \tag{2.3}$$

2.1.3. *Conditioning and concatenation of probability measures*. We also recall that for every probability measure $\mathbb{P}$ on Ω and every $\mathbb{F}$-stopping time τ taking values in $[0, T]$, there exists a family of regular conditional probability distribution (r.c.p.d. for short) $(\mathbb{P}^\tau_\omega)_{\omega\in\Omega}$ (see, e.g., Stroock and Varadhan [42]), satisfying:

(i) For every $\omega \in \Omega$, $\mathbb{P}^\tau_\omega$ is a probability measure on $(\Omega, \mathcal{F}_T)$.
(ii) For every $E \in \mathcal{F}_T$, the mapping $\omega \longmapsto \mathbb{P}^\tau_\omega(E)$ is $\mathcal{F}_\tau$-measurable.
(iii) The family $(\mathbb{P}^\tau_\omega)_{\omega\in\Omega}$ is a version of the conditional probability measure of $\mathbb{P}$ on $\mathcal{F}_\tau$, *that is*, for every integrable $\mathcal{F}_T$-measurable random variable ξ we have $\mathbb{E}^{\mathbb{P}}[\xi|\mathcal{F}_\tau](\omega) = \mathbb{E}^{\mathbb{P}^\tau_\omega}[\xi]$, for $\mathbb{P}$-a.e. $\omega \in \Omega$.
(iv) For every $\omega \in \Omega$, $\mathbb{P}^\tau_\omega(\Omega^\omega_\tau) = 1$, where $\Omega^\omega_\tau := \{\overline{\omega} \in \Omega : \overline{\omega}(s) = \omega(s), 0 \le s \le \tau(\omega)\}$.

Furthermore, given some $\mathbb{P}$ and a family $(\mathbb{Q}_\omega)_{\omega\in\Omega}$ such that $\omega \longmapsto \mathbb{Q}_\omega$ is $\mathcal{F}_\tau$-measurable and $\mathbb{Q}_\omega(\Omega^\omega_\tau) = 1$ for all $\omega \in \Omega$, one can then define a concatenated probability measure $\mathbb{P} \otimes_\tau \mathbb{Q}.$ by

$$\mathbb{P} \otimes_\tau \mathbb{Q}.[A] := \int_\Omega \mathbb{Q}_\omega[A]\mathbb{P}(d\omega) \qquad \forall A \in \mathcal{F}_T.$$

2.1.4. *Hypotheses on* $\mathcal{P}(t,\omega)$. We are given a family $\mathcal{P} = (\mathcal{P}(t,\omega))_{(t,\omega)\in[0,T]\times\Omega}$ of sets of probability measures on $(\Omega, \mathcal{F}_T)$, where $\mathcal{P}(t,\omega) \subset \mathcal{P}^W_t$ for all $(t,\omega) \in [0,T] \times \Omega$. Denote also $\mathcal{P}_t := \bigcup_{\omega\in\Omega} \mathcal{P}(t,\omega)$. We make the following assumption on the family $(\mathcal{P}(t,\omega))_{(t,\omega)\in[0,T]\times\Omega}$.

ASSUMPTION 1. (i) For every $(t,\omega) \in [0,T] \times \Omega$, one has $\mathcal{P}(t,\omega) = \mathcal{P}(t,\omega_{\cdot\wedge t})$ and $\mathbb{P}(\Omega^\omega_t) = 1$ whenever $\mathbb{P} \in \mathcal{P}(t,\omega)$. The graph $[[\mathcal{P}]]$ of $\mathcal{P}$, defined by $[[\mathcal{P}]] := \{(t,\omega,\mathbb{P}) : \mathbb{P} \in \mathcal{P}(t,\omega)\}$, is upper semianalytic in $[0,T] \times \Omega \times \mathbb{M}_1$.

(ii) $\mathcal{P}$ is stable under conditioning, *that is*, for every $(t,\omega) \in [0,T] \times \Omega$ and every $\mathbb{P} \in \mathcal{P}(t,\omega)$ together with an $\mathbb{F}$-stopping time τ taking values in $[t,T]$, there is a family of r.c.p.d. $(\mathbb{P}_{\mathsf{w}})_{\mathsf{w}\in\Omega}$ such that $\mathbb{P}_{\mathsf{w}}$ belongs to $\mathcal{P}(\tau(\mathsf{w}),\mathsf{w})$, for $\mathbb{P}$-a.e. $\mathsf{w} \in \Omega$.

(iii) $\mathcal{P}$ is stable under concatenation, *that is*, for every $(t, \omega) \in [0, T] \times \Omega$ and $\mathbb{P} \in \mathcal{P}(t, \omega)$ together with an $\mathbb{F}$-stopping time τ taking values in $[t, T]$, let $(\mathbb{Q}_{\mathsf{w}})_{\mathsf{w} \in \Omega}$ be a family of probability measures such that $\mathbb{Q}_{\mathsf{w}} \in \mathcal{P}(\tau(\mathsf{w}), \mathsf{w})$ for all $\mathsf{w} \in \Omega$ and $\mathsf{w} \longmapsto \mathbb{Q}_{\mathsf{w}}$ is $\mathcal{F}_\tau$-measurable, then the concatenated probability measure $\mathbb{P} \otimes_\tau \mathbb{Q}_\cdot \in \mathcal{P}(t, \omega)$.

We notice that for $t = 0$, we have $\mathcal{P}_0 := \mathcal{P}(0, \omega)$ for any $\omega \in \Omega$.

2.1.5. *Spaces and norms.* We now give the spaces and norms which will be needed in the rest of the paper. Fix some $t \in [0, T]$ and some $\omega \in \Omega$. In what follows, $\mathbb{G} := (\mathcal{G}_s)_{t \le s \le T}$ will denote an arbitrary filtration on $(\Omega, \mathcal{F}_T)$, and $\mathbb{P}$ an arbitrary element in $\mathcal{P}(t, \omega)$. Denote also by $\mathbb{G}^{\mathbb{P}}$ the $\mathbb{P}$-augmented filtration associated to $\mathbb{G}$.

For $p \ge 1$, $\mathbb{L}^p_{t,\omega}(\mathbb{G})$ (resp., $\mathbb{L}^p_{t,\omega}(\mathbb{G}, \mathbb{P})$) denotes the space of all $\mathcal{G}_T$-measurable scalar random variables ξ with

$$\|\xi\|^p_{\mathbb{L}^p_{t,\omega}} := \sup_{\mathbb{P} \in \mathcal{P}(t,\omega)} \mathbb{E}^{\mathbb{P}}[|\xi|^p] < +\infty \qquad \big(\text{resp., } \|\xi\|^p_{\mathbb{L}^p_{t,\omega}(\mathbb{P})} := \mathbb{E}^{\mathbb{P}}[|\xi|^p] < +\infty\big).$$

$\mathbb{H}^p_{t,\omega}(\mathbb{G})$ (resp., $\mathbb{H}^p_{t,\omega}(\mathbb{G}, \mathbb{P})$) denotes the space of all $\mathbb{G}$-predictable $\mathbb{R}^d$-valued processes Z, which are defined $\widehat{a}_s\, ds$-a.e. on $[t, T]$, with

$$\|Z\|^p_{\mathbb{H}^p_{t,\omega}} := \sup_{\mathbb{P} \in \mathcal{P}(t,\omega)} \mathbb{E}^{\mathbb{P}}\bigg[\bigg(\int_t^T \|\widehat{a}_s^{\frac{1}{2}} Z_s\|^2\, ds\bigg)^{\frac{p}{2}}\bigg] < +\infty$$

$$\bigg(\text{resp., } \|Z\|^p_{\mathbb{H}^p_{t,\omega}(\mathbb{P})} := \mathbb{E}^{\mathbb{P}}\bigg[\bigg(\int_t^T \|\widehat{a}_s^{\frac{1}{2}} Z_s\|^2\, ds\bigg)^{\frac{p}{2}}\bigg] < +\infty\bigg).$$

$\mathbb{M}^p_{t,\omega}(\mathbb{G}, \mathbb{P})$ denotes the space of all $(\mathbb{G}, \mathbb{P})$-optional martingales M with $\mathbb{P}$-a.s. càdlàg paths on $[t, T]$, with $M_t = 0$, $\mathbb{P}$-a.s., and

$$\|M\|^p_{\mathbb{M}^p_{t,\omega}(\mathbb{P})} := \mathbb{E}^{\mathbb{P}}\big[[M]_T^{\frac{p}{2}}\big] < +\infty.$$

Furthermore, we say that a family $(M^{\mathbb{P}})_{\mathbb{P} \in \mathcal{P}(t,\omega)}$ belongs to $\mathbb{M}^p_{t,\omega}((\mathbb{G}^{\mathbb{P}})_{\mathbb{P} \in \mathcal{P}(t,\omega)})$ if, for any $\mathbb{P} \in \mathbb{P}(t, \omega)$, $M^{\mathbb{P}} \in \mathbb{M}^p_{t,\omega}(\mathbb{G}^{\mathbb{P}}, \mathbb{P})$ and

$$\sup_{\mathbb{P} \in \mathcal{P}(t,\omega)} \|M^{\mathbb{P}}\|_{\mathbb{M}^p_{t,\omega}(\mathbb{P})} < +\infty.$$

$\mathbb{I}^p_{t,\omega}(\mathbb{G}, \mathbb{P})$ denotes the space of all $\mathbb{G}$-predictable processes K with $\mathbb{P}$-a.s. càdlàg and nondecreasing paths on $[t, T]$, with $K_t = 0$, $\mathbb{P}$-a.s., and

$$\|K\|^p_{\mathbb{I}^p_{t,\omega}(\mathbb{P})} := \mathbb{E}^{\mathbb{P}}[K_T^p] < +\infty.$$

We will say that a family $(K^{\mathbb{P}})_{\mathbb{P} \in \mathcal{P}(t,\omega)}$ belongs to $\mathbb{I}^p_{t,\omega}((\mathbb{G}^{\mathbb{P}})_{\mathbb{P} \in \mathcal{P}(t,\omega)})$ if, for any $\mathbb{P} \in \mathcal{P}(t, \omega)$, $K^{\mathbb{P}} \in \mathbb{I}^p_{t,\omega}(\mathbb{G}_{\mathbb{P}}, \mathbb{P})$ and

$$\sup_{\mathbb{P} \in \mathcal{P}(t,\omega)} \|K^{\mathbb{P}}\|_{\mathbb{I}^p_{t,\omega}(\mathbb{P})} < +\infty.$$

$\mathbb{D}^p_{t,\omega}(\mathbb{G})$ (resp., $\mathbb{D}^p_{t,\omega}(\mathbb{G},\mathbb{P})$) denotes the space of all $\mathbb{G}$-progressively measurable $\mathbb{R}$-valued processes Y with $\mathcal{P}(t,\omega)$-q.s. (resp., $\mathbb{P}$-a.s.) càdlàg paths on $[t,T]$, with

$$\|Y\|^p_{\mathbb{D}^p_{t,\omega}} := \sup_{\mathbb{P}\in\mathcal{P}(t,\omega)} \mathbb{E}^{\mathbb{P}}\Big[\sup_{t\le s\le T}|Y_s|^p\Big] < +\infty$$

$$\Big(\text{resp., } \|Y\|^p_{\mathbb{D}^p_{t,\omega}(\mathbb{P})} := \mathbb{E}^{\mathbb{P}}\Big[\sup_{t\le s\le T}|Y_s|^p\Big] < +\infty\Big).$$

For each $\xi \in \mathbb{L}^1_{t,\omega}(\mathbb{G})$ and $s\in[t,T]$ denote

$$\mathbb{E}^{\mathbb{P},t,\omega,\mathbb{G}}_s[\xi] := \operatorname*{ess\,sup^{\mathbb{P}}}_{\mathbb{P}'\in\mathcal{P}_{t,\omega}(s,\mathbb{P},\mathbb{G})} \mathbb{E}^{\mathbb{P}'}[\xi|\mathcal{G}_s]$$

$$\text{where } \mathcal{P}_{t,\omega}(s,\mathbb{P},\mathbb{G}) := \{\mathbb{P}'\in\mathcal{P}(t,\omega), \mathbb{P}'=\mathbb{P} \text{ on } \mathcal{G}_s\}.$$

Then we define for each $p\ge\kappa\ge 1$,

$$\mathbb{L}^{p,\kappa}_{t,\omega}(\mathbb{G}) := \{\xi\in\mathbb{L}^p_{t,\omega}(\mathbb{G}), \|\xi\|_{\mathbb{L}^{p,\kappa}_{t,\omega}} < +\infty\},$$

where

$$\|\xi\|^p_{\mathbb{L}^{p,\kappa}_{t,\omega}} := \sup_{\mathbb{P}\in\mathcal{P}(t,\omega)} \mathbb{E}^{\mathbb{P}}\Big[\operatorname*{ess\,sup^{\mathbb{P}}}_{t\le s\le T}\big(\mathbb{E}^{\mathbb{P},t,\omega,\mathbb{F}^+}_s[|\xi|^\kappa]\big)^{\frac{p}{\kappa}}\Big].$$

Similarly, given a probability measure $\overline{\mathbb{P}}$ and a filtration $\overline{\mathbb{G}}$ on the enlarged canonical space $\overline{\Omega}$, we denote the corresponding spaces by $\mathbb{D}^p_{t,\omega}(\overline{\mathbb{G}},\overline{\mathbb{P}})$, $\mathbb{H}^p_{t,\omega}(\overline{\mathbb{G}},\overline{\mathbb{P}})$, $\mathbb{M}^p_{t,\omega}(\overline{\mathbb{G}},\overline{\mathbb{P}})$, ... Furthermore, when $t=0$, there is no longer any dependence on ω, since $\omega_0=0$, so that we simplify the notation by suppressing the ω-dependence and write $\mathbb{H}^p_0(\mathbb{G})$, $\mathbb{H}^p_0(\mathbb{G},\mathbb{P})$, ... Similar notation is used on the enlarged canonical space.

When there is no ambiguity (only one probability measure $\mathbb{P}$, see the Appendix), the Brownian motion will be denoted by W and for simplicity in the notation of integrability spaces, we remove the reference to the filtration $\mathbb{F}$, the probability measure and ω: $\mathbb{D}^p_{0,\omega}(\mathbb{F},\mathbb{P}) = \mathbb{D}^p$ and with the same convention $\mathbb{H}^p$, $\mathbb{M}^p$ and $\mathbb{I}^p$. Moreover, for $\alpha\in\mathbb{R}$, for $(Z,M,K)\in\mathbb{H}^p\times\mathbb{M}^p\times\mathbb{I}^p$, we define

$$\|Z\|^p_{\mathbb{H}^{p,\alpha}} = \mathbb{E}\bigg[\bigg(\int_0^T e^{\alpha s}\|Z_s\|^2\,ds\bigg)^{p/2}\bigg],$$

$$\|M\|^p_{\mathbb{M}^{p,\alpha}} = \mathbb{E}\bigg[\bigg(\int_0^T e^{\alpha s}\,d[M]_s\bigg)^{p/2}\bigg],$$

$$\|K\|^p_{\mathbb{I}^{p,\alpha}} = \mathbb{E}\bigg[\bigg(\int_0^T e^{\alpha s/2}\,dK_s\bigg)^{p}\bigg].$$

The corresponding spaces are denoted $\mathbb{H}^{p,\alpha}$, $\mathbb{M}^{p,\alpha}$ and $\mathbb{I}^{p,\alpha}$.

2.2. *Assumptions on f and ξ.* We shall consider an $\mathcal{F}_T$-measurable random variable $\xi : \Omega \longrightarrow \mathbb{R}$ and a generator function

$$f : (t, \omega, y, z, a, b) \in [0, T] \times \Omega \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{S}_d^{\geq 0} \times \mathbb{R}^d \longrightarrow \mathbb{R}.$$

Define for simplicity

$$\widehat{f}_s^{\mathbb{P}}(y, z) := f\big(s, X_{\cdot\wedge s}, y, z, \widehat{a}_s, b_s^{\mathbb{P}}\big) \quad \text{and} \quad \widehat{f}_s^{\mathbb{P},0} := f\big(s, X_{\cdot\wedge s}, 0, 0, \widehat{a}_s, b_s^{\mathbb{P}}\big).$$

The generator function f is jointly Borel measurable and:

H1. For any (t, ω, z, a, b), the map $y \mapsto f(t, \omega, y, z, a, b)$ is continuous.

H2. f satisfies the monotonicity assumption w.r.t. y: there exists a constant $L_1 \in \mathbb{R}$ such that for every $(t, \omega, y, y', z, a, b)$

$$\big(f(t, \omega, y, z, a, b) - f\big(t, \omega, y', z, a, b\big)\big)\big(y - y'\big) \leq L_1\big(y - y'\big)^2.$$

H3. f is Lipschitz continuous w.r.t. z uniformly w.r.t. all other parameters, that is there exists a nonnegative constant L_2 such that for every $(t, \omega, y, z, z', a, b)$,

$$\big|f(t, \omega, y, z, a, b) - f\big(t, \omega, y, z', a, b\big)\big| \leq L_2\|z - z'\|.$$

H4. The following growth assumption w.r.t. y holds: there exists $\mathfrak{q} > 1$ and a jointly Borel measurable function $\Psi : [0, T] \times \Omega \times \mathbb{S}_d^{\geq 0} \to \mathbb{R}$ such that for any (t, ω, a, b, y)

$$\big|f(t, \omega, y, 0, a, b) - f_t^0\big| \leq \Psi(t, \omega, a)\big(1 + |y|^{\mathfrak{q}}\big).$$

f_t^0 is the notation for $f(t, \omega, 0, 0, a, b)$. We say that f satisfies *Condition* (**H**) if **H1** to **H4** hold. As for the generator, we denote

$$\widehat{\Psi}_s := \Psi(s, X_{\cdot\wedge s}, \widehat{a}_s).$$

Finally, on ξ, f^0 and Ψ we impose:

C1. For some fixed constants $\varrho > 1$ and $\bar{\mathfrak{p}} > \varrho/(\varrho - 1)$, one has for every $(t, \omega) \in [0, T] \times \Omega$,

$$\sup_{\mathbb{P}\in\mathcal{P}(t,\omega)} \mathbb{E}^{\mathbb{P}}\bigg[|\xi|^{\bar{\mathfrak{p}}\mathfrak{q}} + \int_t^T \big|\widehat{f}_s^{\mathbb{P},0}\big|^{\bar{\mathfrak{p}}\mathfrak{q}}\, ds + \int_t^T |\widehat{\Psi}_s|^{\varrho}\, ds\bigg] < +\infty.$$

C2. There is some $\kappa \in (1, \bar{\mathfrak{p}}\mathfrak{q}]$ such that $\xi \in \mathbb{L}_0^{\bar{\mathfrak{p}}\mathfrak{q},\kappa}$ and

$$\phi_f^{\bar{\mathfrak{p}}\mathfrak{q},\kappa} = \sup_{\mathbb{P}\in\mathcal{P}_0} \mathbb{E}^{\mathbb{P}}\bigg[\operatorname*{esssup}^{\mathbb{P}}_{0\leq s\leq T}\bigg(\operatorname*{ess\,sup}^{\mathbb{P}}_{\mathbb{P}'\in\mathcal{P}_0(s,\mathbb{P},\mathbb{F}_+)} \mathbb{E}^{\mathbb{P}'}\bigg[\int_0^T \big|\widehat{f}_t^{\mathbb{P}',0}\big|^{\kappa}\, dt \Big| \mathcal{F}_s^+\bigg]\bigg)^{\frac{\bar{\mathfrak{p}}\mathfrak{q}}{\kappa}}\bigg] < +\infty.$$

NOTATION. In the rest of the paper, p denotes any number larger than 1: $p > 1$; $\mathfrak{q}$ denotes the exponent in Condition **H4** (or **H4′** in the Appendix); $\bar{\mathfrak{p}}$ and ϱ are used in Assumptions **C1** and **C2** and satisfy $\bar{\mathfrak{p}} > \varrho/(\varrho - 1)$ ($\bar{\mathfrak{p}}$ is greater than the Hölder conjugate of ϱ). Finally, we will sometimes assume p verifies

$$1 < p \leq \frac{\varrho \bar{\mathfrak{p}}}{\varrho + \bar{\mathfrak{p}}} < \bar{\mathfrak{p}}. \tag{2.4}$$

Under this condition, $\widehat{p} = \frac{p\bar{\mathfrak{p}}}{(\bar{\mathfrak{p}} - p)} \leq \varrho$.

REMARK 2 (On condition **H2**). It is well known that we can suppose w.l.o.g. that $L_1 = 0$. Indeed if (y, z, m) is a solution of (2.7) below, then $(\bar{y}, \bar{z}, \bar{m})$ with

$$\bar{y}_t = e^{L_1 t} y_t, \qquad \bar{z}_t = e^{L_1 t} z_t, \qquad d\bar{m}_t = e^{L_1 t}\, dm_t$$

satisfies an analogous BSDE with terminal condition $\bar{\xi} = e^{L_1 T} \xi$ and generator

$$\bar{f}(t, y, z) = e^{L_1 t} f\big(t, e^{-L_1 t} y, e^{-L_1 t} z\big) - L_1 y.$$

$\bar{f}$ satisfies assumptions (**H**) with $L_1 = 0$. If we consider a supersolution of a BSDE (see equation (A.15)), the nondecreasing k is replaced by $d\bar{k}_t = e^{L_1 t}\, dk_t$. Hence in the rest of this paper, we will sometimes assume w.l.o.g. that $L_1 = 0$.

REMARK 3 (On condition **H4**). Let us explain why we assume Condition **H4** together with the integrability condition **C1** on $\widehat{\Psi}$, and not some weaker growth condition (as in [33] or [11] for standard BSDE). Indeed to prove the existence of a solution for a 2BSDE we will need that the solution (y, z, m) of the standard BSDE with data $\widehat{f}^{\mathbb{P}}$ and ξ (see equation (2.7)) is obtained by approximation with a sequence of solutions (y^n, z^n, m^n) of Lipschitz BSDEs (see Lemma A.4 in the Appendix). Moreover, the fact that Ψ does not depend on b is used for regularization of the paths in order to control the downcrossings (see Section 2.4.2). Finally, notice that this setting is sufficient to solve our optimal control problem (1.1). Existence under weaker conditions is left for further research.

REMARK 4 (On condition **C1** on ξ and $\widehat{f}^{\mathbb{P},0}$). Compared to the integrability assumption imposed in [11], for example, **C1** looks to be too strong. Note again that it is sufficient to solve our control problem. As in the previous remark, this hypothesis is related to the method we use to obtain existence of the solution of the 2BSDE. In particular, in the Lipschitz approximation procedure and in the proof of existence of the solution of the reflected BSDE (see Section A.3 in the Appendix). Weaker integrability condition is also left for further research.

2.3. *Definition, uniqueness and properties.* We consider the 2BSDE

$$\begin{aligned} Y_t = \xi + \int_t^T \widehat{f}_u^{\mathbb{P}}\big(Y_u, \widehat{a}_u^{\frac{1}{2}} Z_u\big)\, du - \left(\int_t^T Z_u\, dX_u^{c,\mathbb{P}}\right)^{\mathbb{P}} \\ - \int_t^T dM_u^{\mathbb{P}} + \int_t^T dK_u^{\mathbb{P}}. \end{aligned} \tag{2.5}$$

In this equation, $(\int_t^T Z_u \, dX_u^{c,\mathbb{P}})^{\mathbb{P}}$ denotes the stochastic integral of Z w.r.t. $X^{c,\mathbb{P}}$ under $\mathbb{P}$, $M^{\mathbb{P}}$ is a martingale orthogonal to $X^{c,\mathbb{P}}$ and $K^{\mathbb{P}}$ is a nondecreasing process.

In this part, we want to obtain the same result as [37], Theorem 4.1, for a monotone generator. The difference is that our generator is not Lipschitz continuous w.r.t. y. Here, we follow the arguments developed in [37] and we check that all the results contained in [37] still hold in our setting. In other words, we explain how their results can be extended under **H2** and **H4**. When the Lipschitz condition is not used, we simply refer to their paper.

DEFINITION 1. $(Y, Z, M^{\mathbb{P}}, K^{\mathbb{P}})$ is a solution if (2.5) is satisfied $\mathcal{P}$-q.s. and if the family $(K^{\mathbb{P}}, \mathbb{P} \in \mathcal{P})$ satisfies the *minimality condition*:

$$\begin{aligned} \operatorname*{essinf}^{\mathbb{P}}_{\mathbb{P}' \in \mathcal{P}(t,\mathbb{P},\mathbb{F}_+)} \mathbb{E}^{\mathbb{P}'}\left[\int_t^T \exp\left(\int_t^s \lambda_u^{\mathbb{P}'} \, du\right) dK_s^{\mathbb{P}'} \Big| \mathcal{F}_t^+\right] = 0, \\ 0 \le t \le T, \mathbb{P}\text{-a.s.}, \forall \mathbb{P} \in \mathcal{P}, \end{aligned} \tag{2.6}$$

where

$$\lambda_s^{\mathbb{P}'} = \frac{\widehat{f}_s^{\mathbb{P}'}(Y_s, z_s^{\mathbb{P}'}) - \widehat{f}_s^{\mathbb{P}'}(y_s^{\mathbb{P}'}, z_s^{\mathbb{P}'})}{Y_s - y_s^{\mathbb{P}'}} \mathbf{1}_{Y_s \neq y_s^{\mathbb{P}'}} \le L_1.$$

$\mathcal{P}$-q.s. means quasi-surely, that is $\mathbb{P}$-a.s. for any $\mathbb{P} \in \mathcal{P}$. In the above definition and in the rest of this section, $(y^{\mathbb{P}}, z^{\mathbb{P}}, m^{\mathbb{P}})$ is the solution under the probability measure $\mathbb{P}$ of the following BSDE:

$$\begin{aligned} y_t = \xi + \int_t^T f(u, X_{\cdot \wedge u}, y_u, \widehat{a}_u^{\frac{1}{2}} z_u, \widehat{a}_u, b_u^{\mathbb{P}}) \, du - \left(\int_t^T z_u \, dX_u^{c,\mathbb{P}}\right)^{\mathbb{P}} \\ - \int_t^T dm_u, \qquad \mathbb{P}\text{-a.s.}, \end{aligned} \tag{2.7}$$

where again m is an additional martingale, orthogonal to $X^{c,\mathbb{P}}$. Moreover, for $t \le s$ and a $\mathcal{F}_s^+$-measurable random variable ζ, $y_t^{\mathbb{P}}(s, \zeta)$ is the solution of (2.7) with terminal time s and terminal condition ζ.

REMARK 5 (Notation for solution). In the rest of this paper, (Y, Z, M, K) is a solution of a 2BSDE, whereas (y, z, m) denotes a solution of a standard BSDE and $(\widetilde{y}, \widetilde{z}, \widetilde{m}, \widetilde{k})$ stands for the solution of a reflected BSDE. If necessary, the dependence w.r.t. the probability measure will be added as a superscript $(y^{\mathbb{P}}, M^{\mathbb{P}}, \dots)$. $y(\tau, \zeta)$ always denotes the first component of the solution of a BSDE with terminal time τ and terminal condition ζ, where τ is an $\mathbb{F}^+$-stopping time and ζ is $\mathcal{F}_\tau^+$-measurable.

REMARK 6. The BSDE (2.7) is defined on $(\Omega, \mathcal{F}_T^{\mathbb{P}}, \mathbb{P})$ w.r.t. the filtration $\mathbb{F}_+^{\mathbb{P}}$ and is equivalent to the BSDE on $(\overline{\Omega}, \overline{\mathcal{F}}_T^X, \overline{\mathbb{P}})$ w.r.t. the filtration $\overline{\mathbb{F}}_+^{X,\overline{\mathbb{P}}}$:

$$\begin{aligned}\bar{y}_t = \xi(X_\cdot) + \int_t^T \widehat{f}_u^{\mathbb{P}}(\bar{y}_u, \widehat{a}_u^{\frac{1}{2}} \bar{z}_u)\, du - \left(\int_t^T \bar{z}_u\, dX_u^{c,\mathbb{P}}\right)^{\overline{\mathbb{P}}} \\ - \int_t^T d\bar{m}_u, \qquad \overline{\mathbb{P}}\text{-a.s.}\end{aligned} \tag{2.8}$$

Moreover, on the enlarged space $(\overline{\Omega}, \overline{\mathcal{F}}_T, \overline{\mathbb{P}})$, with the filtration $\overline{\mathbb{F}}_+$, one defines the BSDE

$$\begin{aligned}\widetilde{y}_t = \xi(X_\cdot) + \int_t^T \widehat{f}_u^{\mathbb{P}}(\widetilde{y}_u, \widehat{a}_u^{\frac{1}{2}} \widetilde{z}_u)\, du - \left(\int_t^T \widetilde{z}_u \widehat{a}_u^{\frac{1}{2}}\, dW_u^{\mathbb{P}}\right)^{\overline{\mathbb{P}}} \\ - \int_t^T d\widetilde{m}_u, \qquad \overline{\mathbb{P}}\text{-a.s.}\end{aligned} \tag{2.9}$$

The key point is contained in [37], Lemma 2.2, where "equivalence" between the three BSDEs is proved and with straightforward modifications in the proof, this result holds under our conditions (**H**) and **C1**.

Let us begin with the uniqueness result, which corresponds to [37], Theorem 4.2.

THEOREM 1. *Under Conditions* (**H**), **C1** *and* **C2**, *let* $(Y, Z, M^{\mathbb{P}}, K^{\mathbb{P}})$ *be a solution of* (2.5) *and for any* $\mathbb{P} \in \mathcal{P}$, *let* $(y^{\mathbb{P}}, z^{\mathbb{P}}, m^{\mathbb{P}})$ *be the solution of the BSDE* (2.7) *in* $\mathbb{D}_0^{\bar{\mathfrak{p}}\mathfrak{q}}(\mathbb{F}_+^{\mathbb{P}}, \mathbb{P}) \times \mathbb{H}_0^{\bar{\mathfrak{p}}\mathfrak{q}}(\mathbb{F}_+^{\mathbb{P}}, \mathbb{P}) \times \mathbb{M}_0^{\bar{\mathfrak{p}}\mathfrak{q}}(\mathbb{F}_+^{\mathbb{P}}, \mathbb{P})$. *Then for any* $0 \leq t_1 \leq t_2 \leq T$

$$Y_{t_1} = \operatorname*{esssup^{\mathbb{P}}}_{\mathbb{P}' \in \mathcal{P}(t_1, \mathbb{P}, \mathbb{F}_+)} y_{t_1}^{\mathbb{P}'}(t_2, Y_{t_2}). \tag{2.10}$$

Thus uniqueness holds in $\mathbb{D}_0^{\bar{\mathfrak{p}}\mathfrak{q}}(\mathbb{F}_+^{\mathcal{P}_0}) \times \mathbb{H}_0^p(\mathbb{F}_+^{\mathcal{P}_0}) \times \mathbb{M}_0^p((\mathbb{F}_+^{\mathbb{P}})_{\mathbb{P} \in \mathcal{P}_0}) \times \mathbb{I}_0^p((\mathbb{F}_+^{\mathbb{P}})_{\mathbb{P} \in \mathcal{P}_0})$ *for any* $1 < p$ *satisfying the condition* (2.4).

PROOF. In [37], the proof is divided in three steps. There is no modification in the first one. By comparison principle for BSDE (see Lemma A.2):

$$Y_{t_1} \geq \operatorname*{esssup^{\mathbb{P}}}_{\mathbb{P}' \in \mathcal{P}(t_1, \mathbb{P}, \mathbb{F}_+)} y_{t_1}^{\mathbb{P}'}(t_2, Y_{t_2}), \qquad \mathbb{P}\text{-a.s.}$$

For the second step, we have almost the same estimate on $K^{\mathbb{P}'}$. For $p > 1$, satisfying (2.4),

$$\begin{aligned}&(K_{t_2}^{\mathbb{P}'} - K_{t_1}^{\mathbb{P}'})^p \\ &\quad \leq C\left[\sup_{t_1 \leq t \leq t_2} |Y_t|^p + \left(\int_{t_1}^{t_2} |\widehat{f}_s^{\mathbb{P}',0}|\, ds\right)^p + \left(\int_{t_1}^{t_2} |\widehat{a}_s^{\frac{1}{2}} Z_s|\, ds\right)^p\right]\end{aligned}$$

$$
\begin{aligned}
&+ C\left[\left(\int_{t_1}^{t_2} \widehat{\Psi}_s(1+|Y_s|^{\mathfrak{q}})\,ds\right)^p + \left|\int_{t_1}^{t_2} Z_s\,dX_s^{c,\mathbb{P}'}\right|^p + \left|\int_{t_1}^{t_2} dM_s^{\mathbb{P}'}\right|^p\right] \\
&\le C\left[\sup_{t_1\le t\le t_2}|Y_t|^p + \left(\int_{t_1}^{t_2}|\widehat{f}_s^{\mathbb{P}',0}|\,ds\right)^p + \left(\int_{t_1}^{t_2}|\widehat{a}_s^{\frac{1}{2}} Z_s|\,ds\right)^p\right] \\
&+ C\left[\left|\int_{t_1}^{t_2} Z_s\,dX_s^{c,\mathbb{P}'}\right|^p + \left|\int_{t_1}^{t_2} dM_s^{\mathbb{P}'}\right|^p\right] \\
&+ C\left[\left(\int_{t_1}^{t_2}(\widehat{\Psi}_s)^p\,ds\right)\left(1+\sup_{t_1\le t\le t_2}|Y_s|^{p\mathfrak{q}}\right)\right].
\end{aligned}
$$

Hence

$$
\begin{aligned}
&\sup_{\mathbb{P}'\in\mathcal{P}(t_1,\mathbb{P},\mathbb{F}_+)} \mathbb{E}^{\mathbb{P}'}\left[(K_{t_2}^{\mathbb{P}'}-K_{t_1}^{\mathbb{P}'})^p\right] \\
&\qquad \le C\left[\phi_f^{p,\kappa} + \|Y\|_{\mathbb{D}_0^p}^p + \|Z\|_{\mathbb{H}_0^p}^p + \sup_{\mathbb{P}\in\mathcal{P}_0}\mathbb{E}^{\mathbb{P}}\left([M^{\mathbb{P}}]_T\right)^{p/2}\right] \\
&\qquad\quad + C\left[\|\widehat{\Psi}\|_{\mathbb{L}^{\widehat{p}}}^p\left(1+\|Y\|_{\mathbb{D}_0^{\bar{\mathfrak{p}}\mathfrak{q}}}^{\bar{\mathfrak{p}}\mathfrak{q}}\right)\right]
\end{aligned}
$$

for $\widehat{p} = \frac{p\bar{\mathfrak{p}}}{(\bar{\mathfrak{p}}-p)} \le \varrho$. The rest of this step does not change (upward directed family, see also Theorem 4.4 in [40]), and thus

$$
C_{t_1}^{\mathbb{P}} = \operatorname*{esssup^{\mathbb{P}}}_{\mathbb{P}'\in\mathcal{P}(t_1,\mathbb{P},\mathbb{F}_+)} \mathbb{E}^{\mathbb{P}'}\left[(K_{t_2}^{\mathbb{P}'}-K_{t_1}^{\mathbb{P}'})^p|\mathcal{F}_{t_1}^+\right] < +\infty, \qquad \mathbb{P}\text{-a.s.}
$$

The third step can be followed almost exactly. We define for $t\ge t_1$

$$
\Lambda_t^{\mathbb{P}'} = \exp\left(\int_{t_1}^t \lambda_u^{\mathbb{P}'}\,du\right), \qquad \Delta_t^{\mathbb{P}'} = \exp\left[-\int_{t_1}^t \zeta_s^{\mathbb{P}'}\,dW_s^{\mathbb{P}'} - \frac{1}{2}\int_{t_1}^t \|\zeta_s^{\mathbb{P}'}\|^2\,ds\right].
$$

Estimate (4.6) in [37] holds only for Δ and any constant $p>1$:

$$
\mathbb{E}^{\mathbb{P}'\otimes\mathbb{P}_0}\left[\sup_{t_1\le t\le t_2}|\Delta_t^{\mathbb{P}'}|^p|\overline{\mathcal{F}}_{t_1}^+\right] \le C, \qquad \mathbb{P}'\otimes\mathbb{P}_0\text{-a.s.}
$$

since we only have an upper estimate on $\lambda_u^{\mathbb{P}'}\le L_1$. Then the linearization argument shows

$$
\delta\mathcal{Y}_{t_1} = Y_{t_1} - \widetilde{y}_{t_1}^{\mathbb{P}'\otimes\mathbb{P}_0} = \mathbb{E}^{\mathbb{P}'\otimes\mathbb{P}_0}\left[\int_{t_1}^{t_2}\Lambda_s^{\mathbb{P}'}\Delta_s^{\mathbb{P}'}\,dK_s^{\mathbb{P}'}\Big|\overline{\mathcal{F}}_{t_1}^+\right].
$$

Thus by the monotone condition **H2** and for $1<p\le \frac{\varrho\bar{\mathfrak{p}}}{\varrho+\bar{\mathfrak{p}}}$:

$$
\begin{aligned}
\delta\mathcal{Y}_{t_1} &\le \left(\mathbb{E}^{\mathbb{P}'\otimes\mathbb{P}_0}\left[\sup_{t_1\le s\le t_2}(\Delta_s^{\mathbb{P}'})^{\frac{p+1}{p-1}}|\overline{\mathcal{F}}_{t_1}^+\right]\right)^{\frac{p-1}{p+1}} \\
&\quad\times\left(\mathbb{E}^{\mathbb{P}'\otimes\mathbb{P}_0}\left[\left(\int_{t_1}^{t_2}\Lambda_s^{\mathbb{P}'}\,dK_s^{\mathbb{P}'}\right)^{\frac{p+1}{2}}\Big|\overline{\mathcal{F}}_{t_1}^+\right]\right)^{\frac{2}{p+1}}
\end{aligned}
$$

$$\le C\left(\mathbb{E}^{\mathbb{P}'\otimes\mathbb{P}_0}\left[\left(\int_{t_1}^{t_2}\Lambda_s^{\mathbb{P}'}\,dK_s^{\mathbb{P}'}\right)^p\Big|\overline{\mathcal{F}}_{t_1}^+\right]\right)^{\frac{1}{p+1}}$$

$$\times\left(\mathbb{E}^{\mathbb{P}'\otimes\mathbb{P}_0}\left[\int_{t_1}^{t_2}\Lambda_s^{\mathbb{P}'}\,dK_s^{\mathbb{P}'}\Big|\overline{\mathcal{F}}_{t_1}^+\right]\right)^{\frac{1}{p+1}}$$

$$\le C\exp\left(\frac{pL_1T}{p+1}\right)(\mathbb{E}^{\mathbb{P}'\otimes\mathbb{P}_0}[(K_{t_2}^{\mathbb{P}'}-K_{t_1}^{\mathbb{P}'})^p|\overline{\mathcal{F}}_{t_1}^+])^{\frac{1}{p+1}}$$

$$\times\left(\mathbb{E}^{\mathbb{P}'\otimes\mathbb{P}_0}\left[\int_{t_1}^{t_2}\Lambda_s^{\mathbb{P}'}\,dK_s^{\mathbb{P}'}\Big|\overline{\mathcal{F}}_{t_1}^+\right]\right)^{\frac{1}{p+1}}$$

$$\le C\exp\left(\frac{pL_1T}{p+1}\right)(C_{t_1}^{\mathbb{P}})^{\frac{1}{p+1}}\left(\mathbb{E}^{\mathbb{P}'\otimes\mathbb{P}_0}\left[\int_{t_1}^{t_2}\Lambda_s^{\mathbb{P}'}\,dK_s^{\mathbb{P}'}\Big|\overline{\mathcal{F}}_{t_1}^+\right]\right)^{\frac{1}{p+1}}.$$

By arbitrariness of $\mathbb{P}'$, and from the condition (2.6), we deduce

$$Y_{t_1}-\operatorname*{esssup}^{\mathbb{P}}_{\mathbb{P}'\in\mathcal{P}(t_1,\mathbb{P},\mathbb{F}_+)} y_{t_1}^{\mathbb{P}'}(t_2,Y_{t_2})\le 0,\qquad \mathbb{P}\text{-a.s.}$$

This achieves the proof of the theorem. □

The comparison principle ([37], Theorem 4.3), the a priori estimate ([37], Theorem 4.4) and the stability result ([37], Theorem 4.5) for 2BSDE remain unchanged here. Indeed it is a direct consequence of Lemmas A.2 and A.3 and the formula (2.10). The other arguments follow exactly the proofs in [37].

2.4. *Existence of a solution of a solution for second-order BSDE.* In order to obtain a solution for the 2BSDE (2.5), we define for any $(t,\omega)\in[0,T]\times\Omega$,

$$\widehat{\mathcal{Y}}_t(\omega):=\sup_{\mathbb{P}\in\mathcal{P}(t,\omega)}\mathbb{E}^{\mathbb{P}}(y_t^{\mathbb{P}}). \tag{2.11}$$

This quantity $\widehat{\mathcal{Y}}$ is a "candidate" to be a solution of the 2BSDE (2.5).

2.4.1. *Measurability property of $\widehat{\mathcal{Y}}$.* Our aim is to prove that the conclusion of [37], Theorem 2.1, holds in our setting. To avoid to write again, the complete machinery developed in [37], Section 2.4, we will use their Proposition 2.1. We already know that the solution $(y^{\mathbb{P}},z^{\mathbb{P}},m^{\mathbb{P}})$ exists since (**H**) holds. Moreover, from Lemma A.4 and the condition on $\bar{\mathfrak{p}}$ and ϱ, $(y^{\mathbb{P}},z^{\mathbb{P}},m^{\mathbb{P}})$ can be approximated by solutions of Lipschitz BSDE in the space $\mathbb{D}^p\times\mathbb{H}^p\times\mathbb{M}^p$ for any $1<p\le\frac{\varrho\bar{\mathfrak{p}}}{\varrho+\bar{\mathfrak{p}}}<\bar{\mathfrak{p}}$. Moreover, Lemmas A.2 and A.3 (for comparison and stability) hold. Thus the conclusion of [37], Proposition 2.1, is satisfied. Hence, as in [37], Theorem 2.1, the map

$$(s,\omega,\mathbb{P})\mapsto\widehat{\mathcal{Y}}_s(\omega)=\sup_{\mathbb{P}\in\mathcal{P}(s,\omega)}\mathbb{E}^{\mathbb{P}}(y_s^{\mathbb{P}})$$

is measurable and $(t,\omega) \mapsto \widehat{\mathcal{Y}}_t(\omega)$ is $\mathcal{B}([0,T]) \otimes \mathcal{F}_T$-universally measurable. Finally, for all $(t,\omega) \in [0,T] \times \Omega$ and $\mathbb{F}$-stopping times τ taking values in $[t,T]$,

$$\widehat{\mathcal{Y}}_t(\omega) = \sup_{\mathbb{P} \in \mathcal{P}(t,\omega)} \mathbb{E}^{\mathbb{P}}\big(y_t^{\mathbb{P}}(\tau, \widehat{\mathcal{Y}}_\tau)\big),$$

where $y^{\mathbb{P}}(\tau, \widehat{\mathcal{Y}}_\tau)$ denotes the first component of the solution (y,z,m) of the BSDE (2.7) with terminal time τ and terminal condition $\widehat{\mathcal{Y}}_\tau$ under the probability measure $\mathbb{P}$. Notice that for any $\mathbb{P} \in \mathcal{P}(t,\omega)$ and any stopping time τ with values in $[t,T]$

$$\mathbb{E}^{\mathbb{P}}\big(|\widehat{\mathcal{Y}}_\tau|^{\bar{\mathfrak{p}}\mathfrak{q}}\big) < +\infty. \tag{2.12}$$

The proof is contained in [37], Theorem 2.1. Let us recall the main ideas. First, for every $\mathbb{P} \in \mathcal{P}(t,\omega)$ and $\varepsilon > 0$, using the measurable selection theorem (see, e.g., Proposition 7.50 of [7] or Theorem III.82 in [12]), one can choose a family of probability measures $(\mathbb{Q}^\varepsilon_{\mathsf{w}})_{\mathsf{w} \in \Omega}$ such that $\mathsf{w} \longmapsto \mathbb{Q}^\varepsilon_{\mathsf{w}}$ is $\mathcal{F}_\tau$-measurable, and for $\mathbb{P}$-a.e. $\mathsf{w} \in \Omega$,

$$\mathbb{Q}^\varepsilon_{\mathsf{w}} \in \mathcal{P}\big(\tau(\mathsf{w}), \mathsf{w}\big) \quad \text{and} \quad \widehat{\mathcal{Y}}_{\tau(\mathsf{w})}(\mathsf{w}) - \varepsilon \le \mathbb{E}^{\mathbb{Q}^\varepsilon_{\mathsf{w}}}\big[y^{\mathbb{Q}^\varepsilon_{\mathsf{w}}}_{\tau(\mathsf{w})}(T,\xi)\big] \le \widehat{\mathcal{Y}}_{\tau(\mathsf{w})}(\mathsf{w}). \tag{2.13}$$

The integrability of $\widehat{\mathcal{Y}}_\tau$ is a direct consequence of a priori estimates on the solution of BSDE (2.7) (see Lemma A.1 and the estimates below). We can then define the concatenated probability $\mathbb{P}^\varepsilon := \mathbb{P} \otimes_\tau \mathbb{Q}^\varepsilon_\cdot$ so that, by *Assumption* 1(iii), $\mathbb{P}^\varepsilon \in \mathcal{P}(t,\omega)$. Notice that $\mathbb{P}$ and $\mathbb{P}^\varepsilon$ coincide on $\mathcal{F}_\tau$, and hence $\mathbb{E}^{\mathbb{P}^\varepsilon}[y_\tau^{\mathbb{P}^\varepsilon} | \mathcal{F}_\tau] \in \mathbb{L}^{\bar{\mathfrak{p}}\mathfrak{q}}_{t,\omega}(\mathcal{F}_\tau, \mathbb{P})$. It follows then from the inequality in (2.13) that $\mathbb{E}^{\mathbb{P}}[|\widehat{\mathcal{Y}}_\tau|^{\bar{\mathfrak{p}}\mathfrak{q}}] < \infty$ and the upper bound depends on $\|\xi\|^{\bar{\mathfrak{p}}\mathfrak{q}}_{\mathbb{L}_0^{\bar{\mathfrak{p}}\mathfrak{q},\kappa}}$ and $\phi_f^{\bar{\mathfrak{p}}\mathfrak{q},\kappa}$, but not on the choice of τ.

2.4.2. *Path regularization*. As in [37], Section 3, to obtain a solution of the 2BSDE (2.5), we shall characterize a càdlàg modification of $\widehat{\mathcal{Y}}$ defined by (2.11). Again we do not want to write all the details of the proof. Let us only explain the main difficulties due to the monotonicity condition **H2**. The proof of [37], Lemma 3.1, does not use the Lipschitz property of f.

The next step is to prove existence of right- and left-limits for $\widehat{\mathcal{Y}}$ outside a $\mathcal{P}_0$-polar sets ([37], Lemma 3.2). The proof is based on a downcrossing estimate and the Lipschitz constant of f w.r.t. y explicitly appears. Since f is no more Lipschitz continuous w.r.t. y, we show a downcrossing inequality for $\mathcal{Y}$, but under stronger conditions on ξ and $\widehat{f}^{\mathbb{P},0}$. Let us assume that there exists a constant $\mathfrak{C}$ such that for any t and ω,

$$\operatorname*{esssup}_{\mathbb{P} \in \mathcal{P}(t,\omega)} \mathbb{E}^{\mathbb{P}}\Big[|\xi| + \sup_{s \in [t,T]} \big|\widehat{f}_s^{\mathbb{P},0}\big|\Big] \le \mathfrak{C}. \tag{2.14}$$

Under this condition and Estimate (A.4), $y^{\mathbb{P}}$ and $\widehat{\mathcal{Y}}$ are also essentially bounded and we still denote by $\mathfrak{C}$ the upper bound.

An estimate on the downcrossings of $\widehat{\mathcal{Y}}$ under condition (2.14). For simplicity, we assume that $L_1 = 0$ in the monotonicity condition **H2** (see Remark 2) and we keep the same notation and the same scheme as in [37]. For any $a < b$ and for $J_N = \{\tau_0, \dots, \tau_N\}$ with $0 = \tau_0 < \tau_1 < \cdots < \tau_N = T$, a finite family of $\mathbb{F}$-stopping times, we denote by $D_a^b(\widehat{\mathcal{Y}}, J_N)$ the number of downcrossings of the process $(\widehat{\mathcal{Y}}_{\tau_k}, 0 \le k \le N)$ from b to a.

Let us fix $\mathbb{P} \in \mathcal{P}_0$ and consider the solution $(y^i, z^i, m^i) = (y^{i,\overline{\mathbb{P}}_\omega^{\tau_{i-1}(\omega)}}, z^{i,\overline{\mathbb{P}}_\omega^{\tau_{i-1}(\omega)}}, m^{i,\overline{\mathbb{P}}_\omega^{\tau_{i-1}(\omega)}})$ of the BSDE with terminal condition $\widehat{\mathcal{Y}}_{\tau_i}$ and driver $\widehat{f}$ on the enlarged space under the probability measure $\overline{\mathbb{P}}_\omega^{\tau_{i-1}(\omega)} = \mathbb{P}_\omega^{\tau_{i-1}(\omega)} \otimes \mathbb{P}_0$ and on the interval $[\tau_{i-1}, \tau_i]$:

$$y_t^i = \widehat{\mathcal{Y}}_{\tau_i} + \int_t^{\tau_i} \widehat{f}_u^{\mathbb{P}_\omega^{\tau_{i-1}(\omega)}}(y_u^i, \widehat{a}_u^{\frac{1}{2}} z_u^i)\, du - \int_t^{\tau_i} z_u^i \widehat{a}_u^{\frac{1}{2}}\, dW_u^{\mathbb{P}_\omega^{\tau_{i-1}(\omega)}} \\ - \int_t^{\tau_i} dm_u^i, \qquad \overline{\mathbb{P}}_\omega^{\tau_{i-1}(\omega)}\text{-a.s.}$$

We can linearize the previous BSDE (see the arguments before equations (A.6) and (A.7)) to obtain

$$y_t^i = \widehat{\mathcal{Y}}_{\tau_i} + \int_t^{\tau_i} [\widehat{f}_u^{\mathbb{P}_\omega^{\tau_{i-1}(\omega)}}(y_u^i, 0) - \widehat{f}_u^{\mathbb{P}_\omega^{\tau_{i-1}(\omega)},0}]\, du \\ + \int_t^{\tau_i} \widehat{f}_u^{\mathbb{P}_\omega^{\tau_{i-1}(\omega)},0}\, du + \int_t^{\tau_i} \zeta_u^i \widehat{a}_u^{\frac{1}{2}} z_u^i\, du - \int_t^{\tau_i} z_u^i \widehat{a}_u^{\frac{1}{2}}\, dW_u^{\mathbb{P}_\omega^{\tau_{i-1}(\omega)}} - \int_t^{\tau_i} dm_u^i.$$

Note that we do not use the complete linearization of the BSDE. By the very definition of $\widehat{\mathcal{Y}}$, we get

$$\mathbb{E}^{\overline{\mathbb{P}}_\omega^{\tau_{i-1}(\omega)}}(y_{\tau_{i-1}}^i) \le \widehat{\mathcal{Y}}_{\tau_{i-1}}(\omega).$$

Now we consider again on $[\tau_{i-1}, \tau_i]$ and the probability space, the solution of the following BSDE:

$$\widetilde{y}_t^i = \widehat{\mathcal{Y}}_{\tau_i} + \int_t^{\tau_i} [\widehat{f}_u^{\mathbb{P}_\omega^{\tau_{i-1}(\omega)}}(\widetilde{y}_u^i, 0) - \widehat{f}_u^{\mathbb{P}_\omega^{\tau_{i-1}(\omega)},0}] \mathbf{1}_{\widetilde{y}_u^i \ge 0}\, du - \int_t^{\tau_i} |\widehat{f}_u^{\mathbb{P}_\omega^{\tau_{i-1}(\omega)},0}|\, du \\ + \int_t^{\tau_i} \zeta_u^i \widehat{a}_u^{\frac{1}{2}} \widetilde{z}_u^i\, du - \int_t^{\tau_i} \widetilde{z}_u^i \widehat{a}_u^{\frac{1}{2}}\, dW_u^{\mathbb{P}_\omega^{\tau_{i-1}(\omega)}} - \int_t^{\tau_i} d\widetilde{m}_u^i, \qquad \overline{\mathbb{P}}_\omega^{\tau_{i-1}(\omega)}\text{-a.s.}$$

Here, the generator is

$$(t, y, z) \mapsto [\widehat{f}_t^{\mathbb{P}_\omega^{\tau_{i-1}(\omega)}}(y, 0) - \widehat{f}_t^{\mathbb{P}_\omega^{\tau_{i-1}(\omega)},0}] \mathbf{1}_{y \ge 0} - |\widehat{f}_t^{\mathbb{P}_\omega^{\tau_{i-1}(\omega)},0}| + \zeta_t^i \widehat{a}_t^{\frac{1}{2}} z$$

and satisfies Condition (**H**). By the monotonicity condition **H2** with $L_1 = 0$, for $s \in [\tau_{i-1}, \tau_i]$

$$[\widehat{f}_s^{\mathbb{P}_\omega^{\tau_{i-1}(\omega)}}(y, 0) - \widehat{f}_s^{\mathbb{P}_\omega^{\tau_{i-1}(\omega)},0}]\, \mathrm{sign}(y) \le 0.$$

In particular, when $y \geq 0$, the increment is nonpositive. Hence from the comparison principle for BSDEs, we have: $\widetilde{y}^i_{\tau_{i-1}} \leq y^i_{\tau_{i-1}}$.

Let for $t \in [\tau_{i-1}, \tau_i]$

$$L_t := \mathcal{E}\left(\int_{\tau_{i-1}}^t \zeta^i_u \, dW_u^{\mathbb{P}_\omega^{\tau_{i-1}(\omega)}}\right),$$

be the stochastic exponential and

$$\Xi^i_t = -[\widehat{f}_t^{\mathbb{P}_\omega^{\tau_{i-1}(\omega)}}(\widetilde{y}^i_t, 0) - \widehat{f}_t^{\mathbb{P}_\omega^{\tau_{i-1}(\omega)},0}]\mathbf{1}_{\widetilde{y}^i_t \geq 0}.$$

From our previous arguments, the key point is that $\Xi^i_t \geq 0$. Then

$$\mathbb{E}^{\mathbb{P}_\omega^{\tau_{i-1}(\omega)}}\left[L_{\tau_i}\left(\widehat{\mathcal{Y}}_{\tau_i} - \int_{\tau_{i-1}}^{\tau_i} \Xi^i_u \, du - \int_{\tau_{i-1}}^{\tau_i} |\widehat{f}_u^{\mathbb{P}_\omega^{\tau_{i-1}(\omega)},0}| \, du\right)\right] \leq \widehat{\mathcal{Y}}_{\tau_{i-1}}(\omega).$$

And by definition of the r.c.p.d.

$$\mathbb{E}^{\overline{\mathbb{Q}}}\left[\widehat{\mathcal{Y}}_{\tau_i} - \int_{\tau_{i-1}}^{\tau_i} \Xi^i_u \, du - \int_{\tau_{i-1}}^{\tau_i} |\widehat{f}_u^{\mathbb{P}_\cdot^{\tau_{i-1}(\cdot)},0}| \, du \Big| \overline{\mathcal{F}}^+_{\tau_{i-1}}\right] \leq \widehat{\mathcal{Y}}_{\tau_{i-1}}, \qquad \mathbb{P} \otimes \mathbb{P}_0\text{-a.s.},$$

where $\overline{\mathbb{Q}}$ is equivalent to $\mathbb{P} \otimes \mathbb{P}_0$ with density

$$\frac{d\overline{\mathbb{Q}}}{d(\mathbb{P} \otimes \mathbb{P}_0)} = \mathcal{E}\left(\int_{\tau_{i-1}}^t \zeta^i_u \, dW^{\mathbb{P}}_u\right), \qquad t \in [\tau_{i-1}, \tau_i].$$

We define $\Xi_t = \sum_{i=1}^n \Xi^i_t \mathbf{1}_{[\tau_{i-1}, \tau_i)}(t)$ and the discrete process

$$V_i := V_{\tau_i} = \widehat{\mathcal{Y}}_{\tau_i} - \int_0^{\tau_i} (\Xi_s + |\widehat{f}_s^{\mathbb{P},0}|) \, ds.$$

For $b > 0$, let

$$\overline{V}_i := V_i \wedge \left(b - \int_0^{\tau_i} (\Xi_s + |\widehat{f}_s^{\mathbb{P},0}|) \, ds\right).$$

These two processes V and $\overline{V}$ are $\overline{\mathbb{Q}}$-supermartingales relative to $\overline{\mathbb{F}}$ (see also the proof of Lemma A.1 in [8] for more details). Up to this point, we do not change the proof of [37], because we do not use Lipschitz continuity argument.

We also introduce

$$u_t = b - \int_0^t (\Xi_s + |\widehat{f}_s^{\mathbb{P},0}|) \, ds, \qquad \ell_t = -\int_0^t (\Xi_s + |\widehat{f}_s^{\mathbb{P},0}|) \, ds,$$

together with $u_i = u_{\tau_i}$ and $\ell_i = \ell_{\tau_i}$. Remark that $D^b_0(\widehat{\mathcal{Y}}, J_N) \leq D^u_\ell(V, J_N) = D^u_\ell(\overline{V}, J_N)$. Let us now explain how to derive a control on the downcrossings under the monotonicity condition, using the proof of inequality (12.5), page 446 in [13] (see pages 448–449). We define $\theta_0 = 0$,

$$S_1 = \min\{j \geq 0, \overline{V}_j \geq u_j\}, \qquad \theta_1 = S_1 \wedge N$$

and

$$S_k = \begin{cases} \min\{j > \theta_{k-1}, \overline{V}_j \geq u_j\}, & k \text{ odd}, k \geq 3, \\ \min\{j > \theta_{k-1}, \overline{V}_j \leq \ell_j\}, & k \text{ even}, k \geq 2, \end{cases}$$

$\theta_k = S_k \wedge N$. We have

$$\overline{V}_0 - \overline{V}_N = \sum_{j=0}^{N-1} [\overline{V}_{\theta_j} - \overline{V}_{\theta_{j+1}}].$$

Each bracket has a nonnegative expectation (supermartingale inequality). We shall give a lower bound for each bracket with odd j. On the set where the number of downcrossings is k, the first k brackets in the above sum with odd j are larger than: $u_{T_j} - \ell_{T_{j+1}} \geq u_{T_j} - \ell_{T_j} = b$ since ℓ is decreasing. For the other terms (again with odd j), only $\overline{V}_{T_{2k+1}} - \overline{V}_{T_{2k+2}}$ (i.e., $j = 2k+1$) may be nonzero and is bounded from below by $u_T - \overline{V}_T$. Hence we obtain the next estimate:

$$\mathbb{E}^{\overline{\mathbb{Q}}}[\overline{V}_0 - \overline{V}_N] \geq \mathbb{E}^{\overline{\mathbb{Q}}}[bD^u_\ell(\overline{V}, J_N)] + \mathbb{E}^{\overline{\mathbb{Q}}}[(u_T - \overline{V}_T) \wedge 0].$$

Thereby

$$\begin{aligned} b\mathbb{E}^{\overline{\mathbb{Q}}}[D^b_0(\widehat{\mathcal{Y}}, J_N)] &\leq b\mathbb{E}^{\overline{\mathbb{Q}}}[D^u_\ell(\overline{V}, J_N)] \\ &\leq \mathbb{E}^{\overline{\mathbb{Q}}}[\overline{V}_0 - \overline{V}_T - (u_T - \overline{V}_T) \wedge 0] \\ &\leq \mathbb{E}^{\overline{\mathbb{Q}}}\Big[(\widehat{\mathcal{Y}}_0 \wedge b) - (\widehat{\mathcal{Y}}_T \wedge b) + \int_0^T (\Xi_s + |\widehat{f}^{\mathbb{P},0}_s|)\, ds\Big] \\ &\leq \mathbb{E}^{\overline{\mathbb{Q}}}\Big[(\widehat{\mathcal{Y}}_0 \wedge b) + (\widehat{\mathcal{Y}}_T \wedge b)^- + \int_0^T (\Xi_s + |\widehat{f}^{\mathbb{P},0}_s|)\, ds\Big]. \end{aligned}$$

Since ζ^i is a bounded process, using Hölder's inequality, we get that for some $1 < p \leq \varrho$, there exists a constant C depending on p and L_2 such that

$$\begin{aligned} & b\mathbb{E}^{\overline{\mathbb{Q}}}[D^b_0(\widehat{\mathcal{Y}}, J_N)] \\ & \quad \leq C\bigg(\mathbb{E}^{\overline{\mathbb{P}}}\Big[(\widehat{\mathcal{Y}}_0 \wedge b)^p + ((\widehat{\mathcal{Y}}_T \wedge b)^-)^p + \int_0^T ((\Xi_s)^p + |\widehat{f}^{\mathbb{P},0}_s|^p)\, ds\Big]\bigg)^{1/p}. \end{aligned}$$

To complete the proof, from Condition **C2** and estimate (2.12), we only need to control the term Ξ of the right-hand side. Recall that on $[\tau_{i-1}, \tau_i)$,

$$\Xi_t = \Xi^i_t = -[\widehat{f}_t^{\mathbb{P}^{\tau_{i-1}(\omega)}_\omega}(\widetilde{y}^i_t, 0) - \widehat{f}_t^{\mathbb{P}^{\tau_{i-1}(\omega)}_\omega, 0}]\mathbf{1}_{\widetilde{y}^i_t \geq 0}$$

is nonnegative. Using condition (2.14) and estimate (A.4), from Hypothesis **H4**, we deduce that for $t \in [\tau_{i-1}, \tau_i)$

$$|\Xi_t|^p \leq (1 + \mathfrak{C}^{\mathfrak{q}})^p (\widehat{\Psi}_t)^p.$$

Since $p \leq \varrho$, from condition **C1**, there exists a constant C independent of the choice of τ_i such that

$$\mathbb{E}^{\overline{\mathbb{P}}}\left[\int_0^T (\Xi_s)^p \, ds\right] \leq C.$$

Therefore, $D_0^b(\widehat{\mathcal{Y}}, J_N)$ is $\overline{\mathbb{Q}} - a.s.$ finite. Then for $a < b$, we have the same inequality:

$$\mathbb{E}^{\overline{\mathbb{Q}}}[D_a^b(\widehat{\mathcal{Y}}, J_N)] \\ \leq \frac{1}{b-a}\mathbb{E}^{\overline{\mathbb{Q}}}\left[(\widehat{\mathcal{Y}}_0 \wedge (b-a)) + (\widehat{\mathcal{Y}}_T \wedge (b-a))^- + \int_0^T (|\widehat{f}_s^{\mathbb{P},0}| - \Xi_s) \, ds\right].$$

This estimate implies that $D_a^b(\widehat{\mathcal{Y}}, J_N)$ is $\overline{\mathbb{Q}}$-a.s. finite. Since the right-hand side does not depend on N, we can extend this estimate to any countable family of $\mathbb{F}$-stopping times. And the conclusion of [37], Lemma 3.2, still holds under (2.14).

Let us define $\widehat{\mathcal{Y}}^+$ by

$$\widehat{\mathcal{Y}}_t^+ := \limsup_{r \in \mathbb{Q} \cap (t,T], r \downarrow t} \widehat{\mathcal{Y}}_r.$$

Let us stress that [37], Lemmata 3.3 and 3.5, does not use Lipschitz continuity of the generator w.r.t. y. As we did for the downcrossing estimate, we adapt the proof of [37], Lemma 3.4, to obtain that $\widehat{\mathcal{Y}}^+$ is càdlàg, $\mathcal{P}_0$-q.s. Moreover, since the Lipschitz continuity w.r.t. y is not involved, the representation formula of [37], Lemma 3.5, holds, that is for any $0 \leq t \leq T$, for any $\mathbb{P} \in \mathcal{P}_0$, we have $\mathbb{P}$-a.s.

$$\widehat{\mathcal{Y}}_t^+ = \operatorname*{esssup}^{\mathbb{P}}_{\mathbb{P}' \in \mathcal{P}_0(t,\mathbb{P},\mathbb{F}_+)} y_t^{\mathbb{P}'}(T, \xi).$$

From condition (2.14), we deduce that $\widehat{\mathcal{Y}}^+$ is essentially bounded (again by $\mathfrak{C}$), and thus belongs to $\mathbb{D}_0^{\bar{\mathfrak{p}}\mathfrak{q}}(\mathbb{F}^{\mathcal{P}_0+})$. Finally, from our Section A.3 on reflected BSDE, we can argue as in [37], Lemma 3.6, and we obtain the next result.

LEMMA 1. *Under Conditions* (**H**)–**C1**–**C2** *and assumption* (2.14), *this process $\widehat{\mathcal{Y}}^+$ is càdlàg, $\mathcal{P}_0$-q.s. and belongs to $\mathbb{D}_0^{\bar{\mathfrak{p}}\mathfrak{q}}(\mathbb{F}^{\mathcal{P}_0+})$. Moreover it is a semimartingale under any $\mathbb{P} \in \mathcal{P}_0$ with an explicit decomposition: there exists $(Z^{\mathbb{P}}, M^{\mathbb{P}}, K^{\mathbb{P}}) \in \mathbb{H}_0^p(\mathbb{F}^{\mathbb{P}+}, \mathbb{P}) \times \mathbb{M}_0^p(\mathbb{F}^{\mathbb{P}+}, \mathbb{P}) \times \mathbb{I}_0^p(\mathbb{F}^{\mathbb{P}+}, \mathbb{P})$ with $1 < p \leq \frac{\varrho\bar{\mathfrak{p}}}{\varrho+\bar{\mathfrak{p}}}$ and for any $t \in [0, T]$, $\mathbb{P}$-a.s.*

$$\widehat{\mathcal{Y}}_t^+ = \xi + \int_t^T \widehat{f}_s^{\mathbb{P}}(\widehat{\mathcal{Y}}_s^+, \widehat{a}_s^{\frac{1}{2}} Z_s^{\mathbb{P}}) \, ds - \int_t^T Z_s^{\mathbb{P}} \, dX_s^{c,\mathbb{P}} - \int_t^T dM_s^{\mathbb{P}} + \int_t^T dK_s^{\mathbb{P}}.$$

Moreover, there is some $\mathbb{F}^{\mathcal{P}_0}$-predictable process Z which aggregates the family $(Z^{\mathbb{P}})_{\mathbb{P} \in \mathcal{P}_0}$.

2.4.3. *Conclusion*. Now we come to the existence result (equivalent to [37], Theorems 4.1 and 4.4).

THEOREM 2. *Under Conditions* (**H**)–**C1**–**C2**, *there exists a solution* $(Y, Z, M^{\mathbb{P}}, K^{\mathbb{P}})$ *to the 2BSDE* (2.5) *in the space* $\mathbb{D}_0^{\bar{\mathfrak{p}}\mathfrak{q}}(\mathbb{F}_+^{\mathcal{P}_0}) \times \mathbb{H}_0^p(\mathbb{F}_+^{\mathcal{P}_0}) \times \mathbb{M}_0^p((\mathbb{F}_+^{\mathbb{P}})_{\mathbb{P}\in\mathcal{P}_0}) \times \mathbb{I}_0^p((\mathbb{F}_+^{\mathbb{P}})_{\mathbb{P}\in\mathcal{P}_0})$ *for any* $p > 1$ *satisfying* (2.4). *More precisely, there exists a constant* C *depending on* $\bar{\mathfrak{p}}$, $\mathfrak{q}$ T, L_1, L_2 *such that*

$$\begin{aligned}(2.15)\qquad &\|Y\|_{\mathbb{D}_0^{\bar{\mathfrak{p}}\mathfrak{q}}}^{\bar{\mathfrak{p}}\mathfrak{q}} + \|Z\|_{\mathbb{H}_0^p}^p + \sup_{\mathbb{P}\in\mathcal{P}_0} \mathbb{E}^{\mathbb{P}}(K_T^{\mathbb{P}})^p + \sup_{\mathbb{P}\in\mathcal{P}_0} \mathbb{E}^{\mathbb{P}}([M^{\mathbb{P}}]_T)^{p/2} \\ &\quad \leq C\big(\|\xi\|_{\mathbb{L}_0^{\bar{\mathfrak{p}}\mathfrak{q}}}^{\bar{\mathfrak{p}}\mathfrak{q}} + \phi_f^{\bar{\mathfrak{p}}\mathfrak{q},\kappa}\big).\end{aligned}$$

PROOF. For the existence, we argue as in [37], except for the minimality condition on $K^{\mathbb{P}}$, together with a truncation procedure. Let us define for any $n \in \mathbb{N}$

$$\xi^n = -n \vee \xi \wedge n, \qquad \widehat{f}_s^{\mathbb{P},0,n} = -n \vee f\big(s, X_{\cdot\wedge s}, 0, 0, \widehat{a}_s, b_s^{\mathbb{P}}\big) \wedge n.$$

ξ^n and $\widehat{f}^{\mathbb{P},0,n}$ obviously verify condition (2.14) with $\mathfrak{C} = n$. From Lemma 1, we obtain the existence of a solution $(Y^n, Z^n, M^{n,\mathbb{P}}, K^{n,\mathbb{P}})$ to the 2BSDE (2.5) with terminal condition ξ^n and generator f^n defined by

$$f^n(t, \omega, y, z, a, b) = \big(f(t, \omega, y, z, a, b) - \widehat{f}_t^{\mathbb{P},0}\big) + \widehat{f}_t^{\mathbb{P},0,n}.$$

Note that f^n satisfies Conditions (**H**)–**C1**–**C2**. The minimality criterion on $K^{n,\mathbb{P}}$ is proved arguing as in the proof of minimality for $K^{\mathbb{P}}$ (see just below).

Now the stability result shows that the sequence $(Y^n, Z^n, M^{n,\mathbb{P}} - K^{n,\mathbb{P}})$ converges in $\mathbb{D}_0^{\bar{\mathfrak{p}}\mathfrak{q}} \times \mathbb{H}_0^p \times \mathbb{M}_0^p$ to some process $(Y, Z, N^{\mathbb{P}})$. The supermartingale $N^{\mathbb{P}}$ can be decomposed: $N^{\mathbb{P}} = M^{\mathbb{P}} - K^{\mathbb{P}}$, where $M^{\mathbb{P}}$ is a martingale under $\mathbb{P}$, orthogonal to the canonical process and $K^{\mathbb{P}}$ is a nondecreasing process. The limit $(Y, Z, M^{\mathbb{P}}, K^{\mathbb{P}})$ is a solution of the 2BSDE (2.5) if $K^{\mathbb{P}}$ satisfies the required minimality condition.

Let us prove the minimality criterion for $K^{\mathbb{P}}$ (again the proof is the same for $K^{n,\mathbb{P}}$). For $\mathbb{P}' \in \mathcal{P}(t, \mathbb{P}, \mathbb{F}_+)$, let us denote $\delta\widehat{\mathcal{Y}}^+ = \widehat{\mathcal{Y}}^+ - y^{\mathbb{P}'}(T, \xi)$ and use again a linearization argument:

$$\delta\widehat{\mathcal{Y}}_t^+ = \mathbb{E}^{\mathbb{P}'\otimes\mathbb{P}_0}\bigg[\int_t^T \Lambda_s^{\mathbb{P}'} \Delta_s^{\mathbb{P}'} \, dK_s^{\mathbb{P}'} \bigg| \overline{\mathcal{F}}_t^+\bigg]$$

with

$$\Lambda_s^{\mathbb{P}'} = \exp\bigg(\int_t^s \lambda_u^{\mathbb{P}'} \, du\bigg), \qquad \Delta_s^{\mathbb{P}'} = \exp\bigg[-\int_t^s \zeta_s^{\mathbb{P}'} \, dW_s^{\mathbb{P}'} - \frac{1}{2}\int_t^s \|\zeta_s^{\mathbb{P}'}\|^2 \, ds\bigg].$$

Thus $\mathbb{P}$-a.s.

$$\delta\widehat{\mathcal{Y}}_t^+ \geq \mathbb{E}^{\mathbb{P}'\otimes\mathbb{P}_0}\bigg[\inf_{t\leq s\leq T} \Delta_s^{\mathbb{P}'} \int_t^T \Lambda_s^{\mathbb{P}'} \, dK_s^{\mathbb{P}'} \bigg| \overline{\mathcal{F}}_t^+\bigg]$$

and for p satisfying (2.4), let p' be the Hölder conjugate of p:

$$\begin{aligned}
&\mathbb{E}^{\mathbb{P}'\otimes\mathbb{P}_0}\left[\int_t^T \Lambda_s^{\mathbb{P}'}\, dK_s^{\mathbb{P}'}\Big|\overline{\mathcal{F}}_t^+\right] \\
&\quad\le \left\{\mathbb{E}^{\mathbb{P}'\otimes\mathbb{P}_0}\left[\inf_{t\le s\le T}\Delta_s^{\mathbb{P}'}\int_t^T \Lambda_s^{\mathbb{P}'}\, dK_s^{\mathbb{P}'}\Big|\overline{\mathcal{F}}_t^+\right]\right\}^{1/2} \\
&\qquad\times\left\{\mathbb{E}^{\mathbb{P}'\otimes\mathbb{P}_0}\left[e^{pL_1T}\left(K_T^{\mathbb{P}'}-K_t^{\mathbb{P}'}\right)^p|\overline{\mathcal{F}}_t^+\right]\right\}^{1/(2p)} \\
&\qquad\times\left\{\mathbb{E}^{\mathbb{P}'\otimes\mathbb{P}_0}\left[\left(\inf_{t\le s\le T}\Delta_s^{\mathbb{P}'}\right)^{-p'}|\overline{\mathcal{F}}_t^+\right]\right\}^{1/2p'} \\
&\quad\le C_T\left(C_t^{\mathbb{P}}\right)^{1/(2p)}\left(\delta\widehat{\mathcal{Y}}_t^+\right)^{1/2}.
\end{aligned}$$

Hence the condition (2.6) follows now immediately.

To obtain the a priori estimate (2.15) for the solution of the 2BSDE, we use the a priori estimate given in Lemma A.1, the representation formula (2.10) and we argue as in the proof of [37], Theorem 4.4. □

2.5. *Discussion and comparison with* [36]. When f is Lipschitz continuous w.r.t. y, the process λ is bounded also from below. Thus our minimality condition is equivalent to the classical one:

$$\underset{\mathbb{P}'\in\mathcal{P}(t,\mathbb{P},\mathbb{F}_+)}{\operatorname{essinf}^{\mathbb{P}}}\ \mathbb{E}^{\mathbb{P}'}\left[K_T^{\mathbb{P}'}-K_t^{\mathbb{P}'}|\mathcal{F}_t^+\right]=0,\qquad 0\le t\le T,\mathbb{P}\text{-a.s.}, \forall\mathbb{P}\in\mathcal{P}_0. \tag{2.16}$$

In general, we only have that the classical condition (2.16) implies (2.6).

If there is only one probability measure $\mathbb{P}$ in $\mathcal{P}_0$, the minimality condition (2.6) imposed on $K^{\mathbb{P}}$ should imply that $K^{\mathbb{P}}$ is equivalent to zero. In the Lipschitz setting, this is a direct consequence of (2.16). In our setting, it is still true but the arguments are not direct. From the proof of Theorem 1, (2.6) implies uniqueness of the solution. But if $\mathcal{P}_0$ is the singleton, the solution $(y^{\mathbb{P}}, z^{\mathbb{P}}, 0)$ of the classical BSDE (2.7) becomes a solution of the 2BSDE (2.5). By uniqueness, $K^{\mathbb{P}}\equiv 0$.

The monotone case was already studied in [36]. The generator f satisfies Condition (**H**) and is uniformly continuous in y, uniformly in (t,ω,z,a) and has the linear growth property

$$\left|f(t,\omega,y,0,a)\right|\le\left|f(t,\omega,0,0,a)\right|+C\left(1+|y|\right).$$

Then under some integrability condition on ξ and $\widehat{f}_s^{\mathbb{P},0}$, from [36], Theorem 2.2, there exists a unique solution of the 2BSDE (2.5) such that $K^{\mathbb{P}}$ satisfies the minimality condition (2.16). Therefore, if the generator f satisfies the assumptions of [36] and Condition (**H**), then the solution obtained by [36] with minimality condition (2.16) is also the solution given by Theorems 1 and 2 with minimality criterion (2.6). Let us emphasize that the ways to obtain the solution are completely different. Indeed in [36] the generator is approximated by a sequence of

Lipschitz generators f_n. For any fixed n using [40], there exists a unique solution $(Y^n, Z^n, M^{n,\mathbb{P}}, K^{n,\mathbb{P}})$ to the 2BSDE (2.5) with generator f_n and $K^{n,\mathbb{P}}$ verifies (2.16). Then the core of the paper [36] consists to show that the sequence $(Y^n, Z^n, M^{n,\mathbb{P}}, K^{n,\mathbb{P}})$ converges to $(Y, Z, M^{\mathbb{P}}, K^{\mathbb{P}})$ and that (2.16) is preserved through the limit. The uniform continuity and the linear growth conditions of f w.r.t. y are crucial there.

3. Liquidation problem.

3.1. *The standard formulation of* [5, 29]. In [29], the authors consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. The filtration $\mathbb{F}$ is assumed to be complete, right continuous on $[0, T]$ and left-continuous at time T (see [35] for details on this assumption). In [5], $\mathbb{F}$ is generated by a d-dimensional Brownian motion and thus is quasi-left continuous.

Given ξ a $\mathcal{F}_T$-measurable nonnegative random variable such that $\mathcal{S} = \{\xi = +\infty\}$ has a positive probability, η (resp., γ) a positive (resp., nonnegative) process, the studied optimal stochastic control problem is defined as follows. For some $\vartheta > 1$, consider the functional

$$J(t, \mathcal{X}) = \mathbb{E}\left[\int_t^T (\eta_s|\dot{\mathcal{X}}_s|^{\vartheta} + \gamma_s|\mathcal{X}_s|^{\vartheta})\,ds + \xi|\mathcal{X}_T|^{\vartheta}\Big|\mathcal{F}_t\right] \tag{3.1}$$

over all progressively measurable processes $\mathcal{X}$ that satisfy the dynamics

$$\mathcal{X}_s = x + \int_t^s \dot{\mathcal{X}}_u\,du, \qquad s \geq t \tag{3.2}$$

for some $\dot{\mathcal{X}}$ with $\int_t^T |\dot{\mathcal{X}}_s|\,ds < +\infty$ $\mathbb{P}$-a.s., and some $x \in \mathbb{R}$. To have a finite value $J(t, \mathcal{X})$, the terminal state constraint is

$$\mathcal{X}_T \mathbf{1}_{\mathcal{S}} = 0$$

together with the convention $0 \times \infty = 0$. The set of such processes $\mathcal{X}$ is denoted by $\mathcal{A}^0_{\mathcal{S}}(t, x)$. We introduce the random field v that represents for each initial condition (t, x) the minimal value of $J(t, \mathcal{X})$

$$v(t, x) = \operatorname*{essinf}_{\mathcal{X} \in \mathcal{A}^0_{\mathcal{S}}(t,x)} J(t, \mathcal{X}). \tag{3.3}$$

We follow the convention that the infimum over the empty set is equal to ∞. For some $L > 0$, we also consider the unconstrained minimization problem

$$\begin{aligned} v^L(t, x) &= \operatorname*{essinf}_{\mathcal{X} \in \mathcal{A}(t,x)} J^L(t, \mathcal{X}) \\ &= \operatorname*{essinf}_{\mathcal{X} \in \mathcal{A}(t,x)} \mathbb{E}\left[\int_t^T (\eta_s|\dot{\mathcal{X}}_s|^{\vartheta} + (\gamma_s \wedge L)|\mathcal{X}_s|^{\vartheta})\,ds \right. \\ &\qquad \left. + (L \wedge \xi)|\mathcal{X}_T|^{\vartheta}\Big|\mathcal{F}_t\right], \end{aligned} \tag{3.4}$$

where $\mathcal{A}(t,x)$ is the set of all progressively measurable processes $\mathcal{X}$ of the form (3.2). Here, no terminal constraint is imposed on $\mathcal{X}$.

In [5, 29], the authors show that the related singular BSDE is of the following form:

$$dy_t = \frac{y_t^{\mathfrak{q}}}{(\mathfrak{q}-1)\eta_t^{\mathfrak{q}-1}}\,dt - \gamma_t\,dt + z_t\,dW_t + dm_t \tag{3.5}$$

with terminal condition equal to ξ. Here, $\mathfrak{q} > 1$ is the Hölder conjugate of ϑ: $(\vartheta - 1)(\mathfrak{q}-1) = 1$. The processes η and γ satisfy for some $\ell > 1$

$$\mathbb{E}\int_0^T \left[\left(\eta_t + (T-t)^{\vartheta}\gamma_t\right)^{\ell} + \frac{1}{\eta_t^{\mathfrak{q}-1}} \right] dt < \infty.$$

It is proved that the singular BSDE (3.5) has a minimal supersolution (y,z,m) satisfying:

1. for any $t < T$
$$\mathbb{E}\left[\sup_{0\le s\le t} |y_s|^{\ell} + \left(\int_0^t |z_s|^2\,ds\right)^{\ell/2} + [m]_t^{\ell/2} \right] < +\infty;$$
2. $Y_t \ge 0$ for any t, a.s.;
3. for all $0 \le s \le t < T$
$$y_s = y_t + \int_s^t \left[-\frac{y_u^{\mathfrak{q}}}{(\mathfrak{q}-1)\eta_u^{\mathfrak{q}-1}} + \gamma_u \right] du - \int_s^t z_u\,dW_u - \int_s^t dm_u;$$
4. and the singular terminal condition: $\mathbb{P}$-a.s.

$$\liminf_{t\to T} y_t \ge \xi. \tag{3.6}$$

To prove the existence of a minimal solution, a truncation procedure is used. For any $L \ge 0$, we consider the BSDE

$$dy_t^L = \frac{(y_t^L)^{\mathfrak{q}}}{(\mathfrak{q}-1)\eta_t^{\mathfrak{q}-1}}\,dt - (\gamma_t \wedge L)\,dt + z_t^L\,dW_t + dm_t^L \tag{3.7}$$

with the bounded terminal condition $y_T^L = \xi \wedge L$. This BSDE has a unique solution (y^L, z^L, m^L) (see [11]). Moreover, the solution satisfies the a priori estimate

$$0 \le y_t^L \le \frac{1}{(T-t)^{\vartheta}} \mathbb{E}\left[\int_t^T \left(\eta_s + (T-s)^{\vartheta}\gamma_s\right) ds \Big| \mathcal{F}_t \right] \tag{3.8}$$

Next, by passing to the limit $L \to \infty$, the minimal supersolution (y,z,m) of (3.5) with terminal condition (3.6) is obtained. Let us emphasize here that the left-continuity condition on the filtration is used only to obtain the weak terminal condition (3.6). The next result is given in [29], Proposition 8 and Theorem 3.

LEMMA 2. *Let (y^L, z^L, m^L) be the solution to (3.7) with terminal condition $y_T^L = \xi \wedge L$. Then the process $\mathcal{X}^L$ satisfying the linear dynamics*

$$\mathcal{X}_s^L = x - \int_t^s \left(\frac{y_r^L}{\eta_r}\right)^{\mathfrak{q}-1} \mathcal{X}_r^L \, dr,$$

is optimal in (3.4). Moreover, we have $v^L(t,x) = y_t^L |x|^{\vartheta}$.

Let (y, z, m) denote the minimal solution to (3.5) with singular terminal condition (3.6). Then we have $v(t,x) = y_t |x|^{\vartheta}$. Moreover, the process $\mathcal{X}$ satisfying the linear dynamics

$$\mathcal{X}_s = x - \int_t^s \left(\frac{y_u}{\eta_u}\right)^{\mathfrak{q}-1} \mathcal{X}_u \, du,$$

belongs to $\mathcal{A}_{\mathcal{S}}^0(t,x)$ and is optimal in (3.3).

REMARK 7. The sign of $\mathcal{X}$ is equal to the sign of the position x at time 0. The case $x > 0$ (resp., $x < 0$) corresponds to a selling (resp., buying) liquidation strategy. If $x > 0$ (resp., $x < 0$), the trading rate is nonpositive (resp., nonnegative), which is coherent with the absence of transaction-triggered price manipulation (see [1, 2, 16]).

3.2. *The formulation under uncertainty without terminal constraint.* We work under the setting described in Section 2.1. We consider a $\mathcal{F}_T$-Borel measurable random variable ξ such that for any $\mathbb{P} \in \mathcal{P}_0$, ξ is a.s. nonnegative. We denote by $\mathcal{S}$ the singular set $\{\xi = +\infty\}$. We define the two Borel measurable functions:

$$\eta : (t, \omega, a) \in [0,T] \times \Omega \times \mathbb{S}_d^{\geq 0} \longrightarrow \mathbb{R}_+^*,$$
$$\gamma : (t, \omega, a) \in [0,T] \times \Omega \times \mathbb{S}_d^{\geq 0} \longrightarrow \mathbb{R}_+.$$

Here, η and γ (and thus the generator of our BSDE) do not depend on the drift of X. This condition is sufficient to obtain an optimal control independent of the probability measure $\mathbb{P}$ (see Propositions 1 and 4 below). This hypothesis is similar to the setting in [32].

We define for simplicity

$$\widehat{\eta}_s := \eta(s, X_{\cdot\wedge s}, \widehat{a}_s) \quad \text{and} \quad \widehat{\gamma}_s := \gamma(s, X_{\cdot\wedge s}, \widehat{a}_s).$$

Finally, we assume that there exists $\varrho > 1$ such that for any $(t,\omega) \in [0,T] \times \Omega$

$$\sup_{\mathbb{P} \in \mathcal{P}(t,\omega)} \mathbb{E}^{\mathbb{P}} \int_t^T \left(\frac{1}{\widehat{\eta}_s}\right)^{\varrho(\mathfrak{q}-1)} ds < \infty. \tag{3.9}$$

Our generator is

$$f(t, \omega, y, a) = -\frac{y|y|^{\mathfrak{q}-1}}{(\mathfrak{q}-1)(\eta(t,\omega,a))^{\mathfrak{q}-1}} + \big(\gamma(t,\omega,a) \wedge L\big)$$

and satisfies Condition (**H**): for any (t, ω, a, y, y'):

H1. $y \mapsto f(t, \omega, y, a)$ is continuous.

H2. Monotonicity assumption: f is nonincreasing w.r.t. y

$$\big(f(t, \omega, y, a) - f\big(t, \omega, y', a\big)\big)\big(y - y'\big) \le 0.$$

H4. Growth assumption:

$$\big|f(t, \omega, y, a) - f(t, \omega, 0, a)\big| = \frac{1}{(\mathfrak{q}-1)\eta(t, \omega, a)^{\mathfrak{q}-1}}|y|^{\mathfrak{q}}$$

together with (3.9).

Compared to **H4** in our previous section, here

$$\Psi(t, \omega, a) = \frac{1}{(\mathfrak{q}-1)\eta(t, \omega, a)^{\mathfrak{q}-1}}$$

and Assumption (3.9) corresponds to Condition **C1** on $\widehat{\Psi}$. The terminal condition $\xi \wedge L$ and the process $\widehat{f}^0 = (\widehat{f}_t^0 = \widehat{\gamma}_t \wedge L, t \ge 0)$ are bounded. Hence **C1** and **C2** hold for any $\bar{\mathfrak{p}} > 1$. Hence Condition (2.4) becomes here $1 < p < \varrho$. From Theorems 1 and 2, we deduce that there exists a unique solution $(Y^L, Z^L, M^{L,\mathbb{P}}, K^{L,\mathbb{P}})$ to the second-order BSDE: for any $0 \le t \le T$ and any $\mathbb{P}$

$$\begin{aligned} Y_t^L = (\xi \wedge L) - \int_t^T \frac{|Y_u^L|^{\mathfrak{q}-1} Y_u^L}{(\mathfrak{q}-1)(\widehat{\eta}_u)^{\mathfrak{q}-1}}\, du + \int_t^T (\widehat{\gamma}_u \wedge L)\, du \\ - \left(\int_t^T Z_s^L\, dX_s^{c,\mathbb{P}}\right)^{\mathbb{P}} - \int_t^T dM_s^{L,\mathbb{P}} + \big(K_T^{L,\mathbb{P}} - K_t^{L,\mathbb{P}}\big), \qquad \mathbb{P}\text{-a.s.,} \end{aligned} \tag{3.10}$$

such that:

- For any $p > 1$, Y^L belongs to $\mathbb{D}_0^p(\mathbb{F}_+^{\mathcal{P}_0})$.
- For any $1 < p < \varrho$, $(Z^L, M^{L,\mathbb{P}}, K^{L,\mathbb{P}})$ is in $\mathbb{H}_0^p(\mathbb{F}_+^{\mathcal{P}_0}) \times \mathbb{M}_0^p((\mathbb{F}_+^{\mathbb{P}})_{\mathbb{P} \in \mathcal{P}_0}) \times \mathbb{I}_0^p((\mathbb{F}_+^{\mathbb{P}})_{\mathbb{P} \in \mathcal{P}_0})$.
- $K^{L,\mathbb{P}}$ is a $\mathbb{P}$-a.s. nondecreasing process satisfying the minimality condition (2.6).

Moreover, we have the representation formula

$$Y_t^L = \underset{\mathbb{P}' \in \mathcal{P}(t, \mathbb{P}, \mathbb{F}_+)}{\operatorname{esssup}^{\mathbb{P}}}\ y_t^{L,\mathbb{P}'}, \tag{3.11}$$

where $(y^{L,\mathbb{P}}, z^{L,\mathbb{P}}, m^{L,\mathbb{P}})$ is the solution under $\mathbb{P}$ of the BSDE

$$dy_t^{L,\mathbb{P}} = \frac{|y_t^{L,\mathbb{P}}|^{\mathfrak{q}-1} y_t^{L,\mathbb{P}}}{(\mathfrak{q}-1)(\widehat{\eta}_t)^{\mathfrak{q}-1}}\, dt - (\widehat{\gamma}_t \wedge L)\, dt + z_t^{L,\mathbb{P}}\, dX_t^{c,\mathbb{P}} + dm_t^{L,\mathbb{P}}.$$

Note that by comparison principle for standard BSDEs (Lemma A.2), these solutions $y^{L,\mathbb{P}}$ satisfy the inequality: $\mathbb{P}$-a.s.

$$0 \le y_t^{L,\mathbb{P}} \le L(T+1) \qquad \forall t \in [0, T].$$

Thus $\mathcal{P}_0$-q.s.

$$0 \le Y_t^L \le L(T+1) \qquad \forall t \in [0, T].$$

REMARK 8. Note that the generator is locally Lipschitz continuous w.r.t. y and all solutions $y^{L,\mathbb{P}}$ and Y^L are a priori bounded. Hence

$$-\frac{\mathfrak{q}|L(T+1)|^{\mathfrak{q}-1}}{(\mathfrak{q}-1)}\frac{1}{(\widehat{\eta}_t)^{\mathfrak{q}-1}} \leq \lambda_s^{\mathbb{P}'} = \frac{\widehat{f}_s^{\mathbb{P}'}(Y_s, z_s^{\mathbb{P}'}) - \widehat{f}_s^{\mathbb{P}'}(y_s^{\mathbb{P}'}, z_s^{\mathbb{P}'})}{Y_s - y_s^{\mathbb{P}'}}\mathbf{1}_{Y_s \neq y_s^{\mathbb{P}'}} \leq 0.$$

If $\widehat{\eta}$ is bounded away from zero, the condition (2.6) can be reduced to the usual condition (2.16).

First, we define the following control sets:

- $\mathcal{A}(t,x)$ is the set of processes $\mathcal{X} = (\mathcal{X}_s, 0 \leq s \leq T)$ such that $\mathcal{X}_s = x$ if $s \leq t$ and for any $\mathbb{P} \in \mathcal{P}_t$, $\mathbb{P}$-a.s., $\mathcal{X}$ is absolutely continuous, that is: $\mathcal{X}_s(\omega) = x + \int_t^s \dot{\mathcal{X}}_u(\omega)\,du$ with $\int_t^T |\dot{\mathcal{X}}_u(\omega)|\,du < +\infty$.
- For a fixed $\mathbb{P} \in \mathcal{P}_t$, $\mathcal{A}^{\mathbb{P}}(t,x)$ is the set of processes $\mathcal{X} = (\mathcal{X}_s, 0 \leq s \leq T)$ such that $\mathcal{X}_s = x$ if $s \leq t$ and $\mathbb{P}$-a.s., $\mathcal{X}$ is absolutely continuous, that is: $\mathcal{X}_s(\omega) = x + \int_t^s \dot{\mathcal{X}}_u(\omega)\,du$ with $\int_t^T |\dot{\mathcal{X}}_u(\omega)|\,du < +\infty$.

The set $\mathcal{A}^{\mathbb{P}}(t,x)$ depends on $\mathbb{P}$, whereas $\mathcal{A}(t,x)$ depends only on the probability set $\mathcal{P}_t$. Of course $\mathcal{A}(t,x)$ is included in $\mathcal{A}^{\mathbb{P}}(t,x)$. Next for any $L \geq 0$ we define the following unconstrained control problems:

$$\begin{aligned} J^L(t,x) = \operatorname*{essinf}_{\mathcal{X} \in \mathcal{A}(t,x)} \operatorname*{esssup}_{\mathbb{P} \in \mathcal{P}_t} \mathbb{E}^{\mathbb{P}}\biggl[\int_t^T \bigl(\widehat{\eta}_s|\dot{\mathcal{X}}_s|^{\vartheta} + (\widehat{\gamma}_s \wedge L)|\mathcal{X}_s|^{\vartheta}\bigr)\,ds \\ + (L \wedge \xi)|\mathcal{X}_T|^{\vartheta}\Big|\mathcal{F}_t^+\biggr], \end{aligned} \tag{3.12}$$

together with

$$\begin{aligned} I^L(t,x) = \operatorname*{esssup}_{\mathbb{P} \in \mathcal{P}_t} \operatorname*{essinf}_{\mathcal{X} \in \mathcal{A}(t,x)} \mathbb{E}^{\mathbb{P}}\biggl[\int_t^T \bigl(\widehat{\eta}_s|\dot{\mathcal{X}}_s|^{\vartheta} + (\widehat{\gamma}_s \wedge L)|\mathcal{X}_s|^{\vartheta}\bigr)\,ds \\ + (L \wedge \xi)|\mathcal{X}_T|^{\vartheta}\Big|\mathcal{F}_t^+\biggr] \end{aligned}$$

and

$$\begin{aligned} H^L(t,x) = \operatorname*{esssup}_{\mathbb{P} \in \mathcal{P}_t} \operatorname*{essinf}_{\mathcal{X} \in \mathcal{A}^{\mathbb{P}}(t,x)} \mathbb{E}^{\mathbb{P}}\biggl[\int_t^T \bigl(\widehat{\eta}_s|\dot{\mathcal{X}}_s|^{\vartheta} + (\widehat{\gamma}_s \wedge L)|\mathcal{X}_s|^{\vartheta}\bigr)\,ds \\ + (L \wedge \xi)|\mathcal{X}_T|^{\vartheta}\Big|\mathcal{F}_t^+\biggr]. \end{aligned}$$

Immediately, $H^L(t,x) \leq I^L(t,x) \leq J^L(t,x)$. From the standard formulation (see Section 3.1), we have

$$H^L(t,x) = |x|^{\vartheta} \operatorname*{esssup}_{\mathbb{P} \in \mathcal{P}_t} y_t^{L,\mathbb{P}} = |x|^{\vartheta} Y_t^L.$$

LEMMA 3. *For any* (t,x), $J^L(t,x) \le H^L(t,x)$.

PROOF. We assume that $x \ge 0$, such that $\mathcal{X}_s^*$ is nonnegative for all $s \in [t,T]$. The same arguments lead to the same result for $x < 0$, using that $|\mathcal{X}_s^*|^\vartheta = (-\mathcal{X}_s^*)^\vartheta$. For simplicity, we do not write the constant L in this proof. Let us define

$$\beta_s = -(Y_s/\widehat{\eta}_s)^{\mathfrak{q}-1}, \qquad d\mathcal{X}_s^* = \beta_s \mathcal{X}_s^* \, ds = \dot{\mathcal{X}}_s^* \, ds.$$

Let us apply the Itô formula under the probability $\mathbb{P}$:

$$\begin{aligned}
d\big(Y_s(\mathcal{X}_s^*)^\vartheta\big) &= (\mathcal{X}_s^*)^\vartheta \, dY_s + Y_s \, d\big((\mathcal{X}_s^*)^\vartheta\big) \\
&= (\mathcal{X}_s^*)^\vartheta \frac{Y_s^{\mathfrak{q}}}{(\mathfrak{q}-1)(\widehat{\eta}_s)^{\mathfrak{q}-1}} \, ds - (\widehat{\gamma}_s \wedge L)(\mathcal{X}_s^*)^\vartheta \, ds + pY_s\beta_s(\mathcal{X}_s^*)^p \, ds \\
&\quad + (\mathcal{X}_s^*)^\vartheta Z_s \, dX_s^{c,\mathbb{P}} + (\mathcal{X}_s^*)^\vartheta \, dM_s^{\mathbb{P}} - (\mathcal{X}_s^*)^\vartheta \, dK_s^{\mathbb{P}} \\
&= -(\mathcal{X}_s^*)^\vartheta \left(\frac{Y_s}{\widehat{\eta}_s}\right)^{\mathfrak{q}} \widehat{\eta}_s \, ds - (\widehat{\gamma}_s \wedge L)(\mathcal{X}_s^*)^\vartheta \, ds + (\mathcal{X}_s^*)^\vartheta Z_s \, dX_s^{c,\mathbb{P}} \\
&\quad + (\mathcal{X}_s^*)^\vartheta \, dM_s^{\mathbb{P}} - (\mathcal{X}_s^*)^\vartheta \, dK_s^{\mathbb{P}}.
\end{aligned}$$

Since $(\mathfrak{q}-1)\vartheta = \mathfrak{q}$,

$$\begin{aligned}
d\big(Y_s(\mathcal{X}_s^*)^\vartheta\big) &= -\left(\mathcal{X}_s^* \frac{Y_s^{\mathfrak{q}-1}}{(\widehat{\eta}_s)^{\mathfrak{q}-1}}\right)^\vartheta \widehat{\eta}_s \, ds - (\widehat{\gamma}_s \wedge L)(\mathcal{X}_s^*)^\vartheta \, ds + (\mathcal{X}_s^*)^\vartheta Z_s \, dX_s^{c,\mathbb{P}} \\
&\quad + (\mathcal{X}_s^*)^\vartheta \, dM_s^{\mathbb{P}} - (\mathcal{X}_s^*)^\vartheta \, dK_s^{\mathbb{P}} \\
&= -\big[\widehat{\eta}_s(|\dot{\mathcal{X}}_s^*|)^\vartheta + (\widehat{\gamma}_s \wedge L)(\mathcal{X}_s^*)^\vartheta\big] \, ds + (\mathcal{X}_s^*)^\vartheta Z_s \, dX_s^{c,\mathbb{P}} \\
&\quad + (\mathcal{X}_s^*)^\vartheta \, dM_s^{\mathbb{P}} - (\mathcal{X}_s^*)^\vartheta \, dK_s^{\mathbb{P}}.
\end{aligned}$$

Now integrate this from t to T:

$$\begin{aligned}
Y_T(\mathcal{X}_T^*)^\vartheta - Y_t(\mathcal{X}_t^*)^\vartheta &= (\xi \wedge L)(\mathcal{X}_T^*)^\vartheta - Y_t x^\vartheta \\
&= -\int_t^T \big[\widehat{\eta}_s(\dot{\mathcal{X}}_s^*)^\vartheta + (\widehat{\gamma}_s \wedge L)(\mathcal{X}_s^*)^\vartheta\big] \, ds \\
&\quad + \left(\int_t^T (\mathcal{X}_s^*)^\vartheta Z_s \, dX_s^{c,\mathbb{P}}\right)^{\mathbb{P}} \\
&\quad + \left(\int_t^T (\mathcal{X}_s^*)^\vartheta \, dM_s^{\mathbb{P}}\right)^{\mathbb{P}} - \int_t^T (\mathcal{X}_s^*)^\vartheta \, dK_s^{\mathbb{P}}.
\end{aligned}$$

And taking the conditional expectation w.r.t. $\mathbb{P}$,

$$\begin{aligned}
&\mathbb{E}^{\mathbb{P}}\left[(\xi \wedge L)(\mathcal{X}_T^*)^\vartheta + \int_t^T \big[\widehat{\eta}_s(\dot{\mathcal{X}}_s^*)^\vartheta + (\widehat{\gamma}_s \wedge L)(\mathcal{X}_s^*)^\vartheta\big] \, ds \,\Big|\, \mathcal{F}_t^+\right] \\
&\quad = Y_t x^\vartheta - \mathbb{E}^{\mathbb{P}}\left[\int_t^T (\mathcal{X}_s^*)^\vartheta \, dK_s^{\mathbb{P}} \,\Big|\, \mathcal{F}_t^+\right] \le Y_t x^\vartheta
\end{aligned}$$

since $K^{\mathbb{P}}$ is nondecreasing. Therefore,

$$\operatorname*{esssup}_{\mathbb{P}\in\mathcal{P}_t} \mathbb{E}^{\mathbb{P}}\left[(\xi \wedge L)(\mathcal{X}_T^*)^{\vartheta} + \int_t^T [\widehat{\eta}_s(\dot{\mathcal{X}}_s^*)^{\vartheta} + (\widehat{\gamma}_s \wedge L)(\mathcal{X}_s^*)^{\vartheta}]\, ds \Big| \mathcal{F}_t^+\right] \le Y_t x^{\vartheta}.$$

Moreover, the process $\mathcal{X}^*$ is in $\mathcal{A}(t,x)$:

$$\mathcal{X}_s^* = x - \int_t^s \left(\frac{Y_u}{\widehat{\eta}_u}\right)^{\mathfrak{q}-1} \mathcal{X}_u^*\, du.$$

This implies that

$$J(t,x) \le Y_t x^{\vartheta} = H(t,x). \qquad \square$$

Therefore, we deduce that $H^L(t,x) \le I^L(t,x) \le J^L(t,x) \le H^L(t,x)$ and our first result:

PROPOSITION 1. *The unconstrained problem* (3.12) *satisfies*

$$\operatorname*{essinf}_{\mathcal{X}\in\mathcal{A}(t,x)} \operatorname*{esssup}_{\mathbb{P}\in\mathcal{P}_t} \mathbb{E}^{\mathbb{P}}\left[\int_t^T (\widehat{\eta}_s|\dot{\mathcal{X}}_s|^{\vartheta} + \widehat{\gamma}_s|\mathcal{X}_s|^{\vartheta})\, ds + (\xi \wedge L)|\mathcal{X}_T|^{\vartheta} \Big| \mathcal{F}_t^+\right]$$
$$= \operatorname*{esssup}_{\mathbb{P}\in\mathcal{P}_t} \operatorname*{essinf}_{\mathcal{X}\in\mathcal{A}^{\mathbb{P}}(t,x)} \mathbb{E}^{\mathbb{P}}\left[\int_t^T (\widehat{\eta}_s|\dot{\mathcal{X}}_s|^{\vartheta} + \widehat{\gamma}_s|\mathcal{X}_s|^{\vartheta})\, ds + (\xi \wedge L)|\mathcal{X}_T|^{\vartheta} \Big| \mathcal{F}_t^+\right]$$

and the solution of the 2BSDE (3.10), *denoted by* Y^L, *gives the optimal process* $\mathcal{X}^{*,L}$:

$$d\mathcal{X}_s^{*,L} = [-(Y_s^L/\widehat{\eta}_s)^{\mathfrak{q}-1}\mathcal{X}_s^{*,L}]\, ds, \qquad (\mathfrak{q}-1)(\vartheta-1) = 1.$$

3.3. *The constrained problem under uncertainty.* We denote by $\mathcal{A}_0(t,x)$ the set of admissible controls $\mathcal{X} \in \mathcal{A}(t,x)$ such that $\mathcal{X}_T \mathbf{1}_{\mathcal{S}} = 0$, $\mathcal{P}_t$-q.s. ($\mathcal{P}_t$-q.s. means $\mathbb{P}$-a.s. $\forall \mathbb{P} \in \mathcal{P}_t$) and $\mathcal{A}_0^{\mathbb{P}}(t,x)$ the set of admissible controls $\mathcal{X} \in \mathcal{A}^{\mathbb{P}}(t,x)$ such that $\mathcal{X}_T \mathbf{1}_{\mathcal{S}} = 0$ $\mathbb{P}$-a.s. Now consider

$$J(t,x) = \operatorname*{essinf}_{\mathcal{X}\in\mathcal{A}_0(t,x)} \operatorname*{esssup}_{\mathbb{P}\in\mathcal{P}_t} \mathbb{E}^{\mathbb{P}}\left[\int_t^T (\widehat{\eta}_s|\dot{\mathcal{X}}_s|^{\vartheta} + \widehat{\gamma}_s|\mathcal{X}_s|^{\vartheta})\, ds + \xi|\mathcal{X}_T|^{\vartheta} \Big| \mathcal{F}_t^+\right]. \tag{3.13}$$

Again we use the convention that $0 \times \infty = 0$. As mentioned for the standard formulation, a left-continuity condition is imposed on the underlying filtration to have the terminal condition (3.6).[4] In our present setting, we add the next assumption on our set of probability measures $\mathcal{P}_t^W$:

- *Left-continuity condition*: for any probability measure $\mathbb{P} \in \mathcal{P}_t^W$, the filtration $\mathbb{F}_+^{\mathbb{P}}$ is left continuous at time T.

[4]This technical condition can be avoided if ξ is $\mathcal{F}_{T-}$-measurable (see, e.g., [6]).

This hypothesis implies that a martingale cannot have a jump at time T. For example, this assumption holds if $\mathcal{P} = \{\mathbb{P}^a, a \in \mathbb{A}\}$ is the set of all probability measures $\mathbb{P}^a$ given by

$$\mathbb{P}^a = \mathbb{P}_0 \circ (X^a)^{-1}, \qquad X_t^a = \int_0^t a_s^{\frac{1}{2}} \, dX_s$$

for all processes $a \in \mathbb{A}$ of the form $a = \sum_{n=0}^{\infty} \sum_{i=1}^{\infty} a_i^n \mathbf{1}_{E_i^n} \mathbf{1}_{[\tau_n, \tau_{n+1})}$, where $(a_i^n)_{i,n} \in \mathbb{A}_0$, $(\tau_n)_n$ is a nondecreasing sequence of stopping times with $\tau_0 = 0$ and:

- $\inf\{n, \tau_n = +\infty\} < +\infty$, $\tau_n < \tau_{n+1}$ whenever $\tau_n < +\infty$ and each τ_n takes at most countably many values,
- for each n, $\{E_i^n, i \geq 1\} \subset \mathcal{F}_{\tau_n}$ forms a partition of Ω.

$\mathbb{A}_0$ is the class of all deterministic mappings such that $0 < \underline{a} \leq a_t$ for any $t \geq 0$ (see [39], Section 4.4). Every $\mathbb{P}^a$, $a \in \mathbb{A}_0$, satisfies the martingale representation property. Then from [23], Proposition A.1, every $\mathbb{P} \in \mathcal{P}$ verifies this property, also. Thereby any martingale is continuous, which implies the required argument for the filtration (see [21], Proposition 25.19, for example).

For $L \leq L'$ and any $\mathbb{P} \in \mathcal{P}_0$, we have $\mathbb{P}$-a.s. for any $t \in [0, T]$

$$y_t^{L,\mathbb{P}} \leq y_t^{L',\mathbb{P}} \leq Y_t^{L'}.$$

Hence $\mathcal{P}_0$-q.s., $Y_t^L \leq Y_t^{L'}$ for $t \in [0, T]$ (see also the comparison result [37], Theorem 4.3).

Let us now assume that there exists $\ell > 1$ and $\kappa \in (1, \ell)$ such that for any (t, ω)

$$\sup_{\mathbb{P} \in \mathcal{P}(t,\omega)} \mathbb{E}^{\mathbb{P}} \left[\int_t^T [\widehat{\eta}_s + (T-s)^{\vartheta} \widehat{\gamma}_s]^{\ell} \, ds \right] < \infty, \tag{3.14}$$

and

$$\sup_{\mathbb{P} \in \mathcal{P}_0} \mathbb{E}^{\mathbb{P}} \left[\operatorname*{ess\,sup}^{\mathbb{P}}_{0 \leq t \leq T} \left(\mathbb{E}^{\mathbb{P}} \left[\int_0^T [\widehat{\eta}_s + (T-s)^{\vartheta} \widehat{\gamma}_s]^{\kappa} \, ds \Big| \mathcal{F}_t^+ \right] \right)^{\frac{\ell}{\kappa}} \right] < \infty. \tag{3.15}$$

LEMMA 4 (A priori estimate). *There exists $U \in \mathbb{D}_0^{\ell}(\mathbb{F}_+^{\mathcal{P}_0})$ such that for any $0 \leq t \leq T$, $\mathcal{P}_0$-q.s.*

$$0 \leq Y_t^L \leq \frac{1}{(T-t)^{\vartheta}} U_t. \tag{3.16}$$

Let us emphasize that the right-hand side does not depend on L and is finite on $[0, T)$.

PROOF. The estimate (3.8) gives for any $\mathbb{P} \in \mathcal{P}_0$

$$0 \leq y_t^{L,\mathbb{P}} \leq \frac{1}{(T-t)^{\vartheta}} \mathbb{E}^{\mathbb{P}} \left[\int_t^T (\widehat{\eta}_s + (T-s)^{\vartheta} \widehat{\gamma}_s) \, ds \Big| \mathcal{F}_t^+ \right] = \frac{1}{(T-t)^{\vartheta}} u_t^{\mathbb{P}}.$$

The process $(u^{\mathbb{P}}, v^{\mathbb{P}}, n^{\mathbb{P}})$ is the solution of the BSDE

$$u_t^{\mathbb{P}} = \int_t^T (\widehat{\eta}_s + (T-s)^{\vartheta}\widehat{\gamma}_s)\,ds - \left(\int_t^T v_s^{\mathbb{P}}\,dX_s^{c,\mathbb{P}}\right)^{\mathbb{P}} - \int_t^T dn_s^{\mathbb{P}}.$$

Then using (3.14), (3.15) and [37], Theorem 4.1, there exists a unique solution $(U, V, \mathcal{N}^{\mathbb{P}}, \mathcal{K}^{\mathbb{P}})$ to the 2BSDE

$$U_t = \int_t^T (\widehat{\eta}_s + (T-s)^{\vartheta}\widehat{\gamma}_s)\,ds - \left(\int_t^T V_s\,dX_s^{c,\mathbb{P}}\right)^{\mathbb{P}} - \int_t^T d\mathcal{N}_s^{\mathbb{P}} + (\mathcal{K}_T^{\mathbb{P}} - \mathcal{K}_t^{\mathbb{P}}),$$

such that $U \in \mathbb{D}_0^{\ell}(\mathbb{F}_+^{\mathcal{P}_0})$ and $(V, \mathcal{N}^{\mathbb{P}}, \mathcal{K}^{\mathbb{P}})$ is in $\mathbb{H}_0^{\ell}(\mathbb{F}_+^{\mathcal{P}_0}) \times \mathbb{M}_0^{\ell}((\mathbb{F}_+^{\mathbb{P}})_{\mathbb{P}\in\mathcal{P}_0}) \times \mathbb{I}_0^{\ell}((\mathbb{F}_+^{\mathbb{P}})_{\mathbb{P}\in\mathcal{P}_0})$. Moreover, for any $\mathbb{P} \in \mathcal{P}_0$ and any $t \in [0, T]$, we have the representation formula

$$\underset{\mathbb{P}'\in\mathcal{P}(t,\mathbb{P},\mathbb{F}_+)}{\operatorname{esssup}^{\mathbb{P}}} u_t^{\mathbb{P}'} = U_t, \qquad \mathbb{P}\text{-a.s.}$$

Thus we obtain the desired a priori estimate since

$$Y_t^L = \underset{\mathbb{P}'\in\mathcal{P}(t,\mathbb{P},\mathbb{F}_+)}{\operatorname{esssup}^{\mathbb{P}}} y_t^{L,\mathbb{P}'} \leq \underset{\mathbb{P}'\in\mathcal{P}(t,\mathbb{P},\mathbb{F}_+)}{\operatorname{esssup}^{\mathbb{P}}} u_t^{\mathbb{P}'} = U_t.$$

This achieves the proof of the lemma. □

LEMMA 5. *For any* $\varepsilon > 0$, *the sequence* $(Y^L, Z^L, M^{L,\mathbb{P}}, K^{L,\mathbb{P}})$ *converges, when* L *goes to* $+\infty$, *to* $(Y, Z, M^{\mathbb{P}}, K^{\mathbb{P}})$ *in the space* $\mathbb{D}_0^{\ell}(\mathbb{F}_+^{\mathcal{P}_0}) \times \mathbb{H}_0^{\ell}(\mathbb{F}_+^{\mathcal{P}_0}) \times \mathbb{M}_0^{\ell}((\mathbb{F}_+^{\mathbb{P}})_{\mathbb{P}\in\mathcal{P}_0}) \times \mathbb{I}_0^{\ell}((\mathbb{F}_+^{\mathbb{P}})_{\mathbb{P}\in\mathcal{P}_0})$ *on* $[0, T-\varepsilon]$, *which means that all processes are restricted on this time interval. Moreover*, $(Y, Z, M^{\mathbb{P}}, K^{\mathbb{P}})$ *satisfies the dynamics*: *for any* $\mathbb{P} \in \mathcal{P}_0$, *and any* $0 \leq s \leq t < T$,

$$\begin{aligned} Y_s = Y_t - \int_s^t \frac{Y_u^{\mathfrak{q}}}{(\mathfrak{q}-1)(\widehat{\eta}_u)^{\mathfrak{q}-1}}\,du + \int_s^t \widehat{\gamma}_u\,du - \left(\int_s^t Z_u\,dX_u^{c,\mathbb{P}}\right)^{\mathbb{P}} \\ - \int_s^t dM_u^{\mathbb{P}} + K_t^{\mathbb{P}} - K_s^{\mathbb{P}}. \end{aligned} \tag{3.17}$$

Finally, Y *satisfies the representation property*: *for any* $t < T$ *and any* $\mathbb{P} \in \mathcal{P}_0$,

$$Y_t = \underset{\mathbb{P}'\in\mathcal{P}(t,\mathbb{P},\mathbb{F}_+)}{\operatorname{esssup}^{\mathbb{P}}} y_t^{\mathbb{P}'}, \qquad \mathbb{P}\text{-}a.s.$$

PROOF. Fix $\varepsilon > 0$ and define

$$\psi_{L,L'}^{\ell,\varepsilon} := \sup_{\mathbb{P}\in\mathcal{P}_0} \mathbb{E}^{\mathbb{P}}\left[\int_0^{T-\varepsilon} \left|(\widehat{\gamma}_s \wedge L) - (\widehat{\gamma}_s \wedge L')\right|^{\ell} ds\right],$$

$$\phi_{L,L'}^{\ell,\kappa,\varepsilon} := \sup_{\mathbb{P}\in\mathcal{P}_0} \mathbb{E}^{\mathbb{P}}\left[\underset{0\leq t\leq T-\varepsilon}{\operatorname{esssup}^{\mathbb{P}}} \mathbb{E}^{\mathbb{P}}\left[\left(\int_0^{T-\varepsilon} \left|(\widehat{\gamma}_s^{\mathbb{P}} \wedge L) - (\widehat{\gamma}_s^{\mathbb{P}} \wedge L')\right|^{\kappa} ds\right)^{\frac{\ell}{\kappa}} \middle| \mathcal{F}_t^+\right]\right].$$

From our conditions (3.14) and (3.15) on $\widehat{\gamma}$, $\psi^{\ell,\varepsilon}_{L,L'}$ and $\phi^{\ell,\kappa,\varepsilon}_{L,L'}$ tend to zero when L and L' go to $+\infty$. From [37], Theorem 4.5 (stability result for 2BSDE), for any L and L' we have on $[0, T-\varepsilon]$

$$\|Y^L - Y^{L'}\|^{\ell}_{\mathbb{D}^{\ell}_0} \leq C\big[\|Y^L_{T-\varepsilon} - Y^{L'}_{T-\varepsilon}\|^{\ell}_{\mathbb{L}^{\ell}_0} + \psi^{\ell,\varepsilon}_{L,L'}\big].$$

From the uniform w.r.t. L and $\mathcal{P}_0$-q.s. bound (3.16), and from the monotonicity of the sequence Y^L, we deduce that

$$\|Y^L_{T-\varepsilon} - Y^{L'}_{T-\varepsilon}\|^{\ell}_{\mathbb{L}^{\ell}_0}$$

tends to zero when L and L' go to $+\infty$. Hence there exists $Y \in \mathbb{D}^{\ell}_0(\mathbb{F}^{\mathcal{P}_0}_+)$ defined on $[0, T-\varepsilon]$ such that on $[0, T-\varepsilon]$, Y^L converges strongly to Y. From (3.16), we have: $\mathcal{P}_0$-q.s. for $t \in [0, T)$

$$0 \leq Y_t \leq \frac{1}{(T-t)^{\vartheta}} U_t.$$

By the representation of Y^L, for any $t \in [0, T)$, any $\mathbb{P} \in \mathcal{P}_0$ and any $\mathbb{P}' \in \mathcal{P}(t, \mathbb{P}, \mathbb{F}_+)$

$$y^{L,\mathbb{P}'}_t \leq \operatorname*{esssup}^{\mathbb{P}}_{\mathbb{P}' \in \mathcal{P}(t,\mathbb{P},\mathbb{F}_+)} y^{L,\mathbb{P}'}_t = Y^L_t \leq Y_t, \qquad \mathbb{P}\text{-a.s.}$$

The (minimal) supersolution $y^{\mathbb{P}'}$ of the singular BSDE (3.5) is obtained as the increasing limit of $y^{L,\mathbb{P}'}$. Thus for any $\mathbb{P}'$

$$y^{L,\mathbb{P}'}_t \leq y^{\mathbb{P}'}_t \leq Y_t \quad \Rightarrow \quad \operatorname*{esssup}^{\mathbb{P}}_{\mathbb{P}' \in \mathcal{P}(t,\mathbb{P},\mathbb{F}_+)} y^{\mathbb{P}'}_t \leq Y_t, \qquad \mathbb{P}\text{-a.s.}$$

Moreover, for any L,

$$Y^L_t = \operatorname*{esssup}^{\mathbb{P}}_{\mathbb{P}' \in \mathcal{P}(t,\mathbb{P},\mathbb{F}_+)} y^{L,\mathbb{P}'}_t \leq \operatorname*{esssup}^{\mathbb{P}}_{\mathbb{P}' \in \mathcal{P}(t,\mathbb{P},\mathbb{F}_+)} y^{\mathbb{P}'}_t \quad \Rightarrow \quad Y_t \leq \operatorname*{esssup}^{\mathbb{P}}_{\mathbb{P}' \in \mathcal{P}(t,\mathbb{P},\mathbb{F}_+)} y^{\mathbb{P}'}_t.$$

We deduce the representation formula for Y.

Now from the stability property for 2BSDE ([37], Theorem 4.5), if $N^{L,\mathbb{P}} = M^{L,\mathbb{P}} - K^{L,\mathbb{P}}$, then on $[0, T-\varepsilon]$

$$\begin{aligned}
&\|Z^L - Z^{L'}\|^{\ell}_{\mathbb{H}^{\ell}_0} + \sup_{\mathbb{P} \in \mathcal{P}_0} \mathbb{E}^{\mathbb{P}}\big[[N^{L,\mathbb{P}} - N^{L',\mathbb{P}}]^{\frac{\ell}{2}}_{T-\varepsilon}\big] \\
&\quad \leq C\big[\|Y^L_{T-\varepsilon} - Y^{L'}_{T-\varepsilon}\|^{\ell}_{\mathbb{L}^{\ell}_0} + \|Y^L_{T-\varepsilon} - Y^{L'}_{T-\varepsilon}\|^{\frac{\ell}{2}\wedge(\ell-1)}_{\mathbb{L}^{\ell}_0} \\
&\qquad + \phi^{\ell,\kappa,\varepsilon}_{L,L'} + \big(\phi^{\ell,\kappa,\varepsilon}_{L,L'}\big)^{\frac{\ell}{2}\wedge(\ell-1)}\big].
\end{aligned}$$

Thereby the sequences Z^L and $N^{L,\mathbb{P}}$ have a limit Z and $N^{\mathbb{P}}$. The process $(Y, Z, N^{\mathbb{P}})$ satisfies the dynamics: for any $\mathbb{P} \in \mathcal{P}_0$, and any $0 \leq s \leq t < T$,

$$Y_s = Y_t - \int_s^t \frac{Y^{\mathfrak{q}}_u}{(\mathfrak{q}-1)(\widehat{\eta}_u)^{\mathfrak{q}-1}}\, du + \int_s^t \widehat{\gamma}_u \, du - \left(\int_s^t Z_u \, dX^{c,\mathbb{P}}_u\right)^{\mathbb{P}} - \int_s^t dN^{\mathbb{P}}_u.$$

Then we decompose the process $N_t^{\mathbb{P}} = M_t^{\mathbb{P}} - K_t^{\mathbb{P}}$ and we check that $(Z, M^{\mathbb{P}}, K^{\mathbb{P}})$ is the desired space and that (3.17) holds. $\square$

Now we come to the main result concerning singular 2BSDEs.

THEOREM 3. *Under Conditions* (3.14) *and* (3.15), *the* 2*BSDE*

$$
\begin{aligned}
Y_t = \xi - \int_t^T \frac{(Y_u)^{\mathfrak{q}}}{(\mathfrak{q}-1)(\widehat{\eta}_u)^{\mathfrak{q}-1}}\,du + \int_t^T (\widehat{\gamma}_u)\,du \\
- \left(\int_t^T Z_s\,dX_s^{c,\mathbb{P}}\right)^{\mathbb{P}} - \int_t^T dM_s^{\mathbb{P}} + (K_T^{\mathbb{P}} - K_t^{\mathbb{P}}), \qquad \mathbb{P}\text{-}a.s.
\end{aligned}
$$

for any $0 \le t \le T$ *and any* $\mathbb{P} \in \mathcal{P}_0$, *with the singular terminal condition* ξ, *admits a nonnegative supersolution* $(Y, Z, M^{\mathbb{P}}, K^{\mathbb{P}})$ *satisfying*:

- *the dynamics* (3.17) *for any* $0 \le s \le t < T$;
- *the integrability property for any* $\varepsilon > 0$:

$$
\|Y\|^{\ell}_{\mathbb{D}_0^{\ell}(0,T-\varepsilon)} + \|Z\|^{\ell}_{\mathbb{H}_0^{\ell}(0,T-\varepsilon)} + \sup_{\mathbb{P}\in\mathcal{P}_0} \mathbb{E}^{\mathbb{P}}\big[[M^{\mathbb{P}}]_{T-\varepsilon}^{\frac{\ell}{2}}\big] + \sup_{\mathbb{P}\in\mathcal{P}_0} \mathbb{E}^{\mathbb{P}}\big(K_{T-\varepsilon}^{\mathbb{P}}\big)^{\ell} < +\infty;
$$

- *the minimality condition* (3.19);
- *the weak terminal condition*: $\mathcal{P}_0$-*q.s.*

$$
\liminf_{s\to T} Y_s \ge \xi. \tag{3.18}
$$

Moreover, this solution is the nonnegative minimal supersolution, that is, if $(\overline{Y}, \overline{Z}, \overline{M}^{\mathbb{P}}, \overline{K}^{\mathbb{P}})$ *satisfies the four previous conditions together with* $\overline{Y}_t \ge 0$ *for any* $t \in [0,T]$ $\mathcal{P}_0$-*q.s., then* $\overline{Y}_t \ge Y_t$ *for any* $t \in [0,T]$ $\mathcal{P}_0$-*q.s.*

PROOF. The first two points are direct consequences of the previous lemma. Since Y is the essential supremum of the supersolutions $y^{\mathbb{P}}$, following the same arguments as in the proof of Theorem 2, we deduce that $K^{\mathbb{P}}$ satisfies the minimality condition: for any $\mathbb{P} \in \mathcal{P}_0$

$$
\begin{aligned}
\underset{\mathbb{P}'\in\mathcal{P}(t,\mathbb{P},\mathbb{F}_+)}{\operatorname{essinf}^{\mathbb{P}}} \mathbb{E}^{\mathbb{P}'}\left[\int_t^{T-\varepsilon} \exp\left(\int_t^s \lambda_u^{\mathbb{P}'}\,du\right) dK_s^{\mathbb{P}'} \Big| \mathcal{F}_t^+\right] = 0, \\
0 \le t \le T-\varepsilon, \mathbb{P}\text{-a.s.},
\end{aligned} \tag{3.19}
$$

where

$$
\lambda_u^{\mathbb{P}'} = -\frac{1}{(\mathfrak{q}-1)(\widehat{\eta}_u)^{\mathfrak{q}-1}} \frac{Y_u^{\mathfrak{q}} - (y_u^{\mathbb{P}'})^{\mathfrak{q}}}{Y_u - y_u^{\mathbb{P}'}} \mathbf{1}_{Y_u \neq y_u^{\mathbb{P}'}}.
$$

Moreover, our left-continuity condition on the filtration implies that for any $\mathbb{P} \in \mathcal{P}_0$

$$
\liminf_{s\to T} y_s^{\mathbb{P}} \ge \xi, \qquad \mathbb{P}\text{-a.s.}
$$

Hence from the representation formula, the same inequality holds for Y.

Let us prove minimality of this supersolution. Let us consider $(\overline{Y}, \overline{Z}, \overline{M}^{\mathbb{P}}, \overline{K}^{\mathbb{P}})$ satisfying the dynamics (3.17) for any $0 \leq s \leq t < T$, the minimality condition (3.19) and the weak terminal condition (3.18). We also assume that $\overline{Y}$ is $\mathcal{P}_0 - q.s.$ nonnegative and $(\overline{Y}, \overline{Z}, \overline{M}^{\mathbb{P}}, \overline{K}^{\mathbb{P}})$ verifies some integrability property similar to $(Y, Z, M^{\mathbb{P}}, K^{\mathbb{P}})$ (with maybe a different power $\overline{\ell}$). From the proof of Theorem 1 (uniqueness for 2BSDE), we deduce that for any $\varepsilon > 0$, the representation property holds on $[0, T-\varepsilon]$, that is, for any $t \in [0, T-\varepsilon]$ and $\mathbb{P} \in \mathcal{P}_0$,

$$\overline{Y}_t = \operatorname*{esssup^{\mathbb{P}}}_{\mathbb{P}' \in \mathcal{P}(t,\mathbb{P},\mathbb{F}_+)} \bar{y}_t^{\mathbb{P}'}(T-\varepsilon, \overline{Y}_{T-\varepsilon}), \qquad \mathbb{P}\text{-a.s.}, \tag{3.20}$$

where $\bar{y}^{\mathbb{P}'} = \bar{y}^{\mathbb{P}'}(T-\varepsilon, \overline{Y}_{T-\varepsilon})$ is the first part of the solution $(\bar{y}^{\mathbb{P}'}, \bar{z}^{\mathbb{P}'}, \bar{m}^{\mathbb{P}'})$ of the BSDE (3.5) on $[0, T-\varepsilon]$ with terminal condition $\overline{Y}_{T-\varepsilon}$ at time $T-\varepsilon$ under $\mathbb{P}'$.

Recall that Y^L satisfies (3.10) and (3.11). Fix L and any probability $\mathbb{P}$ and consider $\bar{y}^{\mathbb{P}}$ and $y^{L,\mathbb{P}}$ on the time interval $[0, T-\varepsilon]$. Set

$$\widetilde{y} = \bar{y}^{\mathbb{P}} - y^{L,\mathbb{P}}, \qquad \widetilde{z} = \bar{z}^{\mathbb{P}} - z^{L,\mathbb{P}}, \qquad \widetilde{m} = \bar{m}^{\mathbb{P}} - m^{L,\mathbb{P}}.$$

We have

$$\begin{aligned} f(t, \bar{y}_t^{\mathbb{P}}) - f(t, y^{L,\mathbb{P}}) &= -\frac{1}{(\mathfrak{q}-1)(\widehat{\eta}_t)^{\mathfrak{q}-1}} \left(\frac{(\bar{y}_t^{\mathbb{P}})^{\mathfrak{q}} - (y_t^{L,\mathbb{P}})^{\mathfrak{q}}}{\bar{y}_t^{\mathbb{P}} - y_t^{L,\mathbb{P}}} \right) \mathbf{1}_{\bar{y}_t^{\mathbb{P}} \neq y_t^{L,\mathbb{P}}} (\bar{y}_t^{\mathbb{P}} - y_t^{L,\mathbb{P}}) \\ &\quad + \widehat{\gamma}_t - (\widehat{\gamma}_t \wedge L) \\ &= \lambda_t^{\mathbb{P}} (\bar{y}_t^{\mathbb{P}} - y_t^{L,\mathbb{P}}) + \widehat{\gamma}_t - (\widehat{\gamma}_t \wedge L) \end{aligned}$$

with $\lambda_t^{\mathbb{P}} \leq 0$. Thus the process $(\widetilde{y}, \widetilde{z}, \widetilde{m})$ solves the BSDE

$$d\widetilde{y}_t = [-\lambda_t^{\mathbb{P}} \widetilde{y}_t - \widehat{\gamma}_t \mathbf{1}_{\widehat{\gamma}_t \geq L}]\, dt + \widetilde{z}_t\, dW_t + d\widetilde{m}_t$$

on $[0, T-\varepsilon]$ with terminal condition $\widetilde{y}_{T-\varepsilon} = \overline{Y}_{T-\varepsilon} - y_{T-\varepsilon}^{L,\mathbb{P}}$. Thereby

$$\widetilde{y}_s = \mathbb{E}^{\mathbb{P}}\left[\widetilde{y}_{T-\varepsilon} \Gamma_{s,T-\varepsilon} + \int_s^{T-\varepsilon} \Gamma_{s,u} \widehat{\gamma}_u \mathbf{1}_{\widehat{\gamma}_u \geq L}\, du \Big| \mathcal{F}_s^{\mathbb{P},+} \right] \geq \mathbb{E}^{\mathbb{P}}[\widetilde{y}_{T-\varepsilon} \Gamma_{s,T-\varepsilon} | \mathcal{F}_s^{\mathbb{P},+}],$$

where $\Gamma_{s,t} = \exp(\int_s^t \lambda_u^{\mathbb{P}}\, du)$. Note that we have $y_t^{L,\mathbb{P}} \leq (1+T)L$, and hence $\widetilde{y}_t \geq -(1+T)L$. Thus $\widetilde{y}\Gamma_{s,\cdot}$ is bounded from below. We can apply Fatou's lemma to obtain

$$\widetilde{y}_s = \liminf_{\varepsilon \downarrow 0} \mathbb{E}^{\mathbb{P}}[\widetilde{y}_{T-\varepsilon} \Gamma_{s,T-\varepsilon} | \mathcal{F}_s^{\mathbb{P},+}] \geq \mathbb{E}^{\mathbb{P}}\left[\liminf_{\varepsilon \downarrow 0} (\widetilde{y}_{T-\varepsilon} \Gamma_{s,T-\varepsilon}) \Big| \mathcal{F}_s^{\mathbb{P},+} \right].$$

The process $(\Gamma_{s,t}, s \leq t \leq T)$ is càdlàg, nonnegative and bounded by one. Hence a.s.

$$\liminf_{\varepsilon \downarrow 0} (\widetilde{y}_{T-\varepsilon} \Gamma_{s,T-\varepsilon}) = \left(\liminf_{\varepsilon \downarrow 0} \widetilde{y}_{T-\varepsilon} \right) \Gamma_{s,T^-} \geq (\xi - \xi \wedge L) \Gamma_{s,T^-} \geq 0.$$

Here, again we use left-continuity of the filtration to exclude jumps for the orthogonal martingale $m^{L,\mathbb{P}}$. Finally, $\bar{y}_s^{\mathbb{P}} \geq y_s^{L,\mathbb{P}}$ for any $s \in [0,T)$. Since it holds for any $\mathbb{P} \in \mathcal{P}_0$, we deduce that $\overline{Y}_s \geq Y_s^L$ for any $L \geq 0$. Taking the limit as L goes to ∞ yields the claim. $\square$

REMARK 9. The previous theorem holds true for more general generators satisfying Conditions (**H**) together with the growth condition: there exists a constant $\mathfrak{q} > 1$ and a positive process η such that for any $y \geq 0$

$$\widehat{f}^{\mathbb{P}}(t,y,z) \leq -\frac{1}{(\mathfrak{q}-1)\widehat{\eta}_t^{\mathfrak{q}-1}}|y|^{\mathfrak{q}} + \widehat{f}^{\mathbb{P}}(t,0,z)$$

and the conditions (3.14) and (3.15) hold replacing $\widehat{\gamma}$ by $\widehat{f}^{0,\mathbb{P}}$. See [29] for details for singular BSDEs.

We can now obtain an optimal solution for the control problem (3.13).

THEOREM 4. *The constrained problem* (3.13) *has an optimal state process* $\mathcal{X}^*$ *defined by*

$$\mathcal{X}_s^* = x - \int_t^s \left(\frac{Y_u}{\widehat{\eta}_u}\right)^{\mathfrak{q}-1} \mathcal{X}_u^* \, du.$$

Moreover, the value function is given by $J(t,x) = |x|^{\vartheta} Y_t$.

PROOF. Remark that the sign of $\mathcal{X}_s^*$ is equal to the sign of x. Thus if $x < 0$, $|\mathcal{X}_s^*|$ simply denotes $-\mathcal{X}_s^*$.

If we define for $t \leq s < T$

$$\theta_s := \left[Y_s|\mathcal{X}_s^*|^{\vartheta-1} - Y_t|\mathcal{X}_t^*|^{\vartheta-1} + \int_t^s |\mathcal{X}_u^*|^{\vartheta-1}\widehat{\gamma}_u \, du + \int_t^s |\mathcal{X}_u^*|^{\vartheta-1} \, dK_u^{\mathbb{P}}\right],$$

then we can easily show that under each $\mathbb{P} \in \mathcal{P}_0$, θ is a nonnegative local martingale, and thus a nonnegative supermartingale. Thereby θ_s has a limit, $\mathbb{P}$-a.s. when s goes to T. Hence since $\mathbb{P}$-a.s.

$$\liminf_{s \to T} Y_s \mathbf{1}_{\mathcal{S}} = +\infty,$$

we obtain that

$$\begin{aligned}|\mathcal{X}_s^*| &= \left(\frac{\theta_s + Y_t|x|^{\vartheta-1} - \int_t^s |\mathcal{X}_u^*|^{\vartheta-1}(\widehat{\gamma}_u \, du + dK_u^{\mathbb{P}})}{Y_s}\right)^{\mathfrak{q}-1} \\ &\leq \left(\frac{\theta_s + Y_t|x|^{\vartheta-1}}{Y_s}\right)^{\mathfrak{q}-1}\end{aligned}$$

tends to zero on $\mathcal{S}$, $\mathbb{P}$-a.s. In other words, $\mathcal{X}^* \in \mathcal{A}_0(t,x)$.

As in Lemma 3, we have

$$d(Y_s|\mathcal{X}_s^*|^\vartheta) = -[\widehat{\eta}_s|\dot{\mathcal{X}}_s^*|^\vartheta + (\widehat{\gamma}_s)|\mathcal{X}_s^*|^\vartheta]\,ds + |\mathcal{X}_s^*|^\vartheta Z_s\,dX_s^{c,\mathbb{P}} \\ + |\mathcal{X}_s^*|^\vartheta\,dM_s^{\mathbb{P}} - |\mathcal{X}_s^*|^\vartheta\,dK_s^{\mathbb{P}}.$$

Thus, for any $\varepsilon > 0$,

$$Y_t|x|^\vartheta = \mathbb{E}^{\mathbb{P}}\left[Y_{T-\varepsilon}|\mathcal{X}_{T-\varepsilon}^*|^\vartheta + \int_t^{T-\varepsilon}[\widehat{\eta}_s|\dot{\mathcal{X}}_s^*|^\vartheta + (\widehat{\gamma}_s)|\mathcal{X}_s^*|^\vartheta]\,ds\middle|\mathcal{F}_t^+\right] \\ + \mathbb{E}^{\mathbb{P}}\left[\int_t^{T-\varepsilon}|\mathcal{X}_s^*|^\vartheta\,dK_s^{\mathbb{P}}\middle|\mathcal{F}_t^+\right].$$

From the definition of θ, it follows that also the limit $\lim_{t\uparrow T} Y_t|\mathcal{X}_t^*|^{\vartheta-1} \in \mathbb{R}$ exists and that $|\mathcal{X}_T^*| = \lim_{t\uparrow T}|\mathcal{X}_t^*| = 0$ if $\liminf_{t\uparrow T} Y_t = \infty$. Recall that $\liminf_{t\uparrow T} Y_s \geq \xi$ and let us distinguish two cases. First, assume that $\liminf_{t\uparrow T} Y_t = \infty$. Then $\liminf_{t\uparrow T} Y_t|\mathcal{X}_t^*|^\vartheta = (\lim_{t\uparrow T} Y_t|\mathcal{X}_t^*|^{\vartheta-1})(\lim_{t\uparrow T}|\mathcal{X}_t^*|) = 0 = \xi|\mathcal{X}_T^*|^\vartheta$ (for the last equality we use that $\infty\cdot 0 := 0$). Next assume that $\liminf_{t\uparrow T} Y_t < \infty$. Then it follows that $\liminf_{t\uparrow T} Y_t|\mathcal{X}_t^*|^\vartheta \geq \xi|\mathcal{X}_T^*|^\vartheta$. Hence for any $\mathbb{P}$, $\liminf_{t\uparrow T} Y_t|\mathcal{X}_t^*|^\vartheta \geq \xi|\mathcal{X}_T^*|^\vartheta$, $\mathbb{P}$-a.s. By Fatou's lemma and since $K^{\mathbb{P}}$ is nondecreasing, we have

$$Y_t|x|^\vartheta \geq \mathbb{E}^{\mathbb{P}}\left[\xi|\mathcal{X}_T^*|^\vartheta + \int_t^T[\widehat{\eta}_s|\dot{\mathcal{X}}_s^*|^\vartheta + (\widehat{\gamma}_s)|\mathcal{X}_s^*|^\vartheta]\,ds\middle|\mathcal{F}_t^+\right].$$

Thereby

$$Y_t|x|^\vartheta \geq \operatorname*{esssup}_{\mathbb{P}\in\mathcal{P}_t}\mathbb{E}^{\mathbb{P}}\left[\int_t^T(\widehat{\eta}_s|\dot{\mathcal{X}}_s^*|^\vartheta + \widehat{\gamma}_s|\mathcal{X}_s^*|^\vartheta)\,ds + \xi|\mathcal{X}_T^*|^\vartheta\middle|\mathcal{F}_t^+\right] \geq J(t,x).$$

Now we have obviously $J(t,x) \geq J^L(t,x)$ and

$$|x|^\vartheta Y_t = \lim_{L\nearrow+\infty}|x|^\vartheta Y_t^L = \lim_{L\nearrow+\infty} J^L(t,x) \leq J(t,x) \leq Y_t|x|^\vartheta.$$

This gives the optimality for $\mathcal{X}^*$ and the value function of our problem. □

Discussion around the examples of [5]. Even in the classical case, that is for a fixed probability $\mathbb{P}$, there is in general no explicit solution of the BSDE (3.5). But when $\xi = +\infty$ $\mathbb{P}$-a.s. and $\gamma = 0$, then in [5], Section 5, an explicit solution y is given provided that η has uncorrelated multiplicative increments. This condition is equivalent to the property that the process $(\eta_t/\mathbb{E}^{\mathbb{P}}(\eta_t), t \geq 0)$ is a $\mathbb{P}$-martingale (see [5], Lemma 5.1). Under this condition, the value function and an optimal state process are given in [5], Proposition 5.3.

Assume that $\widehat{\eta}$ is given by $\widehat{\eta}_t = \eta_0\exp(X_t - \frac{1}{2}\int_0^t \widehat{a}_s\,ds)$ and that under $\mathbb{P}$, the drift $b^{\mathbb{P}}$ of X is deterministic. Note that the concatenation property of the family $\mathcal{P}$ implies that $b = b^{\mathbb{P}}$ should not depend on $\mathbb{P}$. Then under each $\mathbb{P}$, $\widehat{\eta}$ satisfies

$d\widehat{\eta}_t = \widehat{\eta}_t \, dX_t$ and has uncorrelated multiplicative increments. The solution $y^{\mathbb{P}}$ is explicitly given by

$$y_t^{\mathbb{P}} = \widehat{\eta}_t \frac{1}{(A_t \int_t^T \frac{1}{A_s} ds)^{\vartheta - 1}}, \qquad A_t = \exp\left(\int_0^s (\mathfrak{q} - 1) b_r \, dr\right) = [\mathbb{E}^{\mathbb{P}}(\widehat{\eta}_s)]^{\mathfrak{q}-1}.$$

Hence we have an explicit formula for the solution of the 2BSDE,

$$Y_t = \widehat{\eta}_t \frac{1}{(A_t \int_t^T \frac{1}{A_s} ds)^{\vartheta - 1}}.$$

An optimal state process is deterministic:

$$\mathcal{X}_t = \frac{1}{\int_0^T \frac{1}{A_s} ds} \int_t^T \frac{1}{A_s} ds.$$

In particular, if the canonical process is a local martingale under each $\mathbb{P}$ ($b = 0$), $Y_t = \frac{\widehat{\eta}_t}{(T-t)}$ and $\mathcal{X}_t = \frac{(T-t)}{T}$. Roughly speaking, since $\widehat{\eta}$ models the cost (price impact) and $\widehat{\gamma}$ the risk, then the drift is important for η (average cost) and the volatility is important for $\widehat{\gamma}$. That is why when taking $\widehat{\gamma}$ is equal to 0, the volatility uncertainty can not be seen in the generator.

APPENDIX: (REFLECTED) BSDE WITH MONOTONE GENERATOR

A.1. Notation and Conditions (H) and (H′). In this section, the setting is the same as in [9] or [28]. Let $T > 0$ be fixed and let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, equipped with a filtration $\mathbb{F} = \{\mathcal{F}_t, 0 \leq t \leq T\}$ satisfying the usual conditions and carrying a standard d-dimensional $\mathbb{F}$-Brownian motion W. In the Section 2.1.5, for any $p > 1$ and any $\alpha \in \mathbb{R}$ we have defined the spaces $\mathbb{L}^p$, $\mathbb{D}^p$, $\mathbb{H}^{p,\alpha}$, $\mathbb{M}^{p,\alpha}$, $\mathbb{I}^{p,\alpha}$. The spaces $\mathbb{H}^{p,0}$, $\mathbb{M}^{p,0}$ and $\mathbb{I}^{p,0}$ will be denoted $\mathbb{H}^p$, $\mathbb{M}^p$ and $\mathbb{I}^p$. In the Itô formula for $p > 1$, we will use the constant

$$c(p) = \frac{p}{2}\big((p-1) \wedge 1\big) \tag{A.1}$$

and the function $\phi_p(x) = |x|^{p-1} \operatorname{sgn}(x) \mathbf{1}_{x \neq 0}$ (see [11], Corollary 2.3, or [28], Corollary 1).

We consider a generator function

$$f : (t, \omega, y, z) \in [0, T] \times \Omega \times \mathbb{R} \times \mathbb{R}^d \longrightarrow \mathbb{R}.$$

The generator f satisfies Condition (**H**). In **H4**, the process Ψ depends on t and ω and is supposed to satisfy

$$\mathbb{E} \int_0^T (\Psi_t)^{\varrho} \, dt < +\infty$$

(see Condition **C1**). Sometimes we will use the stronger condition **H4′**: there exists $\mathfrak{q} \geq 1$ and a constant $L_{\mathfrak{q}}$ such that

$$\big| f(t, \omega, y, 0) - f_t^0 \big| \leq L_{\mathfrak{q}}\big(1 + |y|^{\mathfrak{q}}\big).$$

If f verifies **H1**, **H2**, **H3** and **H4′**, we will say that (**H′**) holds.

A.2. Results for monotone BSDE. We consider the BSDE

$$y_t = \xi + \int_t^T f(u, y_u, z_u)\, du - \int_t^T z_u\, dW_u - \int_t^T dm_u. \tag{A.2}$$

Let us now recall several classical results on the BSDE (A.2). If (**H**) holds[5] and if for some $p > 1$

$$\mathbb{E}\left[|\xi|^p + \int_0^T |f_t^0|^p\, dt\right] < +\infty, \tag{A.3}$$

then there is a unique solution $(y, z, m) \in \mathbb{D}^p \times \mathbb{H}^p \times \mathbb{M}^p$ (see [11, 14] or [28]). Let us emphasize that the quasi-left continuity property of the filtration assumed in [28] is in fact unnecessary (see the Introduction of [9] or [35]). The next a priori estimate on (y, z, m) will be crucial.

LEMMA A.1 (A priori estimate). *Under Condition* (***H***), *for any* $\alpha > pL_1 + \frac{L_2^2}{(p-1)\wedge 1}$ *there exists a constant* C_p *such that*

$$\|e^{\alpha \cdot} y\|_{\mathbb{D}^p} + \|z\|_{\mathbb{H}^{p,\alpha}} + \|m\|_{\mathbb{M}^{p,\alpha}} \le C_p \mathbb{E}\left[e^{\alpha p T}|\xi|^p + \int_0^T |e^{\alpha t} f_t^0|^p\, dt\right].$$

PROOF. See, for example, Propositions 2 and 3 in [28]. $\square$

Moreover, from the proof of this lemma, we get the next classical estimates: for any stopping time $0 \le \tau \le T$,

$$|y_\tau| \le C\left(\mathbb{E}\left[|\xi|^\kappa + \int_\tau^T |f_s^0|^\kappa\, ds \Big| \mathcal{F}_\tau\right]\right)^{1/\kappa} \tag{A.4}$$

for any $1 < \kappa \le p$ and

$$\mathbb{E}\int_t^T |y_s|^p\, ds \le C\mathbb{E}\left(|\xi|^p + \int_t^T |f_s^0|^p\, ds\right). \tag{A.5}$$

The constant C depends only on κ or p, T, L_1 and L_2.

The next trick (*linearization procedure*) is used several times in this paper. If (y, z, m) satisfies the BSDE (A.2), then

$$y_t = \xi + \int_t^T f_s^0\, ds + \int_t^T \lambda_s y_s\, ds + \int_t^T \zeta_s z_s\, ds - \int_t^T z_s\, dW_s - \int_t^T dm_s,$$

where

$$\lambda_s = \frac{f(s, y_s, 0) - f_s^0}{y_s}\mathbf{1}_{y_s \neq 0}$$

[5] In fact in this section, Hypothesis **H4** could be replaced by a more general condition (see [28], Assumption (H2)).

and

$$\zeta_t^i = (z_t^i)^{-1}\big(f\big(t, y_t, z_t^{(i)}\big) - f\big(t, y_t, z_t^{(i-1)}\big)\big)\mathbf{1}_{z_t^i \neq 0},$$

where for $1 \le i \le d$, $z_t^{(i)}$ is the d-dimensional vector in which the first i-components are equal to the ones of z_t and the $d-i$ others are equal to zero. From **H3**, ζ is a bounded (by L_2) vector-valued process. From the monotone condition **H2**, $\lambda_s \le L_1$ a.s. If $\Lambda_t = \exp(\int_0^t \lambda_s\, ds)$,

$$\Lambda_t y_t = \Lambda_T \xi + \int_t^T \Lambda_s f_s^0\, ds + \int_t^T \zeta_s \Lambda_s z_s\, ds - \int_t^T \Lambda_s z_s\, dW_s - \int_t^T \Lambda_s\, dm_s. \tag{A.6}$$

Hence if $\mathbb{Q}$ is the probability measure equivalent to $\mathbb{P}$ defined by the density

$$\mathcal{E}\bigg(-\int_0^\cdot \zeta_s\, dW_s\bigg) = \exp\bigg(-\int_0^\cdot \zeta_s\, dW_s - \frac{1}{2}\int_0^\cdot \|\zeta_s\|^2\, ds\bigg), \tag{A.7}$$

we obtain

$$y_t = \mathbb{E}^{\mathbb{Q}}\bigg[\frac{\Lambda_T}{\Lambda_t}\xi + \int_t^T \frac{\Lambda_s}{\Lambda_t} f_s^0\, ds \Big| \mathcal{F}_t\bigg].$$

And for any $0 \le t \le s \le T$,

$$0 < \frac{\Lambda_s}{\Lambda_t} = \exp\bigg(\int_t^s \lambda_u\, du\bigg) \le \exp\big(L_1(t-s)\big).$$

This implies immediately that if ξ and f^0 are $\mathbb{P}$-a.s. bounded, then y is bounded.

Moreover, comparison principle ([28], Proposition 4) and stability property ([28], Propositions 2 and 3) hold for monotone BSDE.

LEMMA A.2 (Comparison). *We consider two generators f_1 and f_2 satisfying* (**H**). *Let ξ^1 and ξ^2 be two terminal conditions for BSDEs* (A.2) *driven respectively by f_1 and f_2. Denote by (y^1, z^1, m^1) and (y^2, z^2, m^2) the respective solutions in some $\mathbb{D}^p \times \mathbb{H}^p \times \mathbb{M}^p$ with $p > 1$. If $\xi^1 \le \xi^2$ and $f_1(t, y_t^1, z_t^1) \le f_2(t, y_t^1, z_t^1)$, then a.s. for any $t \in [0, T]$, $y_t^1 \le y_t^2$.*

Note that a strict comparison principle does not hold in general (see [34], Proposition 5.34, and the comments just after).

LEMMA A.3 (Stability). *Let now (ξ, f) and (ξ', f') be two sets of data each satisfying the assumptions* (**H**). *Let (y, z, m)* (*resp*., (y', z', m')) *denote the solution of the BSDE* (A.2) *with data (ξ, f)* (*resp*., (ξ', f')). *Define*

$$(\Delta y, \Delta z, \Delta m, \Delta \xi, \Delta f) = \big(y - y', z - z', m - m', \xi - \xi', f - f'\big).$$

Then there exists a constant C depending on L_1, L_2, p and T, such that

$$\mathbb{E}\bigg[\sup_{t\in[0,T]}|\Delta y_t|^p+\bigg(\int_0^T|\Delta z_s|^2\,ds\bigg)^{p/2}+[\Delta m]_T^{p/2}\bigg]$$
$$\leq C\mathbb{E}\bigg[|\Delta\xi|^p+\int_0^T\big|\Delta f\big(t,y'_t,z'_t\big)\big|^p\,dt\bigg].$$

Let us describe why we assume Condition **H4**, and not some weaker growth condition. Indeed for second order BSDE (Section 2) we used that the solution (y,z,m) is obtained by approximation with a sequence of solutions (y^n,z^n,m^n) of Lipschitz BSDE.[6] This is the reason why polynomial growth of f w.r.t. y is assumed in **H4**, as in the paper [10]. In their work, the filtration is generated by the Brownian motion and the generator f satisfies Condition (**H′**). Hence we extend it to our setting.

LEMMA A.4 (Lipschitz approximation). *Assume that* (**H**) *holds and that $\xi\in\mathbb{L}^{\bar{\mathfrak{p}}\mathfrak{q}}$ and $f^0\in\mathbb{H}^{\bar{\mathfrak{p}}\mathfrak{q}}$ for some $\bar{\mathfrak{p}}>\varrho/(\varrho-1)$. The solution (y,z,m) of the BSDE* (A.2) *belongs to $\mathbb{D}^{\bar{\mathfrak{p}}\mathfrak{q}}\times\mathbb{H}^{\bar{\mathfrak{p}}\mathfrak{q}}\times\mathbb{M}^{\bar{\mathfrak{p}}\mathfrak{q}}$ and is obtained as the limit in $\mathbb{D}^p\times\mathbb{H}^p\times\mathbb{M}^p$ of a sequence (y^n,z^n,m^n) solution of Lipschitz BSDEs for p satisfying* (2.4), *that is*, $1<p\leq\frac{\varrho\bar{\mathfrak{p}}}{\varrho+\bar{\mathfrak{p}}}<\bar{\mathfrak{p}}$.

PROOF. The first part of the result is a direct consequence of Lemma A.1. Let us now only explain the second assertion. We will adapt the result of [10] and we refer to this paper for the details. We only give the main arguments. W.l.o.g. we can assume in this proof that $L_1=0$ (just consider $\bar{y}_t=e^{-L_1t}y_t$, $\bar{z}_t=e^{-L_1t}z_t$, $\bar{m}_t=e^{-L_1t}m_t$, instead of (y,z,m)).

Step 1. First, we consider the following BSDE;

$$y_t=\xi+\int_t^T f(s,y_s,v_s)\,ds-\int_t^T z_s\,dW_s-\int_t^T dm_s, \tag{A.8}$$

where v belongs to $\mathbb{H}^{\bar{\mathfrak{p}}\mathfrak{q}}$ and f satisfies (**H′**). We denote by h the function $h(t,y)=f(t,y,v_t)$. This function h satisfies Conditions **H1**–**H2** and **H4′**. And $h_t^0=h(t,0)\in\mathbb{H}^{\bar{\mathfrak{p}}\mathfrak{q}}$. We construct a sequence of Lipschitz functions h_n which approximate h. Let $\varsigma:\mathbb{R}\mapsto\mathbb{R}_+$ be a nonnegative function with the unit ball for support and such that $\int\varsigma(u)\,du=1$ and define for each integer $n>1$, $\varsigma_n(u)=n\varsigma(nu)$. We denote also for each integer n, by Θ_n, a C^∞ function from $\mathbb{R}$ to $\mathbb{R}_+$ such that $0\leq\Theta_n\leq 1$, $\Theta_n(u)=1$ for $|u|\leq n$ and $\Theta_n(u)=0$ as soon as $|u|>n+1$. We set

$$\xi_n=\frac{n\xi}{n\vee|\xi|},\qquad \widetilde{h}_n(t,y)=\frac{nh(t,y)}{n\vee|h_t^0|}.$$

[6] Note that this setting is sufficient to solve our control problem. Weaker conditions could be introduced using Mazur's Lemma and this technical point is left for further research.

Moreover,

$$\varpi(n) = \lfloor e^{1/2}(n+2L_{\mathfrak{q}})\sqrt{1+T^2}\rfloor + 1,$$

where $\lfloor r\rfloor$ is the integer part of r, $L_{\mathfrak{q}}$ coming from **H4′**, and we define as the convolution product

$$h_n(t,\cdot) = \varsigma_n * \big(\Theta_{\varpi(n+1)}\widetilde{h}_n(t,\cdot)\big), \qquad t\in[0,T]. \tag{A.9}$$

This function h_n is globally Lipschitz w.r.t. y uniformly in t and ω with $|h_n(t,0)| \le n\wedge |h_t^0| + 2L_{\mathfrak{q}}$. Moreover, for any (t,ω,y),

$$yh_n(t,y) \le \big((n\wedge |h_t^0|) + 2L_{\mathfrak{q}}\big)|y|,$$

and for any y and y' in the ball $\overline{B(0,\varpi(n))}$ then

$$(y-y')\big(h_n(t,y)-h_n(t,y')\big)\le 0.$$

In other words, h_n is only locally monotone (only in a given ball with the radius depending on n). Let (y^n,z^n,m^n) be the unique solution of the BSDE

$$y_t^n = \xi_n + \int_t^T h_n(u,y_u^n)\,du - \int_t^T z_u^n\,dW_u - \int_t^T dm_u^n \tag{A.10}$$

in $\mathbb{D}^{\bar{\mathfrak{p}}\mathfrak{q}}\times\mathbb{H}^{\bar{\mathfrak{p}}\mathfrak{q}}\times\mathbb{M}^{\bar{\mathfrak{p}}\mathfrak{q}}$. This solution verifies (see [10], Proposition 2.1)

$$\sup_{t\in[0,T]}|y_t^n| \le (n+2L_{\mathfrak{q}})e^{1/2}\sqrt{1+T^2}.$$

Hence y_t^n is in $\overline{B(0,\varpi(n))}$. And from Lemma A.1 we also have for any $1<p\le\bar{\mathfrak{p}}$ and for some α large enough

$$\sup_{n\in\mathbb{N}^*}\big[\|e^{\alpha\cdot}y^n\|_{\mathbb{D}^{p\mathfrak{q}}} + \|z^n\|_{\mathbb{H}^{p\mathfrak{q},\alpha}} + \|m^n\|_{\mathbb{M}^{p\mathfrak{q},\alpha}}\big]$$
$$\le C_p\mathbb{E}\bigg[e^{p\alpha\mathfrak{q}T}|\xi|^{p\mathfrak{q}} + \int_0^T e^{\alpha t}\big(|h_t^0|+2L_{\mathfrak{q}}\big)^{p\mathfrak{q}}\,dt\bigg].$$

Now we fix two integers ℓ and n such that $\ell\ge n$ and

$$\delta y = y^\ell - y^n,\qquad \delta z = z^\ell - z^n,\qquad \delta m = m^\ell - m^n,\qquad \delta\xi = \xi_\ell - \xi_n.$$

For $1<p\le\bar{\mathfrak{p}}$, with $c(p)$ defined by (A.1), since $|y_t^n|\le\varpi(n)\le\varpi(\ell)$, we use Itô's formula and the local monotonicity of h_ℓ to obtain

$$|\delta y_t|^p + c(p)\int_t^T |\delta y_{u-}|^{p-2}\mathbf{1}_{\delta y_{u-}\neq 0}|\delta z_u|^2\,du$$
$$+ c(p)\int_t^T |\delta y_{u-}|^{p-2}\mathbf{1}_{\delta y_{u-}\neq 0}\,d[\delta m]_u^c$$
$$+ \sum_{t<u\le T}\big[|\delta y_{u-}+\Delta(\delta m)_u|^p - |\delta y_{u-}|^p - p\phi_p(\delta y_{u-})\Delta(\delta m)_u\big]$$

$$\leq |\delta\xi|^p + p\int_t^T \big[h_\ell(u, y_u^n) - h_n(u, y_u^n)\big]\phi_p(\delta y_{u-})\,du$$
$$- p\int_t^T \phi_p(\delta y_{u-})\delta z_u\,dW_u - p\int_t^T \phi_p(\delta y_{u-})\,d(\delta m)_u.$$

Since the set $\{\delta y_u \neq \delta y_{u-}\}$ is countable, arguing as in the proof of [28], Proposition 3, we deduce that there exists a constant C such that

$$\begin{aligned} &\mathbb{E}\Big[\sup_{t\in[0,T]}|\delta y_t|^p + \Big(\int_0^T |\delta z_u|^2\,du\Big)^{p/2} + \big([\delta m]_T\big)^{p/2}\Big] \\ &\quad \leq C\mathbb{E}\Big[|\delta\xi|^p + \int_0^T \big|h_\ell(u, y_u^n) - h_n(u, y_u^n)\big||\delta y_u|^{p-1}\,du\Big]. \end{aligned} \tag{A.11}$$

Since $\xi \in \mathbb{L}^{\bar{\mathfrak{p}}\mathfrak{q}}$, then $\delta\xi$ goes to zero in $\mathbb{L}^p$ as n and ℓ tend to $+\infty$. For any given number k, we put

$$S_n^\ell = \mathbb{E}\Big[\int_0^T \mathbf{1}_{\{(|y_u^n|+|y_u^\ell|)\leq k\}}\big|h_\ell(u, y_u^n) - h_n(u, y_u^n)\big||\delta y_u|^{p-1}\,du\Big],$$
$$R_n^\ell = \mathbb{E}\Big[\int_0^T \mathbf{1}_{\{(|y_u^n|+|y_u^\ell|)\geq k\}}\big|h_\ell(u, y_u^n) - h_n(u, y_u^n)\big||\delta y_u|^{p-1}\,du\Big].$$

With this notation, we have

$$\begin{aligned} &\mathbb{E}\Big[\int_0^T \big|h_\ell(u, y_u^n) - h_n(u, y_u^n)\big||\delta y_u|^{p-1}\,du\Big] \\ &\quad = S_n^\ell + R_n^\ell \leq C_p k^{p-1}\mathbb{E}\Big[\sup_{|y|\leq k}\int_0^T \big|h_\ell(u, y) - h_n(u, y)\big|\,du\Big] + R_n^\ell. \end{aligned} \tag{A.12}$$

Since $h(s, \cdot)$ is continuous ($\mathbb{P}$-a.s., for every s), $h_n(s, \cdot)$ converges toward $h(s, \cdot)$ uniformly on compact sets. Taking into account that

$$\sup_{|y|<k}\big|h_n(s, y)\big| \leq \big|h(s, 0)\big| + 2^{\mathfrak{q}}L_{\mathfrak{q}}\big(1 + k^{\mathfrak{q}}\big),$$

Lebesgue's dominated convergence theorem implies that for any fixed number k, the quantity

$$C_p k^{p-1}\mathbb{E}\Big[\sup_{|y|\leq k}\int_0^T \big|h_\ell(u, y) - h_n(u, y)\big|\,du\Big]$$

goes to 0 as n tends to infinity uniformly with respect to ℓ. The proof will be completed if we prove the convergence of R_n^ℓ. Using Hölder's and Chebychev's inequalities, we get

$$
\begin{aligned}
R_n^\ell &\le \left[\mathbb{E}\int_0^T \mathbf{1}_{\{(|y_u^n|+|y_u^\ell|)\ge k\}}\,du\right]^{\frac{(\mathfrak{q}-1)(p-1)}{p\mathfrak{q}}} \\
&\quad\times\left[\mathbb{E}\int_0^T \big|h_\ell(u,y_u^n)-h_n(u,y_u^n)\big|^{\frac{p\mathfrak{q}}{\mathfrak{q}+p-1}}|\delta y_u|^{\frac{p(p-1)\mathfrak{q}}{\mathfrak{q}+p-1}}\,du\right]^{\frac{\mathfrak{q}+p-1}{p\mathfrak{q}}} \\
&\le k^{(1-\mathfrak{q})(p-1)}2^{p\mathfrak{q}-1}T^{\frac{(\mathfrak{q}-1)(p-1)}{p\mathfrak{q}}}\left[\sup_{n\in\mathbb{N}^*}\mathbb{E}\Big(\sup_{t\in[0,T]}\big(|y_u^n|\big)^{p\mathfrak{q}}\Big)\right]^{\frac{(\mathfrak{q}-1)(p-1)}{p\mathfrak{q}}} \\
&\quad\times\left[\mathbb{E}\int_0^T \big|h_\ell(u,y_u^n)-h_n(u,y_u^n)\big|^{\frac{p\mathfrak{q}}{\mathfrak{q}+p-1}}|\delta y_u|^{\frac{p(p-1)\mathfrak{q}}{\mathfrak{q}+p-1}}\,du\right]^{\frac{\mathfrak{q}+p-1}{p\mathfrak{q}}}.
\end{aligned}
\tag{A.13}
$$

Remember that the above expectation is bounded uniformly w.r.t. n. Thus the first term of the right-hand side of (A.13) is uniformly bounded. We have to control

$$
A_n^\ell = \mathbb{E}\int_0^T \big|h_\ell(u,y_u^n)-h_n(u,y_u^n)\big|^{\frac{p\mathfrak{q}}{\mathfrak{q}+p-1}}|\delta y_u|^{\frac{p(p-1)\mathfrak{q}}{\mathfrak{q}+p-1}}\,du. \tag{A.14}
$$

By Young's inequality,

$$
A_n^\ell \le 2C\sup_{n\in\mathbb{N}^*}\mathbb{E}\Big(\sup_{t\in[0,T]}\big(|y_u^n|\big)^{p\mathfrak{q}}\Big)+C\mathbb{E}\int_0^T\big(|f_u^0|^p+\|v_u\|^{p\mathfrak{q}}\big)\,du.
$$

Thus A_n^ℓ remains bounded w.r.t. n and ℓ. Collecting (A.11), (A.12), (A.13) with (A.14), we deduce that there exists a constant C such that for any k and $\varepsilon>0$, there exists N such that for $\ell\ge N$ and $n\ge N$

$$
\mathbb{E}\left[\sup_{t\in[0,T]}|\delta y_t|^p+\left(\int_0^T|\delta z_u|^2\,du\right)^{p/2}+[\delta m]_T^{p/2}\right]\le C\frac{1}{k^{(\mathfrak{q}-1)(p-1)}}+\varepsilon.
$$

Since we can fix k large enough to ensure that the right-hand side is smaller than 2ε, we deduce the convergence result.

Step 2. We consider now the general BSDE (A.2), but with Condition (**H′**). The Lipschitz approximation will be obtained by a fixed point argument in $\mathbb{D}^p\times\mathbb{H}^p\times\mathbb{M}^p$, arguing as in the proof of [10], Theorem 3.6, with straightforward modifications.

Step 3. For the more general growth condition **H4**, consider

$$
f_n(t,y,z)=\big(f(t,y,z)-f(t,0,0)\big)\frac{n}{\Psi(t)\vee n}+f(t,0,0).
$$

Then f_n is still Lipschitz continuous w.r.t. z, continuous and monotone w.r.t. y and satisfies

$$
\big|f_n(t,y,0)-f_n(t,0,0)\big|\le\frac{n\Psi(t)}{\Psi(t)\vee n}\big(1+|y|^{\mathfrak{q}}\big)\le n\big(1+|y|^{\mathfrak{q}}\big).
$$

f_n satisfies **H4′** with $L_{\mathfrak{q}} = n$. Thus there is a sequence (y^n, z^n, m^n) of solutions for the BSDE with generator f_n. As before let us define

$$\delta y = y^\ell - y^n, \qquad \delta z = z^\ell - z^n, \qquad \delta m = m^\ell - m^n.$$

Note that

$$\left| f_\ell(u, y_u^n, z_u^n) - f_n(u, y_u^n, z_u^n) \right| \le \left(\Psi(u) \mathbf{1}_{\Psi(u) \ge n \wedge \ell}\right)\left(1 + |y_u^n|^{\mathfrak{q}}\right).$$

Young's inequality implies that for any $\mathfrak{k} > 0$

$$\begin{aligned} &\left| e^{\alpha u} [f_\ell(u, y_u^n, z_u^n) - f_n(u, y_u^n, z_u^n)] \phi_p(\delta y_u) \right| \\ &\quad \le \frac{e^{\alpha u}}{p \mathfrak{k}^{p-1}} \left(\Psi(u) \mathbf{1}_{\Psi(u) \ge n \wedge \ell}\right)^p \left(1 + |y_u^n|^{\mathfrak{q}}\right)^p + \mathfrak{k} \frac{p-1}{p} e^{\alpha u} |\delta y_u|^p. \end{aligned}$$

Therefore, using Hölder's inequality and Itô's formula for $1 < p \le \frac{\varrho \bar{\mathfrak{p}}}{\varrho + \bar{\mathfrak{p}}}$ and $\alpha > \frac{p}{(p-1)} L_2^2 + \mathfrak{k} \frac{p-1}{p}$, taking the expectation and leads to

$$\begin{aligned} &\frac{c(p)}{2} \mathbb{E} \int_0^T e^{\alpha u} |\delta y_{u-}|^{p-2} \mathbf{1}_{\delta Y_{u-} \neq 0} |\delta z_u|^2 \, du \\ &\quad + c(p) \mathbb{E} \int_0^T e^{\alpha u} |\delta y_{u-}|^{p-2} \mathbf{1}_{\delta y_{u-} \neq 0} d[\delta m]_u^c \\ &\quad + \mathbb{E} \sum_{0 < u \le T} \left[\left| \delta y_{u-} + \Delta(\delta m)_u \right|^p - |\delta y_{u-}|^p - p \phi_p(\delta y_{u-}) \Delta(\delta m)_u \right] \\ &\quad \le \frac{1}{p \mathfrak{k}^{p-1}} \left(\mathbb{E} \int_0^T e^{\alpha u} \left(\Psi(u) \mathbf{1}_{\Psi(u) \ge n \wedge \ell}\right)^{p \bar{\mathfrak{p}}/(\bar{\mathfrak{p}} - p)} \, du \right)^{(\bar{\mathfrak{p}} - p)/\bar{\mathfrak{p}}} \\ &\qquad \times \left(\mathbb{E} \int_0^T e^{\alpha u} \left(1 + |y_u^n|^{\mathfrak{q}}\right)^{\bar{\mathfrak{p}}} \, du \right)^{p/\bar{\mathfrak{p}}}. \end{aligned}$$

But for $p \le \frac{\bar{\mathfrak{p}} \varrho}{\bar{\mathfrak{p}} + \varrho} < \bar{\mathfrak{p}}$, we have $p \bar{\mathfrak{p}} / (\bar{\mathfrak{p}} - p) \le \varrho$. Hence the right-hand side of the previous inequality goes to zero as ℓ and n tend to $+\infty$. Now we proceed as in the proof of Proposition 3 in [28] and we deduce that the sequence (y^n, z^n, m^m) converges in $\mathbb{D}^p \times \mathbb{H}^p \times \mathbb{M}^p$ to (y, z, m). □

REMARK A.1. The condition $\bar{\mathfrak{p}} > \varrho / (\varrho - 1)$ is equivalent to $1 < \frac{\varrho \bar{\mathfrak{p}}}{\varrho + \bar{\mathfrak{p}}}$.

- If **H4′** holds, then $\varrho = +\infty$ and $p = \bar{\mathfrak{p}}$.
- If everything is bounded (ξ and f_s^0), then y is also bounded and we only need $\varrho > 1$.

Finally, let us recall a technical but crucial lemma, called Lemma A.2 in [37]. The result is the same, but the proof has to be modified since f is not Lipschitz continuous in y anymore.

LEMMA A.5. *For any $\mathbb{F}$-stopping times $0 \le r \le \mathfrak{u} \le \tau \le T$, any decreasing sequence of $\mathbb{F}$-stopping times $(\tau_n)_{n\ge 1}$ converging $\mathbb{P}$-a.s. to τ and any $\mathbb{F}$-progressively measurable and right-continuous process $V \in \mathbb{D}^{\bar{\mathfrak{p}}\mathfrak{q}}$, if $y(\cdot, V_\cdot)$ denotes the first component of the solution to the BSDE with terminal condition $V_\cdot$ and some generator f satisfying* (**H**), *we have*

$$\lim_{n\to+\infty} \mathbb{E}\big[\big|y_{\mathfrak{u}}(\tau, V_\tau) - y_{\mathfrak{u}}(\tau_n, V_{\tau_n})\big|\big] = 0.$$

PROOF. By classical stability result, for any $\kappa < p$, there exists a constant C depending only on T, κ, L_1 and L_2 such that

$$\begin{aligned}\mathbb{E}\big[\big|y_{\mathfrak{u}}(\tau, V_\tau) - y_{\mathfrak{u}}(\tau_n, V_{\tau_n})\big|\big] &= \mathbb{E}\big[\big|y_{\mathfrak{u}}(\tau, V_\tau) - y_{\mathfrak{u}}\big(\tau, y_\tau(\tau_n, V_{\tau_n})\big)\big|\big] \\ &\le C\mathbb{E}\big[\big|V_\tau - y_\tau(\tau_n, V_{\tau_n})\big|^\kappa\big].\end{aligned}$$

Compared to the proof of [37], Lemma A.2, we do not use the complete linearization argument. But we strongly use the growth condition **H4** with the a priori estimate given in Lemma A.1. Indeed we only write that

$$y_\tau(\tau_n, V_{\tau_n}) = \mathbb{E}\bigg[\mathcal{E}\bigg(\int_\tau^{\tau_n} \zeta_s\,dW_s\bigg)\bigg(V_{\tau_n} + \int_\tau^{\tau_n} f\big(s, y_s(\tau_n, V_{\tau_n}), 0\big)\,ds\bigg)\bigg|\mathcal{F}_\tau\bigg].$$

Then

$$\begin{aligned}&\mathbb{E}\big[\big|y_{\mathfrak{u}}(\tau, V_\tau) - y_{\mathfrak{u}}(\tau_n, V_{\tau_n})\big|\big] \\ &\quad\le C\mathbb{E}\bigg[\mathcal{E}\bigg(\int_\tau^{\tau_n} \zeta_s\,dW_s^{\mathbb{P}}\bigg)^\kappa |V_\tau - V_{\tau_n}|^\kappa\bigg] \\ &\qquad + C\mathbb{E}\bigg[\mathcal{E}\bigg(\int_\tau^{\tau_n} \zeta_s\,dW_s\bigg)^\kappa \int_\tau^{\tau_n} \big|f\big(s, y_s(\tau_n, V_{\tau_n}), 0\big)\big|^\kappa\,ds\bigg] \\ &\quad\le C\mathbb{E}\bigg[\mathcal{E}\bigg(\int_\tau^{\tau_n} \zeta_s\,dW_s\bigg)^\kappa |V_\tau - V_{\tau_n}|^\kappa\bigg] \\ &\qquad + C\mathbb{E}\bigg[\mathcal{E}\bigg(\int_\tau^{\tau_n} \zeta_s\,dW_s\bigg)^\kappa \int_\tau^{\tau_n} |f(s,0,0)|^\kappa\,ds\bigg] \\ &\qquad + C\mathbb{E}\bigg[\mathcal{E}\bigg(\int_\tau^{\tau_n} \zeta_s\,dW_s\bigg)^\kappa \int_\tau^{\tau_n} \Psi_s^\kappa\big(1 + \big|y_s(\tau_n, V_{\tau_n})\big|^{\mathfrak{q}}\big)^\kappa\,ds\bigg].\end{aligned}$$

Since ζ is bounded (by L_2), the Doléans–Dade exponential appearing above has finite moments of any order. Now since we have an a priori estimate on $y_\cdot(\tau_n, V_{\tau_n})$ in $\mathbb{D}^{\bar{\mathfrak{p}}\mathfrak{q}}$, uniformly in n, we argue as in [37] to conclude. $\square$

A.3. Reflected BSDE with monotone driver. In this section, we extend the results contained in [9], where the driver f is supposed to be Lipschitz continuous w.r.t. y and z. One of the main contributions of [9] is the existence of a solution for reflected BSDE in a general filtration, without quasi-left continuity condition.

Here, we follow the same scheme but for monotone generators satisfying hypothesis (**H**). Thus we do not give all details but we point out the differences.

Let us remark that our condition **H4** on the growth of the driver f or the integrability assumption **C1** on the terminal value ξ and f^0 are not optimal, compared to the conditions imposed in [24], for example (see also among others [25, 26, 31], for reflected BSDE with monotone generator). This improvement of our result would be quite long and is left for further research.

Estimates on supersolution. We first consider supersolution of the BSDE

$$y_t = \xi + \int_t^T f(u, y_u, z_u)\, du - \int_t^T z_u\, dW_u - \int_t^T dm_u + \int_t^T dk_u. \tag{A.15}$$

The generator f satisfies Condition (**H**). Let us begin with some a priori estimates. Here, we use the notation $z \star W$ to denote the stochastic integral of z w.r.t. W and

$$n = z \star W + m - k, \qquad \ell = m - k.$$

LEMMA A.6 (Equivalent to Lemma 2.1 in [9]). *Assume that the condition* (2.4) *holds. For all* $\alpha > 0$, *there exists a constant* C *depending only on* L_1, L_2, p *and* T *such that*

$$\|k\|^p_{\mathbb{I}^{p,\alpha}} \le C\big(\|e^{\frac{\alpha}{2}\cdot} y\|^p_{\mathbb{D}^p} + \|e^{\frac{\alpha}{2}\cdot}(1+|y|^{\mathfrak{q}})\|^p_{\mathbb{D}^{\bar{\mathfrak{p}}}} \|\Psi\|^p_{\mathbb{L}^{\widehat{p}}} + \|z\|^p_{\mathbb{H}^{p,\alpha}} + \|f^0\|^p_{\mathbb{H}^{p,\alpha}}\big) \tag{A.16}$$

with

$$\widehat{p} = \frac{p\bar{\mathfrak{p}}}{(\bar{\mathfrak{p}} - p)} \le \varrho. \tag{A.17}$$

Moreover, for any $\varepsilon > 0$, *there exists* $\alpha > 0$ *and* $C^{\varepsilon,\alpha}$ *such that, if* $p \ge 2$,

$$\begin{aligned} &\|y\|^p_{\mathbb{H}^{p,\alpha}} + \|n\|^p_{\mathbb{M}^{p,\alpha}} \\ &\quad \le \varepsilon \|f^0\|^p_{\mathbb{H}^{p,\alpha}} \\ &\qquad + C^{\varepsilon,\alpha}\Big[\|\xi\|^p_{\mathbb{L}^p} + \|e^{\alpha\cdot} y\|^p_{\mathbb{D}^p} + \|(e^{\alpha\cdot} y_- \star n)_T\|^{\frac{p}{2}}_{\mathbb{L}^{\frac{p}{2}}} \mathbf{1}_{p>2} \\ &\qquad + \mathbb{E}\Big(\int_0^T e^{\alpha s}\phi_p(y_{s-})\, dk_s\Big)^+ \mathbf{1}_{p=2}\Big]. \end{aligned} \tag{A.18}$$

and if $p \in (1,2)$

$$\begin{aligned} \|n\|^p_{\mathbb{M}^{p,\alpha}} &\le \varepsilon \|f^0\|^p_{\mathbb{H}^{p,\alpha}} \\ &\quad + C^{\varepsilon,\alpha}\Big[\|\xi\|^p_{\mathbb{L}^p} + \|e^{\alpha\cdot} y\|^p_{\mathbb{D}^p} + \mathbb{E}\Big(\int_0^T e^{\alpha s}\phi_p(y_{s-})\, dk_s\Big)^+\Big]. \end{aligned} \tag{A.19}$$

PROOF. The arguments are similar to the proof of [9], Lemma A.1, and of the estimate (2.9) in [9]. The inequality (A.19) is proved using Itô's formula and the arguments of [28], proof of Proposition 3, Step 2 to control the martingale terms. □

From this lemma, we can copy the arguments in the proof of [9], Theorem 2.1, and we deduce the following.

PROPOSITION A.1 ([9], Theorem 2.1). *If (y, z, m, k) is a supersolution of (A.15) in the space $\mathbb{D}^{\bar{\mathfrak{p}}\mathfrak{q}} \times \mathbb{H}^p \times \mathbb{M}^p \times \mathbb{I}^p_+$ with $\bar{\mathfrak{p}} > \varrho/(\varrho - 1)$ then for any $1 < p \leq \frac{\varrho\bar{\mathfrak{p}}}{\varrho+\bar{\mathfrak{p}}}$ and for α large enough, there exists a constant C such that*

$$\begin{aligned}
&\|z\|^p_{\mathbb{H}^{p,\alpha}} + \|m\|^p_{\mathbb{M}^{p,\alpha}} + \|k\|^p_{\mathbb{I}^{p,\alpha}} \\
&\qquad \leq C\big(\|\xi\|^p_{\mathbb{L}^p} + \|y\|^p_{\mathbb{D}^p} + \big\|\big(1 + |y|^{\mathfrak{q}}\big)\big\|^p_{\mathbb{D}^{\bar{\mathfrak{p}}}} \|\Psi\|^p_{\mathbb{L}^\varrho} + \|f^0\|^p_{\mathbb{H}^{p,\alpha}}\big).
\end{aligned}$$

REMARK A.2. As in Lemma A.4 and Remark A.1, if Ψ is bounded (Condition **H4′**), the result holds for $\bar{\mathfrak{p}} = p > 1$. Note that C may depend on α.

The results of [9], Theorem 2.2, hold. More precisely, if we have two solutions $(y^i, z^i, m^i, k^i) \in \mathbb{D}^{\bar{\mathfrak{p}}\mathfrak{q}} \times \mathbb{H}^{\bar{\mathfrak{p}}} \times \mathbb{M}^{\bar{\mathfrak{p}}} \times \mathbb{I}^{\bar{\mathfrak{p}}}_+$ of (A.15) with terminal condition ξ^i and generator f^i, we define

$$\begin{aligned}
\delta y = y^1 - y^2, &\qquad \delta z = z^1 - z^2, \\
\delta m = m^1 - m^2, &\qquad \delta k = k^1 - k^2, \\
\delta f(t, \omega, y, z) = f^1(t&, \omega, y, z) - f^2(t, \omega, y, z).
\end{aligned}$$

f^2 satisfies Conditions (**H**). Then for any $\alpha \geq 0$ and p satisfying (2.4), namely $1 < p \leq \frac{\varrho\bar{\mathfrak{p}}}{\varrho+\bar{\mathfrak{p}}}$, there exists a constant C such that

$$\begin{aligned}
&\|\delta z\|^p_{\mathbb{H}^{p,\alpha}} + \big\|\delta(m - k)\big\|^p_{\mathbb{M}^{p,\alpha}} \\
&\qquad \leq C\big(\|\delta\xi\|^p_{\mathbb{L}^p} + \|\delta y\|^p_{\mathbb{D}^p} + \|\delta y\|^{\frac{p}{2}\wedge(p-1)}_{\mathbb{D}^p} + \big\|\delta f\big(y^1, z^1\big)\big\|^p_{\mathbb{H}^{p,\alpha}}\big).
\end{aligned}$$

Here, the constant C depends on L_1, L_2, p, α and also on $\|\Psi\|_{\mathbb{L}^\varrho}$, $\|y^i\|_{\mathbb{D}^p}$, $\|y^i\|_{\mathbb{D}^{\bar{\mathfrak{p}}\mathfrak{q}}}$, $\|\xi^i\|_{\mathbb{L}^p}$, $\|f^i(0, 0)\|_{\mathbb{H}^{p,\alpha}}$ for $i = 1, 2$. To prove this inequality, we argue as in the proof of [9], Theorem 2.2.

Application to reflected monotone BSDE. Now we study the reflected BSDE

$$\widetilde{y}_t = \xi + \int_t^T f(u, \widetilde{y}_u, \widetilde{z}_u)\, du - \int_t^T \widetilde{z}_u\, dW_u - \int_t^T d\widetilde{m}_u + \int_t^T d\widetilde{k}_u \tag{A.20}$$

with $\widetilde{y}_t \geq S_t$ and $\int_0^T (\widetilde{y}_{t-} - S_{t-})d\widetilde{k}_t = 0$, $\mathbb{P}$-a.s. (*Skorokhod condition*). S is a càdlàg process such that $S^+ = S \vee 0$ belongs to $\mathbb{D}^p$.

Using again the linearization procedure (A.6) and the new probability measure $\mathbb{Q}$ defined by (A.7), if $(\widetilde{y}, \widetilde{z}, \widetilde{m}, \widetilde{k})$ is a solution, then

$$\Lambda_t \widetilde{y}_t = \Lambda_T \xi + \int_t^T \Lambda_s f_s^0 \, ds + \int_t^T \zeta_s \Lambda_s \widetilde{z}_s \, ds - \int_t^T \Lambda_s \widetilde{z}_s \, dW_s - \int_t^T \Lambda_s \, d\widetilde{m}_s + \int_t^T \Lambda_s \, d\widetilde{k}_s$$

with $\Lambda_t \widetilde{y}_t \geq \Lambda_t S_t$ and $\int_0^T (\Lambda_{u-} \widetilde{y}_{u-} - \Lambda_{u-} S_{u-}) \, d\widetilde{k}_u = 0$ a.s. Again the key point is that for $0 \leq s \leq t \leq T$, $0 < \Lambda_s / \Lambda_t \leq \exp(L_1(t-s))$. As in [9], Proposition 3.1, the following representation holds:

$$\Lambda_t \widetilde{y}_t = \operatorname*{esssup}_{\tau \in \mathcal{T}_{t,T}} \mathbb{E}^{\mathbb{Q}} \left[\int_t^\tau \Lambda_s f_s^0 \, ds + \Lambda_T \xi \mathbf{1}_{\tau = T} + \Lambda_\tau S_\tau \mathbf{1}_{\tau < T} \middle| \mathcal{F}_t \right]. \tag{A.21}$$

Now we denote by (y, z, m) the unique solution of the BSDE (A.2) (or (A.15) with $k = 0$).

$$y_t = \xi + \int_t^T f(u, y_u, z_u) \, du - \int_t^T z_u \, dW_u - \int_t^T dm_u.$$

From the comparison principle, a.s. for any $0 \leq t \leq T$, $\widetilde{y}_t \geq y_t$. Let us begin with two technical lemmas corresponding to [9], Propositions 3.2 and 3.3.

LEMMA A.7. *For $p > 1$, if $(\widetilde{y}, \widetilde{z}, \widetilde{m}, \widetilde{k})$ is a solution of* (A.20), *then*

$$\begin{aligned} \| e^{\alpha \cdot} \widetilde{y} \|_{\mathbb{D}^p}^p \leq C_{\alpha, L_1, L_2, T} \left[\| \xi \|_{\mathbb{L}^p}^p + \| S^+ \|_{\mathbb{D}^p}^p + \mathbb{E} \left(\int_0^T |f_s^0| \, ds \right)^p \right] \\ + \widehat{C}_{\alpha, L_1, T} \| e^{\alpha \cdot} y \|_{\mathbb{D}^p}^p. \end{aligned} \tag{A.22}$$

Moreover, if we have two solutions $(\widetilde{y}^i, \widetilde{z}^i, \widetilde{m}^i, \widetilde{k}^i)$ of the reflected BSDE (A.20) *with terminal condition ξ^i, generator f^i and barrier S^i, then*

$$\begin{aligned} &\| e^{\alpha \cdot} \delta \widetilde{y} \|_{\mathbb{D}^p}^p \\ &\quad \leq \bar{C}_{\alpha, L_1, L_2, T} \left[\| \delta \xi \|_{\mathbb{L}^p}^p + \| \delta S \|_{\mathbb{D}^p}^p + \mathbb{E} \left(\int_0^T |\delta f(s, \widetilde{y}_s^1, \widetilde{z}_s^1)| \, ds \right)^p \right]. \end{aligned} \tag{A.23}$$

PROOF. Fix $p > 1$. Using the representation (A.21), for any $\alpha > 0$, we obtain

$$\begin{aligned} \sup_{t \in [0,T]} (e^{\alpha t} |\widetilde{y}_t|) \leq e^{L_1 T} \sup_{t \in [0,T]} e^{\alpha t} \mathbb{E}^{\mathbb{Q}} \left(\int_0^T |f_s^0| \, ds + \sup_{u \in [t,T]} S_u^+ + |\xi| \middle| \mathcal{F}_t \right) \\ + \sup_{t \in [0,T]} (e^{\alpha t} |y_t|) \end{aligned}$$

$$\leq C \sup_{t\in[0,T]} \left[\mathbb{E}\left(\left(\int_0^T |f_s^0|\,ds\right)^p + \sup_{u\in[t,T]} (S_u^+)^p + |\xi|^p \Big| \mathcal{F}_t\right)\right]^{1/p} + \sup_{t\in[0,T]} (e^{\alpha t}|y_t|)$$

with $C = C_{p,L_1,L_2,T,\alpha}$. Using Doob's inequality, we deduce (A.22).

The second point in [9], Proposition 3.2, is the stability of solutions for reflected BSDE. We assume that we have two solutions $(\widetilde{y}^i, \widetilde{z}^i, \widetilde{m}^i, \widetilde{k}^i)$ of the reflected BSDE (A.20) with terminal condition ξ^i, generator f^i and barrier S^i. The functions f^i satisfy Assumptions (**H**) and again we can assume that the monotonicity constant L_1 is nonpositive. Then (A.23) can be obtained with the same proof. In fact, we only need that f^2 satisfies **H3** with $L_1 \leq 0$. No particular condition on L_1 of the generator f^1 is used here. Hence [9], Proposition 3.2, again holds under our setting. □

LEMMA A.8. *If we have two solutions $(\widetilde{y}^i, \widetilde{z}^i, \widetilde{m}^i, \widetilde{k}^i)$ of the reflected BSDE (A.20) with terminal condition ξ^i, generator f^i and barrier S^i, then*

$$\begin{aligned} &\|\delta\widetilde{z}\|^p_{\mathbb{H}^{p,\alpha}} + \|\delta(\widetilde{m}-\widetilde{k})\|^p_{\mathbb{M}^{p,\alpha}} \\ &\quad\leq \varepsilon \|\delta f(\widetilde{y}^1,\widetilde{z}^1)\|^p_{\mathbb{H}^{p,\alpha}} + C^\alpha(\|\delta\xi\|^p_{\mathbb{L}^p} + \|\delta S\|^p_{\mathbb{D}^p}), \end{aligned} \tag{A.24}$$

where the constant C^α depends on L_2, p, α, ε and $\|\Psi\|^p_{\mathbb{L}^{\widehat{p}}}$, $\|\widetilde{y}^i\|^p_{\mathbb{D}^p}$, $\|\widetilde{y}^i\|^{p\mathfrak{q}}_{\mathbb{D}^{p\mathfrak{q}}}$, $\|\xi^i\|^p_{\mathbb{L}^p}$, and $\|f^i(0,0)\|^p_{\mathbb{H}^{p,\alpha}}$ for $i=1,2$ and $\widehat{p}$ is defined by (A.17).

PROOF. The arguments are the same as [9], Proposition 3.3. Indeed we can use estimates (A.18) and (A.19) and the Skorokhod condition

$$\mathbb{E}\left[\int_0^T e^{\alpha s}\phi_p(\delta\widetilde{y}_{s-})\,d(\delta\widetilde{k}_s)\right] \leq \mathbb{E}\left[\int_0^T e^{\alpha s}\phi_p(\delta S_{s-})\,d(\delta\widetilde{k}_s)\right], \tag{A.25}$$

the function $\phi_p(x) = |x|^{p-1}\operatorname{sgn}(x)$ being nondecreasing. We conclude using Hölder's inequality. □

The right-hand side of (A.24) is finite if we have the same condition as in Lemma A.4.

PROPOSITION A.2 (Theorem 3.1 of [9]). *Assume that $\xi \in \mathbb{L}^{\bar{\mathfrak{p}}\mathfrak{q}}$, $S^+ \in \mathbb{D}^{\bar{\mathfrak{p}}\mathfrak{q}}$ and $f^0 \in \mathbb{H}^{\bar{\mathfrak{p}}\mathfrak{q}}$ for some $\bar{\mathfrak{p}} > 1$ with $\bar{\mathfrak{p}} > \varrho/(\varrho-1)$. There exists a unique solution (Y,Z,M,K) to the reflected BSDE (A.20) in $\mathbb{D}^{\bar{\mathfrak{p}}\mathfrak{q}} \times \mathbb{H}^p \times \mathbb{M}^p \times \mathbb{I}^p$ for any p such that*

$$1 < p \leq \frac{\varrho\bar{\mathfrak{p}}}{\varrho+\bar{\mathfrak{p}}} < \bar{\mathfrak{p}}.$$

PROOF. Uniqueness is a direct consequence of estimates (A.23) and (A.24). Without loss of generality, we can assume that $L_1 = 0$. For the existence of a solution, we proceed in several steps.

- *Step* 1. Assume that $(\mathbf{H'})$ holds and that ξ, f^0 and S^+ are bounded: there exists a constant L_∞ such that a.s.

$$|\xi| + \sup_{t\in[0,T]} |f_t^0| + \sup_{t\in[0,T]} S_t^+ \leq L_\infty. \tag{A.26}$$

Then the estimate of Proposition A.1 holds for any $p > 1$. We denote $\mathbb{H}^\infty = \bigcap_{\mathfrak{e}\geq\bar{\mathfrak{p}}\mathfrak{q}} \mathbb{H}^{\mathfrak{e}}$. We proceed in two substeps.

⋆ *Substep* i. Let us take $V \in \mathbb{H}^\infty$ and we denote by $g(t, y)$ the function $f(t, y, V_t)$.

The generator g satisfies the same condition $(\mathbf{H'})$ as f, with $g^0 \in \mathbb{H}^\infty$. From [9], Theorem 3.1, there exists a unique solution $(\widetilde{y}^n, \widetilde{z}^n, \widetilde{m}^n, \widetilde{k}^n) \in \mathbb{S}^p \times \mathbb{H}^p \times \mathbb{M}^p \times \mathbb{I}_+^p$ to the reflected BSDE (A.20) where g is replaced by g_n:

$$\widetilde{y}_t^n = \xi + \int_t^T g_n(u, \widetilde{y}_u^n)\,du - \int_t^T \widetilde{z}_u^n\,dW_u - \int_t^T d\widetilde{m}_u^n + \int_t^T d\widetilde{k}_u^n$$

with $\widetilde{y}_t^n \geq S_t$ and $\int_0^T (\widetilde{y}_{t-}^n - S_{t-})d\widetilde{k}_t^n = 0$, $\mathbb{P}$-a.s. Here, g_n is defined as in [10] (see equation (A.9)), that is by the convolution product

$$g_n(t,\cdot) = \varsigma_n * (\Theta_{n+1} g(t,\cdot)), \qquad t \in [0,T], \tag{A.27}$$

where ς and Θ are the same as in the proof (Step 1) of Lemma A.4. This function g_n is globally Lipschitz w.r.t. y uniformly in t and ω with $|g_n(t,0)| \leq |f_t^0| + 2L_{\mathfrak{q}}$. Moreover, for any (t, ω, y),

$$y g_n(t, y) \leq (|f_t^0| + 2L_{\mathfrak{q}})|y|. \tag{A.28}$$

From (A.28), (A.21) and (A.22),

$$\sup_n \sup_{t\in[0,T]} |\widetilde{y}_t^n| \leq C_{L_{\mathfrak{q}},L_\infty,T} < +\infty.$$

Hence we consider only the case $n \geq C_{L_{\mathfrak{q}},L_\infty,T}$. Let us take $n' \geq n \geq C_{L_{\mathfrak{q}},L_\infty,T}$ and

$$\delta\widetilde{y} = \widetilde{y}^{n'} - \widetilde{y}^n, \qquad \delta\widetilde{z} = \widetilde{z}^{n'} - \widetilde{z}^n, \qquad \delta\widetilde{m} = \widetilde{m}^{n'} - \widetilde{m}^n, \qquad \delta\widetilde{k} = \widetilde{k}^{n'} - \widetilde{k}^n.$$

We apply Itô's formula to $(\delta\widetilde{y})^2$ and since $|\widetilde{y}_t^n| \leq n \leq n'$, we use the local monotonicity of g_l and we obtain

$$\begin{aligned}
&|\delta\widetilde{y}_t|^2 + \int_t^T |\delta\widetilde{z}_u|^2\,du + \int_t^T d[\delta(\widetilde{m} - \widetilde{k})]_u \\
&\quad \leq 2\int_t^T [g_\ell(u, \widetilde{y}_u^n) - g_n(u, \widetilde{y}_u^n)](\delta\widetilde{y}_{u-})\,du \\
&\qquad - 2\int_t^T (\delta\widetilde{y}_{u-})\delta\widetilde{z}_u\,dW_u - 2\int_t^T (\delta\widetilde{y}_{u-})\,d(\delta(\widetilde{m} - \widetilde{k}))_u.
\end{aligned}$$

The Skorokhod condition implies that

$$\mathbb{E}\left[\int_t^T e^{\alpha u}(\delta \widetilde{y}_{u-})d(\delta \widetilde{k})_u\right] \le \mathbb{E}\left[\int_t^T e^{\alpha u}(\delta S_{u-})d(\delta \widetilde{k})_u\right] = 0.$$

Since the set $\{\delta \widetilde{y}_u \neq \delta \widetilde{y}_{u-}\}$ is countable, classical arguments (using BDG inequality) imply that there exists a constant $C = C_{L_f, L_\infty, T}$ such that

$$\mathbb{E}\left[\sup_{t\in[0,T]} |\delta \widetilde{y}_t|^2 + \int_0^T |\delta \widetilde{z}_u|^2\,du + \int_0^T d\big[\delta(\widetilde{m} - \widetilde{k})\big]_u\right]$$

$$\le C\mathbb{E}\left[\sup_{|y|\le C} \int_0^T \big|g_{n'}(u, y) - g_n(u, y)\big|\,du\right].$$

We argue as in Step 1 of the proof of Lemma A.4 and we obtain that $(\widetilde{y}^n, \widetilde{z}^n, \widetilde{\mu}^n = \widetilde{m}^n - \widetilde{k}^n) \in \mathbb{D}^2 \times \mathbb{H}^2 \times \mathbb{M}^2$ is a Cauchy sequence. But we also have

$$\widetilde{\mu}_t^{n'} - \widetilde{\mu}_t^n = \int_0^t \big(g_{n'}(u, \widetilde{y}_u^{n'}) - g_n(u, \widetilde{y}_u^n)\big)\,du - \int_0^t \big(\widetilde{z}_u^{n'} - \widetilde{z}_u^n\big)\,dW_u$$

$$+ \big(\widetilde{y}_t^{n'} - \widetilde{y}_t^n\big) - \big(\widetilde{y}_0^{n'} - \widetilde{y}_0^n\big).$$

Hence $\widetilde{\mu}^n$ converges also in $\mathbb{D}^2$. Arguing as step (iii) in the proof of [9], Theorem 3.1, we deduce that the limit $(\widetilde{y}, \widetilde{z}, \widetilde{m}, \widetilde{k})$ satisfies the reflected BSDE

$$\widetilde{y}_t = \xi + \int_t^T f(u, \widetilde{y}_u, V_u)\,du - \int_t^T \widetilde{z}_u\,dW_u - \int_t^T d\widetilde{m}_u + \int_t^T d\widetilde{k}_u \tag{A.29}$$

with $\widetilde{y}_t \ge S_t$ and $\int_0^T (\widetilde{y}_{t-} - S_{t-})d\widetilde{k}_t = 0$, $\mathbb{P}$-a.s. From Proposition A.1 with $\varrho = +\infty$ and $\bar{\mathfrak{p}} = +\infty$, we obtain for any $p \ge 1$,

$$\|\widetilde{z}\|_{\mathbb{H}^p}^p + \|\widetilde{m}\|_{\mathbb{M}^p}^p + \|\widetilde{k}\|_{\mathbb{I}^p}^p < +\infty.$$

Hence for $V \in \mathbb{H}^\infty$, we have a unique solution $(\widetilde{y}, \widetilde{z}, \widetilde{m}, \widetilde{k}) \in \mathbb{S}^\infty \times \mathbb{H}^\infty \times \mathbb{M}^\infty \times \mathbb{I}^\infty$ to (A.29).

⋆ *Substep* ii. General case under boundedness conditions.

We use a fixed-point argument. Let $(\widetilde{y}^0, \widetilde{z}^0, \widetilde{m}^0, \widetilde{k}^0) = (0, 0, 0, 0)$ and let $(\widetilde{y}^{n+1}, \widetilde{z}^{n+1}, \widetilde{m}^{n+1}, \widetilde{k}^{n+1}) \in \mathbb{D}^\infty \times \mathbb{H}^\infty \times \mathbb{M}^\infty \times \mathbb{I}^\infty$ be the unique solution of the reflected BSDE

$$\widetilde{y}_t^{n+1} = \xi + \int_t^T f\big(u, \widetilde{y}_u^{n+1}, \widetilde{z}_u^n\big)\,du - \int_t^T \widetilde{z}_u^{n+1}\,dW_u - \int_t^T d\widetilde{m}_u^{n+1} + \int_t^T d\widetilde{k}_u^{n+1}$$

with $\widetilde{y}_t^{n+1} \ge S_t$ and $\int_0^T (\widetilde{y}_{t-}^{n+1} - S_{t-})d\widetilde{k}_t^{n+1} = 0$, $\mathbb{P}$-a.s. From our first substep, the sequence is well defined. For any i, we denote

$$\delta \widetilde{y}^{n,i} = \widetilde{y}^{n+i} - \widetilde{y}^n, \qquad \delta \widetilde{z}^{n,i} = \widetilde{z}^{n+i} - \widetilde{z}^n,$$

$$\delta \widetilde{m}^{n,i} = \widetilde{m}^{n+i} - \widetilde{m}^n, \qquad \delta \widetilde{k}^{n,i} = \widetilde{k}^{n+i} - \widetilde{k}^n$$

and $\widetilde{\ell}^n := \widetilde{m}^n - \widetilde{k}^n$. We apply Itô's formula to $e^{\alpha t}|\delta\widetilde{y}_t^{n,i}|^2$ with $\alpha = 1 + 2L_1 + 2L_2^2$ and we use Young's inequality to obtain

$$\mathbb{E}\int_0^T e^{\alpha u}|\delta\widetilde{y}_u^{n,i}|^2\,du + \mathbb{E}\int_0^T e^{\alpha u}|\delta\widetilde{z}_u^{n,i}|^2\,du + \mathbb{E}\int_t^T e^{\alpha u}\,d[\delta\widetilde{\ell}^{n,i}]_u$$
$$\leq \frac{1}{2}\mathbb{E}\int_0^T e^{\alpha u}|\delta\widetilde{z}_u^{n-1,i}|^2\,du.$$

Thus $(\widetilde{y}^n, \widetilde{z}^n, \widetilde{\ell}^n)$ is a Cauchy sequence in $\mathbb{H}^2\times\mathbb{H}^2\times\mathbb{M}^2$, and using BDG inequality we will have convergence in $\mathbb{D}^2\times\mathbb{H}^2\times\mathbb{M}^2$. Then the conclusion follows by the same arguments as in [9], Theorem 3.1. Then since (A.26) holds, from Lemmas A.7 and A.8, we deduce that the limit is also in $\mathbb{D}^\infty\times\mathbb{H}^\infty\times\mathbb{M}^\infty\times\mathbb{I}^\infty$.

- *Step* 2. Assume that for some $\bar{\mathfrak{p}} > \varrho/(\varrho-1)$, ξ, f^0 and S^+ are in $\mathbb{L}^{\bar{\mathfrak{p}}\mathfrak{q}}\times\mathbb{H}^{\bar{\mathfrak{p}}\mathfrak{q}}\times\mathbb{D}^{\bar{\mathfrak{p}}\mathfrak{q}}$. We fix $1 < p \leq \varrho\bar{\mathfrak{p}}/(\varrho+\bar{\mathfrak{p}})$.

For any $n\in\mathbb{N}^*$,

$$\xi_n = \frac{n\xi}{n\vee|\xi|},\qquad S_t^n = \frac{nS_t}{n\vee S_t},\qquad f_n(t,y,z) = \big(f(t,y,z) - f_t^0\big) + \frac{nf_t^0}{n\vee|f_t^0|}.$$

Then $|f_n(t,0,0)| = |\frac{nf_t^0}{n\vee|f_t^0|}| \leq n$. We apply the result of Step 1: there exists a unique solution $(\widetilde{y}^n, \widetilde{z}^n, \widetilde{m}^n, \widetilde{k}^n)\in\mathbb{D}^{\bar{\mathfrak{p}}\mathfrak{q}}\times\mathbb{H}^p\times\mathbb{M}^p\times\mathbb{I}^p$ to the reflected BSDE

$$\widetilde{y}_t^n = \xi_n + \int_t^T f_n(u,\widetilde{y}_u^n,\widetilde{z}_u^n)\,du - \int_t^T \widetilde{z}_u^n\,dW_u - \int_t^T d\widetilde{m}_u^n + \int_t^T d\widetilde{k}_u^n$$

with $\widetilde{y}_t^n \geq S_t^n$ and $\int_0^T(\widetilde{y}_{t-}^n - S_{t-}^n)\,d\widetilde{k}_t^n = 0$, $\mathbb{P}$-a.s. From Lemma A.7 and estimate (A.22),

$$\sup_{n\in\mathbb{N}}\|e^{\alpha\cdot}\widetilde{y}^n\|_{\mathbb{D}^{\bar{\mathfrak{p}}\mathfrak{q}}} < +\infty.$$

Thus from Proposition A.1, the $\mathbb{H}^p\times\mathbb{M}^p\times\mathbb{I}^p$-norm of the sequence $(\widetilde{z}^n,\widetilde{m}^n,\widetilde{k}^n)$ is bounded uniformly w.r.t. n. We denote $\widetilde{\ell} := \widetilde{m} - \widetilde{k}$, we take $n \leq n'$, we define again

$$\delta\widetilde{y} = \widetilde{y}^{n'} - \widetilde{y}^n,\qquad \delta\widetilde{z} = \widetilde{z}^{n'} - \widetilde{z}^n,\qquad \delta\widetilde{m} = \widetilde{m}^{n'} - \widetilde{m}^n,$$
$$\delta\widetilde{k} = \widetilde{k}^{n'} - \widetilde{k}^n,\qquad \delta\widetilde{\ell} = \widetilde{\ell}^{n'} - \widetilde{\ell}^n.$$

We apply Itô's formula for $t\in[0,T]$ and for $\alpha \geq pL_1 + \frac{p}{(p-1)\wedge 1}L_2^2$ and by standard arguments and from (A.25) we have

$$e^{\alpha t}|\delta\widetilde{y}_t|^p + \frac{c(p)}{2}\int_t^T |\delta\widetilde{y}_{s-}|^{p-2}\mathbf{1}_{\delta\widetilde{y}_{s-}\neq 0}(\delta\widetilde{z}_s)^2\,ds$$
$$+ c(p)\int_t^T |\delta\widetilde{y}_{s-}|^{p-2}\mathbf{1}_{\delta\widetilde{y}_{s-}\neq 0}\,d[\delta\widetilde{\ell}]_s^c$$

$$+ c(p) \sum_{t<s\le T} e^{\alpha s} |\Delta(\delta\widetilde{\ell})_s|^2 [|\delta\widetilde{y}_{s-}|^2 \vee |\delta\widetilde{y}_{s-} + \Delta(\delta\widetilde{\ell})_s|^2]^{\frac{p}{2}-1}$$

$$\times \mathbf{1}_{|\delta\widetilde{y}_{s-}|\vee|\delta\widetilde{y}_{s-}+\Delta(\delta\widetilde{\ell})_s|\neq 0}$$

$$\le e^{\alpha T}|\delta\xi|^p + p\int_t^T e^{\alpha u}\phi_p(\delta\widetilde{y}_{u-})[f_{n'}(u, \widetilde{y}_u^n, \widetilde{z}_u^n) - f_n(u, \widetilde{y}_u^n, \widetilde{z}_u^n)]\,du$$

$$- p\int_t^T e^{\alpha u}\phi_p(\delta\widetilde{y}_{u-})(\delta\widetilde{z}_u\,dW_u + d\delta\widetilde{m}_u) + p\int_t^T e^{\alpha u}\phi_p(\delta S_{u-})\,d(\delta\widetilde{k})_u.$$

Since the $\mathbb{I}^{p,\alpha}$-norm of $(\widetilde{k}^n)$ is bounded uniformly w.r.t. n, we deduce that there exists a constant C such that

$$\|e^{\alpha\cdot}\delta\widetilde{y}^n\|_{\mathbb{D}^p} + \|e^{\alpha\cdot}\delta\widetilde{z}^n\|_{\mathbb{H}^p} + \|e^{\alpha\cdot}\delta(\widetilde{m}-\widetilde{k})^n\|_{\mathbb{M}^p}$$
$$\le C(\|\delta\xi\|_{\mathbb{L}^p} + \|e^{\alpha\cdot}\delta S\|_{\mathbb{D}^p} + \|e^{\alpha\cdot}\delta f^0\|_{\mathbb{H}^p}).$$

Note that the constant C depends in particular on $\|\Psi\|_{\mathbb{L}^{\widehat{p}}}$ which will be finite since $\widehat{p} \le \varrho$. Thus we have a Cauchy sequence in $\mathbb{D}^p \times \mathbb{H}^p \times \mathbb{M}^p$, and in $\mathbb{D}^{\bar{\mathfrak{p}}\mathfrak{q}} \times \mathbb{H}^p \times \mathbb{M}^p$, which converges to $(\widetilde{y}, \widetilde{z}, \widetilde{\nu})$. We argue as in the Step 1, (iii) of the proof of [9], Theorem 3.1, to obtain the desired result. □

Remark A.1 also holds for this last result. In particular, if $(\mathbf{H}')$ holds then $\bar{\mathfrak{p}} = p$.

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REFERENCES

[1] ALFONSI, A. and SCHIED, A. (2010). Optimal trade execution and absence of price manipulations in limit order book models. *SIAM J. Financial Math.* **1** 490–522. MR2669402

[2] ALFONSI, A., SCHIED, A. and SLYNKO, A. (2012). Order book resilience, price manipulation, and the positive portfolio problem. *SIAM J. Financial Math.* **3** 511–533. MR2968044

[3] ALMGREN, R. (2012). Optimal trading with stochastic liquidity and volatility. *SIAM J. Financial Math.* **3** 163–181. MR2968031

[4] ALMGREN, R. and CHRISS, N. (2001). Optimal execution of portfolio transactions. *J. Risk* **3** 5–40.

[5] ANKIRCHNER, S., JEANBLANC, M. and KRUSE, T. (2014). BSDEs with singular terminal condition and a control problem with constraints. *SIAM J. Control Optim.* **52** 893–913. MR3180837

[6] BANK, P. and VOSS, M. (2018). Linear quadratic stochastic control problems with stochastic terminal constraint. *SIAM J. Control Optim.* **56** 672–699. MR3769694

[7] BERTSEKAS, D. P. and SHREVE, S. E. (1978). *Stochastic Optimal Control*: *The Discrete Time Case*. *Mathematics in Science and Engineering* **139**. Academic Press, New York. MR0511544

[8] BOUCHARD, B., POSSAMAÏ, D. and TAN, X. (2016). A general Doob–Meyer–Mertens decomposition for g-supermartingale systems. *Electron. J. Probab.* **21** Paper No. 36, 21. MR3508683

[9] BOUCHARD, B., POSSAMAÏ, D., TAN, X. and ZHOU, C. (2018). A unified approach to *a priori* estimates for supersolutions of BSDEs in general filtrations. *Ann. Inst. Henri Poincaré Probab. Stat.* **54** 154–172. MR3765884

[10] BRIAND, P. and CARMONA, R. (2000). BSDEs with polynomial growth generators. *J. Appl. Math. Stoch. Anal.* **13** 207–238. MR1782682

[11] BRIAND, P., DELYON, B., HU, Y., PARDOUX, E. and STOICA, L. (2003). L^p solutions of backward stochastic differential equations. *Stochastic Process. Appl.* **108** 109–129. MR2008603

[12] DELLACHERIE, C. and MEYER, P.-A. (1978). *Probabilities and Potential*. *North-Holland Mathematics Studies* **29**. North-Holland, Amsterdam. MR0521810

[13] DOOB, J. L. (2001). *Classical Potential Theory and Its Probabilistic Counterpart*. *Classics in Mathematics*. Springer, Berlin. Reprint of the 1984 edition. MR1814344

[14] EL KAROUI, N. and HUANG, S.-J. (1997). A general result of existence and uniqueness of backward stochastic differential equations. In *Backward Stochastic Differential Equations* (*Paris*, 1995–1996). *Pitman Res. Notes Math. Ser.* **364** 27–36. Longman, Harlow. MR1752673

[15] FORSYTH, P. A., KENNEDY, J. S., TSE, S. T. and WINDCLIFF, H. (2012). Optimal trade execution: A mean quadratic variation approach. *J. Econom. Dynam. Control* **36** 1971–1991. MR2982963

[16] GATHERAL, J. and SCHIED, A. (2011). Optimal trade execution under geometric Brownian motion in the Almgren and Chriss framework. *Int. J. Theor. Appl. Finance* **14** 353–368. MR2804102

[17] GRAEWE, P., HORST, U. and QIU, J. (2015). A non-Markovian liquidation problem and backward SPDEs with singular terminal conditions. *SIAM J. Control Optim.* **53** 690–711. MR3319844

[18] GRAEWE, P., HORST, U. and SÉRÉ, E. (2018). Smooth solutions to portfolio liquidation problems under price-sensitive market impact. *Stochastic Process. Appl.* **128** 979–1006. MR3758345

[19] HORST, U. and NAUJOKAT, F. (2014). When to cross the spread? Trading in two-sided limit order books. *SIAM J. Financial Math.* **5** 278–315. MR3206984

[20] JACOD, J. and SHIRYAEV, A. N. (2003). *Limit Theorems for Stochastic Processes*, 2nd ed. *Grundlehren der Mathematischen Wissenschaften* [*Fundamental Principles of Mathematical Sciences*] **288**. Springer, Berlin. MR1943877

[21] KALLENBERG, O. (2002). *Foundations of Modern Probability*, 2nd ed. *Probability and Its Applications* (*New York*). Springer, New York. MR1876169

[22] KARANDIKAR, R. L. (1995). On pathwise stochastic integration. *Stochastic Process. Appl.* **57** 11–18. MR1327950

[23] KAZI-TANI, N., POSSAMAÏ, D. and ZHOU, C. (2015). Second-order BSDEs with jumps: Formulation and uniqueness. *Ann. Appl. Probab.* **25** 2867–2908. MR3375890

[24] KLIMSIAK, T. (2012). Reflected BSDEs with monotone generator. *Electron. J. Probab.* **17** no. 107, 25. MR3015691

[25] KLIMSIAK, T. (2013). BSDEs with monotone generator and two irregular reflecting barriers. *Bull. Sci. Math.* **137** 268–321. MR3043077

[26] KLIMSIAK, T. (2015). Reflected BSDEs on filtered probability spaces. *Stochastic Process. Appl.* **125** 4204–4241. MR3385601

[27] KRATZ, P. and SCHÖNEBORN, T. (2015). Portfolio liquidation in dark pools in continuous time. *Math. Finance* **25** 496–544. MR3358592

[28] KRUSE, T. and POPIER, A. (2016). BSDEs with monotone generator driven by Brownian and Poisson noises in a general filtration. *Stochastics* **88** 491–539. MR3473849

[29] KRUSE, T. and POPIER, A. (2016). Minimal supersolutions for BSDEs with singular terminal condition and application to optimal position targeting. *Stochastic Process. Appl.* **126** 2554–2592. MR3522293

[30] KRUSE, T. and POPIER, A. (2017). L^p-solution for BSDEs with jumps in the case $p < 2$: Corrections to the paper 'BSDEs with monotone generator driven by Brownian and Poisson noises in a general filtration' [MR3473849]. *Stochastics* **89** 1201–1227. MR3742328

[31] LEPELTIER, J.-P., MATOUSSI, A. and XU, M. (2005). Reflected backward stochastic differential equations under monotonicity and general increasing growth conditions. *Adv. in Appl. Probab.* **37** 134–159. MR2135157

[32] MATOUSSI, A., POSSAMAÏ, D. and ZHOU, C. (2015). Robust utility maximization in nondominated models with 2BSDE: The uncertain volatility model. *Math. Finance* **25** 258–287. MR3321250

[33] PARDOUX, É. (1999). BSDEs, weak convergence and homogenization of semilinear PDEs. In *Nonlinear Analysis, Differential Equations and Control* (*Montreal, QC*, 1998). *NATO Sci. Ser. C Math. Phys. Sci.* **528** 503–549. Kluwer Academic, Dordrecht. MR1695013

[34] PARDOUX, E. and RĂŞCANU, A. (2014). *Stochastic Differential Equations, Backward SDEs, Partial Differential Equations. Stochastic Modelling and Applied Probability* **69**. Springer, Cham. MR3308895

[35] POPIER, A. (2016). Limit behaviour of BSDE with jumps and with singular terminal condition. *ESAIM Probab. Stat.* **20** 480–509. MR3581831

[36] POSSAMAÏ, D. (2013). Second order backward stochastic differential equations under a monotonicity condition. *Stochastic Process. Appl.* **123** 1521–1545. MR3027889

[37] POSSAMAÏ, D., TAN, X. and ZHOU, C. (2018). Stochastic control for a class of nonlinear kernels and applications. *Ann. Probab.* **46** 551–603. MR3758737

[38] SCHIED, A. (2013). A control problem with fuel constraint and Dawson–Watanabe superprocesses. *Ann. Appl. Probab.* **23** 2472–2499. MR3127942

[39] SONER, H. M., TOUZI, N. and ZHANG, J. (2011). Quasi-sure stochastic analysis through aggregation. *Electron. J. Probab.* **16** 1844–1879. MR2842089

[40] SONER, H. M., TOUZI, N. and ZHANG, J. (2012). Wellposedness of second order backward SDEs. *Probab. Theory Related Fields* **153** 149–190. MR2925572

[41] SONER, H. M., TOUZI, N. and ZHANG, J. (2013). Dual formulation of second order target problems. *Ann. Appl. Probab.* **23** 308–347. MR3059237

[42] STROOCK, D. W. and VARADHAN, S. R. S. (1979). *Multidimensional Diffusion Processes. Grundlehren der Mathematischen Wissenschaften* [*Fundamental Principles of Mathematical Sciences*] **233**. Springer, Berlin. MR0532498

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