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# WRIGHT–FISHER DIFFUSIONS IN STOCHASTIC SPATIAL EVOLUTIONARY GAMES WITH DEATH–BIRTH UPDATING[1]

BY YU-TING CHEN

*University of Tennessee*

We investigate stochastic spatial evolutionary games with death–birth updating in large finite populations. Within growing spatial structures subject to appropriate conditions, the density processes of a fixed type are proven to converge to the one-dimensional Wright–Fisher diffusions. Convergence in the Wasserstein distance of the laws of the occupation measures also holds. The proofs study the convergences under certain voter models by an equivalence between their laws and the laws of the evolutionary games. In particular, the additional growing dimensions in minimal systems that close the dynamics of the game density processes are cut off in the limit.

As another application of this equivalence of laws, we consider a first-derivative test among the major methods for these evolutionary games in a large population of size $N$. Requiring only the assumption that the stationary probabilities of the corresponding voting kernel are comparable to uniform probabilities, we prove that the test is applicable at least up to weak selection strengths in the usual biological sense [i.e., selection strengths of the order $\mathcal{O}(1/N)$].

## CONTENTS



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**1. Introduction and main results.** The goal of this paper is to investigate diffusion approximations of the interacting particle systems which are known in the biological literature as evolutionary games with death–birth updating. In the Supplementary Information [34], SI, of their seminal work on evolutionary games, Ohtsuki et al. analyze the density processes of a fixed type in the evolutionary games with death–birth updating on random regular graphs. They find that these processes approximate the (one-dimensional) Wright–Fisher diffusions with drift in the limit of large population size, and the key argument there follows the physics method of pair approximation. This method goes back to Matsuda et al. [28] for the Lotka–Volterra model and has been applied extensively to spatial models in biology.

The first main result of this work is a mathematical proof of the diffusion approximation in [34], SI, which is followed by its refinements for the other main results. The proof of the diffusion approximation follows the viewpoint in [6, 13], where the evolutionary games are regarded as perturbations of certain reference voter models, and is built on the assumption that the diffusion approximation of the evolutionary games on large finite sets holds in the special case of voter models. This assumption is supported by the recent results in [8, 9]. There it is proven that the diffusion approximation of voter models on large finite sets requires only mild conditions of the underlying spatial structures and that the Wright–Fisher diffusions appear as the universal limiting processes. The approach in this paper thereby develops along an equivalence between the probability laws of the evolutionary games on finite sets and the reference voter models.

1.1. *The evolutionary games and voter models.* Throughout this paper, we consider evolutionary games on finite sets to be defined as follows. On a finite set $E$ with size $N \geq 2$, each of the sites is occupied by an individual with one of the two types, 1 and 0. Individuals engage in pairwise interaction. Payoffs from this interaction follow a given payoff matrix $\Pi = (\Pi(\sigma, \tau))_{\sigma,\tau\in\{1,0\}}$ with real entries. Whenever an individual with type $\sigma$ and an individual with type $\tau$ interact, the individual with type $\sigma$ receives payoff $\Pi(\sigma, \tau)$. With respect to a given transition probability $q$ on $E$, the total payoff of the individual at site $x$ is given by the following weighted average, provided that the population configuration is $\xi \in \{1, 0\}^E$:

$$\sum_{y\in E} q(x, y)\Pi\big(\xi(x), \xi(y)\big). \tag{1.1}$$

Here and throughout this paper, we require that $q$ have trace zero [i.e., $q(x,x) \equiv 0$] and be irreducible and reversible. An individual's total payoff enters its fitness (i.e., reproductive rate). The fitness is given by a convex combination of baseline fitness 1 and the total payoff:

$$\text{fitness} = (1-w)\cdot 1 + w \cdot (\text{total payoff}). \tag{1.2}$$

Here, *selection strength* $w$ is the constant weight applied to the total payoff of every individual throughout time. It is understood to be sufficiently small, relative to the entries of the payoff matrix $\Pi$, to ensure that all the fitness values are positive. This positivity is required since these values define rates in the dynamics of the evolutionary game.

In this evolutionary game, players in the population are updated indefinitely according to the following rule: At the unit rate, the individual at $x$ is chosen to die. Then the individuals at all the other sites compete for reproduction to occupy the vacant site $x$ in a random fashion; the probability of successful reproduction of the parent at $y$, $y \neq x$, is proportional to the following product:

$$q(x,y) \cdot (\text{fitness of the individual at } y), \tag{1.3}$$

where the same kernel in (1.1) is used. See equation (2.4) for the Markov generator of the evolutionary game and, for example, [3, 15, 17, 32] for more about this evolutionary game.

For the purpose of this Introduction, we remark that in the above scenario, the entire population fixates at either the all-1 state or the all-0 state after a sufficiently large amount of time. This results from the assumed irreducibility of $q$ and the fact that the underlying population size is finite. On the other hand, in certain biological contexts (cf. [33]), significant interest in including mutation in evolutionary game dynamics exists. To introduce the main results of this paper in terms of the earlier background, we will only consider models without mutations unless otherwise mentioned until Section 2. The main result on the diffusion approximation of the evolutionary games allows for mutations.

In the special case of zero selection strength, the evolutionary game introduced above simplifies to a reference voter model with *voting kernel* $q$, where all the individuals in the population have the same fitness one. A voter model is an over-simplified model for death and birth of species in biological systems and should be regarded as a generalization of the Moran process from population genetics [30] on a structured population. The underlying spatial structure is defined in the natural way by the nonzero entries of $q$. The canonical example that the reader may keep in mind in reading the rest of this paper is the case where $q$ is the transition kernel of a random walk on a finite, connected, simple graph. In this case, the individuals chosen to die are replaced by the children of their neighbors.

The study of voter models allows for several classical approaches of interacting particle systems to start with (cf. [26]), including attractiveness and a duality

by coalescing Markov chains driven by voting kernels both in the sense of the Feynman–Kac representation (cf. Section 8) and in the pathwise sense of identity by descent in population genetics (cf. [26], Section III.6 and [20, 27, 39]). By contrast, the game transition probabilities at positive selection strengths $w > 0$ show configuration-dependent asymmetry arising from the differences in individuals' payoffs. In this case, attractiveness is absent, and exact evaluations of basic quantities in population genetics (e.g., absorbing probabilities and expected times to absorption) become difficult. See [22] for computational perspectives of this issue and [10] for particular properties of the evolutionary games arising from the lack of attractiveness.

1.2. *Pair approximation for the evolutionary games.* The primary focus of this paper is an approximation method in [34], SI, for the evolutionary games with death–birth updating. With the goal of quantifying the absorbing probabilities of the evolutionary games under weak selection, the analysis in [34], SI, invokes the corresponding density processes and conditional density processes. Here, weak selection is usually understood in the biological literature as requiring $w \leq \mathcal{O}(1/N)$ for $N$ being the population size. In addition, with respect to a voting kernel $q$ with stationary distribution $\pi$ and types $\sigma, \tau \in \{1, 0\}$, the *density* of $\sigma$'s in $\xi \in \{1, 0\}^E$ is given by the following weighted average:

$$p_\sigma(\xi) = \sum_{x \in E} \pi(x) \mathbb{1}_{\{\sigma\}}\big(\xi(x)\big), \tag{1.4}$$

and, with $p_{\tau\sigma}(\xi)$ defined by the weighted average

$$p_{\tau\sigma}(\xi) = \sum_{x,y \in E} \pi(x) q(x, y) \mathbb{1}_{\{\tau\}}\big(\xi(x)\big) \mathbb{1}_{\{\sigma\}}\big(\xi(y)\big), \tag{1.5}$$

the conditional densities of $\tau$ given $\sigma$ are defined by the ratios

$$p_{\tau|\sigma}(\xi) = \frac{p_{\tau\sigma}(\xi)}{p_\sigma(\xi)} \tag{1.6}$$

$(0/0 = 0$ by convention).

The analysis in [34], SI, provides diffusion approximations of the absorbing probabilities of the evolutionary game $(\xi_t)$ by means of the same probabilities of the one-dimensional process $p_1(\xi_t)$, where the underlying spatial structure is assumed to be a large random regular graph of degree $k \geq 3$. (The case $k = 2$ admits exact solutions for the absorbing probabilities [35].) As we will discuss in more detail below, the implication of pair approximation is nontrivial and is a key step to make further analysis possible in [34], SI. It leads to the property that the two processes $p_1(\xi_t)$ and $p_{1|0}(\xi_t)$ form a *closed* system. Moreover, the two processes decouple in the limit of large population size, whereas the density

process $p_1(\xi_t)$ approximates a "self-consistent" Wright–Fisher diffusion with drift coefficient and squared noise coefficient given by

$$\begin{aligned} &w \cdot \frac{k-2}{k^2(k-1)} p_1(\xi)\big[1-p_1(\xi)\big]\big[\alpha p_1(\xi)+\beta\big] \quad \text{and} \\ &\frac{2(k-2)}{N(k-1)} p_1(\xi)\big[1-p_1(\xi)\big], \end{aligned} \tag{1.7}$$

respectively. Here, the constants $\alpha$ and $\beta$ entering the drift coefficient are defined by the following equations:

$$\begin{aligned} \alpha &= (k+1)(k-2)\big[\Pi(1,1)-\Pi(1,0)-\Pi(0,1)+\Pi(0,0)\big], \\ \beta &= (k+1)\Pi(1,1)+\big(k^2-k-1\big)\Pi(1,0) \\ &\quad -\Pi(0,1)-\big(k^2-1\big)\Pi(0,0). \end{aligned} \tag{1.8}$$

See [34], equation (18) in SI, for the coefficients in equation (1.7). (The differences between the coefficients in [34], equation (18) in SI, and those in equation (1.7) are only attributable to the definition of total payoffs of individuals and the choice of time scales in this paper. We will explain the connection in Remark 4.10 in more detail.) Notice that in equation (1.7), only the drift coefficient depends on the game payoffs. More importantly, the approximate diffusion process is not mean-field because it contains some information about the underlying spatial structure as a large random $k$-regular graph, albeit only through the simple parameter degree $k$.

If we understand correctly the application of pair approximation in [34], SI, to close $p_1(\xi_t)$ and $p_{1|0}(\xi_t)$, then it can be summarized as two major hypotheses to be discussed below (they are adapted to the setup in this paper). In particular, both of these hypotheses involve reductions of the local frequencies

$$p_\tau(y,\xi)=\sum_{z\in E} q(y,z)\mathbb{1}_{\{\tau\}}\big(\xi(z)\big)=\frac{\#\{z;\, z\sim y, \xi(z)=\tau\}}{k} \tag{1.9}$$

of individuals with type $\tau$ to the conditional densities $p_{\tau|\sigma}(\xi)$, where $y$'s are sites occupied by individuals with type $\sigma$ and $z\sim y$ means that $z$ and $y$ are neighbors to each other. (The second equality above follows since the voting weight between a site and any of its neighboring sites is $1/k$ on a $k$-regular graph.)

Now, we condition on the event that an individual randomly chosen from the entire population, called a *focal* individual, is an individual with type 0 located at site $x$. Then the first major hypothesis states that the types of its neighbors, whose numbers are given by $k$ multiplies of the local frequencies $p_1(x,\xi)$ and $p_0(x,\xi)$ as in equation (1.9), are i.i.d. Bernoulli distributed. Moreover, the probability of finding an individual with type 1 is $p_{1|0}(\xi)$. Next, recall that the fitness of a neighbor, say at site $y$ and with type $\sigma$, of the focal individual is by definition a convex

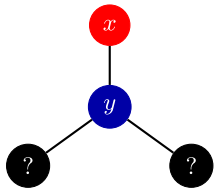


FIG. 1. *Site $x$ is occupied by a focal individual which is red (type 0). Site $y$ is occupied by a blue individual (type 1). The number of types among the neighbors of $y$, excluding the one at $x$, are left to be statistically estimated.*

combination of baseline fitness 1 and the total payoff that it receives [see (1.2)]. In this case, it can be written as follows:

$$f_\sigma(y) = (1-w) + w\left(\frac{1}{k}\Pi(\sigma,0) + p_1(y,\xi)\Pi(\sigma,1) + \left(p_0(y,\xi) - \frac{1}{k}\right)\Pi(\sigma,0)\right). \tag{1.10}$$

Here in equation (1.10), the first payoff $\Pi(\sigma,0)$ on the right-hand side results from the interaction between the focal individual, which has type 0, and the neighbor at $y$ with type $\sigma$ under consideration. Then the second major hypothesis states that the fitness in equation (1.10) satisfies the following approximate equality:

$$f_\sigma(y) \simeq (1-w) + w\left(\frac{1}{k}\Pi(\sigma,0) + \frac{(k-1)p_{1|\sigma}(\xi)}{k}\Pi(\sigma,1) + \frac{(k-1)p_{0|\sigma}(\xi)}{k}\Pi(\sigma,0)\right). \tag{1.11}$$

Hence, in (1.11), the numbers of types of the remaining $k-1$ neighbors of the individual at $y$ are now *statistically estimated* by the conditional densities $p_{1|\sigma}(\xi)$ and $p_{0|\sigma}(\xi)$ defined in equation (1.6). See Figure 1 for an example, where we visualize types 0 and 1 by red and blue, respectively. A similar hypothesis is in force if the focal individual is conditioned to be an individual with type 1. The argument in [34], SI, further uses the locally tree-like property of a large random $k$-regular graph (cf. [29] and the references therein for this property) so that the neighbors of the individual at site $y$, excluding the focal individual, can be neglected when fitnesses of the other neighbors of the focal individual are calculated.

The above application of pair approximation in [34], SI, is closely related to the standard probabilistic technique of characterizing the scaling limits of stochastic processes by the corresponding martingale problems. This technique requires the closure of the dynamical equations under consideration. In a general population, however, a typical statistic of the evolutionary game depends on state of the entire evolving population and the number of equations required to close its dynamics

appears to grow with the population size. This fact should make clear a nontrivial mathematical issue underlying the two hypotheses discussed above. Yet it is not clear to us how to verify the hypotheses.

Before the present work, mathematical proofs are provided to support arguably the most important finding in [34], SI, implied by the above diffusion approximation. That finding applies explicit solutions of the absorbing probabilities of the approximate diffusion processes, and these processes are defined by the coefficients in equation (1.7). Then games of the generalized prisoner's dilemma with payoff matrices given as follows are considered:

$$\Pi = \begin{array}{c} \\ 1 \\ 0 \end{array} \begin{array}{c} \begin{array}{cc} 1 & 0 \end{array} \\ \begin{pmatrix} b-c & -c \\ b & 0 \end{pmatrix} \end{array}, \qquad b, c \in \mathbb{R}. \tag{1.12}$$

Here, $b$ and $c$ are interpreted as benefit and cost, respectively, when they are strictly positive. Accordingly, individuals of type 1 are called cooperators and those of type 0 are defectors based on game theoretical considerations.

The aforementioned finding in [34], SI, states that for $k \geq 3$, the degree $k$ of a large random $k$-regular graph approximates a critical value concerning whether the emergence of cost-benefit effective game interactions can improve the survival of individuals with type 1 under weak selection: If $b > ck$, the survival probability of individuals with type 1 is strictly larger than the same probability under the reference voter model. If $b < ck$, the strict inequality between the probabilities is reversed.

The work of Cox, Durrett and Perkins in [13] obtains the rescaled limits of general voter model perturbations on any integer lattice of dimension $d \geq 3$ and proves related deep results. There these results are used to study long-term behavior of the interacting particle systems. In particular, it is proven in [13] that on any integer lattice of dimension $d \geq 3$, the graph degree $2d$ is exactly the critical value for the evolutionary game. More precisely, the critical values in [13] are defined in terms of the fixation of types in finite regions in space after a large amount of time, instead of the global fixation of types after a large amount of time. (To accommodate the transient nature of these infinite lattices, this definition becomes necessary.) Other progress related to mathematical proofs of the prediction in [34], SI, has been within the scope of finite, simple, regular graphs and considers a first-derivative test that is used to compare absorbing probabilities at all *arbitrarily small* selection strengths by signs of their derivatives at zero selection strength (e.g., [6]). In contrast to the method in [34], SI, the major investigation along that first-derivative test focuses on exact evaluations of the derivatives in general enough finite populations (see Section 1.5 for more details). These evaluations under all initial conditions are now complete in [10] by the duality between voter models and coalescing Markov chains. In particular, within the spatial structures of $k$-regular graphs, [6, 10] recover the critical value $k$ predicted in [34], SI, in the limit of large population size.

1.3. *Weak convergence of the game density processes.* The first main result of this paper is a general theorem for diffusion approximations of the game density processes $p_1(\xi_t)$. [Recall (1.4) for the definition of $p_1(\xi)$.] After a constant time change of the order $N$, we prove the convergence of the game density processes subject to a selection strength of the order $1/N$. Moreover, when a payoff matrix as in equation (1.12) is in use, the limiting process is a Wright–Fisher diffusion where the drift coefficient and noise coefficient are explicitly defined by the limiting spatial structure. See Theorem 4.6 for the precise statement.

The generality of applicable spatial structure we pursue in Theorem 4.6 is in the spirit of the evolutionary game theory that evolutionary game behaviors are ubiquitous in biological and social systems. On the other hand, an application of the theorem leads to a mathematical proof of the prediction in [34], SI, on large random regular graphs discussed above. In this particular case, the approximate diffusion process defined by the coefficients in equation (1.8) coincides with the limiting diffusion process obtained in this paper. See Theorem 4.9 and Remarks 4.10 and 4.12 for more details.

The proof of Theorem 4.6 does not invoke pathwise duality for the evolutionary game dynamics as in [13], in which certain branching coalescing Markov chains are used as the dual processes. By contrast, we proceed with the fact that the laws of the evolutionary games are equivalent to the laws of the reference voter models (Section 2). In this way, we can view the game density processes in terms of the voter models, even with the limit of large population size. The condition we need here is that the corresponding Radon–Nikodým derivative processes satisfy appropriate tightness. Then the proof of Theorem 4.6 turns to, and thereby, relies heavily on the main result in [8] that diffusion approximations of the voter density processes hold on large spatial structures where the underlying voting kernels are subject to appropriate, but mild, mixing conditions; an extension in [9] is used when mutation is present. In these cases, the limiting voter density processes are given by the Wright–Fisher diffusions, which originally arise from the Moran processes, namely, the voter models on complete graphs (cf. [16]). The spatial structures are encoded in the time scales for these diffusion approximations and are not present in the coefficients of the limiting diffusions. See Theorem 4.3 for a restatement of these results in [8, 9]. (See also the pioneer works [13, 31] for diffusion approximations of voter models. They are for voter models defined on integer lattices and characterize the limiting processes as solutions to stochastic partial differential equations.) By Girsanov's theorem, characterizing subsequential limits of the game density processes is thus reduced to characterizing the covariations between the limiting voter density process and subsequential limits of the Radon–Nikodým derivative processes (see Theorems 4.6 and 4.7). A certain spatial homogeneity of the voting kernels (Assumption 4.4) is then introduced to close the covariations by the limiting voter density process.

Let us give two remarks for the present method. First, it allows for the possibility of explicitly characterizing the limiting Radon–Nikodým derivative process.

This limiting process can take the form of a Doléans–Dade exponential martingale explicitly defined in terms of the limiting voter density process [Theorem 4.6(3°)]. We stress that the Radon–Nikodým derivative processes under consideration are used to change the laws of the voter models to the laws of the evolutionary games, not just to relate the laws of their density processes. Second, the only reason why we restrict our attention to the particular payoff matrices in equation (1.12) is because we do not know how to explicitly close the covariations between subsequential limits of the Radon–Nikodým derivative processes and the limiting voter density process, except possibly on complete graphs, finite cycles and star graphs. (These graphs are well known for allowing exact solutions under several closely related spatial stochastic processes. See also a comment below Assumption 4.1 on star graphs.) Nevertheless, we never try to derive the explicit coefficients carefully in those cases, because it is not clear to us if these simple graphs can help us gain any insight for the present diffusion approximation problems on graphs with rich enough structures, such as large random regular graphs considered in [34], SI, or large discrete tori with general degrees. On the other hand, the game density processes under general payoff matrices are proven to be tight if the voting kernels are subject to appropriate conditions. Moreover, any subsequential limit is a continuous semimartingale with a Wright–Fisher martingale part [Theorem 4.6(1°)]. This result proves the presence of a Wright–Fisher noise coefficient in the approximate diffusion process obtained in [34], SI, when individuals play games according to a general payoff matrix.

1.4. *Occupation measures of the game density processes*. As an application of the diffusion approximation of the game density processes, we investigate the use of the absorbing probabilities of the limiting diffusions as approximate solutions for the absorbing probabilities of the evolutionary games; this appears in [34], SI, as the main application of the approximate diffusion processes. A similar method is used in [40] to approximate the expected times to absorption of the evolutionary games. For these two approximations, the reader may recall the fact that the weak convergence of absorbing processes does not guarantee the weak convergence of their times to absorption in general.

By proving a stronger tightness property of the Radon–Nikodým derivative processes at selection strengths of the order $\mathcal{O}(1/N)$, we show that Oliveira's result on the convergence in the Wasserstein distance of order 1 of times to absorption in [36, 37] and the convergence of absorbing probabilities in [8] under voter models can be carried to the corresponding convergences under the evolutionary games. These are included in the second main result of this paper, Theorem 5.2, where the major theme is around convergences of occupation measures of the game density processes. See also Cox and Perkins [14] for a closely related result of voter models on integer lattices.

1.5. *The game absorbing probabilities*. The third main result of this paper, Theorem 6.6, proves that in a large finite population, the first-derivative test discussed by the end of Section 1.2 for the comparison of the game absorbing probabilities and the voter absorbing probabilities is applicable at all selection strengths at least up to the order $\mathcal{O}(1/N)$. We stress that this result does not require particular forms of payoff matrices and assumes much milder assumptions on the underlying spatial structures than those above for the diffusion approximations.

Theorem 6.6 is a long overdue result motivated by a seminar inquiry from Omer Angel several years ago when the author was a Ph.D. student. An answer to this inquiry can be used to quantify the scope of the first-derivative test in terms of the strength of selection, but we were unable to produce results sharp enough to match the weak selection strengths in [34] until now. (See below for a brief discussion of our earlier unpublished method.) Here in this paper, the answer is used to reinforce the comparison of the game absorbing probabilities and the voter absorbing probabilities by diffusion approximations. Indeed, the limiting diffusions in Theorem 4.6 can capture game interactions among individuals only if the selection strengths are asymptotically nonzero constant multiples of $1/N$.

To find selection strengths eligible for the first-derivative test, our earlier unpublished method uses some power series of the game absorbing probabilities in selection strength, which appear in [6], Proposition 3.2. Coefficients in the series are represented as explicit functionals of the voter models. In particular, an exact computation of the first-order coefficients is possible by the duality between voter models and coalescing Markov chains, and then, calculations of the coalescing Markov chains. This suggests similar arguments for all the higher-order coefficients, and so finding appropriate bounds for them should be turned to. Here in this paper, we obtain the bound $\mathcal{O}(1/N)$ for the eligible selection strengths by the equivalence of laws, since this bound seems to pose technical difficulties for that method by the power series in [6]. After all, the dual presentations for the higher-order coefficients appear highly intricate.

*Organization of the paper*. In Section 2, we discuss the dynamics of the evolutionary games with death–birth updating in more detail. In Section 3, we prove some technical a priori bounds for the Radon–Nikodým derivative processes between the laws of the evolutionary games and the laws of the reference voter models. Section 4 investigates convergences of the game density processes. We reinforce this result to a convergence in the Wasserstein distance of occupation measures of the game density processes in Section 5. In Section 6, we present the proof that the first-derivative test discussed above is applicable for selection strengths up to $\mathcal{O}(1/N)$ for suitable voting kernels. In Section 7, we calculate first-order expansions of some covariation processes of the evolutionary games in selection strength. Section 8 gives a brief account of the Feynman–Kac duality between voter models and coalescing Markov chains. Finally, Section 8 is followed by a list of frequent notation and a list of frequent asymptotics.

**2. Stochastic integral equations for the evolutionary games.** In this section, we describe the Markovian dynamics of an evolutionary game with death–birth updating in more detail and give a construction of the evolutionary game by Poisson calculus. We write $S = \{1, 0\}$ from now on. Recall that voting kernels are assumed to have zero traces and be irreducible and reversible throughout this paper.

First, let us specify the generator of an evolutionary game defined by a voting kernel $(E, q)$ and a payoff matrix $\Pi = (\Pi(\sigma, \tau))_{\sigma,\tau\in S}$ in the presence of mutation. In this case, the fitness of an individual at $x \in E$ under population configuration $\xi \in S^E$ is given by

$$f^w(x, \xi) = (1 - w) + w \sum_{y\in E} q(x, y)\Pi\big(\xi(x), \xi(y)\big). \tag{2.1}$$

Here and throughout the rest of this paper, we assume that selection strengths $w$ satisfy the constraint $w \in [0, \overline{w}]$, where

$$\overline{w} = \Big(2 + 2 \max_{\sigma,\tau\in S}\big|\Pi(\sigma, \tau)\big|\Big)^{-1}. \tag{2.2}$$

Hence, $f^w(x, \xi) > 0$ for all these $w$'s. We also define the population configurations $\xi^x$ and $\xi^{x|\sigma}$ as the ones obtained from $\xi$ by changing only the type at $x$, with the type $\xi(x)$ at $x$ changed to

$$\widehat{\xi}(x) = 1 - \xi(x) \tag{2.3}$$

for $\xi^x$ and to $\sigma \in S$ for $\xi^{x|\sigma}$. Then given a mutation measure $\mu$ on $S$, the generator of the evolutionary game is defined as follows:

$$\begin{aligned} \mathsf{L}^{w,\mu}F(\xi) &= \sum_{x\in E} c^w(x, \xi)\big(F\big(\xi^x\big) - F(\xi)\big) \\ &\quad + \sum_{x\in E}\int_S \big(F\big(\xi^{x|\sigma}\big) - F(\xi)\big)\, d\mu(\sigma) \end{aligned} \tag{2.4}$$

for $F : S \longrightarrow \mathbb{R}$, where the rates $c^w(x, \xi)$'s are given by

$$q^w(x, y, \xi) = \frac{q(x, y) f^w(y, \xi)}{\sum_{z\in E} q(x, z) f^w(z, \xi)}, \tag{2.5}$$

$$\begin{aligned} c^w(x, \xi) &= \sum_{y\in E} q^w(x, y, \xi)\mathbb{1}_{\{\xi(x)\neq\xi(y)\}} \\ &= \sum_{y\in E} q^w(x, y, \xi)\big(\xi(x)\widehat{\xi}(y) + \widehat{\xi}(x)\xi(y)\big). \end{aligned} \tag{2.6}$$

Notice that the function $q^w$ defined by (2.5) reduces to the voting kernel $q$ if $w = 0$; in this case, $\mathsf{L}^{0,\mu}$ is the generator of an $(E, q, \mu)$-voter model.

Now we recall a coupling of the $(E, q, \mu)$-voter model as solutions to stochastic integral equations driven by Poisson processes, which have been used in, for example, [31] and [12], Lemma 2.1 (see also the references therein). We introduce the following independent $(\mathscr{F}_t)$-Poisson processes:

$$\begin{aligned} &\Lambda_t(x, y) \qquad \text{with rate } \mathbb{E}\big[\Lambda_1(x, y)\big] = q(x, y) \quad \text{and} \\ &\Lambda_t^{\sigma}(x) \qquad \text{with rate } \mathbb{E}\big[\Lambda_1^{\sigma}(x)\big] = \mu(\sigma), x, y \in E, \sigma \in S, \end{aligned} \tag{2.7}$$

which are defined on a complete filtered probability space $(\Omega, \mathscr{F}, (\mathscr{F}_t), \mathbb{P})$. The filtration $(\mathscr{F}_t)$ is assumed to satisfy the usual conditions, and we set $\mathscr{F}_\infty = \bigvee_{t\geq 0} \mathscr{F}_t$. Then given an initial condition $\xi \in S^E$, an $(E, q, \mu)$-voter model $(\xi_t)$ can be defined as the pathwise unique $S^E$-valued solution, with càdlàg paths, of the following system of stochastic integral equations:

$$\begin{aligned} \xi_t(x) = \xi(x) &+ \sum_{y\in E} \int_0^t \big(\xi_{s-}(y) - \xi_{s-}(x)\big)\, d\Lambda_s(x, y) \\ &+ \int_0^t \widehat{\xi}_{s-}(x)\, d\Lambda_s^1(x) - \int_0^t \xi_{s-}(x)\, d\Lambda_s^0(x), \qquad x \in E. \end{aligned} \tag{2.8}$$

In (2.8), the sum of Poisson integrals governs removal and adoption of types, and the last two Poisson integrals govern independent mutation of types. To see this, let the state $\xi_{t-}$ right before time $t$ and $x$ chosen for update be given. Then a jump of $\Lambda(x, y)$ at time $t$ means that the type at $x$ is changed to

$$\xi_{t-}(x) + \big[\xi_{t-}(y) - \xi_{t-}(x)\big]\Delta\Lambda_t(x, y) = \xi_{t-}(x) + \big[\xi_{t-}(y) - \xi_{t-}(x)\big] \cdot 1 = \xi_{t-}(y).$$

Similarly, a jump of $\Lambda^1(x)$ at time $t$ shows that the type at $x$ is changed to

$$\xi_{t-}(x) + \widehat{\xi}_{t-}(x)\Delta\Lambda_t^1(x) = \xi_{t-}(x) + \widehat{\xi}_{t-}(x) \cdot 1 = 1,$$

and a jump of $\Lambda^0(x)$ at time $t$ gives

$$\xi_{t-}(x) - \xi_{t-}(x)\Delta\Lambda_t^0(x) = \xi_{t-}(x) - \xi_{t-}(x) \cdot 1 = 0.$$

We can use the system in (2.8) to couple the above evolutionary game by a change of measures. This coupling uses a generalization of the elementary change of intensities for Poisson processes which the reader may recall: For a Poisson process $(N_t)_{0\leq t\leq T}$ with $\mathbb{E}^{\mathbb{P}}[N_1] = 1$, its law under

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \exp\big\{N(T)\log\lambda - (\lambda - 1)T\big\} = \exp\left\{\int_0^T \log\frac{\lambda}{1}\, dN_t - \int_0^T (\lambda - 1)\, dt\right\}$$

is a Poisson process with $\mathbb{E}^{\mathbb{Q}}[N_1] = \lambda$ now that $\mathbb{E}^{\mathbb{P}}[e^{aN_t}] = \exp\{t(e^a - 1)\}$.

For the construction of the evolutionary game, we introduce the following $(\mathscr{F}_t, \mathbb{P})$-martingale:

$$\begin{aligned} D_t^w(x, y) = \exp\Bigg\{ &\int_0^t \log\frac{q^w(x, y, \xi_{s-})}{q(x, y)}\, d\Lambda_s(x, y) \\ &- \int_0^t \big(q^w(x, y, \xi_s) - q(x, y)\big)\, ds \Bigg\} \end{aligned} \tag{2.9}$$

to change the intensity of $\Lambda(x,y)$ under $\mathbb{P}$ whenever $q(x,y)>0$, and set $D_t^w(x,y)\equiv 1$ otherwise (see [38], p. 473, for the fact that $D^w(x,y)$ defines an $(\mathscr{F}_t,\mathbb{P})$-martingale). A global change of intensities is done through the following $(\mathscr{F}_t,\mathbb{P})$-martingale:

$$D_t^w \overset{\text{def}}{=} \prod_{(x,y)\in E\times E} D_t^w(x,y). \tag{2.10}$$

Define a probability measure $\mathbb{P}^w$ on $(\Omega,\mathscr{F}_\infty)$, with expectation $\mathbb{E}^w$, by

$$d\mathbb{P}^w|_{\mathscr{F}_t} = D_t^w\, d\mathbb{P}|_{\mathscr{F}_t}. \tag{2.11}$$

Then the process $(\xi_t)$ satisfying (2.8) defines an evolutionary game under $\mathbb{P}^w$, and its generator is given by $\mathsf{L}^{w,\mu}$; recall the definition of $c^w(x,y,\xi)$ in (2.6). In more detail, it follows from Girsanov's theorem (cf. [23], Theorem III.3.11 and Theorem III.3.17) that for any $x,y\in E$, the jump process $\Lambda(x,y)$ under $\mathbb{P}^w$ is an $(\mathscr{F}_t)$-doubly-stochastic Poisson process with an $(\mathscr{F}_t)$-predictable intensity $(q^w(x,y,\xi_{t-});t\ge 0)$ in the sense of Watanabe's characterization [42]. That is, it holds that

$$\mathbb{E}^w\left[\int_0^\infty C_t\, d\Lambda_t(x,y)\right] = \mathbb{E}^w\left[\int_0^\infty C_t q^w(x,y,\xi_t)\, dt\right]$$

for all nonnegative $(\mathscr{F}_t)$-predictable processes $(C_t)$. Notice that under $\mathbb{P}^w$, $\Lambda^\sigma(x)$ remains an $(\mathscr{F}_t)$-Poisson process with rate $\mu(\sigma)$, so that the mutation mechanism is not affected by this change of measure.

The following proposition gives a summary of the above construction.

PROPOSITION 2.1. *For any $w\in[0,\overline{w}]$ and initial condition $\xi\in S^E$, the pathwise unique solution $(\xi_t)$ of the system (2.8) under $\mathbb{P}^w$ is a jump Markov process with its generator given by $\mathsf{L}^{w,\mu}$.*

In the sequel, we write $\mathbb{P}_\xi^w$ and $\mathbb{E}_\xi^w$ whenever the solution to (2.8) is subject to the initial condition $\xi\in S^E$. The notation $\mathbb{P}_\lambda^w$ and $\mathbb{E}_\lambda^w$, for $\lambda$ being a probability measure on $S^E$, are understood similarly. We drop the superscripts $w$ in these notations if $w=0$ and there is no risk of confusion.

**3. The Radon–Nikodým derivative processes.** In this section, we prove some a priori bounds for the $(\mathscr{F}_t,\mathbb{P})$-martingales $(D_t^w)$ defined by (2.10). These bounds will play a crucial role in Section 4 and Section 5 for the proofs of limit theorems of the game density processes.

We begin with the stochastic integrals equations of $D^w$. Recall that, for each fixed $w\in[0,\overline{w}]$, $D^w$ is a Doléans–Dade exponential martingale:

$$\begin{aligned} D_t^w = \mathcal{E}(L^w)_t &\overset{\text{def}}{=} \exp\left(L_t^w - \frac{1}{2}\langle (L^w)^c,(L^w)^c\rangle_t\right)\\ &\quad\times \prod_{s:s\le t}(1+\Delta L_s^w)\exp(-\Delta L_s^w). \end{aligned} \tag{3.1}$$

Here, the stochastic logarithm $L^w$ of $D^w$ is defined with respect to the compensated $(\mathscr{F}_t, \mathbb{P})$-Poisson processes:

$$\widehat{\Lambda}_t(x,y) \equiv \Lambda_t(x,y) - q(x,y)t, \qquad x,y \in E, \tag{3.2}$$

as the following $(\mathscr{F}_t, \mathbb{P})$-martingale:

$$L_t^w = \sum_{x,y\in E} \int_0^t \left( \frac{q^w(x,y,\xi_{s-})}{q(x,y)} - 1 \right) d\widehat{\Lambda}_s(x,y), \tag{3.3}$$

which has a zero continuous part $(L^w)^c \equiv 0$. In (3.3) and what follows, we use the convention that $0/0 = 0$. Equation (3.1) implies that $D^w$ is the pathwise unique solution to the linear equation

$$\begin{aligned} D_t^w &= 1 + \int_0^t D_{s-}^w \, dL_s^w \\ &= 1 + \sum_{x,y\in E} \int_0^t D_{s-}^w \left( \frac{q^w(x,y,\xi_{s-})}{q(x,y)} - 1 \right) d\widehat{\Lambda}_s(x,y), \end{aligned} \tag{3.4}$$

where the last equality follows from (3.3). See [23], Theorem I.4.61, for these properties of $D^w$.

In the sequel, $\mathscr{P}(U)$ denotes the set of probability measures defined on a Polish space $U$. Also, recall that $\pi$ denotes the unique stationary distribution of a voting kernel $q$.

PROPOSITION 3.1. *For every $a \in [1,\infty)$, there is a positive constant $C_{(3.5)}$ depending only on $(\Pi, a)$ such that for all $w \in [0, \overline{w}]$ and $\lambda \in \mathscr{P}(S^E)$,*

$$(D_t^w)^a \exp\left( -C_{(3.5)} w^2 \pi_{\min}^{-1} \sum_{\ell=1}^4 \int_0^t W_\ell(\xi_s)\, ds \right) \tag{3.5}$$

*is an $(\mathscr{F}_t, \mathbb{P}_\lambda)$-supermartingale,*

*where $\pi_{\min} = \min_{x\in E} \pi(x)$ and $W_\ell(\xi)$'s are weighted two-point density functions defined by*

$$\begin{aligned} W_\ell(\xi) &= \sum_{x,y\in E} \pi(x) q^\ell(x,y) \xi(x) \widehat{\xi}(y) \\ &= \sum_{x,y\in E: x\neq y} \pi(x) q^\ell(x,y) \xi(x) \widehat{\xi}(y). \end{aligned} \tag{3.6}$$

*In particular, there is a positive constant $C_{(3.7)}$ depending only on $(\Pi, a)$ such that for all $w \in [0, \overline{w}]$, $\lambda \in \mathscr{P}(S^E)$ and $(\mathscr{F}_t)$-stopping times $T'$, we have*

$$\mathbb{E}_\lambda\left[ (D_{T'}^w)^a \right] \le \mathbb{E}_\lambda\left[ \exp\left( C_{(3.7)} w^2 \pi_{\min}^{-1} \sum_{\ell=1}^4 \int_0^{T'} W_\ell(\xi_t)\, dt \right) \right]^{1/2}. \tag{3.7}$$

PROOF. The method of this proof is to use stochastic calculus to obtain a tight bound for the growth of $(D^w)^a$.

Fix $w \in [0, \overline{w}]$. Note that we have

$$\sup_{s\in[0,t]} \mathbb{E}_\xi\big[(D_s^w)^a\big] < \infty \qquad \forall a \in [1,\infty), t \in (0,\infty), \xi \in S^E, \tag{3.8}$$

which follows from the fact that $q^w(x,y,\xi), q(x,y)$ and $|\log(q^w(x,y,\xi)/q(x,y))|$ are uniformly bounded in $x, y, \xi$ by the choice of the maximal selection strength $\overline{w}$ in (2.2).

To obtain the required supermartingale property in (3.5), we work with the stochastic integral equation in (3.4) satisfied by $D^w$. By the chain rule for Stieltjes' integrals [38], Proposition 0.4.6, and (3.4), we have

$$\begin{aligned}
(D_t^w)^a &= 1 + \sum_{x,y\in E} \int_0^t a\big(D_{s-}^w\big)^{a-1} \cdot D_{s-}^w \left(\frac{q^w(x,y,\xi_{s-})}{q(x,y)} - 1\right) d\widehat{\Lambda}_s(x,y) \\
&\quad + \sum_{s:0<s\le t} \big((D_s^w)^a - (D_{s-}^w)^a - a(D_{s-}^w)^{a-1}\Delta D_s^w\big) \\
&= 1 + \sum_{x,y\in E} \int_0^t a\big(D_{s-}^w\big)^a \left(\frac{q^w(x,y,\xi_{s-})}{q(x,y)} - 1\right) d\widehat{\Lambda}_s(x,y) \\
&\quad + \sum_{x,y\in E} \int_0^t (D_{s-}^w)^a \\
&\quad \times \left[\left(\frac{q^w(x,y,\xi_{s-})}{q(x,y)}\right)^a - 1 - a\left(\frac{q^w(x,y,\xi_{s-})}{q(x,y)} - 1\right)\right] d\Lambda_s(x,y),
\end{aligned} \tag{3.9}$$

where the last equality follows since $D_s^w/D_{s-}^w = q^w(x,y,\xi_{s-})/q(x,y)$ if $\Delta\Lambda_s(x,y) > 0$. The first sum in (3.9) is a martingale by (3.8) and the fact that $q^w(x,y,\xi)/q(x,y)$ are uniformly bounded in $x, y, \xi$ [see (2.5)].

Now we handle the integrands in the last sum in (3.9) with the following three observations. First, if $q(x,y) > 0$, it follows from the definition (2.5) of $q^w$ that $q^w/q$ satisfies the following series expansion in $w$:

$$\begin{aligned}
\frac{q^w(x,y,\xi)}{q(x,y)} &= \frac{1 - wB(y,\xi)}{1 - wA(x,\xi)} \\
&= 1 + \sum_{i=1}^{\infty} w^i A(x,\xi)^{i-1}\big[A(x,\xi) - B(y,\xi)\big] \qquad (3.10) \\
&= 1 + w\big[A(x,\xi) - B(y,\xi)\big] + w^2 R^w(x,y,\xi), \qquad (3.11)
\end{aligned}$$

where $A$, $B$ and $R^w$ are functions defined by

$$A(x,\xi) = 1 - \sum_{z\in E} q(x,z) \sum_{z'\in E} q\big(z,z'\big)\Pi\big(\xi(z),\xi\big(z'\big)\big), \tag{3.12}$$

$$B(y,\xi) = 1 - \sum_{z\in E} q(y,z)\Pi\big(\xi(y),\xi(z)\big), \tag{3.13}$$

$$R^w(x,y,\xi) = \frac{A(x,\xi)[A(x,\xi) - B(y,\xi)]}{1 - wA(x,\xi)}. \tag{3.14}$$

Second, observe that we have the following inequality:

$$\sum_{x,y\in E} \pi(x)q(x,y)\big|A(x,\xi) - B(y,\xi)\big| \le C_{(3.15)} \sum_{\ell=1}^{4} W_\ell(\xi), \tag{3.15}$$

where the constant $C_{(3.15)} \in (0,\infty)$ depends only on $\Pi$ and $W_\ell(\xi)$'s are defined by (3.6). To see (3.15), first we fix a population configuration $\xi$ and recall the reversibility of $q$. Observe that for any $x, y$ such that $q(x,y) > 0$ and

$$A(x,\xi) - B(y,\xi) \neq 0,$$

we must have $\Pi(\xi(z),\xi(z')) \neq \Pi(\xi(y),\xi(z''))$ for some $z, z', z''$ such that $q(x,z)q(z,z') > 0$ and $q(y,z'') > 0$. Hence,

$$\big(\xi(z),\xi(z')\big) \neq \big(\xi(y),\xi(z'')\big),$$

which implies either (1) $1 \in \{\xi(z),\xi(z')\}$ and $0 \in \{\xi(y),\xi(z'')\}$ or (2) $0 \in \{\xi(z),\xi(z')\}$ and $1 \in \{\xi(y),\xi(z'')\}$. This pathwise consideration shows the following: Up to a multiplicative constant depending only on $\Pi$, we can bound the right-hand side of (3.15) by the probability that the stationary, reversible $q$-Markov chain can find at least two sites with different types in $\xi$ along the path $(z', z, x, y, z'')$ taking 4 steps. Since the probability for the stationary, reversible $q$-Markov chain to see two sites which are occupied by different types under $\xi$ and are apart by $\ell$ steps is $W_\ell(\xi)$, the inequality (3.15) of our claim follows.

Third, we can use (3.10) to obtain the first-order Taylor expansions around 0 of the following two functions:

$$w \longmapsto \left(\frac{q^w(x,y,\xi)}{q(x,y)}\right)^a - 1 \quad \text{and} \quad w \longmapsto a\left(\frac{q^w(x,y,\xi)}{q(x,y)} - 1\right). \tag{3.16}$$

Both of these expansions take the same form as follows:

$$wa\big[A(x,\xi) - B(y,\xi)\big] + \mathcal{O}(w^2), \qquad w \longrightarrow 0+.$$

Since the derivatives of $w \mapsto q^w/q$ at zero of all orders are bounded by $|A - B|$ up to multiplicative constants depending only on $\Pi$ by (3.10), we obtain from (3.15) and these Taylor expansions for the functions in (3.16) that, for all $\xi \in S^E$,

$$\begin{aligned}\sum_{x,y\in E} &\left|\left(\frac{q^w(x,y,\xi)}{q(x,y)}\right)^a - 1 - a\left(\frac{q^w(x,y,\xi)}{q(x,y)} - 1\right)\right| q(x,y) \\ &\le C_{(3.17)} w^2 \pi_{\min}^{-1} \sum_{\ell=1}^{4} W_\ell(\xi),\end{aligned} \tag{3.17}$$

where the constant $C_{(3.17)} \in (0, \infty)$ depends only on $(\Pi, a)$.

We are ready to prove the required supermartingale property in (3.5) with the choice $C_{(3.5)} = C_{(3.17)}$. We define a continuous process $A$ by

$$A_t = \int_0^t C_{(3.5)} w^2 \pi_{\min}^{-1} \sum_{\ell=1}^4 W_\ell(\xi_s)\, ds$$

to subdue $(D^w)^a$. By the integration by parts for Stieltjes' integrals (cf. [38], Proposition 0.4.5) and (3.9), we get

$$\begin{aligned}
&\bigl(D_t^w\bigr)^a e^{-A_t} \\
&\quad = 1 + \int_0^t \bigl(D_s^w\bigr)^a e^{-A_s} \biggl(-C_{(3.5)} w^2 \pi_{\min}^{-1} \sum_{\ell=1}^4 W_\ell(\xi_s)\biggr)\, ds \\
&\qquad + \int_0^t \bigl(D_{s-}^w\bigr)^a e^{-A_s} \\
&\qquad \times \sum_{x,y\in E} \biggl[\biggl(\frac{q^w(x,y,\xi_{s-})}{q(x,y)}\biggr)^a - 1 - a\biggl(\frac{q^w(x,y,\xi_{s-})}{q(x,y)} - 1\biggr)\biggr] q(x,y)\, ds \\
&\qquad + \sum_{x,y\in E} \int_0^t a\bigl(D_{s-}^w\bigr)^a e^{-A_s} \biggl(\frac{q^w(x,y,\xi_{s-})}{q(x,y)} - 1\biggr)\, d\widehat{\Lambda}_s(x,y) \\
&\qquad + \sum_{x,y\in E} \int_0^t \bigl(D_{s-}^w\bigr)^a e^{-A_s} \\
&\qquad \times \biggl[\biggl(\frac{q^w(x,y,\xi_{s-})}{q(x,y)}\biggr)^a - 1 - a\biggl(\frac{q^w(x,y,\xi_{s-})}{q(x,y)} - 1\biggr)\biggr] d\widehat{\Lambda}_s(x,y),
\end{aligned}$$

where the sum of the two Riemann-integral terms is nonpositive by (3.17) and the choice $C_{(3.5)} = C_{(3.17)}$, and the last two sums are both finite sums of $(\mathscr{F}_t, \mathbb{P})$-martingales. The foregoing equality is enough for (3.5).

The second assertion of the proposition is a simple application of the first assertion. We use the supermartingale in (3.5) with $a$ replaced by $2a$ and get from the Cauchy–Schwarz inequality that

$$\begin{aligned}
\mathbb{E}_\lambda\bigl[\bigl(D_{T'}^w\bigr)^a\bigr] &\le \mathbb{E}_\lambda\biggl[\biggl(\bigl(D_{T'}^w\bigr)^a \exp\biggl(-\frac{C_{(3.5)}(2a)}{2} w^2 \pi_{\min}^{-1} \sum_{\ell=1}^4 \int_0^{T'} W_\ell(\xi_s)\, ds\biggr)\biggr)^2\biggr]^{1/2} \\
&\quad \times \mathbb{E}_\lambda\biggl[\exp\biggl(\frac{C_{(3.5)}(2a)}{2} w^2 \pi_{\min}^{-1} \sum_{\ell=1}^4 \int_0^{T'} W_\ell(\xi_s)\, ds\biggr)^2\biggr]^{1/2}.
\end{aligned}$$

The required inequality follows from the foregoing inequality and the optional stopping theorem [38], Theorem II.3.3 [this leads to the choice $C_{(3.7)} = C_{(3.5)}(2a)$]. The proof is complete. $\square$

In the rest of this section, we turn to the predictable covariation between $D^w$ and the density process

$$Y_t \overset{\text{def}}{=} p_1(\xi_t) \tag{3.18}$$

as well as their own predictable quadratic variations, where the function $p_1(\xi)$ is defined by (1.4). Recall that $\widehat{\Lambda}_t(x, y)$'s denote the compensated $(\mathscr{F}_t, \mathbb{P})$-Poisson processes defined by (3.2) and $\widehat{\Lambda}^{\sigma}_t(x)$ are similarly defined from $\Lambda^{\sigma}_t(x)$ in (2.7). With the stochastic integral equation satisfied by $D^w$ in (3.4), the other process $Y$ satisfies the following equation by (2.8) and the reversibility of $q$:

$$Y_t = Y_0 + \int_0^t \big[\mu(1)(1 - Y_s) - \mu(0)Y_s\big]\,ds + M_t, \tag{3.19}$$

where $M$ is an $(\mathscr{F}_t, \mathbb{P})$-martingale defined by

$$\begin{aligned} M_t &= \sum_{x,y\in E} \pi(x) \int_0^t \big[\xi_{s-}(y) - \xi_{s-}(x)\big]\,d\widehat{\Lambda}_s(x, y) \\ &\quad + \sum_{x\in E} \pi(x) \int_0^t \widehat{\xi}_{s-}(x)\,d\widehat{\Lambda}^1_s(x) - \sum_{x\in E} \pi(x) \int_0^t \xi_{s-}(x)\,d\widehat{\Lambda}^0_s(x). \end{aligned} \tag{3.20}$$

LEMMA 3.2. *Fix $w \in [0, \overline{w}]$. Then under $\mathbb{P}$, we have*

$$\begin{aligned} \langle M, M\rangle_t &= \int_0^t \sum_{x,y\in E} \nu(x, y)\big[\widehat{\xi}_s(x)\xi_s(y) + \xi_s(x)\widehat{\xi}_s(y)\big]\,ds \\ &\quad + \int_0^t \sum_{x\in E} \pi(x)^2\big[\widehat{\xi}_s(x)\mu(1) + \xi_s(x)\mu(0)\big]\,ds, \end{aligned} \tag{3.21}$$

$$\big\langle M, D^w\big\rangle_t = w \int_0^t D^w_s \overline{D}(\xi_s)\,ds + w^2 \int_0^t D^w_s R^w_1(\xi_s)\,ds, \tag{3.22}$$

$$\begin{aligned} \big\langle D^w, D^w\big\rangle_t &= w^2 \int_0^t \big(D^w_s\big)^2 \sum_{x,y\in E} q(x, y)\big[A(x, \xi_s) - B(y, \xi_s)\big]^2\,ds \\ &\quad + w^3 \int_0^t \big(D^w_s\big)^2 R^w_2(\xi_s)\,ds, \end{aligned} \tag{3.23}$$

*where*

$$\nu(x, y) = \pi(x)^2 q(x, y)\mathbb{1}_{x\neq y} = \pi(x)^2 q(x, y), \qquad x, y \in E, \tag{3.24}$$

*and, for $A$, $B$ and $R^w$ defined by* (3.12), (3.13) *and* (3.14), *the functions $\overline{D}$, $R^w_1$, $R^w_2$ in* (3.22) *and* (3.23) *are defined by*

$$\overline{D}(\xi) = \sum_{x,y\in E} \pi(x)q(x, y)\big[\xi(y) - \xi(x)\big]\big[A(x, \xi) - B(y, \xi)\big], \tag{3.25}$$

$$R_1^w(\xi) = \sum_{x,y\in E} \pi(x)q(x,y)\big[\xi(y)-\xi(x)\big]R^w(x,y,\xi), \tag{3.26}$$

$$R_2^w(\xi) = \sum_{x,y\in E} q(x,y)\big\{2\big[A(x,\xi)-B(y,\xi)\big]R^w(x,y,\xi) + wR^w(x,y,\xi)^2\big\}.$$

PROOF. Recall that the rates of the driving Poisson processes $\Lambda(x,y)$ and $\Lambda^\sigma(x)$ under $\mathbb{P}$ are given by (2.7). Hence, by (3.4) and (3.20), we have

$$\begin{aligned}\langle M,M\rangle_t &= \int_0^t \sum_{x,y\in E} \pi(x)^2 q(x,y)\big[\xi_s(y)-\xi_s(x)\big]^2\,ds \\ &\quad + \int_0^t \sum_{x\in E} \pi(x)^2\big[\widehat{\xi}_s(x)\mu(1)+\xi_s(x)\mu(0)\big]\,ds,\end{aligned} \tag{3.27}$$

$$\begin{aligned}\big\langle M,D^w\big\rangle_t &= \sum_{x,y\in E} \pi(x)q(x,y) \\ &\quad \times \int_0^t D_s^w\big[\xi_s(y)-\xi_s(x)\big]\bigg(\frac{q^w(x,y,\xi_s)}{q(x,y)}-1\bigg)\,ds,\end{aligned} \tag{3.28}$$

$$\big\langle D^w,D^w\big\rangle_t = \sum_{x,y\in E} q(x,y)\int_0^t (D_s^w)^2\bigg(\frac{q^w(x,y,\xi_s)}{q(x,y)}-1\bigg)^2\,ds. \tag{3.29}$$

The first equation above gives (3.21), upon using the notation in (3.24) and the equality

$$\big[\xi(y)-\xi(x)\big]^2 = \widehat{\xi}(x)\xi(y)+\xi(x)\widehat{\xi}(y). \tag{3.30}$$

For (3.22) and (3.23), we apply the Taylor expansion (3.11) of $q^w/q$ in $w$ to (3.28) and (3.29). □

The following lemma gives some moment bounds for $\langle D^w,D^w\rangle$ and $\langle M,D^w\rangle$ under $\mathbb{P}$.

LEMMA 3.3. *For all $a\in[1,\infty)$, we can find positive constants $C_{(3.31)}$ and $C_{(3.32)}$ depending only on $(\Pi,a)$ such that for all $\lambda\in\mathscr{P}(S^E)$,*

$$\begin{aligned}&\mathbb{E}_\lambda\big[\big\langle D^w,D^w\big\rangle_t^a\big] \\ &\quad\le C_{(3.31)}\sum_{\ell=1}^4 \mathbb{E}_\lambda\bigg[\exp\bigg(C_{(3.31)}w^2\pi_{\min}^{-1}\int_0^t W_\ell(\xi_s)\,ds\bigg)\bigg]^{1/2} \\ &\qquad\times \sum_{\ell=1}^4 \mathbb{E}_\lambda\bigg[\bigg(w^2\pi_{\min}^{-1}\int_0^t W_\ell(\xi_s)\,ds\bigg)^{2a}\bigg]^{1/2}\end{aligned} \tag{3.31}$$

*and*

$$\begin{aligned}
&\mathbb{E}_\lambda\big[\mathrm{Var}\big(\langle M, D^w\rangle\big)_t^a\big] \\
&\quad\le C_{(3.32)} \sum_{\ell=1}^4 \mathbb{E}_\lambda\bigg[\exp\bigg(C_{(3.32)} w^2 \pi_{\min}^{-1} \int_0^t W_\ell(\xi_s)\,ds\bigg)\bigg]^{1/2} \\
&\qquad\times \sum_{\ell=1}^4 \mathbb{E}_\lambda\bigg[\bigg(w \int_0^t W_\ell(\xi_s)\,ds\bigg)^{2a}\bigg]^{1/2},
\end{aligned} \tag{3.32}$$

*where* $\mathrm{Var}(A)$ *denotes the total variation process for* $A$.

PROOF. Applying (3.29) and (3.10) to the first and second lines below, respectively, we obtain

$$\begin{aligned}
&\mathbb{E}_\lambda\big[\langle D^w, D^w\rangle_t^a\big] \\
&\quad= \mathbb{E}_\lambda\bigg[\bigg(\sum_{x,y\in E} q(x,y) \int_0^t \big(D_s^w\big)^2 \bigg(\frac{q^w(x,y,\xi_s)}{q(x,y)} - 1\bigg)^2 ds\bigg)^a\bigg] \\
&\quad\le \mathbb{E}_\lambda\bigg[\bigg(\sum_{x,y\in E} q(x,y) \int_0^t \big(D_s^w\big)^2 \bigg|\sum_{i=1}^\infty w^i A^{i-1}(x,\xi_s)\bigg|^2 \\
&\qquad\times \big|A(x,\xi_s) - B(y,\xi_s)\big|^2 ds\bigg)^a\bigg] \\
&\quad\le C_{(3.33)} \mathbb{E}_\lambda\bigg[\bigg(\sum_{x,y\in E} q(x,y) \int_0^t \big(D_s^w\big)^2 w^2 \frac{\pi(x)}{\pi_{\min}} \\
&\qquad\times \big|A(x,\xi_s) - B(y,\xi_s)\big|^2 ds\bigg)^a\bigg] \qquad (3.33) \\
&\quad\le C_{(3.34)} \sum_{\ell=1}^4 \mathbb{E}_\lambda\bigg[\bigg(\int_0^t \big(D_s^w\big)^2 w^2 \pi_{\min}^{-1} W_\ell(\xi_s)\,ds\bigg)^a\bigg] \qquad (3.34) \\
&\quad\le C_{(3.34)} \sum_{\ell=1}^4 \mathbb{E}_\lambda\big[\big(D_t^w\big)^{4a}\big]^{1/2} \times \mathbb{E}_\lambda\bigg[\bigg(w^2 \pi_{\min}^{-1} \int_0^t W_\ell(\xi_s)\,ds\bigg)^{2a}\bigg]^{1/2} \qquad (3.35) \\
&\quad\le C_{(3.36)} \sum_{\ell=1}^4 \mathbb{E}_\lambda\bigg[\exp\bigg(C_{(3.36)} w^2 \pi_{\min}^{-1} \int_0^t W_\ell(\xi_s)\,ds\bigg)\bigg]^{1/2} \\
&\qquad\times \sum_{\ell=1}^4 \mathbb{E}_\lambda\bigg[\bigg(w^2 \pi_{\min}^{-1} \int_0^t W_\ell(\xi_s)\,ds\bigg)^{2a}\bigg]^{1/2}. \qquad (3.36)
\end{aligned}$$

Here, the positive constants $C_{(3.33)}$, $C_{(3.34)}$ and $C_{(3.36)}$ depend only on $(\Pi, a)$. Also, (3.34) follows from (3.15) and an elementary inequality [similar to the second inequality in (3.37) below]; (3.34) follows from the Cauchy–Schwarz inequality and Doob's strong $L^p$-inequality [38], Theorem II.1.7, since $(D^w)^{4a}$ is a submartingale; finally (3.36) follows from (3.7) and the following elementary inequality: for nonnegative random variables $A_1, A_2, A_3, A_4$,

$$\mathbb{E}\Bigg[\prod_{\ell=1}^{4} A_\ell\Bigg]^{1/2} \le \mathbb{E}\Bigg[\sum_{\ell=1}^{4} A_\ell^4\Bigg]^{1/2} \le \sum_{\ell=1}^{4} 4^{1/2}\mathbb{E}\big[A_\ell^4\big]^{1/2}. \tag{3.37}$$

The inequality (3.31) is then implied by (3.36).

The proof of (3.32) is similar. We use (3.10) and (3.28) in the first inequality and then (3.15) in the second inequality below:

$$\begin{aligned}
&\mathbb{E}_\lambda\big[\mathrm{Var}\big(\langle M, D^w\rangle\big)_t^a\big] \\
&\le C_{(3.38)}\mathbb{E}_\lambda\Bigg[\Bigg(\sum_{x,y\in E}\pi(x)q(x,y)\int_0^t D_s^w\big|A(x,\xi_s)-B(y,\xi_s)\big|\,ds\Bigg)^a\Bigg] \qquad (3.38)\\
&\le C_{(3.39)}\sum_{\ell=1}^{4}\mathbb{E}_\lambda\Bigg[\Bigg(\int_0^t D_s^w w W_\ell(\xi_s)\,ds\Bigg)^a\Bigg] \qquad (3.39)
\end{aligned}$$

for constants $C_{(3.38)}, C_{(3.39)}$ depending only on $(\Pi, a)$. The last inequality leads to (3.32) upon applying the same arguments as those for (3.34) and (3.36). The proof is complete. □

Equation (3.21) and the inequalities in Lemma 3.3 show that the voter potential functions $\int_0^\cdot W_\ell(\xi_s)\,ds$ play a key role in bounding the covariations considered in Lemma 3.2. We will study these functions further in Section 4.4.

**4. Weak convergence of the game density processes.** Our goal in this section is to study the density processes of 1's in the evolutionary games.

Let a sequence of voting kernels $(E_n, q^{(n)})$ and a sequence of mutation measures $\mu_n$ defined on $S$ be given, where $N_n = \#E_n$ increases to infinity. To apply the method of equivalence of laws outlined in Section 1.3, we define vectors of semimartingales which consist of the density processes of 1's in the $(E_n, q^{(n)})$-voter models as in (3.19), the jump martingales in these density processes under the voter models as in (3.20), and the Radon–Nikodým derivative processes as in (2.10) to change the laws of the voter models to the laws of the corresponding evolutionary games.

Formally, we introduce the following vector of semimartingales under $\mathbb{P}^{(n)}$ for each $n \in \mathbb{N}$:

$$Z^{(n)} = \big(Y_t^{(n)}, M_t^{(n)}, D_t^{(n)}\big) = \big(Y_{\gamma_n t}, M_{\gamma_n t}, D_{\gamma_n t}^{w_n}\big), \tag{4.1}$$

where the constants $\gamma_n$ and $w_n$ will be chosen later on such that $\gamma_n$ tends to infinity and $w_n$ tends to zero, respectively, as $n \longrightarrow \infty$. Here in (4.1), for each $n$, $(Y, M, D)$ under $\mathbb{P}^{(n)}$ consists of the processes considered in Section 2 and Section 3 with respect to the $(E_n, q^{(n)}, \mu_n)$-voter model. [Recall (2.10), (3.18) and (3.20). See also (3.19) and (3.4) for the dynamical equations of $Y_{\gamma_n t}$ and $D^{w_n}_{\gamma_n t}$, respectively.]

Notice that since $Z^{(n)}$ is defined by $q^{(n)}$, the unique stationary distribution $\pi^{(n)}$ of $q^{(n)}$ enters its definition. More precisely, $\pi^{(n)}$ enters the definitions of $Y^{(n)}$ and $M^{(n)}$ in a crucial way by (3.18) and (3.20) and will be assumed to satisfy $\max_{x\in E_n} \pi^{(n)}(x) \longrightarrow 0$ later on so that the jump sizes of $Y^{(n)}$ and $M^{(n)}$ are negligible in the limit. Also, it is obvious that the vector of semimartingales $Z^{(n)}$ is adapted to the filtration

$$\mathscr{F}_t^{(n)} = \sigma(\xi_{\gamma_n s}; s \le t), \qquad 0 \le t < \infty, \tag{4.2}$$

where $(\xi_t)$ is understood to be the $(E_n, q^{(n)}, \mu_n)$-voter model.

Similar to the above notation, objects defined with respect to the triplet $(E_n, q^{(n)}, \mu_n)$ will carry either subscripts "$n$" or superscripts "$(n)$" whenever necessary. Those where references to "$n$" are not made are defined under general voter models.

4.1. *Dual equations.* The conditions and proofs for theorems in the sequel will use the duality between voter models and coalescing Markov chains, which we discuss briefly here. Although we choose to work with the Feynman–Kac duality in the present setting (see Section 8 for more details), the reader may also recall the graphical duality in [20]. See also [9], Section 6.

For a triplet $(E, q, \mu)$, the dual process is a system of coalescing $q$-Markov chains $\{B^x; x \in E\}$ on $E$ so that $B^x$'s move along sites of $E$ as rate-1 $q$-Markov chains independently before meeting and together afterward. The dual functions are given by

$$H(\xi; x, y) = \big[\xi(x) - \overline{\mu}(1)\big]\big[\widehat{\xi}(y) - \overline{\mu}(0)\big], \qquad x, y \in E, \tag{4.3}$$

where

$$\overline{\mu}(\sigma) = \mu(\sigma)/\mu(\mathbb{1}) \qquad \text{with the convention that } 0/0 = 0 \tag{4.4}$$

and $\mu(\mathbb{1}) = \int \mathbb{1}\, d\mu$ is the total mass of $\mu$. Then the Feynman–Kac duality between the $(E, q, \mu)$-voter model and the coalescing system $\{B^x\}$ is given by the following equation:

$$\begin{aligned}
&\mathbb{E}_\xi\big[H(\xi_t; x, y)\big] \\
&= -\mu(\mathbb{1})\overline{\mu}(1)\overline{\mu}(0)\mathbb{E}\bigg[\int_0^t \mathbb{1}_{\{B_s^x = B_s^y\}} \exp\bigg(-\mu(\mathbb{1})\int_0^s \big|B_r^{\{x,y\}}\big|\, dr\bigg)\, ds\bigg] \\
&\quad + \mathbb{E}\bigg[H\big(\xi; B_t^x, B_t^y\big)\exp\bigg(-\mu(\mathbb{1})\int_0^t \big|B_s^{\{x,y\}}\big|\, ds\bigg)\bigg],
\end{aligned} \tag{4.5}$$

where $B^{\{x,y\}} = \{B^x, B^y\}$ and $|\{x, y\}|$ is the number of distinct points in $\{x, y\}$ [see (8.4) for the generator equation of (4.5)]. It can be shown that by (4.5), for all $\xi \in S^E$ and $x, y \in E$,

$$\begin{aligned} &\big|\mathbb{E}_\xi\big[\xi_t(x)\widehat{\xi}_t(y)\big] - \mathbb{E}\big[\xi\big(B_t^x\big)\widehat{\xi}\big(B_t^y\big)\big]\big| \\ &\qquad \le C_{(4.6)}\big(1 - e^{-\mu(\mathbb{1})t}\big)\mathbb{P}(M_{x,y} > t) \\ &\qquad\quad + C_{(4.6)}\mu(\mathbb{1})\int_0^t \mathbb{P}(M_{x,y} > s)\,ds, \end{aligned} \tag{4.6}$$

where $C_{(4.6)}$ is a universal constant and $M_{x,y}$ is the first time that $B^x$ and $B^y$ meet. An alternative proof of (4.6) by the pathwise duality between voter models and coalescing Markov chains can be found in [9], Proposition 3.1.

4.2. *Main theorem.* We first state four assumptions for the main theorem, Theorem 4.6, of Section 4. In essence, these assumptions can be summarized by saying that the selection strengths are of the order $1/N$ (Assumption 4.5) and the spatial structures, implied by the voting kernels, satisfy properties which can be informally described as follows. First, the spatial structures are not too singular in the sense that their stationary distributions are comparable to uniform distributions (Assumption 4.1) and admit good mixing of the voting Markov chains (Assumption 4.2 and Theorem 4.3). Also, they are locally symmetric in terms of the return probabilities with small numbers of steps, and small perturbations of these return probabilities are permissible (Assumption 4.4).

ASSUMPTION 4.1 (Uniformity in stationary distributions). The stationary distributions $\pi^{(n)}$'s of the voting kernels $q^{(n)}$ satisfy

$$0 < \liminf_{n\longrightarrow\infty} N_n \pi_{\min}^{(n)} \le \limsup_{n\longrightarrow\infty} N_n \pi_{\max}^{(n)} < \infty, \tag{4.7}$$

where $\pi_{\max}^{(n)} = \max_{x\in E_n} \pi^{(n)}(x)$ and $\pi_{\min}^{(n)} = \min_{x\in E_n} \pi^{(n)}(x)$.

In the case of random walks on graphs where stationary probabilities are proportional to degrees of vertices, Assumption 4.1 rules out graphs in which, informally speaking, there exist few vertices linking to large numbers of vertices. In particular, Assumption 4.1 rules out star graphs.

ASSUMPTION 4.2 (Weak convergence of voter models). We can choose a sequence of constants $\gamma_n$ growing to infinity such that the time-changed density processes $Y^{(n)}$ of 1's in the $(E_n, q^{(n)}, \mu_n)$-voter models defined in (4.1) satisfy:

$$\big(Y^{(n)}, \mathbb{P}_{\lambda_n}^{(n)}\big) \xrightarrow[n\longrightarrow\infty]{\text{(d)}} \big(Y, \mathbb{P}^{(\infty)}\big) \tag{4.8}$$

for some $\lambda_n \in \mathscr{P}(S^{E_n})$. Here, $\xrightarrow[n\longrightarrow\infty]{\text{(d)}}$ denotes convergence in distribution, and under $\mathbb{P}^{(\infty)}$, $Y$ is a Wright–Fisher diffusion obeying the following equation:

$$dY_t = \big[\mu(1)(1-Y_t) - \mu(0)Y_t\big]\,dt + \sqrt{Y_t(1-Y_t)}\,dW_t, \tag{4.9}$$

where $\mu = (\mu(1), \mu(0)) \in \mathbb{R}_+^2$ is a constant vector and $W$ is a standard Brownian motion.

In the case that $\sup_n \gamma_n/N_n = \infty$, we also require that (4.8) apply with respect to the same sequence $\{\gamma_n\}$, when mutation measures are zero and the initial laws are given by the Bernoulli product measures $\beta_u$ with constant densities $\beta_u\{\xi \in S^{E_n}; \xi(x) = 1\} \equiv u$ for all $u \in (0, 1)$.

Assumption 4.2 holds if we impose mild mixing conditions on $(E_n, q^{(n)}, \mu_n)$. This is the content of [8], Theorem 2.2, and a particular consequence of [9], Theorem 4.1, which are restated below as Theorem 4.3. Here and in what follows, $\mathbf{g}_n$ denotes the difference between 1 and the second largest eigenvalue of $q^{(n)}$, and

$$\mathbf{t}_{\text{mix}}^{(n)} = \inf\left\{t \ge 0; \max_{x\in E}\big\|e^{tq^{(n)}}(x,\cdot) - \pi^{(n)}\big\|_{\text{TV}} \le \frac{1}{2e}\right\}$$

stands for the mixing time of the rate-1 $(E_n, q^{(n)})$-chains, where $\|\lambda\|_{\text{TV}}$ is the total variation norm of a signed measure $\lambda$.

THEOREM 4.3 ([8, 9]). *Let $(E_n, q^{(n)}, \mu_n)$ with $N_n \nearrow \infty$, mutation measures $\mu_n$ defined on $S$, and $\lambda_n \in \mathscr{P}(S^{E_n})$ be given such that all of the following three properties are satisfied*:

(i) $\lim_{n\longrightarrow\infty} \sum_{x\in E_n} \pi^{(n)}(x)^2 = 0$,
(ii) $\lim_{n\longrightarrow\infty} \gamma_n\mu_n = \mu$,
(iii) *the sequence $\{\lambda_n(p_1(\xi) \in \cdot)\}$ converges weakly to $\widetilde{\lambda}_\infty$ as probability measures on* $[0, 1]$,

*and at least one of the following two conditions applies*:

(iv-1) $\lim_{n\longrightarrow\infty} \frac{\mathbf{t}_{\text{mix}}^{(n)}}{\gamma_n} = 0$,
(iv-2) $\lim_{n\longrightarrow\infty} \frac{\log(e\vee\gamma_n\pi_{\max}^{(n)})}{\mathbf{g}_n\gamma_n} = 0$,

*with respect to the constant time scales $\gamma_n$ defined by*

$$\gamma_n = \sum_{x,y\in E_n} \pi^{(n)}(x)\pi^{(n)}(y)\mathbb{E}^{(n)}[M_{x,y}]. \tag{4.10}$$

*Then* (4.8) *holds*.

For Theorem 4.3, (iv-1) and (iv-2) are its major conditions. Condition (iv-1) has the informal interpretation that on the time scale $\gamma_n$, any two independent

$(E_n, q^{(n)})$-Markov chains starting at $x \neq y$ reach stationarity very soon without meeting. A similar interpretation applies to (iv-2) if one recalls that inverse spectral gaps are interpreted as relaxation times to stationarity [2], Section 3.4. See [1, 24] for general results of such notions in the classical theory of Markov chains. In addition, notice that, in Theorem 4.3, condition (i) is implied by the fact that there is almost uniformity in stationarity in the sense of (4.7). Condition (ii) of Theorem 4.3 follows from (3.19) and (4.8) since, by solving elementary ordinary differential equations, they imply

$$\lim_{n\longrightarrow\infty}\big(e^{-\gamma_n\mu_n(\mathbb{1})t}\mathbb{E}^{(n)}_{\lambda_n}[Y_0] + \big(1-e^{-\gamma_n\mu_n(\mathbb{1})t}\big)\overline{\mu_n}(1)\big)$$
$$= \lim_{n\longrightarrow\infty}\mathbb{E}^{(n)}_{\lambda_n}\big[Y^{(n)}_t\big] = \mathbb{E}^{(\infty)}[Y_t] = \mathbb{E}^{(\infty)}[Y_0]e^{-\mu(\mathbb{1})t} + \big(1-e^{-\mu(\mathbb{1})t}\big)\overline{\mu}(1),$$

where $\mathbb{E}^{(\infty)}$ denotes expectation under $\mathbb{P}^{(\infty)}$.

The next assumption concerns spatial structures defined by voting kernels.

ASSUMPTION 4.4 (Spatial homogeneity). For fixed $L \in \mathbb{N}$, we can choose a sequence of constants $\gamma_n$ growing to infinity such that the following $L$th-order *spatial homogeneity condition* holds: for constants $R_0 = 1$, $R_1 = 0$, $R_2, \ldots, R_L \in \mathbb{R}_+$,

$$\lim_{n\longrightarrow\infty}\gamma_n\nu_n(\mathbb{1})\pi^{(n)}\big\{x \in E_n; q^{(n),\ell}(x,x) \neq R_\ell\big\} = 0 \qquad \forall 0 \leq \ell \leq L, \tag{4.11}$$

where $\nu_n$ is a measure on $E_n \times E_n$ defined by $\nu_n(x,y) = \pi^{(n)}(x)^2 q^{(n)}(x,y)$ as in (3.24) and $\nu_n(\mathbb{1})$ is the total mass of $\nu_n$.

We have $R_1 = 0$ in Assumption 4.4 since voting kernels are already assumed to have zero traces.

Assumption 4.4 is more general than the notion of walk-regular graphs [19], where $\ell$-step return probabilities of the random walks depend only on $\ell$. The simplest examples include random walks on discrete tori. Assumptions 4.4 is also a slight generalization of the local convergence of spatial structures in the sense of McKay [29] or Benjamini and Schramm [4]. For example, if $\gamma_n = \Theta(N_n)$, that is,

$$C^{-1}_{(4.12)}N_n \leq \gamma_n \leq C_{(4.12)}N_n \tag{4.12}$$

for some constant $C_{(4.12)} \in (1,\infty)$ independent of $n$, and $\pi^{(n)}$'s are comparable to uniform distributions in the sense of (4.7), then (4.11) is equivalent to

$$\lim_{n\longrightarrow\infty}\pi^{(n)}\big\{x \in E_n; q^{(n),\ell}(x,x) \neq R_\ell\big\} = 0 \qquad \forall 0 \leq \ell \leq L. \tag{4.13}$$

See Cox's theorem for discrete tori [11], Theorem 4, [8], Section 8, and Proposition 4.8 for examples where $\gamma_n$ defined by (4.10) satisfy (4.12). The convergence in (4.13) for any $\ell \in \mathbb{N}$ on large random regular graphs is due to McKay [29]. We will recall this result a bit more in Section 4.3.

Furthermore, Assumption 4.4 allows for the possibility that $\gamma_n \nu_n(\mathbb{1})$ tends to infinity and the $\pi^{(n)}$-probabilities in (4.11) decay fast to zero. For example, this could arise from small perturbations of random walk transition probabilities on two-dimensional tori that destroy the exact symmetry of walk regularity. On the other hand, $\gamma_n \nu_n(\mathbb{1})$ tends to infinity on these tori by Cox's theorem [11], Theorem 4.

The last assumption specifies the choice of selections strengths.

ASSUMPTION 4.5 (Weak selection). We choose a sequence of selection strengths $w_n \in [0, \overline{w}]$ satisfying

$$w_\infty = \lim_{n \longrightarrow \infty} \frac{w_n}{\nu_n(\mathbb{1})} \in [0, \infty), \tag{4.14}$$

where $\overline{w}$ is defined by (2.2).

Theorem 4.6 below is the main result of Section 4 for the vectors of semimartingales $Z^{(n)}$ defined in (4.1). First, Theorem 4.6(1°) is a tightness result from which, under general payoff matrices, the presence of Wright–Fisher noise coefficients in the limiting game density processes follows immediately. In contrast, Theorem 4.6(2°) and (3°) have stronger quantitative flavors as they show explicitly how $(Y^{(n)}, M^{(n)}, D^{(n)})$ are correlated in the limit by (4.15) and (4.17).

We equip spaces of Polish-space-valued càdlàg functions indexed by $\mathbb{R}_+$ with the usual Skorokhod topology, namely the $J_1$-topology (cf. [23], Chapter VI).

THEOREM 4.6 (Main theorem). *Suppose that*:

(i) *Assumption 4.1 holds*,
(ii) *Assumption 4.2 holds*, *and*
(iii) *a sequence of selection strengths $w_n$ satisfying Assumption 4.5 is given*.

*Then we have the following results*:

(1°) *The sequence of laws of $Z^{(n)} = (Y^{(n)}, M^{(n)}, D^{(n)})$ under $\mathbb{P}^{(n)}_{\lambda_n}$ is C-tight. Any subsequential limit, say along $(Y^{(n_k)}, M^{(n_k)}, D^{(n_k)})$ under $\mathbb{P}^{(n_k)}_{\lambda_{n_k}}$, is the law of a vector of continuous semimartingales $(Y, M, D)$ under $\mathbb{P}^{(\infty)}$ such that the last two components define a vector martingale with respect to the filtration generated by $(Y, M, D)$. In addition, the sequence of laws of $(Y^{(n_k)}, M^{(n_k)})$ under $\mathbb{P}^{(n_k), w_{n_k}}_{\lambda_{n_k}}$ converges to the law of $(Y, M)$ under the law $D \cdot \mathbb{P}^{(\infty)}$ obtained from $\mathbb{P}^{(\infty)}$ by the Radon–Nikodým derivative process $D$.*

(2°) *If, moreover, $q^{(n)}$ are symmetric kernels, Assumption 4.4 with $L = 2$ with respect to the same sequence $\{\gamma_n\}$ chosen in* (ii) *applies, and the payoff matrix $\Pi$ is given by a prisoner's dilemma type matrix in* (1.12), *then any subsequential limit*

*$(Y, M, D)$ under $\mathbb{P}^{(\infty)}$ satisfies the following covariation equations*:

$$\begin{aligned}\langle Y, D\rangle_t &= \langle M, D\rangle_t \\ &= w_\infty K_1(b,c)\int_0^t D_s Y_1(1-Y_s)\,ds \qquad \text{under } \mathbb{P}^{(\infty)},\end{aligned} \tag{4.15}$$

*where $w_\infty$ is defined by* (4.14) *and, with respect to $R_\ell$ chosen in* (4.11), *$K_1(b,c)$ is defined by*

$$K_1(b,c) = \frac{bR_2 - c}{2}. \tag{4.16}$$

(3°) *If the assumptions of* (2°) *apply and the stronger Assumption* 4.4 *with $L = 3$ is valid as well, then the sequence of laws of $(Y^{(n)}, M^{(n)}, D^{(n)})$ under $\mathbb{P}^{(n)}_{\lambda_n}$ converges weakly toward the law of a vector of semimartingales $(Y, M, D)$ under $\mathbb{P}^{(\infty)}$. The triplet $(Y, M, D)$ under $\mathbb{P}^{(\infty)}$ can be characterized as a solution to the following system of stochastic differential equations*:

$$\begin{cases} dY_t = \big[\mu(1)(1-Y_t) - \mu(0)Y_t\big]\,dt + \sqrt{Y_t(1-Y_t)}\,dW_t^1, \\ dM_t = \sqrt{Y_t(1-Y_t)}\,dW_t^1, \\ dD_t = w_\infty D_t\sqrt{Y_t(1-Y_t)} \\ \qquad\quad \times \big[K_1(b,c)\,dW_t^1 + \sqrt{K_2(b,c) - K_1(b,c)^2}\,dW_t^2\big]. \end{cases} \tag{4.17}$$

*Here, $(W^1, W^2)$ is a two-dimensional standard Brownian motion, $K_1(b,c)$ is given by* (4.16), *and $K_2(b,c)$ is defined by*

$$K_2(b,c) = \frac{b^2(R_3 + R_2) - 2bcR_2 + c^2}{2}. \tag{4.18}$$

The proof of Theorem 4.6 is given in Section 4.4 and Section 4.5. See Proposition 4.14 for Theorem 4.6(1°) and Proposition 4.18 for Theorem 4.6(2°) and 3(°). The reader can find more detailed results in these two propositions.

The following theorem is a straightforward application of Theorem 4.6(1°) and (2°), Girsanov's theorem [38], Theorem VIII.1.7, and the Yamada–Watanabe theorem for pathwise uniqueness in stochastic differential equations [38], Theorem IX.3.5.

THEOREM 4.7 (Diffusions for evolutionary games with death–birth updating). *Let the assumptions of Theorem* 4.6(2°) *be in force, and recall the constants $w_\infty$ and $K_1(b,c)$ defined by* (4.14) *and* (4.16), *respectively. Then we have the following*:

(1°) *The sequence of laws of $(Y^{(n)}, \mathbb{P}^{(n),w_n}_{\lambda_n})$ converges weakly to the law of a Wright–Fisher diffusion $Y$ with initial law $\mathscr{L}(Y_0) = \widetilde{\lambda}_\infty$ under $\mathbb{P}^{(\infty),w_\infty}_{\widetilde{\lambda}_\infty}$, where $\mathbb{P}^{(\infty),w_\infty}_{\widetilde{\lambda}_\infty}$ can be defined as the law $D\cdot\mathbb{P}^{(\infty)}$ obtained from $\mathbb{P}^{(\infty)}$ by the Radon–Nikodým derivative process $D$.*

(2°) *The Wright–Fisher diffusion $Y$ in* (1°) *obeys the following equation*:

$$dY_t = \big[w_\infty K_1(b,c)Y_t(1-Y_t) + \mu(1)(1-Y_t) - \mu(0)Y_t\big]\,dt + \sqrt{Y_t(1-Y_t)}\,dW_t \tag{4.19}$$

*with respect to a standard Brownian motion $W$.*

In the rest of Section 4, we will first prove in Section 4.3 the prediction from [34], SI, for evolutionary games on large random regular graphs. The remaining subsections are then devoted to the proof of Theorem 4.6.

4.3. *Example*: *Evolutionary games on large random regular graphs*. We fix $k \geq 3$ and consider a sequence of random $k$-regular graphs $G_n$ on $N_n$ vertices with $N_n \nearrow \infty$. (For definiteness, we assume that $G_n$'s are given by the uniform models.) One basic property of $\{G_n\}$ states that the second eigenvalues of the adjacency matrices of $G_n$ are bounded away from the largest ones, namely $k$, in the limit of infinite volume (see [5, 18]). In particular, $G_n$'s are connected for all large $n$.

The following proposition can be used to verify Assumption 4.4 with $L = 3$, which is one of the conditions for Theorem 4.6(3°).

PROPOSITION 4.8. *Let $\mathbf{g}(G)$ denote the spectral gap of a random walk on a finite connected unweighted graph $G$. Recall that $M_{x,y}$ denotes the first meeting time of two independent rate-*1 *random walks on $G$ starting from $x$ and $y$. Then*

$$\max_{x,y\in G} \mathbb{E}[M_{x,y}] \leq \max_{y\in G} \frac{2\sum_{x:x\neq y} \deg(x)}{\mathbf{g}(G)\deg(y)}. \tag{4.20}$$

PROOF. Let $H_{x,y}$ denote the first hitting time of $y$ by a rate-1 random walk on $G$ starting from $x$. By [2], Proposition 14.5, Lemma 3.15, Lemma 3.17, we have

$$\max_{x,y\in G} \mathbb{E}[M_{x,y}] \leq \max_{x,y\in G} \mathbb{E}[H_{x,y}] \leq \max_{y\in G} 2\sum_{x\in G} \pi(x)\mathbb{E}[H_{x,y}] \leq \max_{y\in G} \frac{2(1-\pi(y))}{\mathbf{g}(G)\pi(y)},$$

which is enough for the required inequality in (4.20) since $\pi(y)$ is equal to $\deg(y)/\sum_x \deg(x)$ for all $y$. □

The following theorem obtains diffusion approximations of the game density processes on large random regular graphs when payoff matrices are given by (1.12).

THEOREM 4.9. *For fixed $k \geq 3$, consider a sequence of random $k$-regular graphs $G_n$ with $G_n$ carrying $N_n$ vertices and $N_n \nearrow \infty$. Set $\gamma_n$ according to* (4.10) *and then choose $\{w_n\}$ according to Assumption 4.5 and mutation measures $\mu_n$ on $S$ which satisfy Theorem* 4.3(ii). *Finally, assume that Theorem* 4.3(iii) *holds*

*for some* $\lambda_n \in \mathscr{P}(S^{E_n})$. *Then the conclusion of Theorem* 4.6(3°) *holds, and the constants* $K_1(b,c)$ *and* $K_2(b,c)$ *are now given explicitly as follows*:

$$K_1(b,c) = \frac{bk^{-1} - c}{2} \quad \text{and} \quad K_2(b,c) = \frac{b^2k^{-1} - 2bck^{-1} + c^2}{2}. \tag{4.21}$$

*In particular, the limiting Wright–Fisher diffusion* $Y$ *under* $\mathbb{P}^{(\infty),w_\infty}_{\widetilde{\lambda}_\infty}$ *in Theorem* 4.7 *simplifies to the following stochastic differential equation*:

$$\begin{aligned} dY_t = {} & \left(\frac{w_\infty(b-ck)}{2k} Y_t(1-Y_t) + \mu(1)(1-Y_t) - \mu(0)Y_t\right) dt \\ & + \sqrt{Y_t(1-Y_t)}\, dW_t, \end{aligned} \tag{4.22}$$

*where* $W$ *is a standard Brownian motion*.

REMARK 4.10. (1) To convert the diffusion process defined by the coefficients in (1.8) to the diffusion process defined by [34], equation (18) in SI, the reader may notice that, on a $k$-regular graph with $N$ vertices, the generator of the evolutionary game considered in [34], SI, is given by $N^{-1}\mathsf{L}^{w,0}$, where $\mathsf{L}^{w,0}$ is defined by (2.4); compare [34], equation (11) and (12) in SI, to (2.4), (2.6) and (3.11). Hence, speeding up its time scale by the constant factor $N$ recovers the evolutionary game considered in this paper. Also, we have an additional multiplicative factor of $k^{-1}$ in the drift coefficient in (1.7) since total payoffs of individuals are defined by the weighted averages in (1.1), where $q(x,y)$ are equal to $k^{-1}$ for all pairs of vertices $x, y$ adjacent to each other.

(2) Assume that $\mu_n = 0$ for all $n$. Given a payoff matrix $\Pi$ taking the form (1.12), the constants $\alpha, \beta$ defined by (1.8) simplify to $\alpha = 0$ and $\beta = k(b - kc)$. If we speed up time by applying the constant time change $[N(k-1)]/[2(k-2)]$ to (1.7), the diffusion process predicted in [34], SI, has a Wright–Fisher noise coefficient as in (4.22). In addition, by setting selection strength $w$ in (1.7) to be $w_\infty/N$, we recover the drift term in (4.22).

QUESTION 4.11. Is the prediction in [34], SI, precise to the degree that

$$\gamma_n = N_n^{-2} \sum_{x,y \in E_n} \mathbb{E}^{(n)}[M_{x,y}] \sim \frac{N_n(k-1)}{2(k-2)} \qquad \text{as } n \longrightarrow \infty?$$

REMARK 4.12. After the submission of the manuscript, Question 4.11 is resolved in the positive by an extension of the method of Green functions due to Cox and Spitzer in [11]. See [7] for this result.

PROOF OF THEOREM 4.9. Note that $\gamma_n = \Theta(N_n)$ since, for example, [8], (3.21), shows

$$\gamma_n \geq N_n \left(\frac{N_n - 1}{2N_n}\right)^2 \tag{4.23}$$

and Proposition 4.8 applies by the fact that the spectral gaps $\mathbf{g}(G_n)$ are bounded away from zero. Here, we have used the aforementioned property of random regular graphs proven in [5, 18].

To obtain the conclusion of (3°) in Theorem 4.6, it is enough to verify Assumption 4.2 and Assumption 4.4 with $L = 3$ since $q^{(n)}(x, y) \equiv 1/k$ for $x \sim y$, $\pi^{(n)}(x) \equiv N_n^{-1}$, and we have chosen $\{w_n\}$ according to Assumption 4.5. For Assumption 4.2, Theorem 4.3 holds with the present choice of $\gamma_n$. Indeed, (i) of Theorem 4.3 obviously holds and its (ii)–(iii) are valid by the choice of $\gamma_n$, $\mu_n$, and $\lambda_n \in \mathscr{P}(S^{E_n})$. We also know that $\mathbf{g}(G_n)$ are bounded away from zero, so that condition (iv-2) of Theorem 4.3 holds. To satisfy Assumption 4.4 with $L = 3$, notice that $\nu_n(\mathbb{1}) = 1/N_n$ and we have seen at the beginning of this proof that $\gamma_n = \Theta(N_n)$. Then it is enough to check (4.13). But this condition follows immediately from the well-known locally tree-like property of random regular graphs (cf. [29]), which yields the following exact values of $R_1, R_2, R_3$:

$$R_1 = R_3 = 0 \quad \text{and} \quad R_2 = k^{-1}. \tag{4.24}$$

Hence, the conclusion of (3°) in Theorem 4.6 holds. The constants $K_1(b, c)$ and $K_2(b, c)$ are now given by (4.21), and the diffusion process in (4.19) simplifies to the diffusion process in (4.22). □

In Section 5, we will continue this discussion of [34], SI, in the context where mutations are absent and prove diffusion approximations of the game absorbing probabilities. See Corollary 5.3 for the precise statement.

4.4. *Proof of the main theorem*: *Tightness*. In this section, we prove tightness of the sequence of laws of the vectors of semimartingales $Z^{(n)}$ defined in (4.1) and related tightness properties. Before that, we handle predictable covariations between $M^{(n)}$ and $D^{(n)}$ and their own predictable quadratic variations by Lemma 3.2 and Lemma 3.3.

To simplify notation, we introduce two discrete-time $(E, q)$-Markov chains $(X_\ell)$ and $(Y_\ell)$, which satisfy the following three properties: (1) $X_0 = Y_0$; (2) they are independent of the system of $(E, q)$-coalescing chains $\{B^x; x \in E\}$ (defined at the beginning of Section 4); (3) they are independent if conditioned on $X_0$. Notice that we can write the two-point density functions $W_\ell$ defined by (3.6) as $W_\ell(\xi) = \mathbb{E}_\pi[\xi(X_0)\widehat{\xi}(X_\ell)]$, where $(X_\ell)$ starts from stationarity under $\mathbb{E}_\pi$.

PROPOSITION 4.13. *Suppose that Assumption 4.1 and Assumption 4.2 are in force*:

(1°) *For all* $a \in (0, \infty)$, $\ell \in \mathbb{N}$ *and* $t \in (0, \infty)$, *it holds that*

$$\sup_{n\in\mathbb{N}} \sup_{\lambda\in\mathscr{P}(S^{E_n})} \mathbb{E}_\lambda^{(n)}\left[\exp\left\{a\gamma_n\nu_n(\mathbb{1})\int_0^t W_\ell(\xi_{\gamma_n s})\,ds\right\}\right] < \infty, \tag{4.25}$$

$$\lim_{\theta\searrow 0+} \sup_{n\in\mathbb{N}} \sup_{\lambda\in\mathscr{P}(S^{E_n})} \mathbb{E}_\lambda^{(n)}\left[\exp\left\{a\gamma_n\nu_n(\mathbb{1})\int_0^\theta W_\ell(\xi_{\gamma_n s})\,ds\right\}\right] = 1. \tag{4.26}$$

(2°) *If* $\mu_n = 0$ *for all* $n$, *then for every* $\ell \in \mathbb{N}$ *we can find* $a \in (0, \infty)$ *small enough such that the inequality in* (4.25) *with* $t = \infty$ *holds*.

PROOF. (1°) We proceed with the following steps to prove (4.25) and (4.26), which start with three claims.

*Step* 1. First, we claim that, for all $\ell \geq 1$,

$$\sup_{n\in\mathbb{N}} \nu_n(\mathbb{1})\mathbb{E}^{(n)}[M_{X_0,X_\ell}] \leq \ell\Big(\sup_{n\in\mathbb{N}} \nu_n(\mathbb{1})\mathbb{E}^{(n)}[M_{X_0,X_1}]\Big) < \infty. \tag{4.27}$$

To see (4.27) for $\ell = 1$, recall that $\pi^{(n)}$'s are comparable to uniform distributions by Assumption 4.1 and we have

$$\sum_{x,y\in E_n} \pi^{(n)}(x)^2 q^{(n)}(x,y)\mathbb{E}^{(n)}_\pi[M_{x,y}] = \frac{1 - \sum_{x\in E_n} \pi^{(n)}(x)^2}{2} \tag{4.28}$$

(cf. [8], (3.17)). These two facts imply that

$$\begin{aligned}&\sup_{n\in\mathbb{N}} \nu_n(\mathbb{1})\mathbb{E}^{(n)}_\pi[M_{X_0,X_1}]\\ &\quad\leq \sup_{n\in\mathbb{N}}\Big(\frac{\pi^{(n)}_{\max}}{\pi^{(n)}_{\min}}\Big)\Big(\sum_{x,y\in E_n} \pi^{(n)}(x)^2 q^{(n)}(x,y)\mathbb{E}^{(n)}_\pi[M_{x,y}]\Big) < \infty\end{aligned} \tag{4.29}$$

and so the inequality in (4.27) with $\ell = 1$ follows.

To obtain (4.27) for $\ell \geq 2$, first notice that given $F : E \times E \longrightarrow \mathbb{R}$, we have

$$\begin{aligned}&\forall x \neq y,\\ &\quad\mathbb{E}^{(n)}\big[F\big(B^x_t, B^y_t\big)\big]\\ &\qquad= F(x,y)e^{-2t} + \int_0^t e^{-2(t-s)} \sum_{z\in E_n} \big(q^{(n)}(x,z)\mathbb{E}^{(n)}\big[F\big(B^z_s, B^y_s\big)\big]\\ &\qquad\quad + q^{(n)}(y,z)\mathbb{E}^{(n)}\big[F\big(B^x_s, B^z_s\big)\big]\big)\,ds.\end{aligned} \tag{4.30}$$

To see (4.30), we write $J$ for the first jump time of the bivariate chain $(B^x, B^y)$. Then the assumption that $x \neq y$ implies

$$\mathbb{E}\big[F\big(B^x_t, B^y_t\big); t < J\big] = F(x,y)e^{-2t},$$

since $J$ is an exponential variable with mean $1/2$. The integral term in (4.30) follows from the strong Markov property of $(B^x, B^y)$ at $J$, the independence of $J$ and $(B^x_J, B^y_J)$, and the same distributional property of $J$ used for the above equality:

$$\begin{aligned}\mathbb{E}\big[F\big(B^x_t, B^y_t\big); t \geq J\big] = \int_0^t 2e^{-2s} \sum_{z\in E_n} \Big(&\frac{1}{2} q^{(n)}(x,z)\mathbb{E}^{(n)}\big[F\big(B^z_{t-s}, B^y_{t-s}\big)\big]\\ &+ \frac{1}{2} q^{(n)}(y,z)\mathbb{E}^{(n)}\big[F\big(B^x_{t-s}, B^z_{t-s}\big)\big]\Big)\,ds.\end{aligned}$$

Now we take $F(u,v)=\mathbb{1}_{\{u\neq v\}}$ so that $\mathbb{E}^{(n)}[F(B_t^x,B_t^y)]=\mathbb{P}^{(n)}(M_{x,y}>t)$ by the coalescing property of $\{B^x\}$ and the definition that $M_{x,y}$ is the first meeting time of $B^x$ and $B^y$. Then integrating both sides of (4.30) from time 0 to time $T$ and randomizing $(x,y)$ according to the sub-probability $\mathbb{P}((X_0,X_{\ell-1})\in\cdot, X_0\neq X_{\ell-1})$ for $\ell\geq 2$ gives the following:

$$
\begin{aligned}
&\int_0^T \mathbb{P}_\pi^{(n)}(M_{X_0,X_{\ell-1}}>t)\,dt\\
&\quad=\left(\frac{1-e^{-2T}}{2}\right)\mathbb{P}_\pi^{(n)}(X_0\neq X_{\ell-1})\\
&\quad\quad+\int_0^T\left(\frac{1-e^{-2(T-s)}}{2}\right)\mathbb{P}_\pi^{(n)}(M_{Y_1,X_{\ell-1}}>s,X_0\neq X_{\ell-1})\,ds\\
&\quad\quad+\int_0^T\left(\frac{1-e^{-2(T-s)}}{2}\right)\mathbb{P}_\pi^{(n)}(M_{X_0,X_\ell}>s,X_0\neq X_{\ell-1})\,ds\\
&\quad=\left(\frac{1-e^{-2T}}{2}\right)\mathbb{P}_\pi^{(n)}(X_0\neq X_{\ell-1})\\
&\quad\quad+\int_0^T\bigl(1-e^{-2(T-s)}\bigr)\mathbb{P}_\pi^{(n)}(M_{X_0,X_\ell}>s)\,ds\\
&\quad\quad-\int_0^T\bigl(1-e^{-2(T-s)}\bigr)\mathbb{P}_\pi^{(n)}(M_{X_0,X_\ell}>s,X_0=X_{\ell-1})\,ds,
\end{aligned}
\tag{4.31}
\tag{4.32}
$$

where the last equality follows from the reversibility of $q^{(n)}$. To see (4.31), note that the integral on its left-hand side follows since

$$
\begin{aligned}
&\int_0^T \mathbb{P}_\pi^{(n)}\bigl(B_t^{X_0}\neq B_t^{X_{\ell-1}},X_0\neq X_{\ell-1}\bigr)\,dt\\
&\quad=\int_0^T\mathbb{P}_\pi^{(n)}(M_{X_0,X_{\ell-1}}>t,X_0\neq X_{\ell-1})\,dt=\int_0^T\mathbb{P}_\pi^{(n)}(M_{X_0,X_{\ell-1}}>t)\,dt,
\end{aligned}
$$

where the last equality uses $M_{x,x}\equiv 0$. The second term on the right-hand side of (4.31) follows since

$$
\begin{aligned}
&\int_0^T\int_0^t e^{-2(t-s)}\\
&\quad\times\mathbb{E}_\pi^{(n)}\Biggl[\sum_{z\in E_n}q^{(n)}(X_0,z)\mathbb{P}^{(n)}\bigl(B_s^z\neq B_s^y\bigr)|_{y=X_{\ell-1}}\mathbb{1}_{\{X_0\neq X_{\ell-1}\}}\Biggr]\,ds\,dt\\
&\quad=\int_0^T\left(\frac{1-e^{-2(T-s)}}{2}\right)\\
&\quad\quad\times\mathbb{E}_\pi^{(n)}\Biggl[\sum_{z\in E_n}q^{(n)}(X_0,z)\mathbb{P}^{(n)}\bigl(B_s^z\neq B_s^y\bigr)|_{y=X_{\ell-1}}\mathbb{1}_{\{X_0\neq X_{\ell-1}\}}\Biggr]\,ds
\end{aligned}
$$

$$= \int_0^T \left( \frac{1 - e^{-2(T-s)}}{2} \right) \mathbb{E}_\pi^{(n)} \big[ \mathbb{P}^{(n)}(M_{z,y} > s)|_{z=Y_1, y=X_{\ell-1}} \mathbb{1}_{\{X_0 \neq X_{\ell-1}\}} \big] \, ds$$

$$= \int_0^T \left( \frac{1 - e^{-2(T-s)}}{2} \right) \mathbb{P}_\pi^{(n)}(M_{Y_1, X_{\ell-1}} > s, X_0 \neq X_{\ell-1}) \, ds.$$

The third term on the right-hand side of (4.31) follows from almost the same argument as above except that, rather than backtracking by the auxiliary discrete-time $Y$-chain, we now extend the discrete-time $X$-chain by one step:

$$\int_0^T \int_0^t e^{-2(t-s)}$$

$$\times \mathbb{E}_\pi^{(n)} \Big[ \sum_{z \in E_n} q^{(n)}(X_{\ell-1}, z) \mathbb{P}^{(n)}\big(B_s^x \neq B_s^z\big)|_{x=X_0} \mathbb{1}_{\{X_0 \neq X_{\ell-1}\}} \Big] \, ds \, dt$$

$$= \int_0^T \left( \frac{1 - e^{-2(T-s)}}{2} \right)$$

$$\times \mathbb{E}_\pi^{(n)} \Big[ \sum_{z \in E_n} q^{(n)}(X_{\ell-1}, z) \mathbb{P}^{(n)}\big(B_s^x \neq B_s^z\big)|_{x=X_0} \mathbb{1}_{\{X_0 \neq X_{\ell-1}\}} \Big] \, ds$$

$$= \int_0^T \left( \frac{1 - e^{-2(T-s)}}{2} \right) \mathbb{E}_\pi^{(n)} \big[ \mathbb{P}^{(n)}(M_{x,z} > s)|_{x=X_0, y=X_\ell} \mathbb{1}_{\{X_0 \neq X_{\ell-1}\}} \big] \, ds$$

$$= \int_0^T \left( \frac{1 - e^{-2(T-s)}}{2} \right) \mathbb{P}_\pi^{(n)}(M_{X_0, X_\ell} > s, X_0 \neq X_{\ell-1}) \, ds.$$

We pass $T$ to infinity for both sides of (4.32) and then use the integrability of each meeting time $M_{x,y}$, dominated convergence and the stationarity of the chain $(X_\ell)$ under $\mathbb{P}_\pi^{(n)}$. These lead to the following "triangle inequality":

$$\begin{aligned} \mathbb{E}_\pi^{(n)}[M_{X_0, X_\ell}] &\leq \mathbb{E}_\pi^{(n)}[M_{X_0, X_{\ell-1}}] + \mathbb{E}_\pi^{(n)}[M_{X_{\ell-1}, X_\ell}] \\ &= \mathbb{E}_\pi^{(n)}[M_{X_0, X_{\ell-1}}] + \mathbb{E}_\pi^{(n)}[M_{X_0, X_1}], \end{aligned}$$

which is enough for (4.27) for all $\ell \geq 2$ by (4.29) and iteration.

*Step* 2. The second claim is the following uniform continuity:

$$\forall \ell \geq 1 \; \forall \varepsilon \in (0,1) \; \exists \delta \in (0,1),$$

$$\sup_{n \in \mathbb{N}} \gamma_n \nu_n(\mathbb{1}) \int_0^\delta \mathbb{P}^{(n)}(M_{X_0, X_\ell} > \gamma_n s) \, ds \leq \varepsilon. \tag{4.33}$$

To prove (4.33), it suffices to consider the case $\sup_n \gamma_n / N_n = \infty$ thanks to the fact that $\nu_n(\mathbb{1}) = \Theta(N_n^{-1})$ by Assumption 4.1 and the definition (3.24) of $\nu_n$.

We use the part in Assumption 4.2 stating that (4.8) holds for all initial laws as Bernoulli product measures with constant densities in the absence of mutation.

Then it follows from [8], Theorem 4.1, that

$$\begin{aligned}\limsup_{n\to\infty} \gamma_n \nu_n(\mathbb{1}) \int_0^t \mathbb{P}^{(n)}(M_{X_0,X_1} > \gamma_n s)\,ds \\ \le C_{(4.34)}\bigl(1-e^{-t}\bigr) \qquad \forall t \ge 0,\end{aligned} \tag{4.34}$$

where $C_{(4.34)}$ depends only on $\limsup \pi^{(n)}_{\max}/\pi^{(n)}_{\min}$ (this limit superior is finite by Assumption 4.1). The uniform continuity in (4.33) for $\ell = 1$ then follows from (4.34). The proof for general $\ell \ge 2$ can be obtained by iterating (4.32) and using (4.34) since

$$\gamma_n \nu_n(\mathbb{1}) \int_0^T e^{-2\gamma_n(T-s)}\,ds \le \frac{\nu_n(\mathbb{1})}{2} \qquad \text{for every } T \ge 0$$

and $\nu_n(\mathbb{1}) = \Theta(N_n^{-1})$. We have proved (4.33).

*Step* 3. The third claim is the following uniform continuity similar to the one in (4.33):

$$\begin{aligned}&\forall \ell \ge 1\ \forall \varepsilon \in (0,1)\ \exists \delta \in (0,1),\\ &\sup_{n\in\mathbb{N}} \sup_{\lambda\in\mathscr{P}(S^{E_n})} \mathbb{E}^{(n)}_\lambda\biggl[\gamma_n \nu_n(\mathbb{1}) \int_0^\delta W_\ell(\xi_{\gamma_n s})\,ds\biggr] \le \varepsilon.\end{aligned} \tag{4.35}$$

To prove (4.35), first we use the following consequence of (4.6) by integrating that inequality from time 0 to time $t$:

$$\begin{aligned}&\biggl|\int_0^t \mathbb{E}^{(n)}_\xi\bigl[\xi_s(x)\widehat{\xi}_s(y)\bigr]\,ds - \int_0^t \mathbb{E}^{(n)}\bigl[\xi\bigl(B^x_s\bigr)\widehat{\xi}\bigl(B^y_s\bigr)\bigr]\,ds\biggr| \\ &\quad\le C_{(4.6)} \int_0^t \bigl(1-e^{-\mu_n(\mathbb{1})s}\bigr)\mathbb{P}^{(n)}(M_{x,y} > s)\,ds \\ &\qquad + C_{(4.6)}\mu_n(\mathbb{1})\mathbb{E}^{(n)}[M_{x,y}]\cdot t.\end{aligned} \tag{4.36}$$

Then by the definition of $W_\ell$ in (3.6) and the above inequality (4.36), we see that, for all $\lambda \in \mathscr{P}(S^{E_n})$,

$$\begin{aligned}&\mathbb{E}^{(n)}_\lambda\biggl[\gamma_n \nu_n(\mathbb{1}) \int_0^t W_\ell(\xi_{\gamma_n s})\,ds\biggr] \\ &\quad\le (1 + C_{(4.6)})\gamma_n \nu_n(\mathbb{1}) \int_0^t \mathbb{P}^{(n)}(M_{X_0,X_\ell} > \gamma_n s)\,ds \\ &\qquad + C_{(4.6)}\gamma_n \mu_n(\mathbb{1})\cdot \nu_n(\mathbb{1})\mathbb{E}^{(n)}[M_{X_0,X_\ell}]t.\end{aligned} \tag{4.37}$$

By (4.37) and the inequality $\sup_n \gamma_n \mu_n(\mathbb{1}) < \infty$ (implied by Assumption 4.2 according to the explanation below Theorem 4.3), the first two claims above in (4.27) and (4.33) are enough for (4.35).

*Step* 4. In this step, we prove (1°) of the present proposition.

Observe that the Markov property of voter models implies both of the following:

$$\forall m \ge 1, \qquad \sup_{\lambda\in\mathscr{P}(S^{E_n})} \mathbb{E}_\lambda^{(n)}\left[\exp\left\{a\gamma_n \nu_n(\mathbb{1}) \int_0^t W_\ell(\xi_{\gamma_n s})\,ds\right\}\right]$$

$$\le \left(\sup_{\lambda\in\mathscr{P}(S^{E_n})} \mathbb{E}_\lambda^{(n)}\left[\exp\left\{a\gamma_n \nu_n(\mathbb{1}) \int_0^{t/m} W_\ell(\xi_{\gamma_n s})\,ds\right\}\right]\right)^m \tag{4.38}$$

and

$$\forall m \ge 1, \qquad \sup_{\lambda\in\mathscr{P}(S^{E_n})} \mathbb{E}_\lambda^{(n)}\left[\left(\gamma_n \nu_n(\mathbb{1}) \int_0^t W_\ell(\xi_{\gamma_n s})\,ds\right)^m\right]$$

$$= m! \cdot \sup_{\lambda\in\mathscr{P}(S^{E_n})} \mathbb{E}_\lambda^{(n)}\left[\left(\gamma_n \nu_n(\mathbb{1})\right)^m \right. \tag{4.39}$$

$$\left. \times \int_0^t ds_1 \int_{s_1}^t ds_2 \cdots \int_{s_{m-1}}^t ds_m \prod_{i=1}^m W_\ell(\xi_{\gamma_n s_i})\right]$$

$$\le m! \cdot \left(\sup_{\lambda\in\mathscr{P}(S^{E_n})} \mathbb{E}_\lambda^{(n)}\left[\gamma_n \nu_n(\mathbb{1}) \int_0^t W_\ell(\xi_{\gamma_n s})\,ds\right]\right)^m. \tag{4.40}$$

For the proof of (4.25) with fixed $t \in (0,\infty)$ and $\ell \ge 1$, we choose $\delta$ according to the uniform continuity in (4.35) with $\varepsilon = 1/(2a)$ and then $m$ large such that $t/m \le \delta$. Applying (4.38) and the Taylor expansion of the exponential function to the first inequality below and (4.40) to the second, we have

$$\sup_{n\in\mathbb{N}} \sup_{\lambda\in\mathscr{P}(S^{E_n})} \mathbb{E}_\lambda^{(n)}\left[\exp\left\{a\gamma_n \nu_n(\mathbb{1}) \int_0^t W_\ell(\xi_{\gamma_n s})\,ds\right\}\right]$$

$$\le \sup_{n\in\mathbb{N}}\left(\sum_{m'=0}^\infty \frac{a^{m'}}{(m')!} \sup_{\lambda\in\mathscr{P}(S^{E_n})} \mathbb{E}_\lambda^{(n)}\left[\left(\gamma_n \nu_n(\mathbb{1}) \int_0^{t/m} W_\ell(\xi_{\gamma_n s})\,ds\right)^{m'}\right]\right)^m$$

$$\le \left(\sup_{n\in\mathbb{N}} \sum_{m'=0}^\infty a^{m'}\left(\sup_{\lambda\in\mathscr{P}(S^{E_n})} \mathbb{E}_\lambda^{(n)}\left[\gamma_n \nu_n(\mathbb{1}) \int_0^{t/m} W_\ell(\xi_{\gamma_n s})\,ds\right]\right)^{m'}\right)^m$$

$$\le \left(\sum_{m'=0}^\infty \frac{1}{2^{m'}}\right)^m < \infty,$$

where the next to the last inequality follows from (4.35) with the particular choice of $\varepsilon$ and $m$ mentioned above.

The proof of (4.26) follows similarly. We argue as above with $t$ replaced by $\theta$ and $m$ set to be 1 and then apply (4.35) and dominated convergence. We have proved (1°).

(2°) The proof of (2°) follows almost the same line as the proof of (4.26) except that we do not need to handle the second term on the right-hand side of (4.37),

which is due to mutation. In more detail, now we consider

$$\begin{aligned}
&\sup_{n\in\mathbb{N}}\sup_{\lambda\in\mathscr{P}(S^{E_n})}\mathbb{E}_{\lambda}^{(n)}\left[\exp\left\{a\gamma_n\nu_n(\mathbb{1})\int_0^{\infty}W_{\ell}(\xi_{\gamma_n s})\,ds\right\}\right]\\
&\quad\le\sup_{n\in\mathbb{N}}\sum_{m'=0}^{\infty}a^{m'}\left(\sup_{\lambda\in\mathscr{P}(S^{E_n})}\mathbb{E}_{\lambda}^{(n)}\left[\gamma_n\nu_n(\mathbb{1})\int_0^{\infty}W_{\ell}(\xi_{\gamma_n s})\,ds\right]\right)^{m'}\\
&\quad\le\sum_{m'=0}^{\infty}a^{m'}\left((1+C_{(4.6)})\ell\sup_{n\in\mathbb{N}}\nu_n(\mathbb{1})\mathbb{E}^{(n)}[M_{X_0,X_1}]\right)^{m'},
\end{aligned}$$

where the last inequality follows from (4.27) and (4.37), with $t$ sent to infinity in (4.37). By (4.27), we can choose $a>0$ small enough such that the last infinite series is finite. This proves (2°). □

For any vector martingale $A$, we write $\langle A,A\rangle$ for the matrix of predictable covariations between components of $A$. If $A$ is a vector of semimartingales, then $[A,A]$ denotes the matrix of covariations between its components. The following proposition proves (1°) in Theorem 4.6.

PROPOSITION 4.14. *If conditions* (i)–(iii) *of Theorem* 4.6 *are in force, then the following holds*:

(1°) *The sequence of laws of* $(Z^{(n)},\langle(M^{(n)},D^{(n)}),(M^{(n)},D^{(n)})\rangle)$ *under* $\mathbb{P}_{\lambda_n}^{(n)}$ *is* $C$*-tight*.

(2°) *Suppose that, by choosing a subsequence if necessary, the sequence of laws of* $Z^{(n)}$ *under* $\mathbb{P}_{\lambda_n}^{(n)}$ *converges weakly to the law of* $Z=(Y,M,D)$ *under* $\mathbb{P}^{(\infty)}$, *then* $Z$ *is a vector of continuous semimartingales*, $M$ *is the martingale part of* $Y$, $(M,D)$ *is a vector martingale with respect to the filtration generated by* $(Y,M,D)$ *and we have the following convergence*:

$$\begin{aligned}
&\big(Z^{(n)},\langle(M^{(n)},D^{(n)}),(M^{(n)},D^{(n)})\rangle,[(M^{(n)},D^{(n)}),(M^{(n)},D^{(n)})]\big)\\
&\quad\xrightarrow[n\longrightarrow\infty]{\text{(d)}}\big(Z,[(M,D),(M,D)],[(M,D),(M,D)]\big).
\end{aligned}\tag{4.41}$$

(3°) *In the context of* (2°), *the sequence of laws of* $(Y^{(n)},M^{(n)})$ *under* $\mathbb{P}_{\lambda_n}^{(n),w_n}$ *converges weakly to the law of* $(Y,M)$ *under* $D\cdot\mathbb{P}^{(\infty)}$ *obtained from* $\mathbb{P}^{(\infty)}$ *by the Radon–Nikodým derivative process* $D$.

PROOF. (1°) By [16], Proposition 3.2.4, it is enough to prove that all the sequences of laws of components of the multidimensional processes under consideration are $C$-tight.

*$C$-tightness of the sequence* $\{\mathscr{L}(Y^{(n)})\}$. This follows readily from (4.8) in Assumption 4.2.

*C-tightness of the sequence* $\{\mathscr{L}(M^{(n)})\}$. Recall that Assumption 4.2 implies that $\gamma_n\mu_n \longrightarrow \mu$. Then it follows from the decomposition (3.19) of $Y^{(n)}$ and (4.8) in Assumption 4.2 that $M^{(n)}$ are martingales uniformly bounded on compacts and converge in distribution to a continuous martingale. In particular, the required $C$-tightness follows.

*C-tightness of the sequence* $\{\mathscr{L}(\langle M^{(n)}, M^{(n)}\rangle)\}$. By the equivalence of (i) and (iii) in [23], Proposition VI.3.26, $C$-tightness of the sequence under consideration is implied by its tightness. Then by [23], Theorem VI.4.5, we need to verify the following compact containment condition:

$$\forall \varepsilon, t > 0\ \exists K > 0 \text{ such that} \qquad \sup_{n\in\mathbb{N}} \mathbb{P}^{(n)}_{\lambda_n}\big(\langle M^{(n)}, M^{(n)}\rangle_t \geq K\big) \leq \varepsilon, \tag{4.42}$$

and Aldous's condition concerning the uniform modulus of continuity of $\langle M^{(n)}, M^{(n)}\rangle$ at stopping times:

$$\begin{aligned}\forall \varepsilon, K > 0, \qquad \lim_{\theta\longrightarrow 0+} \limsup_{n\longrightarrow\infty} \sup_{\substack{S,T\in\mathscr{T}(n,K)\\ S\leq T\leq S+\theta}} \mathbb{P}^{(n)}_{\lambda_n}\big(\langle M^{(n)}, M^{(n)}\rangle_T \\ - \langle M^{(n)}, M^{(n)}\rangle_S \geq \varepsilon\big) = 0.\end{aligned} \tag{4.43}$$

For (4.42), note that (3.21) implies

$$\langle M^{(n)}, M^{(n)}\rangle_t \leq 2\left(\frac{\pi^{(n)}_{\max}}{\pi^{(n)}_{\min}}\right)\left(\gamma_n \nu_n(\mathbb{1}) \int_0^t W_1(\xi_{\gamma_n s})\, ds\right) + \pi^{(n)}_{\max}\gamma_n\mu_n(\mathbb{1})t,$$

where the function $W_1$ is defined by (3.6). Applying Assumption 4.1, (4.25) and the validity of condition (ii) of Theorem 4.3 to the foregoing inequality, we deduce that $\langle M^{(n)}, M^{(n)}\rangle$ are $L^p$-bounded on compacts for every $p \in [1, \infty)$. This is enough for (4.42).

Next, we verify (4.43). For $J, n, K \geq 1$, define

$$w_J(\alpha, \theta) = \sup_{0\leq t\leq t+\theta\leq J} \sup_{a,b\in[t,t+\theta]} \big|\alpha(a) - \alpha(b)\big|$$

for càdlàg functions $\alpha : [0, \infty) \longrightarrow \mathbb{R}$, and $\mathscr{T}(n, K)$ to be the set of all $(\mathscr{F}^{(n)}_t)$-stopping times bounded by $K$. Recall that $M^{(n)}$'s are uniformly bounded on compacts. Then for all $\theta \in (0, 1]$ and $S, T \in \mathscr{T}(n, K)$ satisfying $0 \leq S \leq T \leq S + \theta$, it follows from the martingale characterization of $\langle M^{(n)}, M^{(n)}\rangle$ (cf. [23], Theorem I.4.2) and the optional stopping theorem [38], Theorem II.3.3, that

$$\begin{aligned}&\mathbb{E}^{(n)}_{\lambda_n}\big[\langle M^{(n)}, M^{(n)}\rangle_T - \langle M^{(n)}, M^{(n)}\rangle_S\big] \\ &\quad= \mathbb{E}^{(n)}_{\lambda_n}\big[\big(M^{(n)}_T\big)^2 - \big(M^{(n)}_S\big)^2\big] = \mathbb{E}^{(n)}_{\lambda_n}\big[\big(M^{(n)}_T - M^{(n)}_S\big)^2\big] \\ &\quad\leq \mathbb{E}^{(n)}_{\lambda_n}\big[w_{K+1}\big(M^{(n)}, \theta\big)^2\big].\end{aligned}$$

By the equivalence of (i) and (ii) in [23], Proposition VI.3.26, on $C$-tightness, dominated convergence and the convergence in distribution of $M^{(n)}$ toward a continuous process [see the above proof for $C$-tightness of the sequence $\{\mathscr{L}(M^{(n)})\}$], the foregoing inequality implies

$$\lim_{\theta\longrightarrow 0+}\limsup_{n\longrightarrow\infty}\sup_{\substack{S,T\in\mathscr{T}(n,K)\\ S\le T\le S+\theta}}\mathbb{E}^{(n)}_{\lambda_n}\big[\langle M^{(n)},M^{(n)}\rangle_T-\langle M^{(n)},M^{(n)}\rangle_S\big]=0. \tag{4.44}$$

Aldous's condition in (4.43) is then satisfied by Chebyshev's inequality and (4.44). The required $C$-tightness follows.

*$C$-tightness of the sequence* $\{\mathscr{L}(\langle D^{(n)},D^{(n)}\rangle)\}$. By the equivalence of (i) and (iii) in [23], Proposition VI.3.26, and the continuity of $\langle D^{(n)},D^{(n)}\rangle$, the $C$-tightness of the sequence is implied by its tightness. Then to obtain the tightness, by [23], Theorem VI.4.5, it is enough to verify the compact containment condition and Aldous's condition for $\langle D^{(n)},D^{(n)}\rangle$, that is, analogues of (4.42) and (4.43) for $\langle D^{(n)},D^{(n)}\rangle$.

First, for Aldous's condition, we notice that

$$w_n^2\big(\pi^{(n)}_{\min}\big)^{-1}\le C_{(4.45)}\nu_n(\mathbb{1}) \tag{4.45}$$

by Assumption 4.1, Assumption 4.5 and the fact that $\nu_n(\mathbb{1})\le\pi^{(n)}_{\max}$ [recall the definition (3.24) of $\nu_n$]. Then we take $\theta\in(0,1]$ and obtain from the strong Markov property of voter models that

$$\begin{aligned}
&\sup_{n\in\mathbb{N}}\sup_{\substack{S,T\in\mathscr{T}(n,K)\\ S\le T\le S+\theta}}\mathbb{E}^{(n)}_{\lambda_n}\big[\langle D^{(n)},D^{(n)}\rangle_T-\langle D^{(n)},D^{(n)}\rangle_S\big]\\
&\quad\le\sup_{n\in\mathbb{N}}\sup_{\lambda\in\mathscr{P}(S^{E_n})}\mathbb{E}^{(n)}_{\lambda}\big[\langle D^{(n)},D^{(n)}\rangle_\theta\big]\\
&\quad\le\sup_{n\in\mathbb{N}}\sup_{\lambda\in\mathscr{P}(S^{E_n})}C_{(4.46)}\\
&\qquad\times\sum_{\ell=1}^4\mathbb{E}^{(n)}_{\lambda}\bigg[\exp\bigg(C_{(4.46)}\gamma_n\nu_n(\mathbb{1})\int_0^\theta W_\ell(\xi_{\gamma_n s})\,ds\bigg)\bigg]^{1/2}\\
&\qquad\times\sum_{\ell=1}^4\mathbb{E}^{(n)}_{\lambda}\bigg[\bigg(\gamma_n\nu_n(\mathbb{1})\int_0^\theta W_\ell(\xi_{\gamma_n s})\,ds\bigg)^2\bigg]^{1/2}\xrightarrow[\theta\longrightarrow 0+]{}0
\end{aligned}\tag{4.46}$$

for some constant $C_{(4.46)}$ depending only on $\Pi$. Here in (4.46), the inequality follows from (3.31) and (4.45), and the convergence follows from (4.25) and (4.26).

The proof for the compact containment condition is similar. Now, (4.25) can be applied to the moment bounds in (3.31) after a time change by $\gamma_n$ as above in (4.46). This is enough for the compact containment condition.

*$C$-tightness of the sequence* $\{\mathscr{L}(\langle M^{(n)},D^{(n)}\rangle)\}$. The proof is similar to the previous one by verifying conditions analogous to (4.42) and (4.43) for $\langle M^{(n)},D^{(n)}\rangle$.

The only difference is that the moment bound in (3.32) for $\mathrm{Var}(\langle M^{(n)}, D^{(n)}\rangle)$ needs to be applied instead. We omit the details.

From the $C$-tightness results obtained so far in the present proof, the $C$-tightness of the sequences of laws of $\langle M^{(n)} \pm D^{(n)}, M^{(n)} \pm D^{(n)}\rangle$ also holds.

*$C$-tightness of the sequence* $\{\mathscr{L}(D^{(n)})\}$. By the $C$-tightness of the sequence $\{\mathscr{L}(\langle D^{(n)}, D^{(n)}\rangle)\}$ proven above, the sequence $\{\mathscr{L}(D^{(n)})\}$ is tight since the two conditions (i) and (ii) [23], Theorem VI.4.13, are now satisfied.

In addition, it follows from (3.10) that, for $(x, y)$ such that $q^{(n)}(x, y) > 0$,

$$\bigl\|q^{(n),w_n}(x, y, \cdot)/q^{(n)}(x, y) - 1\bigr\|_\infty \le C_{(4.47)} w_n \tag{4.47}$$

for some constant $C_{(4.47)}$ depending only on $\Pi$. Hence, for any $\varepsilon > 0$, the definition (2.10) of $D^{(n)}$ [or the dynamical equation (3.4) of $D^{(n)}$] implies

$$\begin{aligned}&\limsup_{n\longrightarrow\infty} \mathbb{P}^{(n)}_{\lambda_n}\Bigl(\sup_{s:0\le s\le t} |\Delta D^{(n)}_s| > \varepsilon\Bigr)\\ &\quad\le \limsup_{n\longrightarrow\infty} \mathbb{P}^{(n)}_{\lambda_n}\Bigl(\sup_{s:0\le s\le t} C_{(4.47)} w_n D^{(n)}_s > \varepsilon\Bigr) = 0,\end{aligned} \tag{4.48}$$

where the last inequality follows from Doob's weak $L^2$-inequality [38], Theorem II.1.7, since

$$\sup_{n\in\mathbb{N}} \mathbb{E}^{(n)}_{\lambda_n}\bigl[\bigl(D^{(n)}_t\bigr)^2\bigr] < \infty$$

by (3.7), (4.25) and (4.45).

By (4.48) and the tightness of the sequence $\{\mathscr{L}(D^{(n)})\}$, the equivalence of (i) and (iii) in [23], Proposition VI.3.26, applies and we get the $C$-tightness of the sequence $\{\mathscr{L}(D^{(n)})\}$.

(2°) First, suppose that

$$\begin{aligned}&D^{(n)}, \langle M^{(n)}, M^{(n)}\rangle, \langle M^{(n)}, D^{(n)}\rangle, \text{ and } \langle D^{(n)}, D^{(n)}\rangle\\ &\qquad\text{are } L^p\text{-bounded on compacts for every } p \in [1, \infty).\end{aligned} \tag{4.49}$$

In the above proof of the $C$-tightness of the sequence $\{\mathscr{L}(M^{(n)})\}$, we have seen that $M$ coincides with the martingale part of $Y$. In addition, it follows from (4.49) that $Z$ is a vector of continuous semimartingales and $(M, D)$ is a vector martingale with respect to the filtration generated by $(Y, M, D)$. Below we prove first that (4.41) holds (the argument is very similar to that for [8], Theorem 5.1 (2) and (3)) and then (4.49).

Recall that $|\Delta M^{(n)}| \le \pi^{(n)}_{\max}$ by (3.20) and $|\Delta D^{(n)}|$ is also uniformly bounded by the argument for (4.48). By (4.49) and [23], Corollary VI.6.30, we obtain

$$\begin{aligned}&\bigl((M^{(n)}, D^{(n)}), \bigl[(M^{(n)}, D^{(n)}), (M^{(n)}, D^{(n)})\bigr]\bigr)\\ &\quad\xrightarrow[n\longrightarrow\infty]{\text{(d)}} \bigl((M, D), \bigl[(M, D), (M, D)\bigr]\bigr).\end{aligned} \tag{4.50}$$

Since $(M, D)$ is a continuous vector martingale,

$$\big[(M, D), (M, D)\big] = \big\langle (M, D), (M, D) \big\rangle.$$

We also have the uniform integrability of $(D^{(n)})^2$ and

$$\big\langle \big(M^{(n)}, D^{(n)}\big), \big(M^{(n)}, D^{(n)}\big) \big\rangle$$

from (4.49), and the fact that any weak subsequential limit of the laws of $\langle (M^{(n)}, D^{(n)}), (M^{(n)}, D^{(n)}) \rangle$ must be the law of a matrix of continuous finite variation processes. The uniform boundedness of $M^{(n)}$ is already explained in the proof of the $C$-tightness of the sequence $\{\mathscr{L}(M^{(n)})\}$. With all of these considerations in mind, we deduce from the martingale characterization of predictable covariations (cf. [23], Theorem I.4.2) that (4.50) implies (4.41).

It remains to prove (4.49). First, for every $a \in (0, \infty)$, $(D^{(n)})^a$ are $L^p$-bounded on compacts by (3.7) and Proposition 4.13(1°). Second, we have seen that $\langle M^{(n)}, M^{(n)} \rangle$ are $L^p$-bounded on compacts by the proof of their $C$-tightness. Finally, the required $L^p$-boundedness on compacts of $\langle M^{(n)}, D^{(n)} \rangle$ and $\langle D^{(n)}, D^{(n)} \rangle$ follows from Lemma 3.3 and Proposition 4.13(1°). We have proved (2°).

(3°) We have seen in the proof of (2°) that $D^{(n)}$ are $L^2$-bounded on compacts. This is enough for the required property. □

4.5. *Proof of the main theorem*: *Identification of limits*. The goal of this section is to complete the proof of Theorem 4.6 by proving its (2°) and (3°). The main step here is to prove a key "moment-closure property" for the processes

$$\big(W_\ell(\xi_{\gamma_n t}), \mathbb{P}^{(n)}_{\lambda_n}\big), \tag{4.51}$$

where $W_\ell$ are defined by (3.6). Roughly speaking, the property shows that we can approximate these processes by polynomial functions of the limiting voter density process and the coefficients of the polynomials are explicitly determined by the limiting return probabilities $R_\ell$ defined in Assumption 4.4.

The results to be discussed below in this subsection are generalizations of some arguments in [6, 10] to compute explicitly the first-order expansions of game fixation probabilities by duality (see Proposition 6.3). Nevertheless, now we have to bring those results, which only concern expectations, to results in the pathwise sense, and moreover, handle some complexity arising from the presence of mutation. For these reasons, we need to resort to arguments finer than those in [6, 10].

To study (4.51), we work with the dual functions $H(\xi; x, y)$ defined by (4.3). These dual functions allow us to invoke the coalescing Markov chains $\{B^x\}$ through the duality equation in (4.5), and thus lead to the use of the following density functions: for $\ell \geq 1$,

$$H_\ell(\xi) = \sum_{x, y \in E : x \neq y} \pi(x) q^\ell(x, y) H(\xi; x, y)$$

$$
\begin{aligned}
&= W_\ell(\xi) + \sum_{x,y\in E:x\neq y} \pi(x)q^\ell(x,y) \\
&\quad \times \bigl[-\overline{\mu}(1)\widehat{\xi}(y) - \overline{\mu}(0)\xi(x) + \overline{\mu}(1)\overline{\mu}(0)\bigr] \qquad (4.52)\\
&= W_\ell(\xi) - \overline{\mu}(1)\bigl[1 - p_1(\xi)\bigr] - \overline{\mu}(0)\,p_1(\xi) \\
&\quad + \sum_{x\in E} \pi(x)q^\ell(x,x)\bigl[\overline{\mu}(1)\widehat{\xi}(x) + \overline{\mu}(0)\xi(x)\bigr] \qquad (4.53)\\
&\quad + \sum_{x,y\in E:x\neq y} \pi(x)q^\ell(x,y)\overline{\mu}(1)\overline{\mu}(0).
\end{aligned}
$$

Equations (4.52) and (4.53) are the central equations to study (4.51). However, when mutation is present, (4.53) shows that applying $H_\ell$ for the purpose of studying $W_\ell$ needs us to handle the other residual terms in (4.53), which is in part responsible for the complexity mentioned above.

As the first step to study (4.51), we compute the dynamics of $H(\xi_t; x, y)$.

LEMMA 4.15. (1°) *Fix $x \neq y$. For any $\lambda \in \mathscr{P}(S^E)$, the process*

$$
\begin{aligned}
M_t^{x,y} &= e^{2(1+\mu(\mathbb{1}))t}H(\xi_t; x, y) - H(\xi; x, y) \\
&\quad - \int_0^t e^{2(1+\mu(\mathbb{1}))s}\biggl(\sum_{z\in E} q(x,z)H(\xi_s; z, y) \qquad (4.54)\\
&\quad + \sum_{z\in E} q(y,z)H(\xi_s; x, z)\biggr)\,ds
\end{aligned}
$$

*is an $(\mathscr{F}_t, \mathbb{P}_\lambda)$-martingale.*

(2°) *For any $\ell \in \mathbb{N}$ and $\lambda \in \mathscr{P}(S^E)$, the $(\mathscr{F}_t, \mathbb{P}_\lambda)$-martingale*

$$
M_t^\ell = \sum_{x,y\in E:x\neq y} \pi(x)q^\ell(x,y)M_t^{x,y} \qquad (4.55)
$$

*satisfies*

$$
\begin{aligned}
\mathbb{E}_\lambda\bigl[(M_t^\ell)^2\bigr] &\le 18\pi_{\max}\int_0^t e^{4(1+\mu(\mathbb{1}))s}\mathbb{E}_\lambda\bigl[W_1(\xi_s)\bigr]\,ds \qquad (4.56)\\
&\quad + \frac{9\mu(\mathbb{1})\pi_{\max}}{4+4\mu(\mathbb{1})}e^{4(1+\mu(\mathbb{1}))t}.
\end{aligned}
$$

PROOF. (1°) The argument for (4.55) is similar to the argument for (4.30). Fix $x \neq y$, and again we let $J$ denote the first jump time of the bivariate Markov chain $B^{\{x,y\}}$, which is exponentially distributed with mean $1/2$. For $t \ge J$, we have

$$
\begin{aligned}
&\int_0^t \mathbb{1}_{\{B_s^x = B_s^y\}} \exp\biggl(-\mu(\mathbb{1})\int_0^s \bigl|B_r^{\{x,y\}}\bigr|\,dr\biggr)\,ds \\
&\quad = \int_J^t \mathbb{1}_{\{B_s^x = B_s^y\}} \exp\biggl(-2\mu(\mathbb{1})J - \mu(\mathbb{1})\int_J^s \bigl|B_r^{\{x,y\}}\bigr|\,dr\biggr)\,ds
\end{aligned}
$$

and

$$H(\xi; B_t^x, B_t^y)\exp\Big(-\mu(\mathbb{1})\int_0^t |B_s^{\{x,y\}}|\,ds\Big)$$
$$= H(\xi; B_t^x, B_t^y)\exp\Big(-2\mu(\mathbb{1})J - \mu(\mathbb{1})\int_J^t |B_s^{\{x,y\}}|\,ds\Big).$$

Then (4.5) and the above two displays imply

$$\begin{aligned}
&\mathbb{E}_\xi\big[H(\xi_t; x, y)\big]\\
&\quad= \mathbb{E}\Big[H(\xi; B_t^x, B_t^y)\exp\Big(-\mu(\mathbb{1})\int_0^t |B_s^{\{x,y\}}|\,ds\Big); t < J\Big]\\
&\qquad+ \mathbb{E}\Big[H(\xi; B_t^x, B_t^y)\exp\Big(-\mu(\mathbb{1})\int_0^t |B_s^{\{x,y\}}|\,ds\Big); t \ge J\Big]\\
&\qquad- \mu(\mathbb{1})\overline{\mu}(1)\overline{\mu}(0)\mathbb{E}\Big[\int_0^t \mathbb{1}_{\{B_s^x = B_s^y\}}\exp\Big(-\mu(\mathbb{1})\int_0^s |B_r^{\{x,y\}}|\,dr\Big)ds; t \ge J\Big]\\
&\quad= e^{-2(1+\mu(\mathbb{1}))t}H(\xi; x, y)\\
&\qquad+ \mathbb{E}\Big[H(\xi; B_t^x, B_t^y)\exp\Big(-2\mu(\mathbb{1})J - \mu(\mathbb{1})\int_J^t |B_s^{\{x,y\}}|\,ds\Big); t \ge J\Big]\\
&\qquad- \mu(\mathbb{1})\overline{\mu}(1)\overline{\mu}(0)\\
&\qquad\times \mathbb{E}\Big[\int_J^t \mathbb{1}_{\{B_s^x = B_s^y\}}\exp\Big(-2\mu(\mathbb{1})J - \mu(\mathbb{1})\int_J^s |B_r^{\{x,y\}}|\,dr\Big)ds; t \ge J\Big]\\
&\quad= e^{-2(1+\mu(\mathbb{1}))t}H(\xi; x, y)\\
&\qquad+ \int_0^t e^{-2(1+\mu(\mathbb{1}))u}\sum_{z\in E} q(x, z)\\
&\qquad\times \mathbb{E}\Big[H(\xi; B_{t-u}^z, B_{t-u}^y)\exp\Big(-\mu(\mathbb{1})\int_0^{t-u} |B_s^{\{z,y\}}|\,ds\Big)\Big]du\\
&\qquad+ \int_0^t e^{-2(1+\mu(\mathbb{1}))u}\sum_{z\in E} q(y, z)\\
&\qquad\times \mathbb{E}\Big[H(\xi; B_{t-u}^x, B_{t-u}^z)\exp\Big(-\mu(\mathbb{1})\int_0^{t-u} |B_s^{\{x,z\}}|\,ds\Big)\Big]du\\
&\qquad- \mu(\mathbb{1})\overline{\mu}(1)\overline{\mu}(0)\int_0^t e^{-2(1+\mu(\mathbb{1}))u}\sum_{z\in E} q(x, z)\\
&\qquad\times \mathbb{E}\Big[\int_0^{t-u} \mathbb{1}_{\{B_s^z = B_s^y\}}\exp\Big(-\mu(\mathbb{1})\int_0^s |B_r^{\{z,y\}}|\,dr\Big)ds\Big]du
\end{aligned}$$

$$
\begin{aligned}
&- \mu(\mathbb{1})\overline{\mu}(1)\overline{\mu}(0)\int_0^t e^{-2(1+\mu(\mathbb{1}))u}\sum_{z\in E} q(y,z)\\
&\times \mathbb{E}\biggl[\int_0^{t-u} \mathbb{1}_{\{B_s^x = B_s^z\}} \exp\biggl(-\mu(\mathbb{1})\int_0^s |B_r^{\{x,z\}}|\,dr\biggr)\,ds\biggr]\,du\\
&= e^{-2(1+\mu(\mathbb{1}))t} H(\xi; x, y) + \int_0^t e^{-2(1+\mu(\mathbb{1}))(t-s)}\\
&\quad\times \biggl(\sum_{z\in E} q(x,z)\mathbb{E}_\xi\bigl[H(\xi_s; z, y)\bigr] + \sum_{z\in E} q(y,z)\mathbb{E}_\xi\bigl[H(\xi_s; x, z)\bigr]\biggr)\,ds,
\end{aligned}
$$

where the third equality uses the Markov property of $(B^x, B^y)$ at $J$ and the independence of $J$ and $(B_J^x, B_J^y)$, and the last equality uses (4.5) again.

The foregoing equality proves $\mathbb{E}_\xi[M_t^{x,y}] = 0$ for all $\xi \in S^E$ and $t$. The required martingale property then follows from the Markov property of $(\xi_t)$.

(2°) By [23], Lemma I.4.14(b), Lemma I.4.51, the quadratic variation of $M^\ell$ is given by

$$
\begin{aligned}
\bigl[M^\ell, M^\ell\bigr]_t &= \sum_{s:s\le t} \bigl(\Delta M_s^\ell\bigr)^2\\
&= \sum_{x,y\in E}\int_0^t \bigl(\Delta M_s^\ell\bigr)^2\,d\Lambda_s(x,y)\\
&\quad + \sum_{\sigma\in S}\sum_{x\in E}\int_0^t \bigl(\Delta M_s^\ell\bigr)^2\,d\Lambda_s^\sigma(x),
\end{aligned}
$$

where the second equality follows from the present coupling of the voter model in (2.8). By the definition (4.55) of $M^\ell$ and the definition (4.52) of $H_\ell$,

$$
\begin{aligned}
\Delta M_s^\ell &= \textstyle\sum_{x,y\in E: x\ne y} \pi(x) q^\ell(x,y) e^{2(1+\mu(\mathbb{1}))s}\,\Delta H(\xi_s; x, y)\\
&= e^{2(1+\mu(\mathbb{1}))s}\,\Delta H_\ell(\xi_s),
\end{aligned}
$$

where the first equality follows from (4.54) and (4.55) and the last one from the definition (4.52) of $H_\ell$. Putting the last two displays together and using the coupling (2.8) of $(\xi_t)$ again, we get

$$
\begin{aligned}
\bigl[M^\ell, M^\ell\bigr]_t &= \sum_{x,y\in E}\int_0^t e^{4(1+\mu(\mathbb{1}))s}\bigl[\xi_{s-}(x)\widehat{\xi}_{s-}(y) + \widehat{\xi}_{s-}(x)\xi_{s-}(y)\bigr]\\
&\quad\times \bigl[H_\ell\bigl((\xi_{s-})^x\bigr) - H_\ell(\xi_{s-})\bigr]^2\,d\Lambda_s(x,y)\\
&\quad + \sum_{\sigma\in S}\sum_{x\in E}\int_0^t e^{4(1+\mu(\mathbb{1}))s}\bigl[H_\ell\bigl((\xi_{s-})^{x|\sigma}\bigr) - H_\ell(\xi_{s-})\bigr]^2\,d\Lambda_s^\sigma(x).
\end{aligned}
\tag{4.57}
$$

To bound the right-hand side of the above equality, notice that the definition of $W_\ell$ in (3.6) implies

$$\begin{aligned} &W_\ell(\xi^x) - W_\ell(\xi) \\ &\quad = \pi(x) \sum_{y\in E} q^\ell(x,y)\big[\xi(x)\xi(y) - \xi(x)\widehat{\xi}(y) + \widehat{\xi}(x)\widehat{\xi}(y) - \widehat{\xi}(x)\xi(y)\big] \\ &\quad \forall x \in E. \end{aligned}$$

The foregoing equality and (4.53) then imply that

$$\big|H_\ell(\xi^x) - H_\ell(\xi)\big| \le 3\pi(x) \qquad \forall x \in E.$$

By Poisson calculus, (4.57) implies

$$\begin{aligned} &\mathbb{E}_\xi\big[(M_t^\ell)^2\big] \\ &\quad = \mathbb{E}_\xi\big[[M^\ell, M^\ell]_t\big] \\ &\quad \le \sum_{x,y\in E} \int_0^t e^{4(1+\mu(\mathbb{1}))s} \mathbb{E}_\xi\big[\xi_{s-}(x)\widehat{\xi}_{s-}(y) + \widehat{\xi}_{s-}(x)\xi_{s-}(y)\big] 9\pi(x)^2 q(x,y)\,ds \\ &\qquad + \sum_{\sigma\in S}\sum_{x\in E} \int_0^t e^{4(1+\mu(\mathbb{1}))s} 9\pi(x)^2 \mu(\sigma)\,ds \\ &\quad \le 18\pi_{\max} \int_0^t e^{4(1+\mu(\mathbb{1}))s} \mathbb{E}_\xi\big[W_1(\xi_s)\big]\,ds + \frac{9\mu(\mathbb{1})\pi_{\max}}{4+4\mu(\mathbb{1})}\big(e^{4(1+\mu(\mathbb{1}))t} - 1\big), \end{aligned}$$

which gives (4.56) upon integrating both sides with respect to $\lambda(d\xi)$. The proof is complete. □

We are ready to prove the moment closure property announced before.

PROPOSITION 4.16. *Under Assumption 4.1, Assumption 4.2 and Assumption 4.4, we have the following*:

(1°) *For all* $1 \le \ell \le 2$, *we have the following convergence in distribution of continuous processes*:

$$\begin{aligned} &\Big(\gamma_n \nu_n(\mathbb{1}) \int_0^t \big[W_{\ell+1}(\xi_{\gamma_n s}) - W_\ell(\xi_{\gamma_n s}) - R_\ell W_1(\xi_{\gamma_n s})\big]\,ds\Big)_{t\ge 0} \\ &\qquad \xrightarrow[n\to\infty]{\text{(d)}} 0, \end{aligned} \tag{4.58}$$

*where* $R_\ell$*'s are chosen in Assumption 4.4 with* $L = 2$. *An analogous result for the convergence in* (4.58) *with* $\ell = 3$ *holds if Assumption 4.4 with* $L = 3$ *applies*.

(2°) *If, moreover, the voting kernels* $q^{(n)}$ *are symmetric, then we have*

$$\left(\gamma_n \nu_n(\mathbb{1}) \int_0^t W_1(\xi_{\gamma_n s})\,ds - \frac{1}{2}\int_0^t Y_s^{(n)}\bigl(1 - Y_s^{(n)}\bigr)\,ds\right)_{t\geq 0} \xrightarrow[n\longrightarrow\infty]{\text{(d)}} 0. \tag{4.59}$$

(3°) *Convergences in distribution of one-dimensional marginals of the processes in* (1°) *and* (2°) *can be reinforced to* $L^p$*-convergences for any* $p \in [1, \infty)$.

The proofs of Proposition 4.16(1°) and its extension in (3°) begin with the proof of a particular convergence in the following lemma.

LEMMA 4.17. *Under Assumption 4.1, Assumption 4.2 and Assumption 4.4, we have, for all* $t \in (0, \infty)$ *and* $1 \leq \ell \leq 2$,

$$\begin{aligned}\lim_{n\longrightarrow\infty} \sup_{\lambda\in\mathscr{P}(S^{E_n})} \mathbb{E}_\lambda^{(n)}\Biggl[\Biggl(\gamma_n \nu_n(\mathbb{1}) \int_0^t \bigl[W_{\ell+1}(\xi_{\gamma_n s}) - W_\ell(\xi_{\gamma_n s})\\ - R_\ell W_1(\xi_{\gamma_n s})\bigr]\,ds\Biggr)^2\Biggr] = 0.\end{aligned} \tag{4.60}$$

*The above convergence for* $\ell = 3$ *holds if Assumption 4.4 with* $L = 3$ *applies.*

PROOF. Recall the functions $H_\ell(\xi)$ and the martingales $M^\ell$ defined in (4.52) and (4.55), respectively. Below we need several steps to derive and "clean up" the main approximate equation (4.63) to get (4.60). Very roughly speaking, these steps intend to use $H_\ell(\xi_{\gamma_n s})$ to approximate $W_\ell(\xi_{\gamma_n s})$ in view of (4.53), with $M^{(n),\ell}$ playing the role of a vanishing noise term by (4.54). In this main approximate equation, there is a particular residual term carrying a coefficient as a return probability, which contributes to the coefficient $R_\ell$ in (4.60). The required limit (4.60) may be reminiscent of (4.30) to the reader, but the nature of (4.60) is a stronger $L_2$-approximation for additive functionals of voter models.

By the reversibility of $q^{(n)}$, the equation satisfied by the martingale $M^{(n),\ell}$ under $\mathbb{P}_\lambda^{(n)}$ [see (4.54)] can be written as

$$\begin{aligned} M_t^{(n),\ell} &= e^{2(1+\mu_n(\mathbb{1}))t} H_\ell(\xi_t) - H_\ell(\xi_0)\\ &\quad - \int_0^t e^{2(1+\mu_n(\mathbb{1}))s} \sum_{x,y\in E_n: x\neq y} \pi^{(n)}(x) q^{(n),\ell}(x,y)\\ &\quad \times \Biggl(\sum_{z\in E_n} q^{(n)}(x,z) H(\xi_s; z, y) + \sum_{z\in E_n} q^{(n)}(y,z) H(\xi_s; x, z)\Biggr) ds\\ &= e^{2(1+\mu_n(\mathbb{1}))t} H_\ell(\xi_t) - H_\ell(\xi_0) - 2\int_0^t e^{2(1+\mu_n(\mathbb{1}))s} H_{\ell+1}(\xi_s)\,ds\\ &\quad + \int_0^t e^{2(1+\mu_n(\mathbb{1}))s} I(\xi_s)\,ds, \end{aligned} \tag{4.61}$$

where the function $I(\xi)$ is given by

$$\begin{aligned} I(\xi) &= -2\sum_{x\in E_n} \pi^{(n)}(x) q^{(n),\ell+1}(x,x) H(\xi;x,x) \\ &\quad + \sum_{x\in E_n} \pi^{(n)}(x) q^{(n),\ell}(x,x) \\ &\quad \times \Bigg( \sum_{z\in E_n} q^{(n)}(x,z) H(\xi;z,x) + \sum_{z\in E_n} q^{(n)}(x,z) H(\xi;x,z) \Bigg). \end{aligned} \tag{4.62}$$

Therefore, from (4.61), we have, for fixed $t \in (0,\infty)$,

$$\begin{aligned} &\int_0^t e^{-2(1+\mu_n(\mathbb{1}))s} M_s^{(n),\ell}\, ds \\ &\quad = \int_0^t H_\ell(\xi_s)\, ds - \int_0^t e^{-2(1+\mu_n(\mathbb{1}))s}\, ds\, H_\ell(\xi_0) \\ &\qquad - \int_0^t \int_0^s 2e^{-2(1+\mu_n(\mathbb{1}))s+2(1+\mu_n(\mathbb{1}))r} H_{\ell+1}(\xi_r)\, dr\, ds \\ &\qquad + \int_0^t \int_0^s e^{-2(1+\mu_n(\mathbb{1}))s+2(1+\mu_n(\mathbb{1}))r} I(\xi_r)\, dr\, ds \\ &\quad = \int_0^t H_\ell(\xi_s)\, ds - \int_0^t e^{-2(1+\mu_n(\mathbb{1}))s}\, ds\, H_\ell(\xi_0) \\ &\qquad - \frac{2}{2+2\mu_n(\mathbb{1})} \int_0^t H_{\ell+1}(\xi_s)\, ds \\ &\qquad + \frac{2}{2+2\mu_n(\mathbb{1})} \int_0^t e^{-2(1+\mu_n(\mathbb{1}))(t-s)} H_{\ell+1}(\xi_s)\, ds \\ &\qquad + \frac{1}{2+2\mu_n(\mathbb{1})} \int_0^t I(\xi_s)\, ds \\ &\qquad - \frac{1}{2+2\mu_n(\mathbb{1})} \int_0^t e^{-2(1+\mu_n(\mathbb{1}))(t-s)} I(\xi_s)\, ds. \end{aligned} \tag{4.63}$$

This is the main approximate equation which we study in this proof.

We multiply both sides of (4.63) by $\nu_n(\mathbb{1})$ and change time scales by replacing $t$ by $\gamma_n t$ for fixed $t \in (0,\infty)$. Suppose that

$$\lim_{n\to\infty} \sup_{\lambda\in\mathscr{P}(S^{E_n})} \mathbb{E}_\lambda^{(n)} \left[ \left( \gamma_n \nu_n(\mathbb{1}) \int_0^t e^{-2(1+\mu_n(\mathbb{1}))\gamma_n s} M_{\gamma_n s}^{(n),\ell}\, ds \right)^2 \right] = 0, \tag{4.64}$$

$$\lim_{n\to\infty} \sup_{\lambda\in\mathscr{P}(S^{E_n})} \mathbb{E}_\lambda^{(n)} \left[ \left( \gamma_n \nu_n(\mathbb{1}) \int_0^t e^{-2(1+\mu_n(\mathbb{1}))\gamma_n s}\, ds\, H_\ell(\xi_0) \right)^2 \right] = 0, \tag{4.65}$$

$$\lim_{n\to\infty}\sup_{\lambda\in\mathscr{P}(S^{E_n})}\mathbb{E}_\lambda^{(n)}\Bigg[\Bigg(\frac{2\gamma_n\nu_n(\mathbb{1})}{2+2\mu_n(\mathbb{1})} \times\int_0^t e^{-2(1+\mu_n(\mathbb{1}))\gamma_n(t-s)}H_{\ell+1}(\xi_{\gamma_n s})\,ds\Bigg)^2\Bigg]=0, \tag{4.66}$$

$$\lim_{n\to\infty}\sup_{\lambda\in\mathscr{P}(S^{E_n})}\mathbb{E}_\lambda^{(n)}\Bigg[\Bigg(\frac{\gamma_n\nu_n(\mathbb{1})}{2+2\mu_n(\mathbb{1})} \times\int_0^t e^{-2(1+\mu_n(\mathbb{1}))\gamma_n(t-s)}I(\xi_{\gamma_n s})\,ds\Bigg)^2\Bigg]=0, \tag{4.67}$$

which are used to handle terms among those on the two sides of (4.63). Then our focus for (4.63) is on the remaining terms, that is the first, third and fifth terms on its right-hand side [after multiplying them by $\nu_n(\mathbb{1})$ and changing $t$ to $\gamma_n t$]. That is, by (4.63) and the assumed identities (4.64)–(4.67), we have

$$\lim_{n\to\infty}\sup_{\lambda\in\mathscr{P}(S^{E_n})}\mathbb{E}_\lambda^{(n)}\Bigg[\Bigg(\gamma_n\nu_n(\mathbb{1})\int_0^t H_\ell(\xi_{\gamma_n s})\,ds - \frac{2\gamma_n\nu_n(\mathbb{1})}{2+2\mu_n(\mathbb{1})}\int_0^t H_{\ell+1}(\xi_{\gamma_n s})\,ds + \frac{\gamma_n\nu_n(\mathbb{1})}{2+2\mu_n(\mathbb{1})}\int_0^t I(\xi_{\gamma_n s})\,ds\Bigg)^2\Bigg]=0.$$

Equations (4.64)–(4.67) are verified at the end of this proof.

The foregoing limiting equality can be simplified a bit as follows. Recall that $\sup_n \gamma_n\mu_n(\mathbb{1})<\infty$ by Assumption 4.2, $H_\ell$'s are uniformly bounded by their definitions in (4.52), and $\pi^{(n)}$'s are comparable to uniform distributions by Assumption 4.1 so that $\nu_n(\mathbb{1})=\Theta(N_n^{-1})$. Hence, the foregoing equality implies

$$\lim_{n\to\infty}\sup_{\lambda\in\mathscr{P}(S^{E_n})}\mathbb{E}_\lambda^{(n)}\Bigg[\Bigg(\gamma_n\nu_n(\mathbb{1}) \times\int_0^t\Bigg(H_\ell(\xi_{\gamma_n s})-H_{\ell+1}(\xi_{\gamma_n s})+\frac{1}{2}I(\xi_{\gamma_n s})\Bigg)ds\Bigg)^2\Bigg]=0. \tag{4.68}$$

We show that the foregoing limit implies the required limit (4.60). We use (4.53) and (4.62) to write out the following integral in (4.68):

$$\begin{aligned}&\gamma_n\nu_n(\mathbb{1})\int_0^t\Bigg(H_\ell(\xi_{\gamma_n s})-H_{\ell+1}(\xi_{\gamma_n s})+\frac{1}{2}I(\xi_{\gamma_n s})\Bigg)ds\\&\quad=\gamma_n\nu_n(\mathbb{1})\int_0^t W_\ell(\xi_{\gamma_n s})\,ds-\gamma_n\nu_n(\mathbb{1})\int_0^t W_{\ell+1}(\xi_{\gamma_n s})\,ds\\&\qquad+\gamma_n\nu_n(\mathbb{1})\int_0^t\sum_{x\in E_n}\pi^{(n)}(x)\big(q^{(n),\ell}(x,x)-q^{(n),\ell+1}(x,x)\big)\\&\qquad\times\big(\overline{\mu}_n(1)\widehat{\xi}_{\gamma_n s}(x)+\overline{\mu}_n(0)\xi_{\gamma_n s}(x)\big)\,ds\end{aligned}$$

$$
\begin{aligned}
&+\gamma_n \nu_n(\mathbb{1}) \int_0^t \sum_{x,y\in E_n : x\neq y} \pi^{(n)}(x)\big(q^{(n),\ell}(x,y)\\
&-q^{(n),\ell+1}(x,y)\big)\overline{\mu}_n(1)\overline{\mu}_n(0)\,ds\\
&-\gamma_n \nu_n(\mathbb{1}) \int_0^t \sum_{x\in E_n} \pi^{(n)}(x) q^{(n),\ell+1}(x,x)\\
&\times\big(-\overline{\mu}_n(1)\widehat{\xi}_{\gamma_n s}(x)-\overline{\mu}_n(0)\xi_{\gamma_n s}(x)+\overline{\mu}_n(1)\overline{\mu}_n(0)\big)\,ds\\
&+\frac{\gamma_n \nu_n(\mathbb{1})}{2}\int_0^t \sum_{x\in E_n} \pi^{(n)}(x) q^{(n),\ell}(x,x)\\
&\times\sum_{z\in E_n} q^{(n)}(x,z)\big(\widehat{\xi}_{\gamma_n s}(x)\xi_{\gamma_n s}(z)+\widehat{\xi}_{\gamma_n s}(z)\xi_{\gamma_n s}(x)\big)\,ds\\
&+\frac{\gamma_n \nu_n(\mathbb{1})}{2}\int_0^t \sum_{x\in E_n} \pi^{(n)}(x) q^{(n),\ell}(x,x)\\
&\times\Bigg(\sum_{z\in E_n} q^{(n)}(x,z)\big(-\overline{\mu}(1)\widehat{\xi}_{\gamma_n s}(x)-\overline{\mu}_n(0)\xi_{\gamma_n s}(z)+\overline{\mu}_n(1)\overline{\mu}_n(0)\big)\\
&+\sum_{z\in E_n} q^{(n)}(x,z)\big(-\overline{\mu}(1)\widehat{\xi}_{\gamma_n s}(z)-\overline{\mu}_n(0)\xi_{\gamma_n s}(x)+\overline{\mu}_n(1)\overline{\mu}_n(0)\big)\Bigg)\,ds\\
&=\gamma_n \nu_n(\mathbb{1})\int_0^t W_\ell(\xi_{\gamma_n s})\,ds-\gamma_n \nu_n(\mathbb{1})\int_0^t W_{\ell+1}(\xi_{\gamma_n s})\,ds\\
&+\frac{\gamma_n \nu_n(\mathbb{1})}{2}\int_0^t \sum_{x\in E_n} \pi^{(n)}(x) q^{(n),\ell}(x,x)\\
&\times\sum_{z\in E_n} q^{(n)}(x,z)\big[\widehat{\xi}_{\gamma_n s}(x)\xi_{\gamma_n s}(z)+\widehat{\xi}_{\gamma_n s}(z)\xi_{\gamma_n s}(x)\big]\,ds\\
&+\frac{\gamma_n \nu_n(\mathbb{1})}{2}\int_0^t \sum_{x\in E_n} \pi^{(n)}(x) q^{(n),\ell}(x,x)\\
&\times\Bigg(\overline{\mu}(0)\sum_{z\in E_n} q^{(n)}(x,z)\big[\xi_{\gamma_n s}(x)-\xi_{\gamma_n s}(z)\big]\\
&+\overline{\mu}(1)\sum_{z\in E_n} q^{(n)}(x,z)\big[\widehat{\xi}_{\gamma_n s}(x)-\widehat{\xi}_{\gamma_n s}(z)\big]\Bigg)\,ds.
\end{aligned}
$$

Now we can use Assumption 4.4 to handle the last equation. For $1 \le \ell \le 2$, it follows from the validity of (4.11) with $L = 2$ and Proposition 4.13 that, with respect to the uniform $L^2$-limit as in (4.60), only the first three terms on the

right-hand side of the above equality can survive and the third term approximates $\gamma_n \nu_n(\mathbb{1}) \int_0^t R_\ell W_1(\xi_{\gamma_n s})\,ds$. Hence, (4.68) implies (4.60).

We still need to verify the limits in (4.64)–(4.67). For (4.64), we use the fact that $M^{(n),\ell}$ is a martingale in the first equality below and then Lemma 4.15(2°) in the first inequality:

$$\begin{aligned}
&\mathbb{E}_\lambda^{(n)}\left[\left(\gamma_n \nu_n(\mathbb{1}) \int_0^t e^{-2(1+\mu_n(\mathbb{1}))\gamma_n s} M_{\gamma_n s}^{(n),\ell}\,ds\right)^2\right] \\
&\quad = 2\gamma_n^2 \nu_n(\mathbb{1})^2 \int_0^t \int_0^s e^{-2(1+\mu_n(\mathbb{1}))\gamma_n r - 2(1+\mu_n(\mathbb{1}))\gamma_n s} \mathbb{E}_\lambda^{(n)}\big[(M_{\gamma_n r}^{(n),\ell})^2\big]\,dr\,ds \\
&\quad \le 36\gamma_n^3 \nu_n(\mathbb{1})^2 \pi_{\max}^{(n)} \int_0^t \int_0^s e^{-2(1+\mu_n(\mathbb{1}))\gamma_n r - 2(1+\mu_n(\mathbb{1}))\gamma_n s} \\
&\qquad \times \int_0^r e^{4(1+\mu_n(\mathbb{1}))\gamma_n q} \mathbb{E}_\lambda^{(n)}\big[W_1(\xi_{\gamma_n q})\big]\,dq\,dr\,ds \\
&\qquad + \frac{9\gamma_n^2 \nu_n(\mathbb{1})^2 \mu_n(\mathbb{1}) \pi_{\max}^{(n)}}{2+2\mu_n(\mathbb{1})} \int_0^t \int_0^s e^{2(1+\mu_n(\mathbb{1}))\gamma_n r - 2(1+\mu_n(\mathbb{1}))\gamma_n s}\,dr\,ds \\
&\quad \le \frac{36\nu_n(\mathbb{1})\pi_{\max}^{(n)}}{8[1+\mu_n(\mathbb{1})]^2}\left(\gamma_n \nu_n(\mathbb{1}) \int_0^t \mathbb{E}_\lambda^{(n)}\big[W_1(\xi_{\gamma_n q})\big]\,dq\right) \\
&\qquad + \frac{9\gamma_n \nu_n(\mathbb{1})^2 \mu_n(\mathbb{1}) \pi_{\max}^{(n)} t}{[2+2\mu_n(\mathbb{1})]^2} \xrightarrow[n\to\infty]{} 0,
\end{aligned}$$

where the convergence is uniform in $\lambda \in \mathscr{P}(S^{E_n})$ and follows from Proposition 4.13(1°) and the fact that $\pi^{(n)}$'s are comparable to uniform distributions and $\sup_n \gamma_n \mu_n(\mathbb{1}) < \infty$ by Assumption 4.2. To obtain the remaining limits (4.65)–(4.67), we only need to note that

$$\left(\nu_n(\mathbb{1}) \int_0^{\gamma_n t} e^{-2(1+\mu_n(\mathbb{1}))s}\,ds\right)^2 \le \nu_n(\mathbb{1})^2 \xrightarrow[n\to\infty]{} 0$$

by Assumption 4.1 since the definition (3.24) of $\nu_n$ shows that

$$\nu_n(\mathbb{1}) = \sum_{x,y\in E_n} \pi^{(n)}(x)^2 q^{(n)}(x,y) \le \pi_{\max}^{(n)} \sum_{x,y\in E_n} \pi^{(n)}(x) q^{(n)}(x,y) = \pi_{\max}^{(n)}.$$

The proof is complete. □

PROOF OF PROPOSITION 4.16. We start with the proof of (1°). The sequence of laws of the processes on the left-hand side of (4.58) is tight by Proposition 4.13(1°), the strong Markov property of voter models, and [23], Theorem VI.4.5, (cf. the proof of Proposition 4.14). By Lemma 4.17, the Markov property of voter models, and [16], Lemma 3.7.8(b), we deduce that the sequence converges in distribution to the zero process, as required in (1°).

Since the voting kernels are assumed to be symmetric, the proof of (2°) can be obtained by the same argument as that of [8], Corollary 5.2. It can be detailed as follows. Since $\langle M, M\rangle_t = \int_0^t Y_s(1-Y_s)\,ds$ and

$$w \longmapsto \left(\int_0^t f\big(w(s)\big)\,ds\right)_{t\geq 0} : D\big(\mathbb{R}_+, [0,1]\big) \longrightarrow D(\mathbb{R}_+, \mathbb{R}) \text{ is continuous} \tag{4.69}$$

for every bounded continuous function $f : \mathbb{R} \longrightarrow \mathbb{R}$ by the proof of [16], Proposition 3.7.1, Assumption 4.2 and Proposition 4.14(2°) imply that

$$\begin{aligned}
&\left(\gamma_n \nu_n(\mathbb{1}) \int_0^t W_1(\xi_{\gamma_n s})\,ds - \frac{1}{2}\int_0^t Y_s^{(n)}\big(1-Y_s^{(n)}\big)\,ds\right)_{t\geq 0}\\
&\quad = \left(\frac{1}{2}\big\langle M^{(n)}, M^{(n)}\big\rangle_t - \frac{1}{2}\int_0^t Y_s^{(n)}\big(1-Y_s^{(n)}\big)\,ds\right)_{t\geq 0}\\
&\quad \xrightarrow[n\longrightarrow\infty]{\text{(d)}} \left(\frac{1}{2}\langle M, M\rangle_t - \frac{1}{2}\int_0^t Y_s(1-Y_s)\,ds\right)_{t\geq 0},
\end{aligned}$$

which is the zero process.

Finally, $L^p$-convergences of the one-dimensional marginals of the processes under consideration are immediate consequences of Proposition 4.13(1°) and a standard result of uniform integrability. This proves (3°). □

With the moment-closure equations in (1°) and (2°) in Proposition 4.16 as well as the integrability property, we are ready to complete the proof of Theorem 4.6 by rather straightforward calculations albeit in the form of weak convergence. The following proposition proves Theorem 4.6(2°) and (3°), and thus, completes the proof of the theorem.

PROPOSITION 4.18. *Suppose that the assumptions for Theorem* 4.6(2°) *are in force, and by choosing a subsequence if necessary,* $(Y^{(n)}, M^{(n)}, D^{(n)})$ *under* $\mathbb{P}_{\lambda_n}^{(n)}$ *converges in distribution to a vector of continuous semimartingales* $(Y, M, D)$ *under* $\mathbb{P}^{(\infty)}$. *Then we have the following results*:

(1°) $(Y, M, D)$ *satisfies the covariation equations in* (4.15).

(2°) *Moreover, if Assumption* 4.4 *with* $L = 3$ *applies, then* $(Y, M, D)$ *can be characterized by the system in* (4.17).

*In addition, there is pathwise uniqueness in the system in* (4.17).

PROOF. In this proof, we write (d)-$\lim_{n\longrightarrow\infty}$ for limits in distribution of continuous processes defined under $\mathbb{P}_{\lambda_n}^{(n)}$, with the limiting objects defined under $\mathbb{P}^{(\infty)}$.

(1°) We start with some limiting identities. First, by Proposition 4.16(1°) and (2°), we deduce that

$$\begin{aligned}(\mathrm{d})\text{-}\lim_{n\longrightarrow\infty}\gamma_n\nu_n(\mathbb{1})\int_0^{\cdot}W_{\ell+1}(\xi_{\gamma_n s})\,ds\\ -\frac{R_\ell+\cdots+R_0}{2}\int_0^{\cdot}Y_s^{(n)}\bigl(1-Y_s^{(n)}\bigr)\,ds=0\end{aligned}\tag{4.70}$$

for all $0\le\ell\le 2$ (recall that $R_0=1$). By the assumed convergence in distribution of $(Y^{(n)}, D^{(n)})$ toward $(Y, D)$, (4.69) and (4.70), we get the following convergence of four-dimensional processes:

$$\begin{aligned}&(\mathrm{d})\text{-}\lim_{n\longrightarrow\infty}\biggl(\gamma_n\nu_n(\mathbb{1})\int_0^{\cdot}W_1(\xi_{\gamma_n s})\,ds,\\ &\quad\gamma_n\nu_n(\mathbb{1})\int_0^{\cdot}W_2(\xi_{\gamma_n s})\,ds,\gamma_n\nu_n(\mathbb{1})\int_0^{\cdot}W_3(\xi_{\gamma_n s})\,ds,D^{(n)}\biggr)\\ &\quad=\biggl(\frac{R_0}{2}\int_0^{\cdot}Y_s(1-Y_s)\,ds,\\ &\qquad\frac{R_1+R_0}{2}\int_0^{\cdot}Y_s(1-Y_s)\,ds,\frac{R_2+R_1+R_0}{2}\int_0^{\cdot}Y_s(1-Y_s)\,ds,D\biggr).\end{aligned}$$

Hence, by [23], Proposition VI.6.12, Theorem VI.6.22, we deduce that, for all integers $m\ge 0$,

$$\begin{aligned}&(\mathrm{d})\text{-}\lim_{n\longrightarrow\infty}\biggl(\gamma_n\nu_n(\mathbb{1})\int_0^{\cdot}\bigl(D_s^{(n)}\bigr)^m W_1(\xi_{\gamma_n s})\,ds,\\ &\quad\gamma_n\nu_n(\mathbb{1})\int_0^{\cdot}\bigl(D_s^{(n)}\bigr)^m W_2(\xi_{\gamma_n s})\,ds,\gamma_n\nu_n(\mathbb{1})\int_0^{\cdot}\bigl(D_s^{(n)}\bigr)^m W_3(\xi_{\gamma_n s})\,ds\biggr)\\ &\quad=\biggl(\frac{R_0}{2}\int_0^{\cdot}D_s^m Y_s(1-Y_s)\,ds,\\ &\qquad\frac{R_1+R_0}{2}\int_0^{\cdot}D_s^m Y_s(1-Y_s)\,ds,\frac{R_2+R_1+R_0}{2}\int_0^{\cdot}D_s^m Y_s(1-Y_s)\,ds\biggr).\end{aligned}\tag{4.71}$$

Second, since

$$\bigl|R_1^{w_n}(\xi_{\gamma_n s})\bigr|\le C_{(4.72)}\sum_{\ell=1}^{4}W_\ell(\xi_{\gamma_n s})\tag{4.72}$$

for some constant $C_{(4.72)}$ depending only on $\Pi$ by (3.15) and the definition (3.26) of $R_1^w(\xi)$, (4.71) and Assumption 4.1 imply that

$$(\mathrm{d})\text{-}\lim_{n\longrightarrow\infty}w_n^2\gamma_n\int_0^{\cdot}D_s^{(n)}R_1^{w_n}(\xi_{\gamma_n s})\,ds=0.\tag{4.73}$$

We are ready to prove (4.15). Recall the definition of $w_\infty$ in (4.14). Then applying Proposition 4.14(2°), (3.22), (4.73), (7.1) and (4.71) in order below shows that

$$\begin{aligned}\langle M, D\rangle &= \text{(d)-}\lim_{n\longrightarrow\infty}\langle M^{(n)}, D^{(n)}\rangle = \text{(d)-}\lim_{n\longrightarrow\infty} w_n\gamma_n\int_0^{\cdot} D_s^{(n)}\overline{D}^{(n)}(\xi_{\gamma_n s})\,ds\\ &= \text{(d)-}\lim_{n\longrightarrow\infty}\biggl(\frac{w_n}{\nu_n(\mathbb{1})}\biggr)\gamma_n\nu_n(\mathbb{1})\\ &\quad\times\int_0^{\cdot} D_s^{(n)}\bigl[b\bigl(W_3(\xi_{\gamma_n s})-W_1(\xi_{\gamma_n s})\bigr)-cW_2(\xi_{\gamma_n s})\bigr]\,ds\\ &= w_\infty\biggl(\frac{b(R_2+R_1)-c(R_1+R_0)}{2}\biggr)\int_0^{\cdot} D_sY_s(1-Y_s)\,ds,\end{aligned}\tag{4.74}$$

as required.

(2°) Suppose that Assumption 4.4 with $L=3$ applies. We use (3.23) and (7.2) in place of (3.22) and (7.1) in the proof of (4.74), respectively, and obtain

$$\begin{aligned}\langle D, D\rangle &= \text{(d)-}\lim_{n\longrightarrow\infty} w_n^2\gamma_n\int_0^{\cdot}\bigl(D_s^{(n)}\bigr)^2\\ &\quad\times\sum_{x,y\in E_n} q^{(n)}(x,y)\bigl[A(x,\xi_{\gamma_n s})-B(y,\xi_{\gamma_n s})\bigr]^2\,ds\\ &= \text{(d)-}\lim_{n\longrightarrow\infty}\biggl(\frac{w_n^2N_n}{\nu_n(\mathbb{1})}\biggr)\gamma_n\nu_n(\mathbb{1})\int_0^{\cdot}\bigl(D_s^n\bigr)^2\bigl[b^2\bigl(W_4(\xi_{\gamma_n s})-W_2(\xi_{\gamma_n s})\bigr)\\ &\quad-2bc\bigl(W_3(\xi_{\gamma_n s})-W_1(\xi_{\gamma s})\bigr)+c^2W_2(\xi_{\gamma_n s})\bigr]\,ds\\ &= w_\infty^2\biggl(\frac{b^2(R_3+R_2)-2bc(R_2+R_1)+c^2(R_1+R_0)}{2}\biggr)\\ &\quad\times\int_0^{\cdot} D_s^2Y_s(1-Y_s)\,ds.\end{aligned}\tag{4.75}$$

Notice that in order to elicit the functions $W_\ell$'s in the second equality and to obtain the limit $w_\infty^2$ in the last equality, we have used the fact that $\pi^{(n)}(x)=\nu_n(\mathbb{1})=N_n^{-1}$ by the assumed symmetry of $q^{(n)}$. Putting (4.8), (4.74) and (4.75) together, we see that the vector of continuous semimartingales $(Y, M, D)$ satisfies the system in (4.17) by an enlargement of the underlying probability space if necessary (cf. [38], Theorem VII.2.7).

It remains to prove pathwise uniqueness in the system defined by (4.17). Notice that $D$ is equal to the Doléans–Dade exponential of $\widetilde{M}$, that is, $D=\mathcal{E}(\widetilde{M})$, by [38], p. 149, where $\widetilde{M}$ is a continuous martingale given by

$$\widetilde{M}_t=\int_0^t w_\infty\sqrt{Y_s(1-Y_s)}\bigl(K_1(b,c)\,dW_s^1+\sqrt{K_2(b,c)-K_1(b,c)^2}\,dW_s^2\bigr).$$

The equality $D = \mathcal{E}(\widetilde{M})$ and pathwise uniqueness in the closed system of $(Y, M)$ by the Yamada–Watanabe theorem [38], Theorem IX.3.5, plainly imply pathwise uniqueness in the system defined by (4.17). The proof is complete. □

**5. Wasserstein convergence of occupation measures of the game density processes.** Throughout this section, we consider the context where mutation is absent. Our goal in this section is to prove Wasserstein convergence of the occupation measures

$$\int_0^{\infty} f\big(Y_t^{(n)}\big)\,dt = \int_0^{T^{(n)}} f\big(Y_t^{(n)}\big)\,dt, \tag{5.1}$$

of the time-changed game density processes $Y^{(n)}$ defined by (4.1) under $\mathbb{P}^{(n),w_n}$. Here, under $\mathbb{P}^{(n),w_n}$, $T^{(n)} = T/\gamma_n$ and $T = T_{\mathbf{1}} \wedge T_{\mathbf{0}}$ is the consensus time of the evolutionary game with its generator given by $\mathsf{L}^{w_n,0}$, where $T_\eta$ is the first hitting time of configuration $\eta$ of the evolutionary game and $\boldsymbol{\sigma}$ is the all-$\sigma$ population configuration. The study below uses a good control of the rescaled times $T^{(n)}$ under voter models due to Oliveira [36, 37], and thus starts with an investigation of links between these random objects and convergences of the occupation measures in (5.1).

Since mutation is absent, the Feynman–Kac duality between voter models and coalescing Markov chains as discussed at the beginning of Section 4 is simpler and gives the following moment duality equations: with respect to a given voting kernel $(E, q)$, we have

$$\mathbb{E}_\xi\Big[\prod_{x\in A}\xi_t(x)\Big] = \mathbb{E}\Big[\prod_{x\in A}\xi\big(B_t^x\big)\Big] \qquad \forall \xi \in S^E,\ A \subseteq E. \tag{5.2}$$

The proof of (5.2) can be found in, for example, [26], Section III.4 or (8.1). To use these coalescing Markov chains, we write $\mathbb{P}_{\boldsymbol{x}}$ for $\boldsymbol{x} = (x_1, \ldots, x_N) \in E^N$ when a system of coalescing $(E, q)$-Markov chains starting from distinct components in $\boldsymbol{x}$ is under consideration. The first time that the number of distinct components in a system of coalescing Markov chains becomes less than or equal to $k$ is denoted by $\mathsf{C}_k$. Finally, we write $\boldsymbol{x}_E \in E^N$ for a vector whose components range over all points of $E$.

LEMMA 5.1. *Let a sequence of voting kernels $(E_n, q^{(n)})$ and a sequence of constants $\gamma_n$ increasing to infinity be given such that, for $\mathsf{C}_1^{(n)} = \mathsf{C}_1/\gamma_n$ under $\mathbb{P}^{(n)}_{\boldsymbol{x}_{E_n}}$, we have*

$$\big(\mathsf{C}_1^{(n)}, \mathbb{P}^{(n)}_{\boldsymbol{x}_{E_n}}\big) \xrightarrow[n\longrightarrow\infty]{\text{(d)}} \big(\mathsf{C}_1, \mathbb{P}^{(\infty)}\big), \tag{5.3}$$

*where the limiting object satisfies* $\mathbb{P}^{(\infty)}(\mathsf{C}_1 \in [0,\infty]) = 1$ *and* $\mathbb{P}^{(\infty)}(\mathsf{C}_1 = \infty) < 1$. *Then for some* $\theta > 0$, *it holds that*

$$\sup_{n\in\mathbb{N}} \sup_{\boldsymbol{x}\in E_n^{N_n}} \mathbb{E}_{\boldsymbol{x}}^{(n)}\big[\exp\{\theta \mathsf{C}_1^{(n)}\}\big] < \infty, \tag{5.4}$$

$$\sup_{n\in\mathbb{N}} \sup_{\lambda\in\mathscr{P}(S^{E_n})} \mathbb{E}_{\lambda}^{(n)}\big[\exp\{\theta T^{(n)}\}\big] < \infty. \tag{5.5}$$

*In particular*, $\mathbb{P}^{(\infty)}(\mathsf{C}_1 < \infty) = 1$.

PROOF. Since $\mathbb{P}^{(\infty)}(\mathsf{C}_1 = \infty) < 1$, we can find $t_0 \in (0,\infty)$ such that $\mathbb{P}^{(\infty)}(\mathsf{C}_1 > t_0) = \mathbb{P}^{(\infty)}(\mathsf{C}_1 \geq t_0) < 1$. Then by (5.3), given $\varepsilon > 0$ such that $\delta = 1 - \mathbb{P}^{(\infty)}(\mathsf{C}_1 > t_0) - \varepsilon > 0$, we can find some large enough integer $N_0 \geq 1$ such that, for all $n \geq N_0$,

$$\sup_{\boldsymbol{x}\in E_n^{N_n}} \mathbb{P}_{\boldsymbol{x}}^{(n)}\big(\mathsf{C}_1^{(n)} > t_0\big) \leq \mathbb{P}_{\boldsymbol{x}_{E_n}}^{(n)}\big(\mathsf{C}_1^{(n)} > t_0\big) \leq \mathbb{P}^{(\infty)}(\mathsf{C}_1 > t_0) + \varepsilon. \tag{5.6}$$

Now the proof of [36], Proposition 4.1, shows (5.4). In detail, first note that the Markov property of coalescing Markov chains and (5.6) imply

$$\sup_{\boldsymbol{x}\in E_n^{N_n}} \mathbb{P}_{\boldsymbol{x}}^{(n)}\big(\mathsf{C}_1^{(n)} > kt_0\big) \leq (1-\delta)^k \qquad \forall k \geq 0. \tag{5.7}$$

Hence, for any $n \geq N_0$, $\boldsymbol{x} \in E_n^{N_n}$ and $\theta > 0$ such that $e^{\theta t_0}(1-\delta) < 1$,

$$\begin{aligned}
&\mathbb{E}_{\boldsymbol{x}}^{(n)}\big[\exp\{\theta \mathsf{C}_1^{(n)}\}\big] \\
&\quad = \int_0^\infty \theta e^{\theta s} \mathbb{P}_{\boldsymbol{x}}^{(n)}\big(\mathsf{C}_1^{(n)} > s\big)\, ds + 1 \\
&\quad \leq \sum_{k=0}^{\infty} \theta t_0 e^{\theta(k+1)t_0} \mathbb{P}_{\boldsymbol{x}}^{(n)}\big(\mathsf{C}^{(n)} > kt_0\big) + 1 \leq \theta t_0 e^{\theta t_0} \sum_{k=0}^{\infty} e^{\theta t_0 k}(1-\delta)^k + 1 < \infty,
\end{aligned}$$

where the second inequality follows from (5.7) and the third inequality follows from the choice of $\theta$. The last inequality proves (5.4).

The inequality (5.5) follows from (5.4) and the stochastic dominance of $(T^{(n)}, \mathbb{P}_{\xi}^{(n)})$ by $(\mathsf{C}_1^{(n)}, \mathbb{P}_{\boldsymbol{x}_{E_n}}^{(n)})$: for any $\xi \in S^{E_n}$ and $t \geq 0$,

$$\begin{aligned}
\mathbb{P}_{\xi}^{(n)}\big(T^{(n)} \leq t\big) &= \mathbb{E}_{\xi}^{(n)}\Big[\prod_{x\in E_n} \xi_{\gamma_n t}(x) + \prod_{x\in E_n} \widehat{\xi}_{\gamma_n t}(x)\Big] \\
&= \mathbb{E}_{\boldsymbol{x}_{E_n}}^{(n)}\Big[\prod_{x\in E_n} \xi\big(B_{\gamma_n t}^x\big) + \prod_{x\in E_n} \widehat{\xi}\big(B_{\gamma_n t}^x\big)\Big] \geq \mathbb{P}_{\boldsymbol{x}_{E_n}}^{(n)}\big(\mathsf{C}_1^{(n)} \leq t\big),
\end{aligned}$$

where the second equality follows from (5.2) and the inequality follows since, when $\mathsf{C}_1^{(n)} \leq t$, the set $\{B_{\gamma_n t}^x; x \in E_n\}$ is singleton. The proof is complete. □

The main result of this section is Theorem 5.2 below. We recall that the Wasserstein distance of order 1 between two probability measures $\mu$ and $\nu$ on $\mathbb{R}$ with finite first moments is defined by

$$\mathcal{W}_1(\mu,\nu)=\inf\biggl\{\int_{\mathbb{R}^2}|x-y|\,d\pi(x,y);\pi\in\mathscr{P}\bigl(\mathbb{R}^2\bigr),\pi(\cdot\times\mathbb{R})=\mu,\pi(\mathbb{R}\times\cdot)=\nu\biggr\}.$$

The proof of Theorem 5.2 uses a standard result of Wasserstein distances in [41], which gives an alternative characterization of $\mathcal{W}_1(\mu,\nu)$ as

$$\mathcal{W}_1(\mu,\nu)=\int_{\mathbb{R}}\bigl|\mu\bigl([x,\infty)\bigr)-\nu\bigl([x,\infty)\bigr)\bigr|\,dx. \tag{5.8}$$

We write $\xrightarrow[n\longrightarrow\infty]{(\mathcal{W}_1)}$ for convergence with respect to the metric $\mathcal{W}_1$.

THEOREM 5.2. *Let the assumptions of Theorem 4.6(3°) with $\mu_n\equiv 0$ be in force, and assume*

$$\bigl(\mathsf{C}_\ell^{(n)},\mathbb{P}_{\boldsymbol{x}_{E_n}}^{(n)}\bigr)\xrightarrow[n\longrightarrow\infty]{(\mathrm{d})}\sum_{m=\ell+1}^{\infty}\frac{\mathbf{e}_m}{m(m-1)/2}\qquad\forall\ell\geq 1, \tag{5.9}$$

*where $\mathbf{e}_m$ are i.i.d. standard exponentials. Recall that $\beta_u$ on $S^{E_n}$ denotes the Bernoulli product measure with density $u\in(0,1)$ defined in Assumption 4.2. Then there exists $\overline{w}_\infty>0$ such that for any $\{w_n\}$ satisfying (4.14) with $w_\infty\in[0,\overline{w}_\infty]$, it holds that*

$$\bigl(Y^{(n)},T^{(n)}\bigr)\text{ under }\mathbb{P}_{\beta_u}^{(n),w_n}\xrightarrow[n\longrightarrow\infty]{(\mathrm{d})}(Y,\widetilde{T})\text{ under }\mathbb{P}_{\delta_u}^{(\infty),w_\infty}, \tag{5.10}$$

$$\mathbb{P}_{\beta_u}^{(n),w_n}(\boldsymbol{T_1}<\boldsymbol{T_0})\xrightarrow[n\longrightarrow\infty]{}\mathbb{P}_{\delta_u}^{(\infty),w_\infty}(\boldsymbol{T_1}<\boldsymbol{T_0}), \tag{5.11}$$

$$\int_0^{T^{(n)}}f\bigl(Y_s^{(n)}\bigr)\,ds\text{ under }\mathbb{P}_{\beta_u}^{(n),w_n}\xrightarrow[n\longrightarrow\infty]{(\mathcal{W}_1)}\int_0^{\widetilde{T}}f(Y_s)\,ds\text{ under }\mathbb{P}_{\delta_u}^{(\infty),w_\infty} \tag{5.12}$$

*for any nonnegative continuous function $f$ on $[0,1]$. Here, $\widetilde{T}$ is the time to absorption of $Y$.*

PROOF. Before proving the required convergences in (5.10)–(5.12), we first claim that

$$\bigl(Y^{(n)},D^{(n)},T^{(n)}\bigr)\text{ under }\mathbb{P}_{\beta_u}^{(n)}\xrightarrow[n\longrightarrow\infty]{(\mathrm{d})}(Y,D,\widetilde{T})\text{ under }\mathbb{P}^{(\infty)}. \tag{5.13}$$

We already have the convergence in distribution of $T^{(n)}$ to $\widetilde{T}$ by (4.8) with $\mu_n=0$, (5.9), and [8], Proposition 2.6. So we only need to show the joint convergence in (5.13).

By taking a subsequence if necessary and using Skorokhod's representation theorem (cf. [16], Theorem 3.1.8), we can reinforce the convergence in (5.13) to almost sure convergence, except that $T^{(n)}$ is only known a priori to converge almost

surely to a random variable $\widehat{T}$ with the same distribution as $\widetilde{T}$. Since $T^{(n)}$ (resp., $\widetilde{T}$) is a.s. equal to the time to absorption of $Y^{(n)}$ (resp., $Y$) and $Y^{(n)}$ converges to $Y$ a.s.,

$$\widehat{T} = \lim_{n\longrightarrow\infty} T^{(n)} \geq \widetilde{T} \qquad \text{a.s.}$$

The fact that $\widetilde{T} \overset{\text{(d)}}{=} \widehat{T}$ then implies $\widetilde{T} = \widehat{T}$ a.s., and we get

$$\bigl(Y^{(n)}, D^{(n)}, T^{(n)}\bigr) \xrightarrow[n\longrightarrow\infty]{\text{a.s.}} (Y, D, \widetilde{T}). \tag{5.14}$$

The claim in (5.13) follows.

Now we prove (5.10) and may assume (5.14). The rest of the proof does not use the particular initial conditions $\beta_u$. It follows from (3.7) and Proposition 4.13(2°) that for some $\overline{w}_\infty > 0$, the sequence $(D^{(n)}_{T^{(n)}}, \mathbb{P}^{(n)}_{\beta_u})$ is uniformly integrable for any $\{w_n\}$ satisfying (4.14) with $w_\infty \leq \overline{w}_\infty$, where $w_\infty$ is defined by (4.14). Therefore, for every bounded continuous function $F$ on $D(\mathbb{R}_+, [0,1]) \times \mathbb{R}_+$, we have

$$\begin{aligned}\mathbb{E}^{(n),w_n}_{\beta_u}\bigl[F\bigl(Y^{(n)}, T^{(n)}\bigr)\bigr] &= \mathbb{E}^{(n)}_{\beta_u}\bigl[F\bigl(Y^{(n)}, T^{(n)}\bigr)D^{(n)}_{T^{(n)}}\bigr] \\ &\xrightarrow[n\longrightarrow\infty]{} \mathbb{E}^{(\infty)}\bigl[F(Y, \widetilde{T})D_{\widetilde{T}}\bigr] = \mathbb{E}^{(\infty),w_\infty}_{\delta_u}\bigl[F(Y, \widetilde{T})\bigr],\end{aligned}$$

which is enough for (5.10).

For the proof of (5.11), notice that $Y^{(n)}_{T^{(n)}}$ under $\mathbb{P}^{(n),w_n}_{\beta_u}$ converges in distribution to $Y_{\widetilde{T}}$ under $\mathbb{P}^{(\infty),w_\infty}_{\delta_u}$ by (5.10) and [16], Proposition 3.6.5. Since $Y^{(n)}_{T^{(n)}}$ take values in $\{1,0\}$, $\{T^{(n)}_{\mathbf{1}} < T^{(n)}_{\mathbf{0}}\} = \{Y^{(n)}_{T^{(n)}} = 1\}$ and a similar equality holds under $Y$, the convergence in (5.11) follows.

For the proof of (5.12), notice that by (4.69), (5.10) and [16], Proposition 3.6.5,

$$\left(\int_0^{T^{(n)}} f\bigl(Y^{(n)}_s\bigr)\,ds, \mathbb{P}^{(n),w_n}_{\beta_u}\right) \xrightarrow[n\longrightarrow\infty]{\text{(d)}} \left(\int_0^{\widetilde{T}} f(Y_s)\,ds, \mathbb{P}^{(\infty),w_\infty}_{\delta_u}\right).$$

Hence, to verify the required convergence in the Wasserstein distance by means of (5.8), it is enough to prove uniformly exponential tails of the distributions of $(T^{(n)}, \mathbb{P}^{(n),w_n}_{\beta_u})$. Notice that the conclusions of Lemma 5.1 apply by (5.9) with $\ell = 1$. Therefore,

$$\begin{aligned}&\sup_{n\in\mathbb{N}} \mathbb{E}^{(n),w_n}_{\beta_u}\bigl[\exp\{\theta T^{(n)}\}\bigr] \\ &\quad= \sup_{n\in\mathbb{N}} \mathbb{E}^{(n)}_{\beta_u}\bigl[\exp\{\theta T^{(n)}\}D^{(n)}_{T^{(n)}}\bigr] \\ &\quad\leq \sup_{n\in\mathbb{N}} \mathbb{E}^{(n)}_{\beta_u}\bigl[\exp\{2\theta T^{(n)}\}\bigr]^{1/2}\mathbb{E}^{(n)}_{\beta_u}\bigl[\bigl(D^{(n)}_{T^{(n)}}\bigr)^2\bigr]^{1/2} < \infty,\end{aligned}$$

where the first inequality follows from the Cauchy–Schwarz inequality and the last inequality follows from from Lemma 5.1, (3.7) and Proposition 4.13(2°), if

$\theta > 0$ is small enough and $w_\infty \le \overline{w}_\infty$ by lowering the constant $\overline{w}_\infty$ chosen above if necessary. The foregoing inequality is enough for the proof of (5.12). The proof of the theorem is complete. □

The following proposition proves the diffusion approximation of absorbing probabilities in [34], SI, when Bernoulli product measures are used as initial conditions.

COROLLARY 5.3. *For any fixed $k \ge 3$, the conclusions of Theorem 5.2 apply to any sequence of random $k$-regular graphs on $N_n$ vertices with $N_n \nearrow \infty$.*

PROOF. Since $q^{(n)}(x, y) \le 1/k$ and $\pi^{(n)}(x) \equiv 1/N_n$, the proofs of [37], Theorem 1.1 and Theorem 1.2, show that (5.9) holds if

$$\mathbf{t}^{(n)}_{\mathrm{mix}}/N_n \xrightarrow[n\to\infty]{} 0+ \tag{5.15}$$

(see [37], Lemma 5.1 and Section 6, in particular). Then to verify that condition in Theorem 4.3, we recall a standard result of Markov chains:

$$\mathbf{t}^{(n)}_{\mathrm{mix}} \le \bigl(\mathbf{g}^\star_n\bigr)^{-1} \log\bigl(2e/\pi^{(n)}_{\min}\bigr) = \bigl(\mathbf{g}^\star_n\bigr)^{-1} \log(2eN_n) \tag{5.16}$$

(cf. [25], Theorem 12.3). Here, $\mathbf{g}^\star_n$ is the absolute spectral gap of $(E_n, q^{(n)})$, and is given by the distance between 1 and the maximal absolute values of eigenvalues of $(E_n, q^{(n)})$ excluding the largest one. The fact that $\mathbf{g}^\star_n$ are bounded away from zero is also contained in the main results of [5, 18], and can be applied to (5.16) to validate (5.15). □

**6. Expansions of the game absorbing probabilities in selection strength.** As in the previous section, we focus on the context where mutation is absent. We use payoff matrices with general entries throughout this section unless otherwise.

Write $\partial_w = \partial/\partial w$. With respect to a voting kernel $(E, q)$ and $\lambda \in \mathscr{P}(S^E)$ such that

$$\partial_w \mathbb{P}^w_\lambda(T_{\mathbf{1}} < T_{\mathbf{0}})|_{w=0} \ne 0, \tag{6.1}$$

we define $w^\star(\lambda; E, q)$ to be the supremum of $w'' \in [0, \overline{w}]$ such that either

$$\operatorname{sgn}\bigl(\partial_w \mathbb{P}^w_\lambda(T_{\mathbf{1}} < T_{\mathbf{0}})|_{w=0}\bigr) = \operatorname{sgn}\bigl(\mathbb{P}^{w'}_\lambda(T_{\mathbf{1}} < T_{\mathbf{0}}) - \mathbb{P}^0_\lambda(T_{\mathbf{1}} < T_{\mathbf{0}})\bigr),$$

for all $w' \in (0, w'']$, or

$$\operatorname{sgn}\bigl(\partial_w \mathbb{P}^w_\lambda(T_{\mathbf{1}} < T_{\mathbf{0}})|_{w=0}\bigr) = -\operatorname{sgn}\bigl(\mathbb{P}^{w'}_\lambda(T_{\mathbf{1}} < T_{\mathbf{0}}) - \mathbb{P}^0_\lambda(T_{\mathbf{1}} < T_{\mathbf{0}})\bigr)$$

for all $w' \in (0, w'']$, where we set $\operatorname{sgn}(0) = 0$. In other words, the interval $(0, w^\star(\lambda; E, q)]$ gives a maximal range of selection strengths $w'$ such that the first-order derivative in (6.1) has the same sign as $\mathbb{P}^{w'}_\lambda(T_{\mathbf{1}} < T_{\mathbf{0}}) - \mathbb{P}^0_\lambda(T_{\mathbf{1}} < T_{\mathbf{0}})$, whereas the derivative in (6.1) can be expressed as a functional of the voter model.

Our goal in this section is to estimate the order of $w^\star(\lambda; E, q)$ relative to $N$. To this end, our focus is on the remainder in the first-order Taylor expansion of $w \mapsto \mathbb{P}^w_\lambda(T_{\mathbf{1}} < T_{\mathbf{0}})$. The idea which we develop along in the following is that if, for $w \geq 0$, we have

$$f(w) = f(0) + wf'(0) + R(w)$$

and $|R(w)| \leq Cw^2$, then $f(w) > f(0)$ [resp. $f(w) < f(0)$] for all $w \in (0, |f'(0)|/C)$ whenever $f'(0) > 0$ [resp., $f'(0) < 0$]. It seems to us that this somewhat naive route could only lead to crude estimates for $w^\star(\lambda; E, q)$ defined above, although the main result proven below (Theorem 6.6) is capable of using some weak selection strengths for the comparison of fixation probabilities. It would be interesting to determine whether the popular first-order derivative test in the biological literature is useful beyond the weak selection regime in general, where almost no mathematical results for fixation probabilities of stochastic spatial evolutionary games are known up to now.

To obtain lower bounds for the thresholds $w^\star(\lambda; E, q)$, the first step we take here is prove the first-order expansion $w \mapsto \mathbb{P}^w_\lambda(T_{\mathbf{1}} < T_{\mathbf{0}})$ with a quantifiable error bound for the remainder. See Proposition 6.3 for this expansion. For the proof, let us introduce some notation for the use of coalescing Markov chains. Write $\mathscr{P}$ for the set of functions $F$ on $S^E$ taking the form

$$F(\xi) = \sum_{(A_1, A_2) \in \mathscr{A}} C(A_1, A_2) \prod_{x \in A_1} \xi(x) \prod_{x \in A_2} \widehat{\xi}(x), \tag{6.2}$$

where $(A_1, A_2)$ are pairs of disjoint nonempty subsets of $E$ and $C(A_1, A_2)$ are constants. $\mathscr{P}_+$ denotes the class of functions $F \in \mathscr{P}$ which admit representations as above in (6.2) such that $C(A_1, A_2) \geq 0$ for all $(A_1, A_2) \in \mathscr{A}$. Notice that $F$ is a polynomial in $\xi(x)$ for $x \in E$ and satisfies $F(\mathbf{1}) = F(\mathbf{0}) = 0$. Also, the representation of $F \in \mathscr{P}$ in (6.2) is not unique.

Certain functions in $\mathscr{P}$ will be used to quantify the error bound in the forthcoming Taylor expansion of $w \mapsto \mathbb{P}^w_\lambda(T_{\mathbf{1}} < T_{\mathbf{0}})$. We need the following two lemmas. Recall the auxiliary discrete-time Markov chains $(X_\ell)$ and $(Y_\ell)$ defined at the beginning of Section 4.4. The first lemma follows from a plain generalization of the argument for (3.15) [see (3.10)], and so its proof is omitted.

LEMMA 6.1. *For all $m \geq 1$, $w \in [0, \overline{w}]$ and $\xi \in S^E$,*

$$\sum_{x,y \in E} \pi(x) \left| q^w(x, y, \xi) - \sum_{j=0}^{m-1} \big(\partial_w^j q^w(x, y, \xi)|_{w=0}\big) w^j \right| \leq w^m Q_m(\xi), \tag{6.3}$$

*where the function $Q_m(\xi) \in \mathscr{P}_+$ is given by*

$$Q_m(\xi) = C_{(6.4)}(m) \sum_{\ell=1}^{4} W_\ell(\xi) \tag{6.4}$$

*for some constant $C_{(6.4)}(m) > 0$ depending only on $m$ and $\Pi$.*

For any nonempty set $A \subseteq E$, we write $B^A$ for the subsystem $\{B^x; x \in A\}$ of coalescing $q$-Markov chains. We also write $M_{A_1,A_2}$ for the first meeting time of the two subsystems $B^{A_1}$ and $B^{A_2}$, or more precisely, the first time $t$ when $B_t^x = B_t^y$ for some $x \in A_1$ and $y \in A_2$. If we follow the usual alternative viewpoint that the processes $B^{A_1}$ and $B^{A_2}$ are set-valued processes, then $M_{A_1,A_2}$ is the first time that the two processes "intersect."

The second lemma gives bounds of the game potentials $\mathbb{E}^w[\int_0^\infty F(\xi_s)\,ds]$ by the voter model potentials $\mathbb{E}^0[\int_0^\infty F(\xi_s)\,ds]$, and thus is a key step to study these game potentials by coalescing Markov chains.

LEMMA 6.2. *For all $w \in [0, \overline{w}]$, $\xi \in S^E$, and $F \in \mathscr{P}$ taking the form (6.2), it holds that*

$$\begin{aligned}&\left|\mathbb{E}_\xi^0\left[\int_0^\infty D_s^w F(\xi_s)\,ds\right] - \mathbb{E}_\xi^0\left[\int_0^\infty F(\xi_s)\,ds\right]\right| \\ &\qquad \le \frac{wC_{(6.4)}(1)\cdot C_{(6.6)}(F)}{\pi_{\min}}\mathbb{E}_\xi^0\left[\int_0^\infty D_s^w Q_1(\xi_s)\,ds\right],\end{aligned} \tag{6.5}$$

*where $Q_1$ is chosen in Lemma 6.1 and the constant $C_{(6.6)}(F)$ is defined with respect to (6.2) by*

$$\begin{aligned}&C_{(6.6)}(F) \\ &\quad = \max_{x\in E}\sum_{(A_1,A_2)\in\mathscr{A}} \big|C(A_1,A_2)\big| \int_0^\infty \mathbb{P}\big(x \in B_t^{A_1} \cup B_t^{A_2}, M_{A_1,A_2} > t\big)\,dt.\end{aligned} \tag{6.6}$$

*In particular, if we choose $F = Q_1$ and selection strength $w$ satisfying*

$$0 \le w \le \min\left\{\overline{w}, \frac{\pi_{\min}}{2C_{(6.4)}(1)\cdot C_{(6.6)}(Q_1)}\right\}, \tag{6.7}$$

*then (6.5) gives*

$$\mathbb{E}_\xi^0\left[\int_0^\infty D_s^w Q_1(\xi_s)\,ds\right] \le \frac{\mathbb{E}_\xi^0[\int_0^\infty Q_1(\xi_s)\,ds]}{1 - wC_{(6.4)}(1)\cdot C_{(6.6)}(Q_1)\pi_{\min}^{-1}}. \tag{6.8}$$

PROOF. By the linear equation (3.4) satisfied by $D^w$, it holds that

$$\begin{aligned}&\int_0^\infty D_s^w F(\xi_s)\,ds \\ &\quad = \int_0^\infty F(\xi_s)\,ds \\ &\qquad + \int_0^\infty \sum_{x,y\in E}\int_0^s D_{r-}^w\left(\frac{q^w(x,y,\xi_{r-})}{q(x,y)} - 1\right)d\Lambda_r(x,y)F(\xi_s)\,ds.\end{aligned} \tag{6.9}$$

Notice that the left-hand side has a finite $\mathbb{E}_\xi^0$-expectation since the time to absorption under $\mathbb{E}_\xi^w$ has a tail distribution which decays exponentially by a standard result of finite absorbing Markov chains and so is integrable. Similar reasons apply to obtain the $\mathbb{E}_\xi^0$-integrability of the other integrals in (6.9). The $\mathbb{E}_\xi^0$-expectation of the second term on the right-hand side of (6.9) satisfies the following equations:

$$
\begin{aligned}
&\mathbb{E}_\xi^0\Bigg[\int_0^\infty \sum_{x,y\in E}\int_0^s D_{r-}^w\bigg(\frac{q^w(x,y,\xi_{r-})}{q(x,y)}-1\bigg)\,d\Lambda_r(x,y)F(\xi_s)\,ds\Bigg]\\
&\quad=\sum_{x,y\in E}\mathbb{E}_\xi^0\Bigg[\int_0^\infty D_{r-}^w\bigg(\frac{q^w(x,y,\xi_{r-})}{q(x,y)}-1\bigg)\int_r^\infty F(\xi_s)\,ds\,d\Lambda_r(x,y)\Bigg]\\
&\quad=\sum_{x,y\in E}\mathbb{E}_\xi^0\Bigg[\int_0^\infty D_{r-}^w\bigg(\frac{q^w(x,y,\xi_{r-})}{q(x,y)}-1\bigg)\\
&\qquad\times\mathbb{E}_{\xi_r}^0\bigg[\int_0^\infty F(\xi_s)\,ds\bigg]\,d\Lambda_r(x,y)\Bigg]\\
&\quad=\sum_{x,y\in E}\mathbb{E}_\xi^0\Bigg[\int_0^\infty D_{r-}^w\bigg(\frac{q^w(x,y,\xi_{r-})}{q(x,y)}-1\bigg)\\
&\qquad\times\mathbb{E}_{\xi_{r-}}^0\bigg[\int_0^\infty F(\xi_s)\,ds\bigg]\,d\Lambda_r(x,y)\Bigg]\\
&\qquad+\sum_{x,y\in E}\mathbb{E}_\xi^0\Bigg[\int_0^\infty D_{r-}^w\bigg(\frac{q^w(x,y,\xi_{r-})}{q(x,y)}-1\bigg)\\
&\qquad\times\bigg(\mathbb{E}_{\xi_r}^0\bigg[\int_0^\infty F(\xi_s)\,ds\bigg]-\mathbb{E}_{\xi_{r-}}^0\bigg[\int_0^\infty F(\xi_s)\,ds\bigg]\bigg)\,d\Lambda_r(x,y)\Bigg]\\
&\quad=\sum_{x,y\in E}\mathbb{E}_\xi^0\Bigg[\int_0^\infty D_{r-}^w\bigg(\frac{q^w(x,y,\xi_{r-})}{q(x,y)}-1\bigg)\\
&\qquad\times\bigg(\mathbb{E}_{\xi_r}^0\bigg[\int_0^\infty F(\xi_s)\,ds\bigg]-\mathbb{E}_{\xi_{r-}}^0\bigg[\int_0^\infty F(\xi_s)\,ds\bigg]\bigg)\,d\Lambda_r(x,y)\Bigg],
\end{aligned}
\tag{6.10}
$$

where the second equality follows from the $(\mathscr{F}_t)$-strong Markov property of $(\xi_t)$ and the last equality follows from Poisson calculus and the fact that

$$
1=\sum_{y\in E}q^w(x,y,\xi)=\sum_{y\in E}q(x,y)\qquad\forall x\in E,\xi\in S^E. \tag{6.11}
$$

[Recall (2.5) for the definition of $q^w(x,y,\xi)$.]

We study the right-hand side of (6.10). Since $F$ is a polynomial in $\xi(x)$ for $x\in E$, Equation (5.2) applies to the evaluation of $\mathbb{E}_\xi^0[\int_0^\infty F(\xi_s)\,ds]$ by $\{B^x\}$. We

also observe that

$$\begin{aligned}(6.12)\qquad &\sup_{\xi\in S^E}\Big|\prod_{y\in A_1}\widehat{\xi^x}(B_t^y)\prod_{y\in A_2}\xi^x(B_t^y)-\prod_{y\in A_1}\widehat{\xi}(B_t^y)\prod_{y\in A_2}\xi(B_t^y)\Big|\\ &\qquad\le \mathbb{1}_{(t,\infty)}(M_{A_1,A_2})\mathbb{1}_{B_t^{A_1}\cup B_t^{A_2}}(x)\qquad \forall x\in E.\end{aligned}$$

Hence, we see that, for all $x\in E$ and $\xi\in S^E$,

$$\begin{aligned}(6.13)\qquad &\Big|\mathbb{E}^0_{\xi^x}\Big[\int_0^\infty F(\xi_s)\,ds\Big]-\mathbb{E}^0_{\xi}\Big[\int_0^\infty F(\xi_s)\,ds\Big]\Big|\\ &\quad\le\sum_{(A_1,A_2)\in\mathscr{A}}C(A_1,A_2)\int_0^\infty\mathbb{P}\big(x\in B_t^{A_1}\cup B_t^{A_2},M_{A_1,A_2}>t\big)\,dt\\ &\quad= C_{(6.6)}(F)\end{aligned}$$

by the definition (6.6) of $C_{(6.6)}(F)$. Now we apply the foregoing inequality to the right-hand side of (6.10) and use Poisson calculus and the inequality (6.3) with $m=1$. Then by (6.10) and (6.13), we get

$$\begin{aligned}&\Big|\mathbb{E}^0_{\xi}\Big[\int_0^\infty\sum_{x,y\in E}\int_0^s D^w_{r-}\Big(\frac{q^w(x,y,\xi_{r-})}{q(x,y)}-1\Big)\,d\Lambda_r(x,y)F(\xi_s)\,ds\Big]\Big|\\ &\quad\le\frac{wC_{(6.4)}(1)\cdot C_{(6.6)}(F)}{\pi_{\min}}\mathbb{E}^0_{\xi}\Big[\int_0^\infty D^w_s Q_1(\xi_s)\,ds\Big].\end{aligned}$$

The required inequality (6.5) follows from the last inequality and (6.9). The proof is complete. □

We are ready to state the first-order expansion of the game absorbing probabilities. This result in (6.14) recovers the first-order expansion in [6] but now the bound of the remainder term has a simple expression.

PROPOSITION 6.3. *Fix a choice of functions $Q_1$ and $Q_2$ defined by (6.4). For any $w\in[0,\overline{w}]$ and $\xi\in S^E$, it holds that*

$$\begin{aligned}(6.14)\qquad &\Big|\mathbb{P}^w_\xi(T_{\mathbf{1}}<T_{\mathbf{0}})-\mathbb{P}^0_\xi(T_{\mathbf{1}}<T_{\mathbf{0}})-w\mathbb{E}^0_\xi\Big[\int_0^\infty\overline{D}(\xi_s)\,ds\Big]\Big|\\ &\quad\le\frac{w^2C_{(6.4)}(1)\cdot C_{(6.6)}(\overline{D})}{\pi_{\min}}\mathbb{E}^0_\xi\Big[\int_0^\infty D^w_sQ_1(\xi_s)\,ds\Big]\\ &\qquad+w^2C_{(6.4)}(2)\mathbb{E}^0_\xi\Big[\int_0^\infty D^w_sQ_2(\xi_s)\,ds\Big],\end{aligned}$$

*where $\overline{D}$ is defined by (3.25).*

PROOF. By the definition of $\mathbb{P}^w$ in (2.11) and the fact that $p_1(\mathbf{1}) = 1$ and $p_1(\mathbf{0}) = 0$, it holds that for all $\xi \in S^E$,

$$\begin{aligned}
&\mathbb{P}_\xi^w(T_{\mathbf{1}} < T_{\mathbf{0}}) \\
&= \lim_{t\longrightarrow\infty} \mathbb{P}_\xi^w\big[p_1(\xi_t)\big] = \lim_{t\longrightarrow\infty} \mathbb{E}_\xi^0\big[D_t^w p_1(\xi_t)\big] \\
&= \lim_{t\longrightarrow\infty} \big(p_1(\xi) + \mathbb{E}_\xi^0\big[\langle D^w, p_1(\xi.)\rangle_t\big]\big) = p_1(\xi) + \mathbb{E}_\xi^0\big[\langle D^w, p_1(\xi.)\rangle_\infty\big],
\end{aligned} \tag{6.15}$$

where the third equality follows from integration by parts since $D^w$ and $p_1(\xi.)$ are both $\mathbb{P}_\xi^0$-martingales [23], Theorem I.4.2, and the last equality follows from dominated convergence by (3.28) since the time to absorption is integrable under $\mathbb{P}_\xi^w$.

Now we expand the last term in (6.15) in $w$. It follows from (3.22) and (3.25) that

$$\langle D^w, p_1(\xi.)\rangle_\infty = w\int_0^\infty D_s^w \overline{D}(\xi_s)\,ds + w^2\int_0^\infty D_s^w R_1^w(\xi_s)\,ds. \tag{6.16}$$

By Lemma 6.2, the first term on the right-hand side of (6.16) satisfies

$$\begin{aligned}
&\left|w\mathbb{E}_\xi^0\left[\int_0^\infty D_s^w \overline{D}(\xi_s)\,ds\right] - w\mathbb{E}_\xi^0\left[\int_0^\infty \overline{D}(\xi_s)\,ds\right]\right| \\
&\qquad \le \frac{w^2 C_{(6.4)}(1)\cdot C_{(6.6)}(\overline{D})}{\pi_{\min}}\mathbb{E}_\xi^0\left[\int_0^\infty D_s^w Q_1(\xi_s)\,ds\right].
\end{aligned}$$

By the definition of $R_1^w$ in (3.26) [see also (3.11)] and the choice of $Q_2$ according to (6.3), we also have

$$\left|w^2\mathbb{E}_\xi^0\left[\int_0^\infty D_s^w R_1^w(\xi_s)\,ds\right]\right| \le w^2 C_{(6.4)}(2)\mathbb{E}_\xi^0\left[\int_0^\infty D_s^w Q_2(\xi_s)\,ds\right].$$

Applying the last three displays to the last term in (6.15), we deduce the required inequality (6.14). The proof is complete. □

Our next step is to prove that the bound for the remainder term in (6.14), that is the sum on its right-hand side, is comparable to the first-derivative term $w\mathbb{E}_\xi^0[\int_0^\infty \overline{D}(\xi_s)\,ds]$ if $w$ is enough small and of the order $1/N$.

We use the following lemma to bound the constant $C_{(6.6)}(\overline{D})$. This amounts to bounding the constants $C_{(6.6)}(W_\ell)$ by a generalization of (7.1) to the case of general payoff matrices. In this generalization, $\overline{D}(\xi)$ remains a finite linear combinations of $W_\ell(\xi)$ for small $\ell$'s.

LEMMA 6.4. *For any $\ell \ge 1$ and $x \ne y$,*

$$\int_0^\infty \mathbb{P}_\pi\big(B_s^{X_0} = x, B_s^{X_\ell} = y\big)\,ds \le C_{(6.17)}\left(\frac{\pi_{\max}}{\pi_{\min}}\right)\frac{\pi(x)\pi(y)}{\nu(\mathbb{1})} \tag{6.17}$$

*for some constant $C_{(6.17)}$ depending only on $\ell$. In particular, we have*

$$C_{(6.6)}(W_\ell) \le 2C_{(6.17)}\left(\frac{\pi_{\max}}{\pi_{\min}}\right)^2. \tag{6.18}$$

PROOF. The required inequality (6.17) for $\ell = 1$ is a particular consequence of Kac's formula (cf. [2], Section 2.5.1). Below we give an alternative proof of (6.17) with $\ell = 1$ by voter model calculations.

To see the proof of (6.17) for $\ell \ge 2$, we notice the following general result. If $F$ is a nonnegative function on $E \times E$ which vanishes on the diagonal, then (4.30) implies that for all $\ell \ge 1$,

$$\begin{aligned}
&\int_0^\infty \mathbb{E}\big[F\big(B_t^{X_0}, B_t^{X_\ell}\big)\big]\,dt \\
&\quad = \frac{\mathbb{E}[F(X_0, X_\ell)]}{2} + \frac{1}{2}\int_0^\infty \mathbb{E}\big[F\big(B_t^{X_0}, B_t^{X_{\ell+1}}\big)\mathbb{1}_{\{X_0 \ne X_\ell\}}\big]\,dt \\
&\qquad + \frac{1}{2}\int_0^\infty \mathbb{E}\big[F\big(B_t^{X_{\ell+1}}, B_t^{X_0}\big)\mathbb{1}_{\{X_0 \ne X_\ell\}}\big]\,dt \\
&\quad = \frac{\mathbb{E}[F(X_0, X_\ell)]}{2} + \int_0^\infty \mathbb{E}\big[F\big(B_t^{X_0}, B_t^{X_{\ell+1}}\big)\big]\,dt \\
&\qquad - \int_0^\infty \mathbb{E}\big[F\big(B_t^{X_\ell}, B_t^{X_{\ell+1}}\big)\mathbb{1}_{\{X_0 = X_\ell\}}\big]\,dt,
\end{aligned}$$

and so

$$\int_0^\infty \mathbb{E}\big[F\big(B_t^{X_0}, B_t^{X_\ell}\big)\big]\,dt + \int_0^\infty \mathbb{E}\big[F\big(B_t^{X_0}, B_t^{X_1}\big)\big]\,dt \ge \int_0^\infty \mathbb{E}\big[F\big(B_t^{X_0}, B_t^{X_{\ell+1}}\big)\big]\,dt.$$

Thus (6.17) for general $\ell \ge 2$ follows from the above inequality and the validity of (6.17) for $\ell = 1$. The inequality (6.18) then follows by writing out $C_{(6.6)}(W_\ell)$:

$$C_{(6.6)}(W_\ell) = \max_{x \in E} \sum_{u,v \in E} \pi(u) q^\ell(u, v) \int_0^\infty \mathbb{P}\big(x \in B_t^{\{u,v\}}, M_{u,v} > t\big)\,dt$$

and using (6.17) and the fact that $\nu(\mathbb{1}) \ge \pi_{\min}$.

Now we give a proof of (6.17) with $\ell = 1$ by voter model calculations. It follows from (3.21) and (3.30) that for all $\xi \in S^E$,

$$\begin{aligned}
&\mathbb{E}_\xi^0\big[p_1(\xi_t)p_0(\xi_t)\big] \\
&\quad = p_1(\xi)p_0(\xi) \\
&\qquad - \sum_{x,y \in E} \pi(x)^2 q(x, y) \int_0^t \mathbb{E}_\xi^0\big[\xi_s(x)\widehat{\xi}_s(y) + \widehat{\xi}_s(x)\xi_s(y)\big]\,ds
\end{aligned} \tag{6.19}$$

(see also [8], Theorem 3.1). Passing $t \longrightarrow \infty$ for both sides of the foregoing equality, we get

$$\sum_{x,y\in E} \pi(x)^2 q(x,y) \int_0^\infty \mathbb{E}^0_\xi\big[\xi_s(x)\widehat{\xi}_s(y) + \widehat{\xi}_s(x)\xi_s(y)\big]\,ds = \frac{p_1 p_0(\xi)}{2} \tag{6.20}$$

for all $\xi \in S^E$. By the duality equation in (5.2),

$$\mathbb{E}^0_\xi\big[\xi_s(x)\widehat{\xi}_s(y)\big] = \mathbb{E}\big[\xi\big(B^x_s\big)\widehat{\xi}\big(B^y_s\big)\big] = \sum_{z,z'\in E} \xi(z)\mathbb{P}\big(B^x_s = z,\, B^y_s = z'\big)\mathbb{1}_{\{z\neq z'\}}\widehat{\xi}(z').$$

Hence, both sides of (6.20) take the form $\xi^\top A\widehat{\xi}$ for a symmetric $N \times N$-matrix $A$ with zero diagonal entries, where population configuration $\xi$ is regarded as a column vector. By the linearity of the map,

$$A \longmapsto \xi^\top A\widehat{\xi} = \sum_{x,y\in E} \xi(x)A_{x,y}\widehat{\xi}(y)$$

and the inequality $\nu(\mathbb{1}) \le \pi_{\max}$, to prove (6.17) for $\ell = 1$, it suffices to show that, for a symmetric $N \times N$ matrix $A$ with zero diagonal entries,

$$\xi^\top A\widehat{\xi} = 0 \qquad \forall \xi \in S^E \Longrightarrow A = 0. \tag{6.21}$$

The following proof of (6.21) is due to Rani Hod [21]. Let $\{e_x; x \in E\}$ be the standard basis of $S^E$. Taking $\xi = e_x$ for $x \in E$ in (6.21), we obtain that $e_x^\top A(\sum_{y:y\neq x} e_y) = 0$ and so $e_x^\top A\mathbb{1} = 0$ by the assumption that $A$ has zero diagonal entries. Hence, taking $\xi = e_x + e_y$ for $x \neq y$ in (6.21), we obtain from (6.21) that

$$\begin{aligned} 0 = -(e_x + e_y)^\top A(\mathbb{1} - e_x - e_y) &= A_{x,x} + A_{x,y} + A_{y,x} + A_{y,y} \\ &= A_{x,y} + A_{y,x} = 2A_{x,y} \end{aligned}$$

since $A$ is symmetric. The last equality proves that $A = 0$. This completes the proof. □

EXAMPLE 6.5 (Uniform initial conditions). We show by an example that, relative to the population size $N$, the bound for $C_{(6.6)}(\overline{D})$ from Lemma 6.4, which is $\mathcal{O}(1)$, is sharp in terms of the order of

$$\mathbb{E}^0_\xi\bigg[\int_0^\infty \overline{D}(\xi_s)\,ds\bigg].$$

Recall that the above term is the first-order coefficient in the expansion (6.14) of the game absorbing probabilities.

Let $(E, q)$ be the random walk transition probability on a finite, simple, connected, $k$-regular graph with $N$ vertices, and let $\mathbf{u}_m$ be the uniform probability measure on the set of $S^E$-valued configurations with exactly $m$ many 1's. Assume

that $\Pi$ is given by the special payoff matrix (1.12). By Lemma 6.4, the moment duality equation (5.2) and the random-walk representation of $\overline{D}$ in (7.1), we deduce that, for any $1 \le m \le N-1$,

$$\begin{aligned}(6.22)\qquad &\left|\mathbb{E}^0_{\mathbf{u}_m}\left[\int_0^\infty \overline{D}(\xi_s)\,ds\right]\right| \\ &\quad\le C_{(6.22)}\frac{\pi_{\max}}{\pi_{\min}\nu(\mathbb{1})}\sum_{x,y\in E}\mathbf{u}_m\big[\xi(x)\widehat{\xi}(y)\big]\pi(x)\pi(y),\end{aligned}$$

where the constant $C_{(6.22)}$ depends only on $\Pi$. In the present case of random walks on regular graphs, the above inequality simplifies to

$$\begin{aligned}(6.23)\qquad \left|\mathbb{E}^0_{\mathbf{u}_m}\left[\int_0^\infty \overline{D}(\xi_s)\,ds\right]\right| &\le C_{(6.22)}\sum_{x,y\in E}\frac{\mathbf{u}_m[\xi(x)\widehat{\xi}(y)]}{N} \\ &= C_{(6.22)}\frac{m(N-m)}{N}\end{aligned}$$

(cf. [10], equation (68), for the last equality). On the other hand, it has been proven that

$$(6.24)\qquad \mathbb{E}^0_{\mathbf{u}_m}\left[\int_0^\infty \overline{D}(\xi_s)\,ds\right] = \frac{m(N-m)}{2N(N-1)}\big[b(N-2k)-ck(N-2)\big]$$

(cf. [6], Theorem 1, or [10], Proposition 10).

From (6.23) and (6.24), we see that Lemma 6.4 gives a sharp estimate of $|\mathbb{E}^0_{\mathbf{u}_m}[\int_0^\infty \overline{D}(\xi_s)\,ds]|$ relative to the population size $N$.

The main result of Section 6 is the following theorem. See Example 6.5 for the choice of the denominators in one of its conditions, (6.25), and recall the notation $w^\star(\lambda; E, q)$ defined at the beginning of Section 6.

THEOREM 6.6. *Let a sequence of voting kernels* $\{(E_n, q^{(n)})\}$ *and* $\lambda_n \in \mathscr{P}(S^{E_n})$ *be given such that Assumption 4.1 holds and*

$$(6.25)\qquad \liminf_{n\longrightarrow\infty}\frac{|\mathbb{E}^{(n)}_{\lambda_n}[\int_0^\infty \overline{D}(\xi_s)\,ds]|}{N_n\sum_{x,y\in E_n}\lambda_n[\xi(x)\widehat{\xi}(y)]\pi^{(n)}(x)\pi^{(n)}(y)} > 0.$$

*Then for some positive constant* $C_{(6.26)}$ *depending only on* $\Pi$, $\limsup \pi^{(n)}_{\max}/\pi^{(n)}_{\min}$, *and the above limit infimum, it holds that*

$$(6.26)\qquad \liminf_{n\longrightarrow\infty} N_n w^\star\big(\lambda_n; E_n, q^{(n)}\big) \ge C_{(6.26)} > 0.$$

PROOF. Constants defined in this proof depend only on $\Pi$ and $\limsup \pi^{(n)}_{\max}/\pi^{(n)}_{\min}$. Now it follows from (3.25) and a straightforward generalization of (7.1) under a general payoff matrix that $\overline{D}^{(n)}(\xi)$ is a linear combination of the three

functions $\mathbb{E}_\pi^{(n)}[\xi(X_0)\widehat{\xi}(X_\ell)] = W_\ell^{(n)}(\xi)$, $1 \le \ell \le 3$, with the coefficients depending only on $\Pi$. We apply (6.18) to this property of $\overline{D}^{(n)}$ and (6.4), and get

$$\sup_{n\in\mathbb{N}} \max\{C_{(6.6)}(\overline{D}^{(n)}), C_{(6.6)}(Q_1^{(n)})\} \le C_{(6.27)}. \tag{6.27}$$

The foregoing inequality allows us to estimate the right-hand side of (6.14) with respect to the $n$th models as follows. We apply (6.5), (6.8) and (6.27) to bound the right-hand side of (6.14) as follows:

$$\begin{aligned}
&\frac{w^2 C_{(6.4)}(1)\cdot C_{(6.6)}(\overline{D}^{(n)})}{\pi_{\min}^{(n)}} \mathbb{E}_{\lambda_n}^{(n)}\left[\int_0^\infty D_s^w Q_1^{(n)}(\xi_s)\,ds\right] \\
&\quad + w^2 C_{(6.4)}(2)\mathbb{E}_{\lambda_n}^{(n)}\left[\int_0^\infty D_s^w Q_2^{(n)}(\xi_s)\,ds\right] \\
&\quad \le C_{(6.28)} N_n w^2 \left(\mathbb{E}_{\lambda_n}^{(n)}\left[\int_0^\infty Q_1^{(n)}(\xi_s)\,ds\right] + \mathbb{E}_{\lambda_n}^{(n)}\left[\int_0^\infty Q_2^{(n)}(\xi_s)\,ds\right]\right) \qquad (6.28)\\
&\quad \le C_{(6.29)} N_n w^2 \left(N_n \sum_{x,y\in E_n} \lambda_n[\xi(x)\widehat{\xi}(y)]\pi^{(n)}(x)\pi^{(n)}(y)\right) \qquad (6.29)
\end{aligned}$$

for all $w \in [0, N_n^{-1} w_{(6.29)}]$.

Let us explain the last two inequalities in more detail. The constant $w_{(6.29)}$ above is chosen for $w$ to meet the constraints (6.7) for all $n$, and so we can validate (6.28) by (6.8). Note that this constant $w_{(6.29)}$ can be chosen to be bounded away from zero by (6.27). Also, (6.29) follows from (6.4) and (6.17), now that $\nu_n(\mathbb{1}) = \Theta(N_n^{-1})$. By decreasing $w_{(6.29)} > 0$ according to $C_{(6.29)}$ and the limit infimum in (6.25) if necessary, we deduce (6.26) from (6.14), (6.25) and (6.29). The constant $C_{(6.26)}$ can be chosen to be this refined $w_{(6.29)}$. □

**7. Expansions of some covariation processes.** In this section, we show some calculations to simplify the first-order coefficients in the expansions of $\langle M, D^w\rangle$ and $\langle D^w, D^w\rangle$ in (3.22) and (3.23). Recall the discrete-time $q$-Markov chains $(X_\ell)$ and $(Y_\ell)$ defined at the beginning of Section 4.4. We write $\mathbb{E}_x$ for the expectation under which the common starting point of $(X_\ell)$ and $(Y_\ell)$ is $x$.

LEMMA 7.1. *If the payoff matrix* $\Pi$ *is given by* (1.12), *then the function* $\overline{D}(\xi)$ *defined by* (3.25), *which enters the first-order coefficient of* $\langle M, D^w\rangle$ *in* (3.22), *can be written as*

$$\begin{aligned}
\overline{D}(\xi) = {}& b\big(\mathbb{E}_\pi\big[\xi(X_0)\widehat{\xi}(X_3)\big] - \mathbb{E}_\pi\big[\xi(X_0)\widehat{\xi}(X_1)\big]\big) \\
& - c\mathbb{E}_\pi\big[\xi(X_0)\widehat{\xi}(X_2)\big].
\end{aligned} \tag{7.1}$$

*For the integrand in the first-order expansion of* $\langle D^w, D^w\rangle$ *in* (3.23), *we have*

$$\begin{aligned}&\sum_{x,y\in E}\pi(x)q(x,y)\big[A(x,\xi)-B(y,\xi)\big]^2\\&\quad=b^2\mathbb{E}_\pi\big[\xi(X_0)\widehat{\xi}(X_4)-\xi(X_0)\widehat{\xi}(X_2)\big]\\&\qquad-2bc\mathbb{E}_\pi\big[\xi(X_0)\widehat{\xi}(X_3)-\xi(X_0)\widehat{\xi}(X_1)\big]+c^2\mathbb{E}_\pi\big[\xi(X_0)\widehat{\xi}(X_2)\big].\end{aligned}\tag{7.2}$$

*Here, the functions* $A(x,\xi)$ *and* $B(y,\xi)$ *are defined by* (3.12) *and* (3.13), *respectively.*

PROOF. The proof of (7.1) is almost identical to the proof of [6], Theorem 1(1). We include its short proof here for the convenience of the reader.

Recall the definitions of $A$ and $B$ in (3.12) and (3.13). Since $\Pi(\xi(x),\xi(y))=b\xi(y)-c\xi(x)$, we have

$$\begin{aligned}A(x,\xi)&=1-\sum_{z\in E}\sum_{z'\in E}q(x,z)q\big(z,z'\big)\big(b\xi\big(z'\big)-c\xi(z)\big)\\&=1-\mathbb{E}_x\big[b\xi(X_2)-c\xi(X_1)\big],\\B(y,\xi)&=1-\sum_{z\in E}q(y,z)\big(b\xi(z)-c\xi(y)\big)=1-\mathbb{E}_y\big[b\xi(X_1)-c\xi(X_0)\big].\end{aligned}$$

The foregoing equations give

$$\begin{aligned}&\sum_{x,y\in E}\pi(x)q(x,y)\big[\xi(y)-\xi(x)\big]\big[A(x,\xi)-B(y,\xi)\big]\\&\quad=\sum_{x,y\in E}\pi(x)q(x,y)\big[\xi(y)-\xi(x)\big]\\&\qquad\times\big\{-\mathbb{E}_x\big[b\xi(X_2)-c\xi(X_1)\big]+\mathbb{E}_y\big[b\xi(X_1)-c\xi(X_0)\big]\big\}\\&\quad=\sum_{x,y\in E}\pi(x)q(x,y)\big[\xi(y)-\xi(x)\big]\big\{b\mathbb{E}_y\big[\xi(X_2)+\xi(X_1)\big]\\&\qquad-c\mathbb{E}_y\big[\xi(X_1)+\xi(X_0)\big]\big\}\\&\quad=\sum_{x,y\in E}\pi(x)q(x,y)\\&\qquad\times\big\{b\mathbb{E}_y\big[\xi(X_0)\xi(X_2)+\xi(X_0)\xi(X_1)\big]-c\mathbb{E}_y\big[\xi(X_0)\xi(X_1)+\xi(X_0)\big]\big\}\\&\qquad-\sum_{x,y\in E}\pi(x)\\&\qquad\times\big\{b\mathbb{E}_x\big[\xi(X_0)\xi(X_3)+\xi(X_0)\xi(X_2)\big]-c\mathbb{E}_x\big[\xi(X_0)\xi(X_2)+\xi(X_0)\xi(X_1)\big]\big\}\\&\quad=b\big\{\mathbb{E}_\pi\big[\xi(X_0)\xi(X_1)\big]-\mathbb{E}_\pi\big[\xi(X_0)\xi(X_3)\big]\big\}\end{aligned}$$

$$
\begin{aligned}
&- c\{\mathbb{E}_\pi[\xi(X_0)] - \mathbb{E}_\pi[\xi(X_0)\xi(X_2)]\} \\
&= b\{\mathbb{E}_\pi[\xi(X_0)\widehat{\xi}(X_3)] - \mathbb{E}_\pi[\xi(X_0)\widehat{\xi}(X_1)]\} - c\mathbb{E}_\pi[\xi(X_0)\widehat{\xi}(X_2)],
\end{aligned}
$$

as required. Notice that we use the reversibility of $q$ in the second equality above so that an $\mathbb{E}_x$-expectation is changed to an $\mathbb{E}_y$-expectation.

The proof of (7.2) is similar and the reversibility of $q$ is used again. We have

$$
\begin{aligned}
&\sum_{x,y\in E} \pi(x)q(x,y)[A(x,\xi) - B(y,\xi)]^2 \\
&\quad= \sum_{x,y\in E} \pi(x)q(x,y)\big(-\mathbb{E}_x[b\xi(X_2) - c\xi(X_1)] + \mathbb{E}_y[b\xi(X_1) - c\xi(X_0)]\big)^2 \\
&\quad= \mathbb{E}_\pi\big[\big(b\xi(X_2) - c\xi(X_1)\big)\big(b\xi(Y_2) - c\xi(Y_1)\big)\big] \\
&\qquad - 2\mathbb{E}_\pi\big[\big(b\xi(X_2) - c\xi(X_1)\big)\big(b\xi(Y_2) - c\xi(Y_1)\big)\big] \\
&\qquad + \mathbb{E}_\pi\big[\big(b\xi(X_1) - c\xi(X_0)\big)\big(b\xi(Y_1) - c\xi(Y_0)\big)\big] \\
&\quad= -\mathbb{E}_\pi\big[\big(b\xi(X_2) - c\xi(X_1)\big)\big(b\xi(Y_2) - c\xi(Y_1)\big)\big] \\
&\qquad + \mathbb{E}_\pi\big[\big(b\xi(X_1) - c\xi(X_0)\big)\big(b\xi(Y_1) - c\xi(Y_0)\big)\big] \\
&\quad= b^2\mathbb{E}_\pi[-\xi(X_0)\xi(X_4) + \xi(X_0)\xi(X_2)] \\
&\qquad + 2bc\mathbb{E}_\pi[\xi(X_0)\xi(X_3) - \xi(X_0)\xi(X_1)] \\
&\qquad + c^2\mathbb{E}_\pi[-\xi(X_0)\xi(X_2) + \xi(X_0)] \\
&\quad= b^2\mathbb{E}_\pi[\xi(X_0)\widehat{\xi}(X_4) - \xi(X_0)\widehat{\xi}(X_2)] \\
&\qquad - 2bc\mathbb{E}_\pi[\xi(X_0)\widehat{\xi}(X_3) - \xi(X_0)\widehat{\xi}(X_1)] \\
&\qquad + c^2\mathbb{E}_\pi[\xi(X_0)\widehat{\xi}(X_2)].
\end{aligned}
$$

This completes the proof. □

**8. Feynman–Kac duality for voter models.** In this section, we give a brief discussion of the Feynman–Kac duality for voter models, which we use in the earlier sections. Although these results should be standard, they seem difficult to find in the literature.

We introduce some functions. First, we set

$$
J_\Sigma(\xi; x) = \mathbb{1}_\Sigma\big(\xi(x)\big)
$$

for any subset $\Sigma$ of $S$ and $x \in E$. Then for any $m$-tuple $A = (x_1, \ldots, x_m)$ of points of $E$ and $k$-tuple $\mathbf{\Sigma} = (\Sigma_1, \ldots, \Sigma_m)$ of subsets of $S$, we define

$$
H_{\mathbf{\Sigma}}(\xi; A) = \prod_{i=1}^{m} [J_{\Sigma_i}(\xi; x_i) - \overline{\mu}(\Sigma_i)],
$$

where $\overline{\mu}$ is defined by (4.4). We write $\mathsf{L}_{B,m}$ for the generator of an $m$-tuple of coalescing $q$-Markov chains, which allows for the possibility that there are less than $m$ distinct points in the system.

PROPOSITION 8.1. *For any $m$-tuple $A$ with distinct entries and $m$-tuple $\mathbf{\Sigma}$ of subsets of $S$, we have*

$$\mathsf{L}^{0,\mu} H_{\mathbf{\Sigma}}(\cdot; A)(\xi) = \big[\mathsf{L}_{B,m} - m\mu(\mathbb{1})\big] H_{\mathbf{\Sigma}}(\xi; \cdot)(A). \tag{8.1}$$

PROOF. Write $A = (x_1, \dots, x_m)$ and $\mathbf{\Sigma} = (\Sigma_1, \dots, \Sigma_m)$. For any $x \in A$ and $y \in E$, define $A^{x,y}$ to be the $m$-tuple obtained from $A$ by replacing the entry $x$ of $A$ with $y$. Then by the definition of $\mathsf{L}^{0,0}$ in (2.4), we have

$$\begin{aligned}
&\mathsf{L}^{0,0} H_{\mathbf{\Sigma}}(\cdot; A)(\xi) \\
&\quad = \sum_{x\in E} \Bigg( \sum_{y\in E} q(x,y)\big[\xi(x)\widehat{\xi}(y) + \widehat{\xi}(x)\xi(y)\big] \Bigg) \big[H_{\mathbf{\Sigma}}\big(\xi^x; A\big) - H_{\mathbf{\Sigma}}(\xi; A)\big] \\
&\quad = \sum_{i=1}^{m} \sum_{y\in E} q(x_i, y)\big[\xi(x_i)\widehat{\xi}(y) + \widehat{\xi}(x_i)\xi(y)\big]\big[\mathbb{1}_{\Sigma_i}\big(\widehat{\xi}(x)\big) - \mathbb{1}_{\Sigma_i}\big(\xi(x_i)\big)\big] \\
&\qquad \times \prod_{j:j\neq i} \big[\mathbb{1}_{\Sigma_j}\big(\xi(x_j)\big) - \overline{\mu}(\Sigma_j)\big] \\
&\quad = \sum_{i=1}^{m} \sum_{y\in E} q(x_i, y)\big[\mathbb{1}_{\Sigma_i}\big(\xi(y)\big) - \mathbb{1}_{\Sigma_i}\big(\xi(x_i)\big)\big] \prod_{j:j\neq i} \big[\mathbb{1}_{\Sigma_j}\big(\xi(x_j)\big) - \overline{\mu}(\Sigma_j)\big] \\
&\quad = \sum_{i=1}^{m} \sum_{y\in E} q(x_i, y)\big[H_{\mathbf{\Sigma}}\big(\xi; A^{x_i,y}\big) - H_{\mathbf{\Sigma}}(\xi; A)\big] = \mathsf{L}_{B,m}\big[H_{\mathbf{\Sigma}}(\xi; \cdot)\big](A),
\end{aligned}$$

where the second equality follows from the assumption that the entries of $A$ are distinct. Also, the mutation part of $\mathsf{L}^{0,\mu}$ is given by

$$\begin{aligned}
&\sum_{x\in E} \int_S \big(H_{\mathbf{\Sigma}}\big(\xi^{x|\sigma}; A\big) - H_{\mathbf{\Sigma}}(\xi; A)\big)\, d\mu(\sigma) \\
&\quad = \sum_{i=1}^{m} \int_S \big[\mathbb{1}_{\Sigma_i}\big(\xi^{x_i|\sigma}(x_i)\big) - \mathbb{1}_{\Sigma_i}\big(\xi(x_i)\big)\big] \prod_{j:j\neq i} \big[\mathbb{1}_{\Sigma_j}\big(\xi(x_j)\big) - \overline{\mu}(\Sigma_j)\big]\, d\mu(\sigma) \\
&\quad = \sum_{i=1}^{m} \big[\mu(\Sigma_i) - \mu(\mathbb{1})\mathbb{1}_{\Sigma_i}\big(\xi(x_i)\big)\big] \prod_{j:j\neq i} \big[\mathbb{1}_{\Sigma_j}\big(\xi(x_j)\big) - \overline{\mu}(\Sigma_j)\big] \\
&\quad = -m\mu(\mathbb{1}) H_{\mathbf{\Sigma}}(\xi; A).
\end{aligned}$$

The above two displays give the required equation in (8.1). □

Proposition 8.1 is enough to solve for $\mathbb{E}_\xi[H_{\mathbf{\Sigma}}(\xi_t; A)]$ by coalescing Markov chains. For example, for $m = 1$, Proposition 8.1 shows that

$$\mathsf{L}^{0,\mu} H_\Sigma(\cdot; x)(\xi) = \big[\mathsf{L}_{B,1} - \mu(\mathbb{1})\big] H_\Sigma(\xi; \cdot)(x). \tag{8.2}$$

The foregoing equation can be used to find $\mathsf{L}^{0,\mu} H_{(\Sigma_1,\Sigma_2)}(\cdot; x, x)(\xi)$ as follows. We write (8.2) as

$$\begin{aligned} \mathsf{L}^{0,\mu} J_\Sigma(\cdot; x)(\xi) &= \mathsf{L}^{0,\mu} H_\Sigma(\cdot; x)(\xi) \\ &= \mathsf{L}_{B,1} J_\Sigma(\xi; \cdot)(x) - \mu(\mathbb{1}) J_\Sigma(\xi; x) + \mu(\Sigma) \end{aligned} \tag{8.3}$$

so that

$$\begin{aligned} &\mathsf{L}^{0,\mu} H_{(\Sigma_1,\Sigma_2)}(\cdot; x, x)(\xi) \\ &\quad = \mathsf{L}^{0,\mu}\big[J_{\Sigma_1\cap\Sigma_2}(\cdot; x) - J_{\Sigma_1}(\cdot; x)\overline{\mu}(\Sigma_2) - J_{\Sigma_2}(\cdot; x)\overline{\mu}(\Sigma_1)\big](\xi) \\ &\quad = \mathsf{L}_{B,1}\big[J_{\Sigma_1\cap\Sigma_2}(\xi; \cdot) - \overline{\mu}(\Sigma_2) J_{\Sigma_1}(\xi; \cdot) \\ &\qquad - \overline{\mu}(\Sigma_1) J_{\Sigma_2}(\xi; \cdot) + \overline{\mu}(\Sigma_1)\overline{\mu}(\Sigma_2)\big](x) \\ &\qquad - \mu(\mathbb{1})\big[J_{\Sigma_1\cap\Sigma_2}(\xi; x) - \overline{\mu}(\Sigma_2) J_{\Sigma_1}(\xi; x) \\ &\qquad - \overline{\mu}(\Sigma_1) J_{\Sigma_2}(\xi; x) + \overline{\mu}(\Sigma_1)\overline{\mu}(\Sigma_2)\big] \\ &\qquad + \mu(\mathbb{1})\big[\overline{\mu}(\Sigma_1 \cap \Sigma_2) - \overline{\mu}(\Sigma_1)\overline{\mu}(\Sigma_2)\big] \\ &\quad = \mathsf{L}_{B,2} H_{(\Sigma_1,\Sigma_2)}(\xi; \cdot, \cdot)(x, x) - \mu(\mathbb{1}) H_{(\Sigma_1,\Sigma_2)}(\xi; x, x) \\ &\qquad + \mu(\mathbb{1})\big[\overline{\mu}(\Sigma_1 \cap \Sigma_2) - \overline{\mu}(\Sigma_1)\overline{\mu}(\Sigma_2)\big]. \end{aligned}$$

We can summarize the last equality and (8.1) with $m = 2$ as the following equation: for all $x, y \in E$,

$$\begin{aligned} &\mathsf{L}^{0,\mu} H_{(\Sigma_1,\Sigma_2)}(\cdot; x, y)(\xi) \\ &\qquad = \big[\mathsf{L}_{B,2} - \mu(\mathbb{1})\big|\{x, y\}|\big] H_{(\Sigma_1,\Sigma_2)}(\xi; \cdot, \cdot)(x, y) \\ &\qquad\quad + \mu(\mathbb{1})\big[\overline{\mu}(\Sigma_1 \cap \Sigma_2) - \overline{\mu}(\Sigma_1)\overline{\mu}(\Sigma_2)\big]\mathbb{1}_{\{x=y\}}, \end{aligned} \tag{8.4}$$

which is enough for (4.5).

**List of frequent notation.**

$S$: the set of types $\{1, 0\}$.
$\Pi = (\Pi(\sigma, \tau))$: $2 \times 2$ payoff matrices with real entries.
$(E, q)$: a kernel on $E$ which has a zero trace and is irreducible and reversible.
$R_\ell$: a limiting return probability of voting kernels defined in Assumption 4.4.
$x \sim y$: two vertices $x$ and $y$ adjacent to each other in the sense that $q(x, y) > 0$.
$\pi$: the stationary distribution of $q$.
$\pi_{\min}, \pi_{\max}$: $\pi_{\min} = \min_x \pi(x)$ and $\pi_{\max} = \max_x \pi(x)$.

$\nu(x, y)$: the measure on $E \times E$ defined by $\nu(x, y) = \pi(x)^2 q(x, y)$ in (3.24).
$\nu(\mathbb{1})$: the total mass $\int \mathbb{1}\, d\nu$ of $\nu$.
$\mu$: a mutation measure defined on $S$ (Section 2).
$\mu(\mathbb{1})$: the total mass $\int \mathbb{1}\, d\mu$ of $\mu$.
$\overline{\mu}(\sigma)$: the ratio $\mu(\sigma)/\mu(\mathbb{1})$ with the convention that $0/0 = 0$ defined in (4.4).
$w$: selection strength (Section 2).
$\overline{w}$: a maximal selection strength defined by $(2 + 2\max_{\sigma,\tau\in S} |\Pi(\sigma, \tau)|)^{-1}$ in (2.2).
$\mathbb{P}^w$, $\mathbb{P}^{(n),w}$: laws of evolutionary games subject to selection strength $w$ (Section 2).
$\mathscr{P}(U)$: the set of probability measures defined on a Polish space $U$.

*Functions of configurations*

$\xi, \eta$: $\{1, 0\}$-valued population configurations.
$\widehat{\xi}(x)$: $1 - \xi(x)$ defined in (2.3).
$p_\sigma(\xi)$: the density of type $\sigma$ in a population configuration $\xi$ defined in (1.4).
$W_\ell(\xi)$: the density function $\sum_{x,y\in E} \pi(x) q^\ell(x, y) \xi(x) \widehat{\xi}(y)$ defined in (3.6).
$H(\xi; x, y)$: the dual function $[\xi(x) - \overline{\mu}(1)][\widehat{\xi}(y) - \overline{\mu}(0)]$ defined in (4.3).

*Processes*

$(\Lambda_t(x, y))$, $(\Lambda_t^\sigma(x))$: the Poisson processes defined in (2.7).
$(Y_t)$ under $\mathbb{P}$ or $\mathbb{P}^{(n)}$: the density process of 1's defined in (3.19).
$(M_t)$ under $\mathbb{P}$ or $\mathbb{P}^{(n)}$: the martingale part of $(Y_t)$ in the decomposition in (3.20).
$(D_t^w)$ under $\mathbb{P}$ or $\mathbb{P}^{(n)}$: the Radon–Nikodým derivative process defined in (2.10).
$(Z_t^{(n)})$ under $\mathbb{P}^{(n)}$: the process $(Y_{\gamma_n t}, M_{\gamma_n t}, D_{\gamma_n t}^{w_n})$ under $\mathbb{P}^{(n)}$ defined in (4.1).
$(Y_t, M_t, D_t)$ under $\mathbb{P}^{(\infty)}$: the limit of $Z^{(n)}$ under $\mathbb{P}^{(n)}$ (Theorem 4.6).
$\{(B_t^x); x \in E\}$: coalescing $q$-Markov chains (Section 4).
$M_{x,y}$: the first meeting time of $B^x$ and $B^y$ (Section 4).
$(X_\ell)$, $(Y_\ell)$: auxiliary discrete-time $q$-Markov chains (Section 4.4).

**List of frequent asymptotics.**

$N_n$: tending to $\infty$.
$\nu_n(\mathbb{1})$: satisfying $\Theta(N_n^{-1})$ [i.e., $C^{-1}N_n^{-1} \le \nu_n(\mathbb{1}) \le CN_n$] under Assumption 4.1.
$\pi_{\max}^{(n)}, \pi_{\min}^{(n)}$: satisfying $\pi_{\min}^{(n)}, \pi_{\max}^{(n)} = \Theta(N_n^{-1})$ under Assumption 4.1.
$w_n$: tending to zero under Assumptions 4.1 and 4.5.
$w_n/\nu_n(\mathbb{1})$: converging in $[0, \infty)$ under Assumption 4.5.
$\gamma_n$: tending to $\infty$ under Assumption 4.2.
$\mu_n(\mathbb{1})$: tending to zero under Assumption 4.2.
$\gamma_n \mu_n(\mathbb{1})$: tending to $\mu(\mathbb{1})$ under Assumption 4.2.

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DEPARTMENT OF MATHEMATICS
UNIVERSITY OF TENNESSEE
KNOXVILLE, TENNESSEE 37996-1320
USA
E-MAIL: ychen134@utk.edu