The Annals of Applied Probability
2018, Vol. 28, No. 5, 3075–3093
https://doi.org/10.1214/18-AAP1383

REAL EIGENVALUES IN THE NON-HERMITIAN ANDERSON MODEL

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The eigenvalues of the Hatano–Nelson non-Hermitian Anderson matrices, in the spectral regions in which the Lyapunov exponent exceeds the non-Hermiticity parameter, are shown to be real and exponentially close to the Hermitian eigenvalues. This complements previous results, according to which the eigenvalues in the spectral regions in which the non-Hermiticity parameter exceeds the Lyapunov exponent are aligned on curves in the complex plane.

1. Introduction and the main result. Let $v_1, v_2, \dots$ be independent, identically distributed random variables (potential) and let g be a real parameter, $g \geq 0$. Consider the $N \times N$ random matrix

$$H_N(g) = \begin{pmatrix} v_1 & e^{-g} & 0 & 0 & \cdots & 0 & e^g \\ e^g & v_2 & e^{-g} & 0 & \cdots & 0 & 0 \\ 0 & e^g & v_3 & e^{-g} & \cdots & 0 & 0 \\ & & & \ddots & & & \\ & & & & \ddots & & \\ 0 & 0 & 0 & 0 & & v_{N-1} & e^{-g} \\ e^{-g} & 0 & 0 & 0 & \cdots & e^g & v_N \end{pmatrix}. \tag{1.1}$$

Non-Hermitian matrices of the form (1.1) were introduced and studied by Hatano and Nelson [21, 22] to describe the reaction of an Anderson-localised quantum particle on a ring to a constant imaginary vector field. For $g = 0$, the matrix $H_N = H_N(0)$ is Hermitian, and the eigenvalues are real. For $g > 0$, the eigenvalues are not necessarily real. The numerical studies of Hatano and Nelson (carried out for the case when the v_j have the uniform $[-1, 1]$ distribution) suggest that there exist critical values $\overline{g}_{\mathrm{cr}} > \underline{g}_{\mathrm{cr}} > 0$ such that the following hold:

(a) For $0 \leq g < \underline{g}_{\mathrm{cr}}$, all the eigenvalues of $H_N(g)$ are real;

(b) for $g \in (\underline{g}_{\mathrm{cr}}, \overline{g}_{\mathrm{cr}})$, some of the eigenvalues remain real, while others align along a smooth curve in the complex plane;

Received December 2017.

[1]Supported in part by the European Research Council starting grant 639305 (SPECTRUM) and by a Royal Society Wolfson Research Merit Award.

MSC2010 subject classifications. Primary 47B80, 47B36.

Key words and phrases. Sample, non-Hermitian, Anderson model, random Schrödinger.

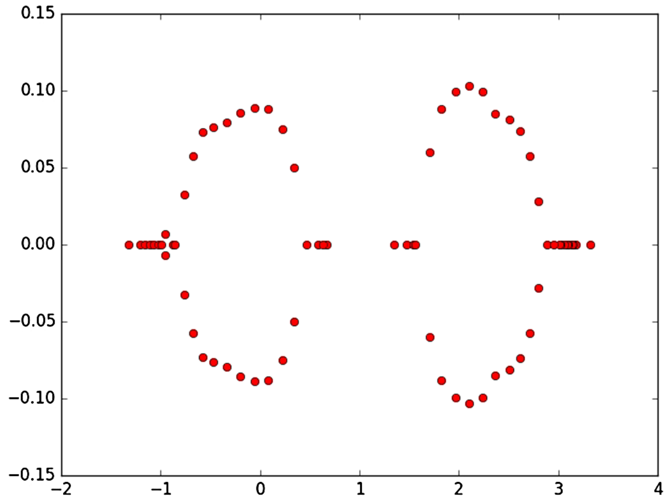

FIG. 1. *A realisation of the spectrum for a variant of* (1.1) *with a* 2*-periodic background* (*cf. Remark* 1.3); $N = 70$ *and* $g = 0.08 \in (\underline{g}_{cr}, \overline{g}_{cr})$. *The axis is split in* 5 *intervals. On the odd ones*, *the Lyapunov exponent is* ≥ 0.08; *on the even ones*, *it is* ≤ 0.08.

(c) essentially all eigenvalues move out of the real axis when $g > \overline{g}_{\mathrm{cr}}$.

A variant of this numerical experiment in the regime (b) is depicted in Figure 1. These observations, and especially (b) and (c), were supported by the subsequent analysis performed on the physical level of rigour; see especially [7, 8, 11, 35]. We refer to these works and also to [28] and references therein for a discussion of the properties of the (left and right) eigenvectors of $H_N(g)$, and for extensions to the strip and to higher dimension, which will mostly remain outside the scope of this paper (see, however, Section 6).

In the mathematical works [15, 17, 18] of Khoruzhenko and the first author, it was shown that the behaviour of the eigenvalues depends crucially on the Lyapunov exponent $\gamma(E)$ associated to the Hermitian operator [see Section 2, equation (2.2)]. Let us label the algebraic spectrum of $H_N(g)\{\lambda_1(g), \ldots, \lambda_N(g)\}$ so that each $\lambda_j(g)$ is a continuous function of g, and $\lambda_1(0) \geq \cdots \geq \lambda_N(0)$ (cf. Lemma 2.3 below).

Fix j; for $g = 0$ the eigenvalue $\lambda_j(0)$ lies on the real axis. It was shown in [15, 17] that for $g < \gamma(\lambda_j(0))$ the eigenvalue $\lambda_j(g)$ remains in the vicinity of the real axis [i.e., it lies in the strip $|\Im\lambda| < \varepsilon$, provided that $N \geq N_0(\varepsilon)$], whereas for $g > \gamma(\lambda_j(0))$ it escapes to certain polynomial curves $\Gamma_g^{(N)}$ in the complex plane. These statements hold simultaneously for all the eigenvalues $\lambda_j(g)$ on an event of asymptotically full probability. As $N \to \infty$, $\Gamma_g^{(N)}$ converges to the curve $\Gamma_g = \{z \in \mathbb{C} \mid \gamma(z) = g\}$.

In [18], these results were extended to a wide class of deterministic potentials, under the mild assumption of existence of the integrated density of states $\mathcal{N}(E)$.

Under this assumption, one defines the Lyapunov exponent via the Thouless formula

$$\gamma(E) = \int \log|E - E'| \, d\mathcal{N}(E').$$

In the case of stationary random sequences, this definition coincides with the usual one, given in (2.2).

Moreover, it was shown in [18] that the eigenvalues near the curves Γ_g boast regular behaviour on a local scale: after re-scaling the eigenvalues near a fixed $z \in \Gamma_g$ by the mean (complex) spacing, these align, in the large N limit, on an arithmetic progression.

Consequently, the critical values should be given by the formulæ

$$\underline{g}_{\mathrm{cr}} = \min\{\gamma(E) \mid E \in \mathcal{S}\}, \qquad \overline{g}_{\mathrm{cr}} = \max\{\gamma(E) \mid E \in \mathcal{S}\},$$

where $\mathcal{S}$ is the support of the limiting eigenvalue distribution of $H_N(0)$ [i.e., the support of the integrated density of states $\mathcal{N}(E)$ defined in (1.3), or equivalently the essential spectrum of the infinite-volume self-adjoint operator].

The results proved in [15, 17, 18] provide a detailed statistical description of the behaviour of the eigenvalue $\lambda_j(g)$ for $g > \gamma(\lambda_j(0))$, both in the global and the local limiting regime; thus one has a complete description of the regime (c), and a partial one of (b).

The description of the behaviour for $g < \gamma(\lambda_j(0))$ remained incomplete. In fact, neither the rigorous analysis of [15, 17, 18] nor the heuristic arguments of [7, 8, 11, 35] provide an indication on whether these eigenvalues are truly real (as suggested by computer simulations such as Figure 1), or they may have a nonzero but asymptotically vanishing imaginary part.

To the best of our knowledge, no progress on this question has been made since the work [18] had been published. We are also not aware of any previous analysis of the spacings between these eigenvalues (the local regime).

In this work, we provide a reasonably complete description of the regime $\gamma(\lambda_j(0)) > g$, thus settling these two questions. We prove that in the case of (1.1) with independent, identically distributed potential the corresponding non-Hermitian eigenvalues $\lambda_j(g)$ do in fact remain on the real axis and, moreover, they are exponentially close to the Hermitian eigenvalues $\lambda_j(0)$. In other words, if j is fixed and g varies from 0 to ∞, the eigenvalue $\lambda_j(g)$ remains real and exponentially close to $\lambda_j(0)$ for $g \le \gamma(\lambda_j(0)) - \varepsilon$ [where $\varepsilon > 0$ is arbitrary small, and $N \ge N_0(\varepsilon)$]. This complements the result of [18], according to which $\lambda_j(g)$ aligns near Γ_g for $g \ge \gamma(\lambda_j(0)) + \varepsilon$. See Figure 2 for an illustration.

In contrast to the potential-theoretic approach of [18], our arguments are based on the properties of products of random matrices.

THEOREM 1. *Assume that (v_j) is a sequence of i.i.d. random variables and that $|v_1| \le A < \infty$ almost surely. Then for any $\varepsilon > 0$,*

$$\mathbb{P}\{\textit{for all } 1 \le j \le N \textit{ and } g \in [0, \gamma(\lambda_j(0)) - \varepsilon] \textit{ one has: } \lambda_j(g) \in \mathbb{R}\} \underset{N\to\infty}{\longrightarrow} 1$$

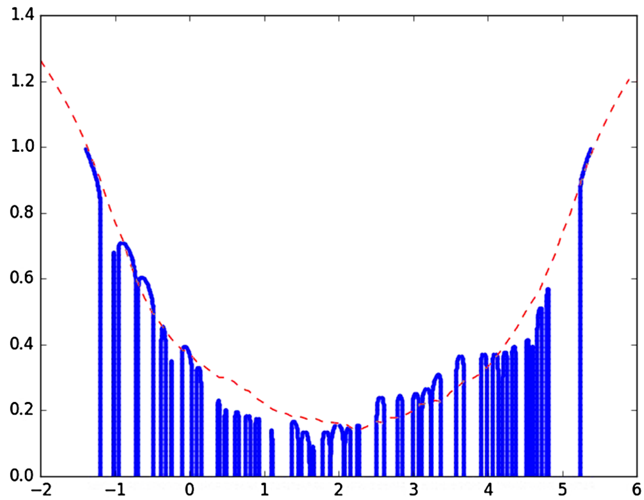

FIG. 2. *The curves connecting the points* $(\lambda_j(g), g) \in \mathbb{R}^2$ *with* $\lambda_j(g) \in \mathbb{R}$, *for* $N = 70$ *and* $v_j \sim \mathrm{Unif}[0,4]$. *In the large* N *limit, the upper envelope of these curves converges to the graph of* $\gamma(E)$ *on* $\mathcal{S}$. *Note that the curves are almost vertical.*

and, moreover, there exists $c = c(\varepsilon) > 0$ *such that*

$$\mathbb{P}\{\forall j\ \forall g \in [0, \gamma(\lambda_j(0)) - \varepsilon] : \lambda_j(g) \in (\lambda_j(0) - e^{-cN}, \lambda_j(0) + e^{-cN})\} \underset{N\to\infty}{\longrightarrow} 1.$$

REMARK 1.1. The first part of the theorem is essentially equivalent to the following statement: if I is an interval, then for any $\varepsilon > 0$,

$$\lim_{N\to\infty} \mathbb{P}\Big\{\text{for all } \lambda_j(0) \in I \text{ and all } g \le \inf_{E\in I} \gamma(E) - \varepsilon \text{ one has } \lambda_j(g) \in I\Big\} = 1.$$

REMARK 1.2. Without invoking new ideas, the theorem can be shown to hold under the weaker assumption $\mathbb{E}|v_1|^\eta < \infty$ for some $\eta > 0$. We restrict ourselves to the case of bounded random variables, to keep the argument reasonably short. On the other hand, we do insist on avoiding any regularity assumptions on the potential.

REMARK 1.3. Only minor adjustments in the argument are required to consider a variant of the model in which v_j is replaced with $v_j + a_j$, where (a_j) is a nonrandom periodic sequence. For cosmetic reasons, we chose to depict this variant in Figure 1, which we included for illustration only.

REMARK 1.4. Similar to [17] and in contrast to [18], we assume that (v_j) is an i.i.d. sequence. While we do not expect the conclusion of the theorem to hold in the generality of [18], additional special cases such as operators with almost periodic potentials merit further consideration.

The theorem implies that the local eigenvalue statistics of $H_N(g)$ in the regime $g < \gamma(\lambda_j(0))$ are the same as for the Hermitian operator $H_N(0)$.

COROLLARY 1.5. *In the setting of Theorem 1, assume that for $g = 0$,*

$$\sum_{j=1}^{N} \delta_{(\lambda_j(g)-E)N\rho} \xrightarrow{\text{distr}} \text{standard Poisson process} \tag{1.2}$$

for some $E \in \mathbb{R}$, $\rho > 0$, as $N \to \infty$. Then (1.2) holds for all $0 \le g < \gamma(E)$.

In the Hermitian case $g = 0$, a limit theorem of the form (1.2) was first proved by Molchanov [27] for a class of (continual) one-dimensional Hermitian random Schrödinger operators. An extension to higher-dimensional operators in the regime of Anderson localisation was proved by Minami [26]; his result implies that (1.2) holds (for $g = 0$) if the cumulative distribution function of v_1 is uniformly Lipschitz, with

$$\rho = \rho(E) = \mathcal{N}'(E), \qquad \mathcal{N}(E) = \lim_{N\to\infty} \frac{1}{N} \#\{\lambda_j(0) < E\}. \tag{1.3}$$

The existence of the density of states $\rho(E) = \mathcal{N}'(E)$ (in the sense of Radon) in this situation follows from an argument of Wegner [34].

Recently, Bourgain showed [5] for the one-dimensional case that the density of states exists (and in fact $\mathcal{N}$ is C^∞ smooth) whenever the cumulative distribution function of v_1 is uniformly Hölder continuous of some order $\nu > 0$. In [6], he showed that (1.2) holds (for $g = 0$) under the same assumptions, for the case of Dirichlet boundary conditions (i.e., the top-right and bottom-left corner matrix elements are set to zero). The argument of [6] can be adjusted to periodic boundary conditions [i.e., to $H_N(0)$]. Combining this with Corollary 1.5, we obtain that, under the same assumption, (1.2) also holds for all $g < \gamma(E)$ (at least, if v_1 is bounded almost surely).

The logical structure of the paper. The key ingredient in the proof of Theorem 1 is a uniform lower bound on the spectral radius of the transfer matrices associated with the Hermitian matrices $H_N(0)$, outside exponentially small neighbourhoods of the bands. This bound, possibly of independent interest, is stated as Proposition 3.1 in Section 3.1, where we also provide its proof. In Section 3.2, we use it to prove Theorem 1.

The proof of Proposition 3.1 makes use of several facts from the theory of random matrix products: particularly, a large deviation bound for the norm (Lemma 2.1) and a comparison between the norm and the spectral radius (Lemma 2.2). While such statements are well known (the former goes back to the work of Le Page [24], and the latter—to the work of Guivarc'h [20] and Reddy [30]), the form in which we found them (and particularly the latter one) in the literature is somewhat weaker than what is needed for our purposes. Therefore, we develop in Sections 4 and 5, an approach (close in spirit to the article [32] of Shubin–Vakilian–Wolff and to unpublished work of the first author on the central limit theorem for eigenvalues of random matrix products) which allows us to reprove these statements in the required form. Two other important ingredients of the proof of Proposition 3.1 are Lemmata 2.5 and 2.4, due to Bourgain [6] and Le Page [25], respectively. The latter lemma lies in the field of random matrix products, and in the short Section 5.2, we deduce it from Lemma 2.1.

In Section 2, we formulate the definitions and the lemmata required to state and prove Proposition 3.1. Possible generalisations and extensions of Theorem 1 are discussed in Section 6.

2. Preliminaries.

Transfer matrices. Let $E \in \mathbb{R}$. For $N = 1, 2, \ldots,$ define

$$\Phi_N(E) = T_N(E) \cdots T_2(E) T_1(E),$$

where

$$T_j(E) = \begin{pmatrix} E - v_j & -1 \\ 1 & 0 \end{pmatrix} \in SL_2(\mathbb{R}). \tag{2.1}$$

More generally, one may consider the matrices $T_j(z)$ and $\Phi_N(z)$ for $z \in \mathbb{C}$. As usual, $\Phi_N(z)$ is associated to the formal solutions ψ of the equation

$$\psi_{j-1} + v_j \psi_j + \psi_{j+1} = z \psi_j, \qquad j \ge 1$$

as follows:

$$\Phi_N \begin{pmatrix} \psi_1 \\ \psi_0 \end{pmatrix} = \begin{pmatrix} \psi_{N+1} \\ \psi_N \end{pmatrix}.$$

Denote

$$\gamma(E) = \lim_{N \to \infty} \frac{1}{N} \mathbb{E} \log \| \Phi_N(E) \|. \tag{2.2}$$

According to a result of Furstenberg and Kesten [14], for any stationary ergodic sequence $v = (v_j)$, the following equality holds with probability one for any (fixed) E:

$$\lim_{N \to \infty} \frac{1}{N} \log \| \Phi_N(E) \| = \gamma(E). \tag{2.3}$$

We emphasise that (2.3) does not hold simultaneously for all E (see [16]); in fact, in the i.i.d. case the left-hand side of (2.2) vanishes on a dense random subset of $\mathcal{S}$.

A fundamental fact which is crucial for our considerations is the positivity of the Lyapunov exponent: in the i.i.d. case, Furstenberg's theorem [13] implies that $\gamma(E) > 0$ for all $E \in \mathbb{R}$. [Formally, we use the quantitative version (4.5) of this fact.]

Large deviations. Large deviations bounds for the norm of a random matrix product go back to the work of Le Page [24]. There are numerous extensions (see particularly the recent work [31]), where a large deviation principle was obtained, and references therein. We need the following upper bound, close to the original work of Le Page; a proof is provided in Section 5.1.

LEMMA 2.1 (Le Page). *If $\mathbb{E}|v_1|^\eta < \infty$ for some $\eta > 0$, then for any $R > 0$ there exist $C, c > 0$ such that for $\varepsilon \in (0, 1/e)$ and $|E| \le R$,*

$$\mathbb{P}\left\{\left|\frac{1}{N}\log\|\Phi_N(E)\| - \gamma(E)\right| \ge \varepsilon\right\} \le \left(C\log\frac{1}{\varepsilon}\right)\exp\left[-\frac{c\varepsilon^2 N}{\log\frac{1}{\varepsilon}}\right]. \tag{2.4}$$

Spectral radius. We shall use the following lemma, which is a variant of the results proved by Guivarc'h [20] and by Reddy [30], Section 2.4. We provide a proof in Section 4.3. For a matrix Φ, we denote by $\rho(\Phi)$ its spectral radius.

LEMMA 2.2. *If $\mathbb{E}|v_1|^\eta < \infty$ for some $\eta > 0$, then for any $R > 0$ there exist $B, B' > 0$ and $b, b' > 0$ such that*

$$\sup_{|E|\le R} \mathbb{P}\{\rho(\Phi_N(E)) \le \delta\|\Phi_N(E)\|\} \le B\delta^b + B'e^{-b'N}, \qquad 0 \le \delta \le 1.$$

Bands and gaps. Consider the transfer matrices corresponding to the potential (v_j) (in this paragraph the potential does not have to be random). The set of E such that $\rho(\Phi_N(E)) = 1$ consists of N disjoint intervals (bands); we denote their interiors, numbered from the rightmost to the leftmost, $I_1, \dots, I_N$. Denote

$$\mathbb{R} = G_0 \uplus I_1 \uplus G_1 \uplus I_2 \uplus \cdots \uplus I_N \uplus G_N,$$

where the G_j (the closures of the gaps) are also ordered from right to left.

The eigenvalues of the periodic operator $H_N(0)$ are exactly the points E at which 1 is an eigenvalue of $\Phi_N(E)$. These are exactly the edges of the gaps $G_0, G_2, \dots$ with even indices. This fact admits the following generalisation to the non-Hermitian case (cf. [17, 18]).

LEMMA 2.3 ([18], Lemma 4.1). *The eigenvalues of $H_N(g)$ are the points $z \in \mathbb{C}$ such that e^{Ng} is an eigenvalue of $\Phi_N(z)$.*

Hölder continuity of the Lyapunov exponent. The local Hölder continuity of the Lyapunov exponent goes back to the work of Le Page [25]. We need the following version, proved in [9] and, by different arguments, in [4, 32]; for the sake of unity of argument, we provide a proof in Section 5.2.

LEMMA 2.4 (Le Page). *If v_j are independent, identically distributed with $\mathbb{E}|v_1|^\eta < \infty$ for some $\eta > 0$, then the Lyapunov exponent $\gamma(E)$ associated to the sequence $T_N(E)$ is uniformly Hölder continuous on any compact interval.*

Gaps between the eigenvalues.

LEMMA 2.5 (Bourgain [6]). *If v_j are independent, identically distributed with $\mathbb{E}|v_1|^\eta < \infty$ for some $\eta > 0$, then for any $R > 0$ there exists $K > 0$ such that*

$$\mathbb{P}\Big\{ \min_{\lambda \neq \lambda'\text{-eigenvalues of } H_N \text{ in } [-R,R]} |\lambda - \lambda'| < N^{-K} \Big\} = 0.$$

REMARK 2.6. In the work [6], the lemma is proved for Dirichlet rather than periodic boundary conditions, and only for the case of Bernoulli potential. However, the argument presented there applies equally well in the current setting.

REMARK 2.7. The argument in [6] relies on Anderson localisation. On the other hand, if the cumulative distribution function of v_1 is uniformly Hölder of order $\nu > 1/2$, the conclusion of the lemma also follows from the Minami estimate; see [26] and further [10, 19]. Thus, for such potentials, the conclusion of Theorem 1 is established using fixed-energy arguments only.

3. Proof of the main result.

3.1. *The key technical statement.* Let $c > 0$ be a sufficiently small constant, to be chosen later. For a gap $G_j = [a_j, b_j]$, denote

$$G_j^{+,c} = [b_j - e^{-cN}, b_j], \qquad G_j^{-,c} = [a_j, a_j + e^{-cN}].$$

The following proposition provides uniform control of the transfer matrices outside exponentially small neighbourhoods of the bands. It is the key ingredient in the proof of the main theorem. Having in mind possible additional applications (in the Hermitian and non-Hermitian setting), we formulate it as an independent statement.

PROPOSITION 3.1. *Let v_j be i.i.d. with $|v_1| \le A$ for some $A > 0$. Then for any $\varepsilon > 0$,*

$$\lim_{N\to\infty} \mathbb{P}\Big\{ \forall 1 \le j < N \ \max_{E\in G_j} \frac{1}{N} \log \rho\big(\Phi_N(E)\big) \ge \max_{E\in G_j} \gamma(E) - \varepsilon \Big\} = 1. \tag{3.1}$$

In addition, if $c > 0$ is small enough and $\mathfrak{s} \in \{+,-\}$,

$$\lim_{N\to\infty} \mathbb{P}\Big\{ \forall 1 \le j < N \ \max_{E\in G_j^{\mathfrak{s},c}} \frac{1}{N} \log \rho\big(\Phi_N(E)\big) \ge \max_{E\in G_j} \gamma(E) - \varepsilon \Big\} = 1. \tag{3.2}$$

PROOF. As customary, we denote $\lfloor x \rfloor = \max_{n\le x} n$ and $\lceil y \rceil = \min_{n \ge x} n$.

If $|v_1| \le A < \infty$ almost surely, then $\|H_N(0)\| \le 2 + A$. Let $(E_l)_{l=1}^{\lfloor e^{cN} \rfloor}$ be a sequence of equally spaced points with $E_1 = -3 - A$ and $E_{\lfloor e^{cN}\rfloor} = 3 + A$.

The large deviation estimate of Lemma 2.1 allows to bound the norm of the transfer matrix from below, outside an event of small probability. Formally, for any $E_l \in [-3-A, 3+A]$, we have

$$\mathbb{P}\{\|\Phi_N(E_l)\| \le e^{N(\gamma(E_l)-\frac{\varepsilon}{4})}\} \le B_1 e^{-b_1 N},$$

where B_1 and b_1 do not depend on $E_l \in [-3-A, 3+A]$.

In turn, Lemma 2.2 allows to compare the spectral radius with the norm: taking $\delta = e^{-N\frac{\varepsilon}{4}}$ in the lemma, we obtain for any E_l:

$$\mathbb{P}\{\rho(\Phi_N(E_l)) \le e^{-N\frac{\varepsilon}{4}}\|\Phi_N(E_l)\|\} \le Be^{-Nb\frac{\varepsilon}{4}} + B'e^{-b'N} \le Ce^{-\bar{b}N},$$

where $C > 0$ and $\bar{b} > 0$ do not depend on $E_l \in [-3-A, 3+A]$. Hence

$$\begin{aligned}\mathbb{P}\left\{\exists l : \frac{1}{N}\log\rho(\Phi_N(E_l)) \le \gamma(E_l) - \frac{\varepsilon}{2}\right\}\\ \le e^{cN}[B_1 e^{-b_1 N} + Ce^{-\bar{b}N}] \le C'e^{-c'N},\end{aligned}$$

where we chose $c > 0$ small enough [$c < \min(b_1, \bar{b})$]. In particular, the probability of the event

$$\Omega_1 = \left\{\forall l \frac{1}{N}\log\rho(\Phi_N(E_l)) \ge \gamma(E_l) - \frac{\varepsilon}{2}\right\}$$

tends to 1 as $N \to \infty$.

Let $m_j \in G_j$ be such that $\max_{E \in G_j} \gamma(E) = \gamma(m_j)$. Denote

$$\Omega_2 = \{\forall 1 \le j < N\, \exists l_j : E_{l_j} \in G_j, |E_{l_j} - m_j| \le Ce^{-cN}\}.$$

By Lemma 2.5, no gap is exponentially short, hence each G_j contains at least one E_l. Therefore, also the probability of Ω_2 tends to 1, for $C > 0$ large enough. Then on the event $\Omega_1 \cap \Omega_2$, for sufficiently large N:

$$\begin{aligned}\forall j \qquad \max_{E \in G_j} \frac{1}{N}\log\rho(\Phi_N(E)) &\ge \frac{1}{N}\log\rho(\Phi_N(E_{l_j}))\\ &\ge \gamma(E_{l_j}) - \frac{\varepsilon}{2} \ge \gamma(m_j) - \varepsilon = \max_{E \in G_j}\gamma(E) - \varepsilon,\end{aligned}$$

where we used the Hölder continuity of the Lyapunov exponent (Lemma 2.4). Similarly,

$$\forall j \qquad \max_{E \in G_j^{\mathfrak{s},c}} \frac{1}{N}\log\rho(\Phi_N(E)) \ge \max_{E \in G_j}\gamma(E) - \varepsilon.$$

□

3.2. *Proof of Theorem* 1. Let $\varepsilon > 0$. By equation (3.1) of Proposition 3.1,

$$\lim_{N\to\infty} \mathbb{P}\left\{\forall 1 \le j < N \max_{E\in G_j} \frac{1}{N}\log\rho\big(\Phi_N(E)\big) \ge \max_{E\in G_j}\gamma(E) - \varepsilon/2\right\} = 1.$$

On this event, we have for any $1 \le j < N$:

$$\max_{E\in G_j} \frac{1}{N}\log\rho\big(\Phi_N(E)\big) \ge \max_{E\in G_j}\gamma(E) - \varepsilon/2.$$

Since $G_{2j} = [\lambda_{2j+1}, \lambda_{2j}]$ [where $\lambda_j = \lambda_j(0)$; in the notation of Proposition 3.1, $a_{2j} = \lambda_{2j+1}$ and $b_{2j} = \lambda_{2j}$], we conclude the following: if, for some g,

$$\max\big(\gamma(\lambda_{2j}), \gamma(\lambda_{2j+1})\big) \ge g + \varepsilon,$$

then

$$\max_{E\in G_{2j}} \frac{1}{N}\log\rho\big(\Phi_N(E)\big) \ge (g + \varepsilon/2).$$

On the other hand, $\frac{1}{N}\log\rho(\Phi_N(\lambda_{2j})) = \frac{1}{N}\log\rho(\Phi_N(\lambda_{2j+1})) = 0$, hence by the intermediate value theorem there are two solutions to $\rho(\Phi_N(E)) = e^{Ng}$ lying in G_j. By Lemma 2.3, these are exactly the eigenvalues $\lambda_{2j}(g)$ and $\lambda_{2j+1}(g)$. As to the eigenvalues $\lambda_1(g)$ and, for even N, $\lambda_N(g)$, these are real. Invoking the second part (3.2) of Proposition 3.1, we obtain that $|\lambda_j(g) - \lambda_j(0)|$ is exponentially small. $\square$

4. On the spectral radius of transfer matrices. The ultimate goal of this section is the proof of Lemma 2.2 in Section 4.3. We start with some auxiliary statements.

Let $T_j = T_j(E)$ with $\mathbb{E}|v_j|^\eta < \infty$ for some $\eta > 0$. Then $\mathbb{E}\|T_j\|^\eta < e^{A\eta}$, where $A > 0$ can be chosen locally uniformly in E. Let $\Phi_N = T_N \cdots T_2T_1$, and further let $\Phi_{N,M} = T_N \cdots T_{M+1}$ for $N > M$. We use the singular value decomposition

$$\Phi_N = U_N \begin{pmatrix} s_N & 0 \\ 0 & s_N^{-1} \end{pmatrix} V_N, \qquad \Phi_{N,M} = U_{N,M} \begin{pmatrix} s_{N,M} & 0 \\ 0 & s_{N,M}^{-1} \end{pmatrix} V_{N,M}, \tag{4.1}$$

where $s_N, s_{N,M} \ge 1$, $U_N, V_N, U_{N,M}, V_{N,M} \in SO(2)$. The application of singular value decomposition in the study of random matrix products goes back at least to the work of Tutubalin [33], who realised that the sequence (U_N) is approximated by a Markov chain whereas V_N converges to a random limit. This idea plays an important role in our analysis as well.

4.1. *A lemma in linear algebra.* Denote by uv^* the rank-one operator taking w to $(w, v)u$, where (u, v) is the inner product. Also, we denote by e_j the jth vector of the standard basis. Although we need the following lemma only for two-dimensional matrices, specialising the argument to this case would only obscure the idea.

LEMMA 4.1. *If* $u, v \in \mathbb{C}^m$ *and* $\mathfrak{h} : \mathbb{C}^m \to \mathbb{C}^m$ *is a linear map such that* $\|\mathfrak{h}\| \le \frac{|(u,v)|^2}{9\|u\|\|v\|}$, *then*

$$\rho\big(uv^* + \mathfrak{h}\big) \ge \frac{1}{2}\big|(u, v)\big|. \tag{4.2}$$

PROOF. Let $0 \le t \le 1$, and let z be a complex number on the circle of radius $\frac{1}{2}|(u, v)|$ about (u, v). We shall show that for such t and z the determinant $\det(z - uv^* - t\mathfrak{h})$ does not vanish. This will imply that the number of eigenvalues of $uv^* + t\mathfrak{h}$ in the disc enclosed by the circle does not change as t varies from 0 to 1. For $t = 0$, the spectrum of uv^* consists of two eigenvalues, 0 (with multiplicity $m - 1$) and (u, v) (with multiplicity 1), of which the second one lies in the disk; thus also for $t = 1$ there is (exactly) one simple eigenvalue in the disc, and in particular (4.2) holds.

Let us factorise

$$\det\big(z\mathbb{1} - uv^* - t\mathfrak{h}\big) = \det\big(z\mathbb{1} - uv^*\big)\det\big(\mathbb{1} - \big(z\mathbb{1} - uv^*\big)^{-1}t\mathfrak{h}\big).$$

The first term is equal to

$$z^{m-1}\big(z - (u, v)\big),$$

and thus does not vanish on the circle. To show that the second term does not vanish, observe that

$$\begin{aligned}\big\|\big(z\mathbb{1} - uv^*\big)^{-1}\big\| &= \left\|\frac{1}{z}\mathbb{1} + \frac{1}{z}\frac{uv^*}{z - (u, v)}\right\| \le \frac{1}{|z|}\left\{1 + \frac{\|u\|\|v\|}{|z - (u, v)|}\right\} \\ &\le \frac{2}{|(u, v)|}\frac{4\|u\|\|v\|}{|(u, v)|} = \frac{8\|u\|\|v\|}{|(u, v)|^2},\end{aligned}$$

hence

$$\big\|\big(z\mathbb{1} - uv^*\big)^{-1}t\mathfrak{h}\big\| \le \frac{8\|u\|\|v\|}{|(u, v)|^2}\frac{|(u, v)|^2}{9\|u\|\|v\|} = \frac{8}{9} < 1. \qquad \square$$

COROLLARY 4.2. *If* $\Phi = U\begin{pmatrix} s & 0 \\ 0 & 1/s\end{pmatrix}V$ *with* $U, V \in SO(2)$ *and* $s \ge 3$, $|(VUe_1, e_1)| \ge \frac{3}{s}$, *then*

$$\rho(\Phi) \ge \frac{s}{2}\big|(VUe_1, e_1)\big|.$$

PROOF. Apply the lemma to $u = Ue_1$, $v = V^*e_1$, observing that

$$U\begin{pmatrix} s & 0 \\ 0 & 0\end{pmatrix}V = uv^*$$

and

$$\left\| U \begin{pmatrix} 0 & 0 \\ 0 & 1/s \end{pmatrix} V \right\| = 1/s. \qquad \square$$

Our goal in the remaining part of the section is to prove the estimate

$$\mathbb{P}\{|(V_N U_N e_1, e_1)| \le \delta\} \le K\delta^\kappa + e^{-cN}, \tag{4.3}$$

which will imply Lemma 2.2, in view of Corollary 4.2.

4.2. *On an important unitary operator.* For each $T \in SL_2(\mathbb{R})$, consider the operator $\pi(T) : L_2(S^1) \to L_2(S^1)$, defined via

$$(\pi(T)f)(x) = f(Tx/\|Tx\|)/\|Tx\|.$$

For any $T \in SL_2(\mathbb{R})$, $\pi(T)$ is unitary.

LEMMA 4.3 (Shubin–Vakilian–Wolff [32]). *If v_1 is not almost surely equal to a constant, then there exists $a > 0$ such that*

$$\sup_E \|(\mathbb{E}\pi(T_1(E)))^2\| \le e^{-a} < 1. \tag{4.4}$$

Denoting by $\mathbf{1} \in L_2(S^1)$ the function identically equal to 1 and parametrising the points on the circle by an argument $\theta \in [0, 2\pi]$, we obtain

$$\mathbb{E}\|\Phi_n(E)\|^{-1} \le \mathbb{E}\int_0^{2\pi} \frac{d\theta}{2\pi} \|\Phi_n(E)e^{i\theta}\|^{-1} = \frac{1}{2\pi}(\mathbb{E}\pi(\Phi_n)\mathbf{1}, \mathbf{1}) \le e^{-a\lfloor \frac{n}{2} \rfloor}, \tag{4.5}$$

and in particular $\gamma \ge a/2$.

Next, by the Oseledec multiplicative ergodic theorem [29], V_N converges almost surely to a random limit V. We use this fact in the following form.

LEMMA 4.4. *Suppose $\mathbb{E}|v_1|^\eta < \infty$ for some $\eta > 0$. Then, for any $R > 0$,*

$$\sup_{|E| \le R} \mathbb{E}\|V_N(E) - V(E)\| \le Be^{-bN}.$$

PROOF. Let $V_N^* e_2 = \cos\theta V_{N-1}^* e_2 + \sin\theta V_{N-1}^* e_1$. Then

$$\begin{aligned} \|\Phi_N\|^{-1} = \|\Phi_N V_N^* e_2\| &= \|T_N \Phi_{N-1} V_N^* e_2\| \ge \|T_N\|^{-1} \|\Phi_{N-1} V_N^* e_2\| \\ &\ge \|T_N\|^{-1} [\cos^2\theta \|\Phi_{N-1}\|^{-2} + \sin^2\theta \|\Phi_{N-1}\|^2]^{1/2} \\ &\ge |\sin\theta| \|T_N\|^{-1} \|\Phi_{N-1}\|, \end{aligned}$$

whence

$$|\sin\theta| \le \frac{\|T_N\|}{\|\Phi_N\| \|\Phi_{N-1}\|},$$

and, using (4.5),

$$\mathbb{E}|\sin\theta| \le \mathbb{E}|\sin\theta|^\eta \le e^{A(R)\eta} B_0 e^{-bN}.$$

Thus an estimate of the same form holds for $\|V_N - V_{N-1}\|$ in place of $|\sin\theta|$. □

4.3. *Conclusion of the proof of Lemma* 2.2. As before, we denote $\lfloor x \rfloor = \max_{n \le x} n$ and $\lceil y \rceil = \min_{n \ge x} n$. In the notation of (4.1), Lemma 4.4 applied to the matrix products $T_N \cdots T_1$ and $T_1^* T_2^* \cdots T_N^*$ implies that for $N \ge C \log A$

$$\mathbb{E}\|V_N - V_{\lfloor N/2 \rfloor}\| \le e^{-b'N}, \qquad \mathbb{E}\|U_N - U_{N,\lceil N/2 \rceil}\| \le e^{-b'N}. \tag{4.6}$$

The matrices $V_{\lfloor N/2 \rfloor}, U_{N,\lceil N/2 \rceil}$ are independent, therefore, by an additional application of Lemma 4.4,

$$\mathbb{E}\|V_N - \tilde{V}\| \le e^{-b'N}, \qquad \mathbb{E}\|U_N - \tilde{U}\| \le e^{-b'N}, \tag{4.7}$$

where $\tilde{U}, \tilde{V} \in SO(2)$ are independent random matrices sampled from the corresponding limiting distributions. To conclude the proof of Lemma 2.2, we state (and prove) the following.

LEMMA 4.5. *Assume that* $\mathbb{E}|v_1|^\eta < \infty$, *and that* v_1 *is not almost surely constant*. *Then for any* E *there exist* $K > 0$ *and* $\kappa > 0$ *such that for any* $0 \le \delta \le 1$ *and* $w, w' \in S^1$,

$$\mathbb{P}\{|(U(E)w, w')| \le \delta\} \le K\delta^\kappa, \qquad \mathbb{P}\{|(V(E)w, w')| \le \delta\} \le K\delta^\kappa. \tag{4.8}$$

The numbers K *and* κ *may be chosen locally uniformly in* E.

PROOF OF LEMMA 2.2. Applying Lemma 4.5 with $\tilde{U}$ and $\tilde{V}$ in place of U and V, we obtain

$$\mathbb{P}\{|(\tilde{V}\tilde{U}e_1, e_1)| \le \delta\} \le K\delta^\kappa. \tag{4.9}$$

Together with (4.7), this implies (4.3), which by Corollary 4.2 implies the conclusion of Lemma 2.2. □

PROOF OF LEMMA 4.5. Since we do not keep track on the dependence of the constants on a (from Lemma 4.3) and η, we may assume that $a < 4\eta$. Let $\mathcal{C}_\delta \subset S^1$ be a cap of angular size δ. Assume: $\mathbb{E}\|T_1\|^\eta \le e^{A\eta}$, and denote $n = \lfloor \log \frac{2(A+1)}{\delta} \rfloor$. Let us show that

$$\mathbb{P}\{V_n^* e_2 \in \mathcal{C}_\delta\} \le C\left\lceil \log \frac{16(A+1)}{a} \right\rceil e^{-\frac{an}{16}}. \tag{4.10}$$

By an additional application of Lemma 4.4, this implies (4.8).

To prove (4.10), we start with the estimates

$$\mathbb{E}\|\Phi_n\|^\eta \le e^{An\eta}, \qquad \mathbb{E}\|\Phi_n\|^{-1} \le (\mathbb{E}\pi(\Phi_n)\mathbf{1}, \mathbf{1}) \le 2\pi e^{-\frac{an}{2}},$$

which imply that

$$\mathbb{P}\{e^{\frac{an}{4}} \le \|\Phi_n\| \le e^{(A+1)n}\} \ge 1 - C_1 e^{-\frac{an}{4}}.$$

If (4.10) fails, there exists $4 \le m \le \lceil \log \frac{16(A+1)}{a} \rceil$ such that

$$\mathbb{P}\{e^{\frac{ma}{16}n} \le \|\Phi_n\| \le e^{\frac{(m+1)a}{16}n}, V_n^* e_2 \in \mathcal{C}_\delta\} \ge C e^{-\frac{an}{16}}. \tag{4.11}$$

Let $\tau = \max(\delta, e^{-\frac{(m+1)a}{16}n})$, let $\mathcal{C}_\tau \supset \mathcal{C}_\delta$ be a cap of size τ, and let $\mathbf{1}_\tau$ be the indicator of $\mathcal{C}_\tau$. Then (still treating $e^{i\theta}$ as a vector on the circle)

$$\mathbb{E} \int_{\mathcal{C}_\tau} \|\Phi_n e^{i\theta}\|^{-1} d\theta = \big(\mathbb{E}\pi(\Phi_n)\mathbf{1}, \mathbf{1}\big) \le C_2 e^{-\frac{an}{2}} \sqrt{\tau}. \tag{4.12}$$

On the other hand, if

$$V_n^* e_2 \in \mathcal{C}_\tau, \qquad e^{\frac{ma}{16}n} \le \|\Phi_n\| \le e^{\frac{(m+1)a}{16}n}, \tag{4.13}$$

then

$$\int_{\mathcal{C}_\tau} \|\Phi_n e^{i\theta}\|^{-1} d\theta \ge \frac{1}{C_3} \min\big(\tau, \|\Phi_n\|^{-2}\big) \|\Phi_n\| \ge \frac{1}{C_3} \tau e^{\frac{(m-1)a}{16}n}, \tag{4.14}$$

therefore by (4.11)

$$\begin{aligned} \int_{\mathcal{C}_\tau} \|\Phi_n e^{i\theta}\|^{-1} d\theta &\ge \frac{1}{C_3} \tau \|\Phi_n\| \ge \frac{1}{C_3} \tau e^{\frac{(m-1)a}{16}n} \mathbb{P}\big(\text{the event } (4.13)\big) \\ &\ge \frac{1}{C_3} \tau e^{\frac{(m-1)a}{16}n} C e^{-\frac{an}{16}}. \end{aligned} \tag{4.15}$$

If we choose $C > C_2 C_3$, the juxtaposition of (4.15) with (4.12) leads to

$$e^{-\frac{(m+1)a}{16}n} \le \sqrt{\tau} \le e^{-\frac{(m-1)a}{16}n} e^{\frac{an}{16}} e^{-\frac{an}{2}},$$

which is a contradiction. □

5. Proofs of the additional lemmata.

5.1. *Large deviations*: *Proof of Lemma* 2.1. We suppress the dependence on the spectral parameter E, on which the estimates below are locally uniform. Fix $x \in S^1$. It will suffice to prove the following: for any $\varepsilon \in (0, 1/e)$,

$$\mathbb{P}\left\{\left|\frac{1}{N} \log \|\Phi_N x\| - \gamma\right| \ge \varepsilon\right\} \le C \log \frac{1}{\varepsilon} \exp\left\{-\frac{c\varepsilon^2 N}{\log \frac{1}{\varepsilon}}\right\}.$$

Let $x_0 = x$, and let $x_{j+1} = T_{j+1} x_j / \|T_{j+1} x_j\|$. The vectors $x_j \in S^1$ form a Markov chain, and

$$\|\Phi_N x\| = \prod_{j=1}^{N} \|T_j x_{j-1}\|.$$

Our strategy from this point (based on two arguments going back to the work of S. N. Bernstein [2, 3]) is as follows. Fix $k = \lceil \log \frac{1}{\varepsilon} \rceil$, and split the product into k sub-products corresponding to the different residues of j modulo k. The terms in each sub-product are almost independent; we make them independent by restarting the Markov chain from an invariant distribution every k steps. Then we obtain a bound on the positive and negative fractional moments of each sub-product, from which the desired estimate follows using the Chebyshev inequality.

Formally, for each j choose (independently) a random vector on the circle, distributed according to the invariant measure of the Markov chain; denote this vector by $y_{j-k,j}$. Then set $y_{n+1,j} = T_{n+1} y_{n,j} / \|T_{n+1} y_{n,j}\|$ for $n \geq j-k$, and, finally, define $y_j = y_{j,j}$. Then, for each r, the random variables $\{y_j \mid j \in I_r\}$, where

$$I_r = (r + k\mathbb{Z}) \cap \{1, \ldots, N\},$$

are jointly independent. The vectors y_j are close to x_j: for $j \geq k$,

$$\mathbb{E}\big\{\|x_j - y_j\| \mid v_1, v_2, \ldots, v_{j-k}\big\} \leq C \exp(-ck), \tag{5.1}$$

as implied by the following consequence of Lemma 4.5:

$$\mathbb{P}\big\{V^*_{j,j-k} e_2 \in \mathcal{C}_\delta\big\} \leq C\big[\delta^c + e^{-ck}\big].$$

Denote

$$A_r = \prod_{j \in I_r} \|T_j x_{j-1}\|, \qquad \text{so that } \|\Phi_N x\| = \prod_{r=1}^{k} A_r,$$

and observe that (for $0 \leq p \leq 1$)

$$\|T_j x_{j-1}\|^p \leq \|T_j y_{j-1}\|^p + \|T_j\|^p \|x_{j-1} - y_{j-1}\|^p \tag{5.2}$$

whereas

$$\begin{aligned} \|T_j x_{j-1}\|^{-p} &= \|T_j y_{j-1}\|^{-p} + \big(\|T_j x_{j-1}\|^{-p} - \|T_j y_{j-1}\|^{-p}\big) \\ &\leq \|T_j y_{j-1}\|^{-p} + \frac{\|T_j y_{j-1}\|^p - \|T_j x_{j-1}\|^p}{\|T_j y_{j-1}\|^p \|T_j x_{j-1}\|^p} \\ &\leq \|T_j y_{j-1}\|^{-p} + \|T_j\|^{3p} \|x_{j-1} - y_{j-1}\|^p. \end{aligned} \tag{5.3}$$

Also observe that (for η from the formulation of the lemma)

$$\mathbb{E} \log \|T_j y_{j-1}\| = \gamma, \qquad \mathbb{E}\|T_j y_{j-1}\|^{\pm\eta} \leq C < \infty. \tag{5.4}$$

From (5.1) and (5.4), we obtain that for $0 < p < \min(\frac{\eta}{10}, 1)$,

$$\begin{aligned} \mathbb{E}\big\{\|T_j x_{j-1}\|^p \mid v_1, v_2, \ldots, v_{j-k}\big\} &\leq 1 + p\gamma + C\big(p^2 + e^{-ck}\big) \\ \mathbb{E}\big\{\|T_j x_{j-1}\|^{-p} \mid v_1, v_2, \ldots, v_{j-k}\big\} &\leq 1 + p\gamma + C\big(p^2 + e^{-ck}\big). \end{aligned} \tag{5.5}$$

Taking the products of each of these inequalities over $j \in I_r$ and using the exponential Chebyshev inequality, we have

$$\mathbb{P}\{|\log A_r - \gamma \# I_r| \geq \varepsilon\} \leq \exp(-c\varepsilon^2 \# I_r), \tag{5.6}$$

whence Lemma 2.1 follows by the union bound. □

REMARK 5.1. Using a slightly longer spectral-theoretic argument, one may dispose of the logarithmic terms in (2.4).

5.2. *Hölder continuity*: *Proof of Lemma* 2.4. Let $R \geq e$, $\delta \in (0, 1/e]$, and $N = \lfloor \log \frac{1}{\delta} \rfloor$. By the large deviation estimate (2.4), for any $|E|, |E'| \leq R$,

$$\mathbb{P}\left\{\left|\log\|\Phi_N(E)\| - \gamma(E)\right| < \frac{\delta}{3}, \qquad \left|\log\|\Phi_N(E')\| - \gamma(E')\right| < \frac{\delta}{3}\right\} \geq \frac{3}{4}.$$

Next, by the assumption $\mathbb{E}|v_1|^\eta < \infty$, we have for sufficiently large C:

$$\mathbb{P}\{\|\Phi_j\| \geq (CR)^j\} \leq e^{-10j},$$

therefore with probability $> 3/4$,

$$\begin{aligned} \forall 1 \leq j \leq N \qquad \|\Phi_j(E)\|, \qquad \|\Phi_j(E')\| &\leq (CR)^j, \\ \|\Phi_{N,j}(E)\|, \qquad \|\Phi_{N,j}(E')\| &\leq (CR)^{N-j}, \end{aligned}$$

and on this event

$$\|\Phi_N(E) - \Phi_N(E')\| \leq |E - E'| N (CR)^N.$$

Therefore, for $|E - E'| \leq (2CR)^{-N} = \delta^{\log(2CR)}$,

$$|\gamma(E) - \gamma(E')| \leq \delta,$$

as claimed. □

6. Outlook. Let us briefly comment on possible extensions and directions for further study.

Other potentials. As mentioned in Remark 1.4, it would be interesting to explore the counterparts of Theorem 1 for other stationary (but nonindependent) potentials. It may be of independent interest to explore the counterparts of Proposition 3.1 in this setting.

Higher dimension. The arguments used in the proof of Theorem 1 can be recast into the language of resolvent estimates. In this form, they are applicable to the following higher-dimensional analogue of (1.1) acting on $\ell_2(\mathbb{Z}^d/N\mathbb{Z}^d)$:

$$\begin{aligned}(H_{N,d}(g)\psi)(x) &= e^{-g}\psi(x+e_1)+e^{g}\psi(x-e_1)\\ &\quad+\sum_{j=2}^{d}\big(\psi(x+e_j)+\psi(x-e_j)\big)+v_x\psi(x),\end{aligned}\tag{6.1}$$

where v_x are i.i.d. We state a sample result that can be proved by these arguments.

PROPOSITION 6.1. *Assume that the cumulative distribution function of v_1 is uniformly Hölder of order $\nu > 1/2$. Let $I \subset \mathbb{R}$ be a bounded interval such that, for some $\eta \in (0, 1)$,*

$$\mathbb{E}\big|\big(H_{N,d}(0)-E\big)^{-1}(x,y)\big|^{\eta} < C\exp\big(-\gamma\eta\min(|x_1-y_1|, N-|x_1-y_1|)\big)\tag{6.2}$$

for all E in I. Then for any $g < \gamma$ there exists $c > 0$ such that

$$\mathbb{P}\big\{\forall j \text{ s.t. } \lambda_j(0)\in I : \lambda_j(g)\in\mathbb{R} \text{ and } \big|\lambda_j(g)-\lambda_j(0)\big| \le e^{-cN}\big\} \underset{N\to\infty}{\longrightarrow} 1.$$

The assumption (6.2) is a signature of Anderson localisation; it was shown by Aizenman and Molchanov [1] to hold for any interval I when the disorder is sufficiently strong, and for intervals I at the spectral edges for any strength of the disorder. A similar result can be proved if (6.2) is replaced with the conclusion of the multiscale analysis of Fröhlich and Spencer [12]. Proposition 6.1 confirms the prediction of Kuwae and Taniguchi [23], which was challenged in some of the subsequent works (see [28] and references therein).

Similar to Theorem 1, the proof of Proposition 6.1 makes use of a mesh $(E_l)_{l=1}^{\lfloor e^{cN}\rfloor}$ in I. Instead of Lemma 2.3, one relies on the following observation: if an interval (E_l, E_{l+1}) between a pair of adjacent points of the mesh contains exactly one eigenvalue $\lambda_j(0)$ of $H_{N,d}(0)$, and for all $0 < g' < g$, the points E_l, E_{l+1} are not eigenvalues of $H_{N,d}(g')$, then also $\lambda_j(g) \in [E_l, E_{l+1}] \subset \mathbb{R}$.

Beyond the smallest Lyapunov exponent. While in dimension 1, the conclusion of Proposition 6.1 is similar to that of Theorem 1, we emphasise a distinction between resolvents and transfer matrices, which becomes essential already for a one-dimensional strip of width ≥ 2: the decay of the resolvent kernel is only sensitive to the smallest Lyapunov exponent, whereas the full description of the eigenvalues of non-Hermitian operators of the form considered here is believed to depend on all the Lyapunov exponents. Similarly, in higher dimension, the matrices (6.1) are believed to have some real eigenvalues in the spectral regions in which Anderson localisation does not hold.

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