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Marie-France Bru and Bernard Bru on Dice Games and Contracts

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Abstract. This note introduces Marie-France and Bernard Bru's forthcoming book on the history of probability, especially its chapter on dice games, translated in this issue of *Statistical Science*, and its commentary on the history of fair price in the settlement of contracts.

As the Brus remind us, the traditions of counting chances in dice games and estimating fair price came together in the correspondence between Pascal and Fermat in 1654. To solve the problem of dividing the stakes in a prematurely halted game, Fermat used combinatorial principles that had been used for centuries to analyze dice games, while Pascal used principles that had been proposed in previous centuries by students of commercial arithmetic.

Key words and phrases: Dice games, emergence of probability, *De vetula*, expectation.

1. INTRODUCTION

Next year a long-awaited history of mathematical probability, *Les jeux de l'infini et du hasard* (games of infinity and chance) by Marie-France Bru and Bernard Bru, will be published by the Presses universitaires de Franche-Comté. This issue of *Statistical Science* includes an English translation of the book's chapter on dice games, which reminds us of the antiquity of dice and of the association of belief with frequency that is forced on those who gamble with dice.

Les jeux de l'infini et du hasard is not a substitute for any of our existing histories of mathematical probability. But with its extensive notes and careful philosophical perspective, it provides an essential complement to those more forward-looking histories. It takes us deep into the times it studies, drawing us into multi-faceted worlds and characters and excavating ideas and motives that never merely mirror and can sometimes enrich the ways we have come to use their mathematics today. This note describes the wide sweep of *Les jeux de l'infini et du hasard* and picks out some points that cast unfamiliar light on familiar issues in the history and philosophy of probability.

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The chapter on dice games is of particular interest because it demonstrates the historical and conceptual depth of the duality of probability—its combination of belief with frequency. But we also learn from the book that Pascal's and Huygens's theory of expectation was rooted in a different tradition, in which this duality was absent or at least contingent.

2. GAMES OF INFINITY AND CHANCE

In the 1990s, the University of Paris mathematician Marie-France Bru began to collaborate with her husband Bernard Bru on the history of probability and statistics. The couple produced two jewels: an erudite edition of two unpublished manuscripts by the mathematical statistician Irénée-Jules Bienaymé [7], and an insightful article, in collaboration with Kai Lai Chung, on the mathematician Émile Borel's long fascination with unbounded martingales [8]. The collaboration was tragically cut short when Marie-France's health deteriorated, and she died in 2012, after a valiant struggle with Lou Gehrig's disease. See Figure 1.

The appearance of *Les jeux de l'infini et du hasard*, which emphasizes the role of infinities in probability theory, gives us a new occasion to celebrate Marie-France's memory. It is a scholarly book, with innumerable references and notes so voluminous that they outweigh the main text. But the main text itself is often playful, avoiding technicalities when possible

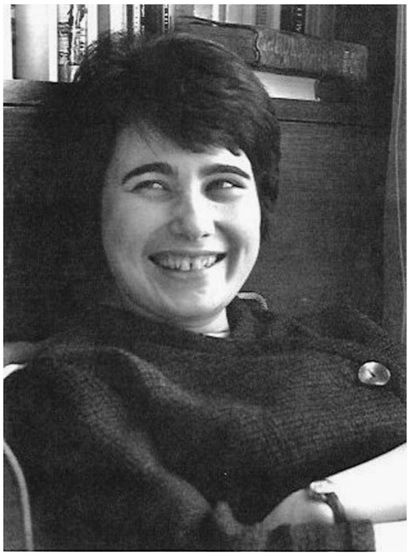

FIG. 1. *Marie-France Bru* (*November* 22, 1943–*January* 30, 2012) *at about* 20 *years of age*, *in her parents' apartment in Paris*.

and sometimes pretending to be accessible even to children—*à la portée de tous*.

The underlying theme of the book is that history can help us understand our ideas more clearly. Loving novelty and aspiring to profundity, we sometimes imagine that we can advance our understanding by making our concepts ever more complicated. The Brus prefer a historical approach. "To penetrate to the reasons of things," they advise, "look at how they have gradually been revealed in the course of time, in their progression and in their ruptures ..."

The book has two parts. Part I, *Les probabilités dénombrables à la portée de tous*, delves into the history of countably infinite probability. Part II, *Les probabilités indénombrables à la portée de tous*, delves into the history of continuous probability.

Part I reminds us how early probability theory became entangled with the countably infinity. We see countable infinities at work in Abraham De Moivre's recurrent series and André-Marie Ampère's theory of the gambler's ruin. We see the roots of "almost sure" in the work of Joseph Bertrand, its emergence in Henri Poincaré's probabilistic mechanics and its first flowering in Émile Borel's pathbreaking denumerable probability calculus, which produced the first strong law of large numbers and led to modern measure-theoretic probability.

Part II emphasizes Laplace's work on continuous probability, beginning with the approximations that we now think of as Laplace's version of the central limit theorem. The authors trace the nineteenth century rise and the fall of Laplace's asymptotic probability calculus, with a particularly touching emphasis on Laplace's efforts to promote its use in the last years of his career. In letters to colleagues across Europe in 1815, Laplace explained how to calibrate the uncertainty in Alexis Bouvard's estimate of the mass of Jupiter. The odds are a million to one, he declared, that the estimate is accurate to within one percent. Alas, there were significant errors in the data Bouvard used. Laplace died in 1827, and by 1832 scientists knew that Bouvard's estimate was in error by about two percent.

The relevance of this history to the philosophy of probability is evident at the outset of the Brus' book. We often fall into thinking that countably additive probability was invented in the twentieth century, "classical probability" being concerned only with problems where there are finitely many equally likely cases. But the Brus remind us of Jacob Bernoulli's use of infinite series in his *Ars conjectandi*, published in 1713 [5]. Suppose Peter and Paul alternately toss a fair coin. If Peter goes first, what is the probability that he will be the first to get a head? We can find the answer by adding the probabilities for the first head coming on the first toss, the third toss and so on:

$$\frac{1}{2}+\left(\frac{1}{2}\right)^3+\left(\frac{1}{2}\right)^5+\cdots=\frac{2}{3}. \tag{2.1}$$

Following Huygens, Bernoulli used the concept of expectation (contingent payoff) rather than the more abstract concept of probability to solve problems of this kind; he assumed that the values we assign to contingencies add. What is the total value of Peter's expectation if the first player to get a head wins some amount of money, say one ducat? To answer such questions, Bernoulli first altered the picture by assuming that there are an infinite number of players, a different player tossing each time. Suppose the first player who gets a head gets the ducat, but he passes it on to Peter if his toss is odd-numbered. The value of the expectation of the player who makes toss n being $1/2^n$, equation (2.1) gives the sum of these values for the odd-numbered tosses, and hence the total value of Peter's expectation.[1]

[1]This problem is a variant of the first of the five problems that Christiaan Huygens posed at the end of his 1657 treatise [18]. Huygens presumably solved the problem using a recursive argument rather than an infinite series, but Bernoulli preferred to use infinite series, and he demonstrated the value of this method by showing that it can handle more complicated rules for alternating play, as when A tosses once, B once, A twice, B twice, then each three

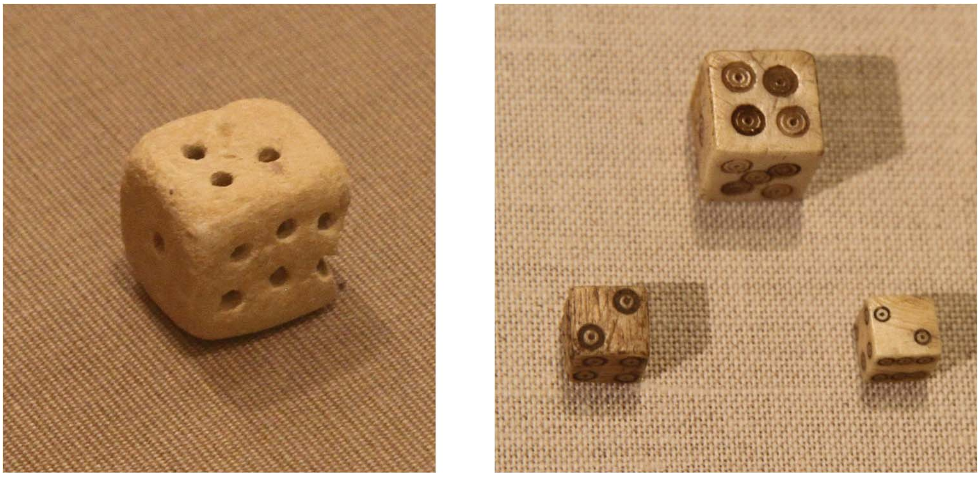

FIG. 2. *Some ancient dice at the Oriental Institute of the University of Chicago. The die on the left was found at Khafajah, northwest of Baghdad, and dates from the Akkadian period,* 2300–2000 *B.C. The dice on the right are from the Roman empire,* 100–150 *A.D; see* [36]. *Photographs used with permission of Stephen M. Stigler.*

Instead of translating Bernoulli's idea into an argument that adds probabilities, we could translate it into an argument that adds gambling strategies. For each n, we have a strategy that costs $1/2^n$ and pays 1 if the first head is on the nth toss. Adding the strategies for odd n, we obtain a strategy with total cost $2/3$ that pays 1 if the first head is on an odd-numbered toss. As this illustrates, the axiom of countable additivity can usually be replaced, in a game-theoretic formulation of probability theory, with the more concrete notion of adding strategies [35].

3. THE PSEUDO-OVID'S *DE VETULA*

The most enduring legend about mathematical probability is that it began with the exchange of letters between Pascal and Fermat in 1654. As Laplace put it, these two French mathematicians were the first to give principles and methods for the calculus of chances and to solve probability problems of any complexity.[2] In his influential book, *The Emergence of Probability* [15], Ian Hacking embellished this legend by arguing that the late seventeenth century invented not only the mathematical theory of probability but also a philosophical concept of probability—a concept that combined, for the first time, the ideas of belief and frequency. These legends have been endlessly repeated and have a hold on our minds, even when we readily admit that they are simplifications or worse.

The Brus' chapter on dice games challenges the legend of a seventeenth century invention of probability by taking us step by step through calculations in *De vetula*, a long medieval poem by an unknown author who claims to be Ovid. The pseudo-Ovid correctly counts the 216 ways three dice can fall and calculates how these 216 chances are distributed over the 16 different possible sums, 3 to 18, of the points on the faces that fall upward.

De vetula touches on many topics and was used in European universities for centuries, copied and recopied and eventually printed. Its calculations for dice have hardly been unknown to modern historians of probability. Maurice G. Kendall commented on them in his 1956 article in *Biometrika* on the beginnings of the probability calculus [19], and Florence Nightingale David noted them in her 1962 *Games, Gods and Gambling* [11]. Modern editions of the Latin text appeared in the 1960s [20, 28], and Ivo Schneider translated the relevant passages into German in his 1988 reader on the history of probability [31]. Yet this knowledge has done little to dampen the legend of Pascal and Fermat. As David Bellhouse noted in 2000, the poem "has for

times and so on. Bernoulli posed these problems in the *Journal des Sçavans* in 1685 (August 26, p. 314) and published his solutions in the May 1690 number of Leibniz's *Acta eruditorum*. He explained his method in his commentary on Huygens's first problem, on pages 49–57 of *Ars conjectandi*. The 1685 and 1690 publications are reproduced on pages 91–98 of Volume 3 of [6].

[2]Here, translated from the French, is what Laplace writes at the beginning of his sketch of the history of probability in the last section of his *Essai philosophique sur les probabilités* [22]: "Since long ago, people determined ratios of chances for and against players in the simplest games, using them to fix stakes and bets. But before Pascal and Fermat, no one had given the principles and methods for reducing this topic to calculation, or had solved questions of this type that were of any complexity."

FIG. 3. *Depiction of* triga *from Alfonso the Wise's* Libro de los juegos (1283). *Like all the other dice games in the book, triga involved throwing three dice. It is a two-player game; the first player to throw three of a kind,* 15 *or greater,or* 6 *or less wins. The Brus, who discuss* Libro de los juegos*'s analysis of triga in notes not included in the translation following this introduction, tell us that it falls short of* De vetula*'s analysis. It counts the* 56 punctaturae *but not the* 216 cadentiae. *Using* De vetula*'s tabulation of the cadentiae, we find that the probability of winning on the first throw is* 42/216, *or approximately* 19%. *We then find, using Jacob Bernoulli's method, that the player who throws first has a probability of* 252/455, *or approximately* 55%, *of winning. Photograph used with permission of Charles Knutson of MacGregor Historic Games.*

the most part been passed over by historians of probability" ([4], page 123). Bellhouse gave a much fuller account of *De vetula*'s calculations than had previously appeared in English, but it remains a commonplace that mathematical probability began in the seventeenth century.

The Brus go beyond Bellhouse and other earlier authors on the history of probability by emphasizing not only the pseudo-Ovid's calculations but also his rhetoric. The poet makes clear why one must look deeper than the 56 different arrangements of the three faces: these arrangements do not each have the same force or frequency. So here, four centuries before Pascal and Fermat, we see a concept of probability that ties together betting (belief) and outcome (frequency). How could it be otherwise? And how could it have been otherwise for avid and practiced dice players in the Roman armies or among the Babylonians? See Figure 2.

As the Brus emphasize, we have no specific evidence and yet much reason to conjecture that others, in the Islamic world or earlier, preceded the pseudo-Ovid in calculating chances for three dice. *De vetula* appeared in the first century of the massive flow of Islamic science into Europe through Spain and Italy [2, 23, 29], and we know that many of the ingredients for probability theory, including algebra and combinatorics, came by this route [12, 30]. Among the many books commissioned by Alfonso the Wise of Castile (1221–1284), one of the great patrons of the inflow of knowledge from Islam, we find his book of games, *Libro de los juegos*, where dice games were not overlooked; see Figure 3. We have not yet found Arabic texts that counted the chances for dice. Did they never exist? Were they all destroyed by religious zealots? Or do some lie undiscovered among the thousands of Arabic manuscripts, scattered across libraries and archives, that no one has yet transcribed, translated or even read for centuries? We do not know, and we may yet be surprised. It is only in recent decades that cryptographers have discovered how much had been accomplished in their field by the Arabs [1, 3] and that economists have discerned financial derivatives in cuneiform tablets of ancient Mesopotamia [37].

4. FROM *DE VETULA* TO LAPLACE

As the Brus explain in the chapter translated here, the problem of generalizing *De vetula*'s accomplishment from three to a larger number of dice and from six to an arbitrary number of faces is an important thread in the history of mathematical probability in the eighteenth century. The pseudo-Ovid's combinatorial solution was generalized by Jacob Bernoulli, then by Abraham De Moivre and Pierre Rémond de Montmort, and finally by Laplace. In 1810, forty years after first attacking the problem, Laplace found an approximation to the Montmort-Moivre formula for the problem, the same normal approximation that we now frame as the central limit theorem and that Laplace used to find what we now call Bayesian credence intervals and Bernoullian (frequentist) confidence intervals for large samples.

5. THE PROBLEM OF DIVISION

While we have no record of their considering *De vetula*'s problem of counting the chances for three dice, we know that Pascal and Fermat solved a number of other betting problems. The most celebrated of these was the problem of division: how to divide the stakes in a prematurely halted multi-round game. If Paul is one round short of winning, and Peter is two rounds short, how should they divide the money on the table? The correct answer is that Peter should receive only $1/4$. Pascal and Fermat arrived at this answer in different ways:

- Fermat noted that if the players were to play two more rounds, then there would be four equally possible outcomes; the winners could be (Peter,Peter), (Peter,Paul), (Paul,Peter) or (Paul,Paul), and Paul would win the stakes in 3 out of 4 of these cases.
- Rejecting this argument because the second round would not be played if Paul won the first, Pascal gave a different argument. If Peter won the first round, he would play the second at even odds and at that point have an expectation worth half the stakes. So on the first round he is playing at even odds to obtain either zero or an expectation worth half the stakes and, therefore, has an expectation worth a quarter of the stakes.

In a note to their chapter on dice games, not reproduced in the following translation, the Brus point out that Pascal's reasoning about the problem of division connects with a medieval tradition distinct from the combinatorial tradition in which we can place both Fermat's reasoning about the problem of division and the pseudo-Ovid's analysis of the chances for three dice [10].

Pascal's mathematics certainly involved combinatorics; he saw the problem of division as one application of his marvelous arithmetic triangle [12]. But his principles were about fairness, not about counting chances. In this respect, he was not echoing the age-old experience of dice players and its marriage of betting with frequency. He was echoing instead the equally ancient experience of merchants and tradesmen, forced to settle contracts when circumstances prevent their being fulfilled to the letter.

The problem of division had been discussed by mathematicians for centuries before Pascal proposed it to Fermat. Those discussing it in print included Luca Pacioli in his *Summa* (1494), Gerolamo Cardano (1539) and Niccolò Tartaglia (1556). Were these authors motivated by the concept of chance that we find in the pseudo-Ovid's and Alfonso's dice games? No. These authors mention games of skill: ball games and archery competitions. What motivated their interest in betting in such games? The answer, surely, is that the risks resemble those taken by businessmen. Both luck and skill play a role. As Ivo Schneider convincingly argued in 1988, evidence abounds that Pacioli, Cardano, Tartaglia and other authors of their time who wrote about the problem of division were thinking about settling unfulfilled contracts. In a nutshell, "gaming was understood as a process which recapitulated the activities of merchant adventurers in a condensed time span" ([32], page 220).[3]

Pacioli, Cardano and Tartaglia did not arrive at Pascal's solution, and historians of probability have often treated their alternative solutions as mistakes. Anders Hald, for example, observes that the arguments made by Cardano in 1539 "do not lead to the correct division rule" ([16], page 36). But when we take Schneider's arguments seriously, we may wish to agree with Tartaglia that there is no unique correct solution in general, even if the parties are bound to agree on some solution that they consider fair.

The commercial context of Pacioli's solution of the problem of division is hardly hidden, his *Summa* being a book on commercial mathematics, beginning with arithmetic and algebra and concluding with the mathematics of bookkeeping and finance. It is here that Pacioli introduced double-entry accounting. He drew less on the learning of the universities than on the tradition of the abacus schools of northern Italy, where young men learned the mathematics of trade in the vernacular. For information on the sources of this tradition on both sides of the Mediterranean and its influence on the development and language of probability theory; see Jen Høyrup [17], Edith Sylla [38] and *Probabilités Médié-*

[3]On the other hand, as Schneider has suggested to me in personal correspondence, the widespread prohibition of games of chance by governmental and religious authorities may be at least partly responsible for these authors failing to mention them.

vales, a special issue of the *Electronic Journal for History of Probability and Statistics* (www.jehps.net, Volume 3, Number 1) edited by Norbert Meusnier and Sylvain Piron.

The relevance of the abacus school tradition to the problem of division has come into better focus with the discovery of manuscripts whose authors did arrive at Pascal's solution. In 1985, Laura Toti Rigatelli published a previously unnoticed Italian manuscript, dating from about 1400 and preserved in the National Library in Florence, which addresses the problem of division when the game is a version of chess [41]. One player needs to win only one more match and the other needs to win three. As Schneider has shown, the manuscript's reasoning conforms to Pascal's principles and is completely correct [32].[4] A second, much more extensive Italian manuscript, dating from about the same period and preserved in the Vatican Apostolic Library, was published by Raffaella Franci in 2002 and analyzed in detail by Norbert Meusnier in 2007 [13, 25]. Its unknown author does not say what game is being played, but the manuscript presents an elegant theory, again conforming to Pascal's principles. It instructs its readers to keep the methods secret, so that they will not lose their monopoly in using them, suggesting the existence over some period, beginning at least two centuries before Pascal, of a whole school that understood them.

Pascal did have games of chance in mind. He mentions tossing a coin in his second letter to Fermat (24 August 1654), and he appeals to the idea of a game of pure chance when he explains his two principles in his *Traité du triangle arithmétique* [27]. The first principle is that a player is entitled to an amount that he will get no matter what happens. The second is that when one player wins what the other loses, the amount should be divided equally. The second principle, he argues, is justified if the game is one of pure chance and there is as much chance for the one player as for the other.[5] But players in a game of skill can also use Pascal's principles if they want, for they can agree to bet on equal terms even if they do not agree that their skills are equal.

In our twenty-first century, Pascal's game-theoretic picture remains an alternative to Fermat's combinatorial or measure-theoretic picture, an alternative in which the connection between betting rate and outcome (frequency) is not taken for granted. The fundamental principle is that agreed-on bets can be compounded to produce other bets. This idea has been exploited in modern finance theory, where it is understood that an auxiliary principle—an "efficient market hypothesis"—is needed in order to make the connection with outcomes. It has also been exploited in game-theoretic probability, where the auxiliary principle can be understood as a version of Cournot's principle [33, 34].

6. NAMING THE PROBLEMS

We should pause over one issue of translation. In French, Pascal and Fermat's problem of division is called *le problème des partis*. The French masculine noun *parti* can be translated as "part" or "share," but it can easily be confused with the French feminine noun *partie*, which can be translated as "point" or "round" in a game with multiple rounds. The French mathematician Sylvestre François Lacroix warned against this confusion in his 1816 probability textbook ([21], page 93), but the English mathematicians Lubbock and Drinkwater fell victim to it in their anonymous text on probability ([24], 1830), translating *le problème des partis* as "the problem of points." Their example was followed by Isaac Todhunter in his authoritative history ([40], 1865) and by nearly everyone else writing on the topic in English since.

On the other hand, the Brus have chosen to use *le problème des points* to refer to the pseudo-Ovid's problem of calculating the chances for the different possible sums of points for three (or more) dice. This makes perfect sense in French, and it would make sense in English as well had "problem of points" not become the standard name for the problem of division.

What should the translator do? Ignoring the standard English usage and trusting the reader to note and remember that we are using the term in a nonstandard way, I have translated the Brus' *problème des points* into English as "problem of points."

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[4]The manuscript is also discussed by James Franklin ([14], pages 294–296), who cites Schneider but is puzzled that the game is not a game of chance.

[5]Here, translated from the French, is what Pascal writes in the fifth paragraph of Part III of [27]: "If two players find themselves in a situation where a certain sum belongs to one of them if he wins and to the other if he loses; and if the game is one of pure chance and there is as much chance for the one as for the other, and consequently no more reason why the one should win rather than the other, then if they want to separate without playing and take what legitimately belongs to each, they should divide the sum at hazard in half…"

Schneider, Edith Sylla, Vladimir Vovk, Sandy Zabell and an anonymous referee.

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