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Harnack inequalities for SDEs driven by time-changed fractional Brownian motions*

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Abstract

We establish Harnack inequalities for stochastic differential equations (SDEs) driven by a time-changed fractional Brownian motion with Hurst parameter $H \in (0, 1/2)$. The Harnack inequality is dimension-free if the SDE has a drift which satisfies a one-sided Lipschitz condition; otherwise we still get Harnack-type estimates, but the constants will, in general, depend on the space dimension. Our proof is based on a coupling argument and a regularization argument for the time-change.

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1 Introduction

Throughout this paper, $(\Omega, \mathscr{A}, \mathbb{P})$ is a probability space. Consider the following d-dimensional SDE

$$X_t(x) = x + \int_0^t b_s\big(X_s(x)\big)\,\mathrm{d}s + U_t, \quad t \geq 0,\ x \in \mathbb{R}^d, \tag{1.1}$$

where $b : [0, \infty) \times \mathbb{R}^d \to \mathbb{R}^d$ is measurable, locally bounded in the time variable $t \geq 0$ and continuous in the space variable $x \in \mathbb{R}^d$; the driving noise $U = (U_t)_{t \geq 0}$ is a locally bounded measurable process on $\mathbb{R}^d$ starting at zero $U_0 = 0$. Let us assume, for the time being, that this SDE has a unique non-explosive solution.

In this paper, we want to establish for the solution to the SDE (1.1) a dimension-free Harnack inequality with power, first introduced by Wang [19] for diffusions on

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Riemannian manifolds, and a log-Harnack inequality, considered in [16] for semi-linear SDEs. These two Harnack-type inequalities have many applications, for example when studying the strong Feller property, heat kernel estimates, contractivity properties, entropy-cost inequalities, and many more; for an in-depth explanation we refer to the monograph by Wang [20, Subsection 1.4.1] and the references given there. Both, the power-Harnack and log-Harnack inequalities have been thoroughly investigated for various finite- and infinite-dimensional SDEs and SPDEs driven by Brownian noise; the main tool was a coupling method and the Girsanov transformation, see [20] and the references mentioned there. If the noise is a jump process, it is usually very difficult to construct a successful coupling, and the methods from diffusion processes cannot be directly applied. One notable exception are driving noises which are subordinate to a diffusion process.

Let $\Sigma : [0,\infty) \to \mathbb{R}^d \otimes \mathbb{R}^d$ be a measurable and locally bounded deterministic function, and assume that U is of the following form:

$$U_t = \int_0^t \Sigma_{s-}\,\mathrm{d}W_{S(s)} + V_t, \quad t \geq 0,$$

where $W = (W_t)_{t\geq 0}$ is a standard d-dimensional Brownian motion, $S = (S(t))_{t\geq 0}$ is a subordinator (i.e. a non-decreasing process on $[0,\infty)$ with stationary and independent increments a.k.a. increasing Lévy process) and $V = (V_t)_{t\geq 0}$ is a locally bounded $(\mathscr{B}[0,\infty)\otimes\mathscr{A}/\mathscr{B}(\mathbb{R}^d)$-)measurable process on $\mathbb{R}^d$ with $V_0 = 0$; we will, in addition, assume that the processes W, S and V are stochastically independent.

In this setting, Wang & Wang [21] were able to obtain Harnack and log-Harnack inequalities, using an approximation of the subordinator (as in [23]) and a coupling argument. The following assumptions turned out to be crucial: The coefficient b has to satisfy a so-called one-sided Lipschitz condition, i.e. there exists a locally bounded measurable function $k : [0,\infty) \to \mathbb{R}$ such that

$$\langle b_t(x) - b_t(y), x - y\rangle \leq k(t)|x-y|^2, \quad x, y \in \mathbb{R}^d,\ t \geq 0; \tag{H}$$

moreover, the inverse Σ_t^{-1} exists for each $t \geq 0$, and there exists a non-decreasing function $\lambda : [0,\infty) \to [0,\infty)$ such that $\|\Sigma_t^{-1}\| \leq \lambda_t$ for all $t \geq 0$.

The first-named author used in [6] the same approximation argument and a gradient estimate approach, in order to improve the Harnack inequalities derived in [21]. Recently, in [22] the approximation argument was also used to establish Harnack-type inequalities for SDEs with non-Lipschitz drift and anisotropic subordinated Brownian noise, i.e. with U having the form

$$U_t = \big(W^{(1)}_{S^{(1)}(t)}, \dots, W^{(d)}_{S^{(d)}(t)}\big), \quad t \geq 0,$$

where $\big(W^{(1)}, \dots, W^{(d)}\big)$ is a standard Brownian motion in $\mathbb{R}^d$, and $\big(S^{(1)}, \dots, S^{(d)}\big)$ is an independent d-dimensional Lévy process such that each coordinate process $S^{(i)}$ is a subordinator. Unfortunately, this gives only dimension-dependent Harnack inequalities. Note that the techniques of [21, 6, 22] do not really need that the time-change is a subordinator; we may, as we do here, assume that the time-change is any non-decreasing process on $[0,\infty)$ starting from zero and which is independent of the original process.

It is a natural question to ask whether one can still get Harnack-type inequalities if the driving noise U is a more general, maybe non-Markovian, process. As far as we know, Harnack inequalities were established in [8, 10, 9] for SDEs driven by fractional Brownian motions. Inspired by these papers as well as [21, 6], we will combine general time-change and coupling arguments to obtain Harnack inequalities for SDEs driven by time-changed fractional Brownian motions.

Recall that a fractional Brownian motion $W^H = (W_t^H)_{t\geq 0}$ on $\mathbb{R}^d$ with Hurst parameter $H \in (0,1)$ is a self-similar, mean-zero Gaussian process with stationary increments. The covariance function is given by

$$\mathbb{E}\big(W_t^{H,(i)} W_s^{H,(j)}\big) = \frac{1}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right)\delta_{ij}, \quad t,s \geq 0,\ 1 \leq i,j \leq d, \tag{1.2}$$

(δ_{ij} denotes Kronecker's delta). If $H = 1/2$, then W^H is the classical Brownian motion which will be denoted as W; if $H \neq 1/2$, then W^H does not have independent increments. One can deduce from (1.2) that W^H is self-similar with index H, i.e. for any constant $c > 0$, the processes $(W_{ct}^H)_{t\geq 0}$ and $(c^H W_t^H)_{t\geq 0}$ have the same finite dimensional distributions. Let $Z = (Z(t))_{t\geq 0}$ be a non-decreasing process on $[0,\infty)$ starting from 0, independent of W^H, and introduce the (random) time-changed process $W_Z^H = (W_{Z(t)}^H)_{t\geq 0}$. Typically, Z can be a subordinator or the inverse of a subordinator; since inverse subordinators are constant on some random intervals, W_Z^H is sometimes called a 'delayed' fractional Brownian motion. We refer to [12] for small deviation probabilities of time-changed fractional Brownian motions, while [13, 11] consider large deviations of fractional Brownian motions delayed by inverse α-stable subordinators.

Assume that $U = W_Z^H + V$ where V is a locally bounded measurable process on $\mathbb{R}^d$ starting from zero $V_0 = 0$. In this paper, we restrict ourselves to the case $H \in (0, 1/2)$. In order to ensure the existence and uniqueness of the solution to the SDE (1.1) and to construct a successful coupling, we assume that the coefficient b satisfies the one-sided Lipschitz condition (H). As a direct consequence of the log-Harnack inequality, we obtain a gradient estimate for the associated Markov operator.

As in [22], we can also deal with the anisotropic case, i.e.

$$U_t = \left(W_{Z^{(1)}(t)}^{H_1,(1)}, \dots, W_{Z^{(d)}(t)}^{H_d,(d)}\right) + V_t, \quad ,t \geq 0,$$

where, for each $i = 1, \dots, d$, $W^{H_i,(i)} = (W_t^{H_i,(i)})_{t\geq 0}$ is a real-valued fractional Brownian motion with Hurst index $H_i \in (0, 1/2)$, $Z^{(i)} = (Z^{(i)}(t))_{t\geq 0}$ is a one-dimensional non-decreasing process such that $Z^{(i)}(0) = 0$, and $V = (V_t)_{t\geq 0}$ is a locally bounded measurable process with values in $\mathbb{R}^d$ and $V_0 = 0$; moreover, we assume that these processes are independent. As in [22], we replace the Lipschitz condition for the drift coefficient b by a Yamada–Watanabe-type condition; in general, however, this condition cannot be compared with the one-sided Lipschitz condition.

The remaining part of this paper is organized as follows. We collect some basics on fractional Brownian motions in Section 2. In Section 3 we establish the Harnack inequalities for SDEs driven by a time-changed fractional Brownian motion and with drift coefficient satisfying the one-sided Lipschitz condition (H). More explicit expressions in the Harnack and log-Harnack inequalities are obtained if the time-change Z is (the inverse of) a subordinator; this is a consequence of our moment estimates from [7]; if Z is the inverse of a subordinator, only the log-Harnack inequality holds, since the exponential moment of $Z(t)^{-\theta}$ is usually infinite for $\theta > 0$. The last section is devoted to the case of an anisotropic driving noise; as one would expect from [22], the Harnack inequalities turn out to be dimension-dependent.

2 Basics of fractional Brownian motion

In this section, we recall briefly some basic facts on fractional Brownian motion (fBM) which will be used later on. For further details of fBM and proofs we refer the readers, for instance, to [2, 5] or [14].

Denote by $\Gamma(\cdot)$, resp., $B(\cdot,\cdot)$, the Euler Gamma and Beta functions, and write ${}_2F_1$ for Gauss' hypergeometric function. Let $a, b \in \mathbb{R}$ with $a < b$. For $f \in L^1[a,b]$ and $\alpha > 0$, the left fractional Riemann-Liouville integral of f of order α on (a,b) is given by the expression

$$I^{\alpha}_{a+}f(x) := \frac{1}{\Gamma(\alpha)}\int_a^x (x-y)^{\alpha-1}f(y)\,\mathrm{d}y, \quad x\in(a,b).$$

Let $W^H = (W^H_t)_{t\geq 0}$ be a fractional Brownian motion on $\mathbb{R}^d$ with Hurst parameter $H\in(0,1/2)\cup(1/2,1)$ and define for $0<s<t$ the kernel

$$\mathcal{K}_H(t,s) := \frac{1}{\Gamma\left(H+\frac{1}{2}\right)}(t-s)^{H-\frac{1}{2}}\,{}_2F_1\left(H-\tfrac{1}{2},\tfrac{1}{2}-H,H+\tfrac{1}{2},1-\tfrac{t}{s}\right)$$

Fix $T>0$. It is known that the operator $\mathcal{K}_H$, associated with the kernel $\mathcal{K}_H(\cdot,\cdot)$

$$\mathcal{K}_H f^{(i)}(t) := \int_0^t \mathcal{K}_H(t,s)f^{(i)}(s)\,\mathrm{d}s, \quad i=1,\dots,d,$$

establishes a bijection from $L^2([0,T];\mathbb{R}^d)$ to the space $I^{H+1/2}_{0+}(L^2([0,T];\mathbb{R}^d))$, see e.g. [17, p. 187] or [5]. Moreover, fractional Brownian motion has the following integral representation with respect to a standard d-dimensional Brownian motion $W=(W_t)_{t\geq 0}$:

$$W^H_t = \int_0^t \mathcal{K}_H(t,s)\,\mathrm{d}W_s.$$

In particular, if $0<H<\frac{1}{2}$ and $h\in I^{H+1/2}_{0+}(L^2([0,T];\mathbb{R}^d))$ is absolutely continuous, the inverse operator is given by

$$(\mathcal{K}_H^{-1}h)(s) = s^{H-1/2}I^{1/2-H}_{0+}s^{1/2-H}h'(s), \tag{2.1}$$

cf. [15, Eq. (13), p. 108]. If $H\in(0,1/2)$, then (2.1) implies

$$\int_0^T |h(s)|^2\,\mathrm{d}s<\infty \quad\Longrightarrow\quad \int_0^\bullet h(s)\,\mathrm{d}s\in I^{H+1/2}_{0+}(L^2([0,T];\mathbb{R}^d)), \tag{2.2}$$

see also [15, p. 108].

3 SDEs driven by delayed fractional Brownian motions

Consider the following SDE on $\mathbb{R}^d$

$$X_t(x) = x+\int_0^t b_s\big(X_s(x)\big)\,\mathrm{d}s + W^H_{Z(t)} + V_t, \quad t\geq 0,\ x\in\mathbb{R}^d, \tag{3.1}$$

where $b:[0,\infty)\times\mathbb{R}^d\to\mathbb{R}^d$, $(t,x)\mapsto b_t(x)$ is measurable, locally bounded as a function of $t\geq 0$ and continuous in x. The processes $W^H=(W^H_t)_{t\geq 0}$, $Z=(Z_t)_{t\geq 0}$ and $V=(V_t)_{t\geq 0}$ are stochastically independent and satisfy

$$\begin{cases} W^H \text{ is a fBM on } \mathbb{R}^d \text{ with Hurst parameter } H\in(0,1/2);\\ Z \text{ is a time-change, i.e. a non-decreasing process on } [0,\infty) \text{ with } Z(0)=0;\\ V \text{ is a locally bounded measurable process on } \mathbb{R}^d \text{ with } V_0=0.\end{cases} \tag{3.2}$$

Moreover, we assume that the coefficient b satisfies the one-sided Lipschitz condition (H).

Remark 3.1. The one-sided Lipschitz condition (H) ensures, in particular, the existence, uniqueness and non-explosion of the solution to the SDE (3.1). Indeed, it is well known that the following ordinary differential equation

$$Y_t(x) = x+\int_0^t \tilde{b}_s\left(Y_s(x)\right)\,\mathrm{d}s, \quad t\geq 0,\ x\in\mathbb{R}^d,$$

has a unique solution which does not explode in finite time since the coefficient $\tilde{b}$, defined by $\tilde{b}_s(\cdot) := b_s\big(\,\cdot + W^H_{Z(s)} + V_s\big)$, satisfies the one-sided Lipschitz condition (H) with b replaced by $\tilde{b}$; setting $X_t(x) := Y_t(x) + W^H_{Z(t)} + V_t$, we conclude that the the SDE (3.1) has a unique non-explosive solution.

Throughout this section, we write $|x| := \big(|x^{(1)}|^2 + \cdots + |x^{(d)}|^2\big)^{1/2}$ for the Euclidean norm of $x = \big(x^{(1)}, \ldots, x^{(d)}\big) \in \mathbb{R}^d$. Set

$$P_t f(x) := \mathbb{E} f(X_t(x)), \quad t \geq 0,\ f \in \mathscr{B}_b(\mathbb{R}^d),\ x \in \mathbb{R}^d. \tag{3.3}$$

3.1 Statement of the main result

In order to state our main result, we need the following notation:

$$K(t) := \int_0^t k(s)\,\mathrm{d}s, \quad t \geq 0, \tag{3.4}$$

where $k(s)$ is the constant appearing in (H),

$$\Theta_H := \frac{1}{4(1-H)} \left(\frac{B\left(\frac{3}{2} - H, \frac{1}{2} - H\right)}{\Gamma\left(\frac{1}{2} - H\right)} \right)^2, \tag{3.5}$$

and we denote for any function $f : \mathbb{R}^d \to \mathbb{R}$ the local Lipschitz constant at the point x by

$$|\nabla f|(x) := \limsup_{y \to x} \frac{|f(y) - f(x)|}{|y - x|} \in [0, \infty], \quad x \in \mathbb{R}^d.$$

Theorem 3.2. *We assume that* (3.2) *and* (H) *hold for the SDE* (3.1) *and we denote its unique solution by* $X_t(x)$.

i) *For* $T > 0$, $x, y \in \mathbb{R}^d$ *and any bounded Borel function* $f : \mathbb{R}^d \to [1, \infty)$

$$P_T \log f(y) \leq \log P_T f(x) + \Theta_H \mathbb{E}\left[\frac{Z(T)^{2-2H}}{\left(\int_0^T \mathrm{e}^{-K(t)}\,\mathrm{d}Z(t)\right)^2} \right] |x - y|^2.$$

ii) *For* $T > 0$, $x \in \mathbb{R}^d$ *and any bounded Borel function* $f : \mathbb{R}^d \to \mathbb{R}$

$$|\nabla P_T f|^2(x) \leq \left\{ P_T f^2(x) - \big(P_T f(x)\big)^2 \right\} 2\Theta_H \mathbb{E}\left[\frac{Z(T)^{2-2H}}{\left(\int_0^T \mathrm{e}^{-K(t)}\,\mathrm{d}Z(t)\right)^2} \right].$$

iii) *For* $T > 0$, $x, y \in \mathbb{R}^d$, $p > 1$ *and any bounded Borel function* $f : \mathbb{R}^d \to [0, \infty)$

$$\big(P_T f(y)\big)^p \leq P_T f^p(x) \cdot \left(\mathbb{E} \exp\left[\frac{\frac{p\Theta_H}{(p-1)^2} Z(T)^{2-2H} |x - y|^2}{\left(\int_0^T \mathrm{e}^{-K(t)}\,\mathrm{d}Z(t)\right)^2} \right] \right)^{p-1}.$$

3.2 Proof of Theorem 3.2

For the proof of Theorem 3.2, we need a few preparations. Let $\ell : [0, \infty) \to [0, \infty)$ be a non-decreasing and càdlàg function with $\ell(0) = 0$, and $v : [0, \infty) \to \mathbb{R}$ a locally bounded measurable function with $v(0) = 0$. By Remark 3.1 the following SDE has a unique non-explosive solution

$$X^{\ell,v}_t(x) = x + \int_0^t b_s\big(X^{\ell,v}_s(x)\big)\,\mathrm{d}s + W^H_{\ell(t)} + v_t, \quad t \geq 0,\ x \in \mathbb{R}^d. \tag{3.6}$$

Set for any bounded Borel function $f:\mathbb{R}^d\to\mathbb{R}$

$$P_t^{\ell,v}f(x):=\mathbb{E}f\big(X_t^{\ell,v}(x)\big),\quad t\geq 0,\ x\in\mathbb{R}^d. \tag{3.7}$$

We want to transform the equation (3.6) into an SDE driven by a fractional Brownian motion which will allow us to establish Harnack inequalities using a combination of coupling and the Girsanov transformation, cf. [8, 10, 9]. First, however, we have to approximate the (deterministic) time-change ℓ by an absolutely continuous function. Consider the following regularization of ℓ:

$$\ell_\epsilon(t):=\frac{1}{\epsilon}\int_t^{t+\epsilon}\ell(s)\,\mathrm{d}s+\epsilon t=\int_0^1\ell(\epsilon s+t)\,\mathrm{d}s+\epsilon t,\quad t\geq 0,\ \epsilon\in(0,1).$$

By construction, for each $\epsilon\in(0,1)$ the function ℓ_ϵ is absolutely continuous, strictly increasing and satisfies for any $t\geq 0$

$$\ell_\epsilon(t)\downarrow\ell(t)\quad\text{as }\epsilon\downarrow 0. \tag{3.8}$$

Let $X_t^{\ell_\epsilon,v}(x)$ be the unique non-explosive solution to the SDE

$$X_t^{\ell_\epsilon,v}(x)=x+\int_0^t b_s\big(X_s^{\ell_\epsilon,v}(x)\big)\,\mathrm{d}s+W^H_{\ell_\epsilon(t)-\ell_\epsilon(0)}+v_t,\quad t\geq 0,\ x\in\mathbb{R}^d, \tag{3.9}$$

and define $P^{\ell_\epsilon,v}$ by (3.7) with ℓ_ϵ instead of ℓ.

Lemma 3.3. *We assume that* (3.2) *and* (H) *hold for the SDE* (3.1) *and we denote its unique solution by $X_t(x)$. Fix $\epsilon\in(0,1)$ and let ℓ_ϵ and $X_t^{\ell_\epsilon,v}(x)$ be as above.*

i) *For $T>0$, $x,y\in\mathbb{R}^d$ and any bounded Borel function $f:\mathbb{R}^d\to[1,\infty)$*

$$P_t^{\ell_\epsilon,v}\log f(y)\leq\log P_t^{\ell_\epsilon,v}f(x)+\frac{\Theta_H[\ell_\epsilon(T)-\ell_\epsilon(0)]^{2-2H}}{\big(\int_0^T\mathrm{e}^{-K(t)}\,\mathrm{d}\ell_\epsilon(t)\big)^2}|x-y|^2.$$

ii) *For $T>0$, $x,y\in\mathbb{R}^d$, $p>1$ and any bounded Borel function $f:\mathbb{R}^d\to[0,\infty)$*

$$\big(P_t^{\ell_\epsilon,v}f(y)\big)^p\leq P_t^{\ell_\epsilon,v}f^p(x)\cdot\exp\left[\frac{\frac{p\Theta_H}{p-1}[\ell_\epsilon(T)-\ell_\epsilon(0)]^{2-2H}|x-y|^2}{\big(\int_0^T\mathrm{e}^{-K(t)}\,\mathrm{d}\ell_\epsilon(t)\big)^2}\right].$$

Proof. Fix $T>0$, $x,y\in\mathbb{R}^d$ and denote by $(Y_t)_{t\geq 0}$ a solution of the equation

$$Y_t=y+\int_0^t b_s(Y_s)\,\mathrm{d}s+W^H_{\ell_\epsilon(t)-\ell_\epsilon(0)}+v_t+\xi\int_0^t\mathbb{1}_{[0,\tau)}(s)\frac{X_s^{\ell_\epsilon,v}(x)-Y_s}{|X_s^{\ell_\epsilon,v}(x)-Y_s|}\,\mathrm{d}\ell_\epsilon(s), \tag{3.10}$$

where

$$\xi:=\frac{|x-y|}{\int_0^T\mathrm{e}^{-K(r)}\,\mathrm{d}\ell_\epsilon(r)}$$

and

$$\tau:=\inf\big\{t\geq 0\,:\,X_t^{\ell_\epsilon,v}(x)=Y_t\big\}$$

is the coupling time. Since

$$\mathbb{R}^d\times\mathbb{R}^d\ni(z,z')\mapsto\mathbb{1}_{\{z\neq z'\}}\frac{z-z'}{|z-z'|}\in\mathbb{R}^d$$

is locally Lipschitz continuous off the diagonal, the system of coupled equations (3.9) and (3.10) has a unique solution for $t\in[0,\tau)$. If $\tau<\infty$, we set $Y_t=X_t^{\ell_\epsilon,v}(x)$ for all $t\geq\tau$. In this way, we can construct a unique solution $(Y_t)_{t\geq 0}$ to (3.10).

Let us show that the coupling time satisfies $\tau \le T$. Let $t < \tau$, write ζ_t for the difference of the solutions to the SDEs (3.9) and (3.10), and observe that ζ_t admits a classic differential satisfying $\mathrm{d}|\zeta_t| = \mathbb{1}_{\{\zeta_t \neq 0\}} |\zeta_t|^{-1} \langle \zeta_t, \mathrm{d}\zeta_t \rangle$; therefore, (H) yields

$$\begin{aligned}
&|X_t^{\ell_\epsilon,v}(x) - Y_t| \mathrm{e}^{-K(t)} \\
&= |x-y| + \int_0^t \frac{1}{|X_s^{\ell_\epsilon,v}(x) - Y_s|} \left\langle X_s^{\ell_\epsilon,v}(x) - Y_s,\, b_s(X_s^{\ell_\epsilon,v}(x)) - b_s(Y_s) \right\rangle \mathrm{e}^{-K(s)}\,\mathrm{d}s \\
&\quad - \xi \int_0^t \mathrm{e}^{-K(s)}\,\mathrm{d}\ell_\epsilon(s) - \int_0^t |X_s^{\ell_\epsilon,v}(x) - Y_s|\, k(s) \mathrm{e}^{-K(s)}\,\mathrm{d}s \\
&\le |x-y| - \xi \int_0^t \mathrm{e}^{-K(s)}\,\mathrm{d}\ell_\epsilon(s).
\end{aligned}$$

Now assume that $\tau(\omega) > T$ for some $\omega \in \Omega$. Taking $t = T$ in the above inequality, we get

$$0 < |X_T^{\ell_\epsilon,v}(x,\omega) - Y_T(\omega)| \mathrm{e}^{-K(T)} \le |x-y| - |x-y| = 0,$$

which is absurd. Therefore, we have $\tau \le T$ and $X_T^{\ell_\epsilon,v}(x) = Y_T$.

Denote by $\gamma_\epsilon : [\ell_\epsilon(0), \infty) \to [0,\infty)$ the inverse function of ℓ_ϵ. By definition, $\ell_\epsilon(\gamma_\epsilon(t)) = t$ for $t \ge \ell_\epsilon(0)$, $\gamma_\epsilon(\ell_\epsilon(t)) = t$ for $t \ge 0$, and $t \mapsto \gamma_\epsilon(t)$ is absolutely continuous and strictly increasing. Let

$$g_r := \xi \mathbb{1}_{[0,\ell_\epsilon(\tau)-\ell_\epsilon(0))}(r) \frac{X^{\ell_\epsilon,v}_{\gamma_\epsilon(r+\ell_\epsilon(0))}(x) - Y_{\gamma_\epsilon(r+\ell_\epsilon(0))}}{|X^{\ell_\epsilon,v}_{\gamma_\epsilon(r+\ell_\epsilon(0))}(x) - Y_{\gamma_\epsilon(r+\ell_\epsilon(0))}|}, \quad r \ge 0.$$

A simple calculation shows $\int_0^{\ell_\epsilon(T)-\ell_\epsilon(0)} |g_r|^2\,\mathrm{d}r < \infty$, and this, together with $H < 1/2$ and (2.2), implies that $\int_0^\bullet g_r\,\mathrm{d}r \in I_{0+}^{H+1/2}(L^2([0,\ell_\epsilon(T)-\ell_\epsilon(0)];\mathbb{R}^d))$. Therefore, the following stochastic integral defines a martingale

$$M_t := -\int_0^t \langle \eta_s, \mathrm{d}W_s \rangle, \quad t \ge 0,$$

where $\eta_s := \mathcal{K}_H^{-1}\left(\int_0^\bullet g_r\,\mathrm{d}r\right)(s)$, $s \ge 0$, and $W = (W_t)_{t\ge0}$ is a d-dimensional standard Brownian motion. Because of (2.1) we see

$$\mathcal{K}_H^{-1}\left(\int_0^\bullet g_r\,\mathrm{d}r\right)(s) = s^{H-\frac12} I_{0+}^{\frac12-H} s^{\frac12-H} g_s,$$

and this yields for any $s \in [0, \ell_\epsilon(T)-\ell_\epsilon(0)]$

$$\begin{aligned}
|\eta_s| &= \left| \frac{1}{\Gamma\left(\frac12 - H\right)} s^{H-\frac12} \int_0^s r^{\frac12-H}(s-r)^{-H-\frac12} g_r\,\mathrm{d}r \right| \\
&\le \frac{\xi}{\Gamma\left(\frac12 - H\right)} s^{H-\frac12} \int_0^s r^{\frac12-H}(s-r)^{-H-\frac12}\,\mathrm{d}r \\
&= \frac{B\left(\frac32 - H, \frac12 - H\right)}{\Gamma\left(\frac12 - H\right) \int_0^T \mathrm{e}^{-K(t)}\,\mathrm{d}\ell_\epsilon(t)} |x-y|\, s^{\frac12-H} \\
&=: C_{T,H}|x-y|\, s^{\frac12-H}.
\end{aligned}$$

Thus, the compensator of the martingale M satisfies

$$\begin{aligned}
\langle M \rangle_{\ell_\epsilon(T)-\ell_\epsilon(0)} &= \int_0^{\ell_\epsilon(T)-\ell_\epsilon(0)} |\eta_s|^2\,\mathrm{d}s \\
&\le C_{T,H}^2 |x-y|^2 \int_0^{\ell_\epsilon(T)-\ell_\epsilon(0)} s^{1-2H}\,\mathrm{d}s \\
&= \frac{C_{T,H}^2|x-y|^2}{2(1-H)} \left[\ell_\epsilon(T) - \ell_\epsilon(0)\right]^{2-2H}.
\end{aligned} \tag{3.11}$$

Set

$$R := \exp\left[M_{\ell_\epsilon(T)-\ell_\epsilon(0)} - \frac{1}{2}\langle M\rangle_{\ell_\epsilon(T)-\ell_\epsilon(0)}\right].$$

Since $\mathbb{E}\,\mathrm{e}^{\frac{1}{2}\langle M\rangle_{\ell_\epsilon(T)-\ell_\epsilon(0)}} < \infty$, one can use Novikov's criterion to get $\mathbb{E}R = 1$, and by Girsanov's theorem, the process

$$\widetilde{W}_t := W_t + \int_0^t \eta_s \,\mathrm{d}s, \quad 0 \le t \le \ell_\epsilon(T) - \ell_\epsilon(0),$$

is a d-dimensional Brownian motion under the new probability measure $R\mathbb{P}$. This allows us to rewrite (3.9) and (3.10) as

$$X_t^{\ell_\epsilon,v}(x) = x + \int_0^t b_s\big(X_s^{\ell_\epsilon,v}(x)\big)\,\mathrm{d}s + \int_0^{\ell_\epsilon(t)-\ell_\epsilon(0)} \mathcal{K}_H\left(\ell_\epsilon(t)-\ell_\epsilon(0), s\right)\,\mathrm{d}W_s + v_t$$

and

$$Y_t = y + \int_0^t b_s(Y_s)\,\mathrm{d}s + \int_0^{\ell_\epsilon(t)-\ell_\epsilon(0)} \mathcal{K}_H\left(\ell_\epsilon(t)-\ell_\epsilon(0), s\right)\,\mathrm{d}\widetilde{W}_s + v_t,$$

respectively. Thus, the distribution of $(X_T^{\ell_\epsilon,v}(y))_{0\le t\le T}$ under $\mathbb{P}$ coincides with the law of $(Y_t)_{0\le t\le T}$ under $R\mathbb{P}$; in particular, we get for all bounded Borel functions $f:\mathbb{R}^d\to\mathbb{R}$

$$\mathbb{E}f\big(X_T^{\ell_\epsilon,v}(y)\big) = \mathbb{E}_{R\mathbb{P}}f(Y_T) = \mathbb{E}\left[Rf(Y_T)\right] = \mathbb{E}\big[Rf\big(X_T^{\ell_\epsilon,v}(x)\big)\big]. \tag{3.12}$$

By the Jensen inequality, we get for any random variable $F \ge 1$,

$$\mathbb{E}\left[R\log\frac{F}{R}\right] = \mathbb{E}_{R\mathbb{P}}\left[\log\frac{F}{R}\right] \le \log\mathbb{E}_{R\mathbb{P}}\left[\frac{F}{R}\right] = \log\mathbb{E}F,$$

hence

$$\mathbb{E}\left[R\log F\right] \le \log\mathbb{E}F + \mathbb{E}\left[R\log R\right].$$

Combining this with (3.12) and the observation that

$$\begin{aligned}
\log R &= -\int_0^{\ell_\epsilon(T)-\ell_\epsilon(0)} \langle\eta_s, \mathrm{d}W_s\rangle - \frac{1}{2}\int_0^{\ell_\epsilon(T)-\ell_\epsilon(0)} |\eta_s|^2\,\mathrm{d}s \\
&= -\int_0^{\ell_\epsilon(T)-\ell_\epsilon(0)} \langle\eta_s, \mathrm{d}\widetilde{W}_s\rangle + \frac{1}{2}\langle M\rangle_{\ell_\epsilon(T)-\ell_\epsilon(0)} \\
&\le -\int_0^{\ell_\epsilon(T)-\ell_\epsilon(0)} \langle\eta_s, \mathrm{d}\widetilde{W}_s\rangle + \frac{C_{T,H}^2|x-y|^2}{4(1-H)}\left[\ell_\epsilon(T)-\ell_\epsilon(0)\right]^{2-2H},
\end{aligned}$$

we get for all bounded Borel functions $f:\mathbb{R}^d\to[1,\infty)$ that

$$\begin{aligned}
P_T^{\ell_\epsilon,v}\log f(y) &= \mathbb{E}\log f\big(X_T^{\ell_\epsilon,v}(y)\big) \\
&= \mathbb{E}\big[R\log f\big(X_T^{\ell_\epsilon,v}(x)\big)\big] \\
&\le \log\mathbb{E}f\big(X_T^{\ell_\epsilon,v}(x)\big) + \mathbb{E}[R\log R] \\
&= \log P_T^{\ell_\epsilon,v}f(x) + \mathbb{E}_{R\mathbb{P}}\log R \\
&\le \log P_T^{\ell_\epsilon,v}f(x) + \frac{C_{T,H}^2|x-y|^2}{4(1-H)}\left[\ell_\epsilon(T)-\ell_\epsilon(0)\right]^{2-2H}.
\end{aligned}$$

This completes the proof of the log-Harnack inequality.

Let us now prove part ii) of the lemma. For any non-negative bounded Borel function $f : \mathbb{R}^d \to [0, \infty)$ we find with (3.12) and the Hölder inequality

$$\begin{aligned}\big(P_T^{\ell_\epsilon,v} f(y)\big)^p &= \big(\mathbb{E} f\big(X_T^{\ell_\epsilon,v}(y)\big)\big)^p \\ &= \big(\mathbb{E}\big[R f\big(X_T^{\ell_\epsilon,v}(x)\big)\big]\big)^p \\ &\le P_T^{\ell_\epsilon,v} f^p(x) \cdot \Big(\mathbb{E}[R^{p/(p-1)}]\Big)^{p-1}.\end{aligned} \tag{3.13}$$

Using (3.11) we get

$$\begin{aligned}R^{p/(p-1)} &= \exp\left[\frac{p}{p-1} M_{\ell_\epsilon(T)-\ell_\epsilon(0)} - \frac{p}{2(p-1)}\langle M\rangle_{\ell_\epsilon(T)-\ell_\epsilon(0)}\right] \\ &= \exp\left[\frac{p}{2(p-1)^2}\langle M\rangle_{\ell_\epsilon(T)-\ell_\epsilon(0)}\right] \\ &\qquad \times \exp\left[\frac{p}{p-1} M_{\ell_\epsilon(T)-\ell_\epsilon(0)} - \frac{p^2}{2(p-1)^2}\langle M\rangle_{\ell_\epsilon(T)-\ell_\epsilon(0)}\right] \\ &\le \exp\left[\frac{pC_{T,H}^2|x-y|^2}{4(p-1)^2(1-H)}\left[\ell_\epsilon(T)-\ell_\epsilon(0)\right]^{2-2H}\right] \\ &\qquad \times \exp\left[\frac{p}{p-1} M_{\ell_\epsilon(T)-\ell_\epsilon(0)} - \frac{p^2}{2(p-1)^2}\langle M\rangle_{\ell_\epsilon(T)-\ell_\epsilon(0)}\right].\end{aligned}$$

Noting the fact that $\exp\left[\frac{p}{p-1} M_{\ell_\epsilon(t)-\ell_\epsilon(0)} - \frac{p^2}{2(p-1)^2}\langle M\rangle_{\ell_\epsilon(t)-\ell_\epsilon(0)}\right]$, $0 \le t \le T$, is a martingale with mean 1 – this is due to Novikov's criterion – we get

$$\mathbb{E}\left[R^{p/(p-1)}\right] \le \exp\left[\frac{pC_{T,H}^2|x-y|^2}{4(p-1)^2(1-H)}\left[\ell_\epsilon(T)-\ell_\epsilon(0)\right]^{2-2H}\right].$$

Inserting this expression into (3.13), completes the proof of the power-Harnack inequality. ☐

Proof of Theorem 3.2. By [1, Proposition 2.3], ii) is a direct consequence of i).

Fix $T > 0$. By a standard approximation argument, it is enough to prove the formulae in i) and iii) for $f \in C_b(\mathbb{R}^d)$.

Step 1: Assume that $b_t : \mathbb{R}^d \to \mathbb{R}^d$ is, uniformly for t in compact intervals, a global Lipschitz function, i.e. for any $t > 0$ there is some $C_t > 0$ such that

$$|b_s(x) - b_s(y)| \le C_t|x-y|, \quad 0 \le s \le t,\ x, y \in \mathbb{R}^d. \tag{3.14}$$

This implies that for all $x \in \mathbb{R}^d$ and $\epsilon \in (0,1)$

$$\begin{aligned}\big|X_T^{\ell_\epsilon,v}(x) - X_T^{\ell,v}(x)\big| &\le \int_0^T \big|b_s\big(X_s^{\ell_\epsilon,v}(x)\big) - b_s\big(X_s^{\ell,v}(x)\big)\big|\,\mathrm{d}s + \big|W^H_{\ell_\epsilon(T)-\ell_\epsilon(0)} - W^H_{\ell(T)}\big| \\ &\le C_T \int_0^T \big|X_s^{\ell_\epsilon,v}(x) - X_s^{\ell,v}(x)\big|\,\mathrm{d}s + \big|W^H_{\ell_\epsilon(T)-\ell_\epsilon(0)} - W^H_{\ell(T)}\big|.\end{aligned}$$

Since $X_t^{\ell_\epsilon,v}(x)$ and $X_t^{\ell,v}(x)$ are non-explosive, the integral in the above expression is finite. Therefore, we can apply Gronwall's inequality with $g(\epsilon,t) := \big|W^H_{\ell_\epsilon(t)-\ell_\epsilon(0)} - W^H_{\ell(t)}\big|$ and find

$$\big|X_T^{\ell_\epsilon,v}(x) - X_T^{\ell,v}(x)\big| \le g(\epsilon,T) + C_T\int_0^T g(\epsilon,s)\mathrm{e}^{(T-s)C_T}\,\mathrm{d}s.$$

From (3.8), we conclude that $\lim_{\epsilon\downarrow 0} g(\epsilon,s)=0$ for all $s\geq 0$. Using the dominated convergence theorem, we obtain

$$\lim_{\epsilon\downarrow 0} X_T^{\ell_\epsilon,v}(x)=X_T^{\ell,v}(x),\quad x\in\mathbb{R}^d;$$

hence,

$$\lim_{\epsilon\downarrow 0} P_T^{\ell_\epsilon,v}f=P_T^{\ell,v}f,\quad f\in C_b(\mathbb{R}^d).$$

Since ℓ is of bounded variation, the limit $\ell_\epsilon\downarrow\ell$ also holds for the integrals

$$\lim_{\epsilon\downarrow 0}\int_0^T \mathrm{e}^{-K(t)}\,\mathrm{d}\ell_\epsilon(t)=\int_0^T \mathrm{e}^{-K(t)}\,\mathrm{d}\ell(t).$$

We can now use Lemma 3.3 i) and let $\epsilon\downarrow 0$ to get

$$P_T^{\ell,v}\log f(y)\leq \log P_T^{\ell,v}f(x)+\frac{\Theta_H\ell(T)^{2-2H}}{\left(\int_0^T \mathrm{e}^{-K(t)}\,\mathrm{d}\ell(t)\right)^2}|x-y|^2$$

for $x,y\in\mathbb{R}^d$ and all $f\in C_b(\mathbb{R}^d)$ with $f\geq 1$. Similarly, Lemma 3.3 ii) yields

$$\left(P_T^{\ell,v}f(y)\right)^p\leq P_T^{\ell,v}f^p(x)\cdot\exp\left[\frac{\frac{p}{p-1}\Theta_H\ell(T)^{2-2H}}{\left(\int_0^T \mathrm{e}^{-K(t)}\,\mathrm{d}\ell(t)\right)^2}|x-y|^2\right]$$

for $x,y\in\mathbb{R}^d$ and all non-negative $f\in C_b(\mathbb{R}^d)$.

Since the processes W, V and Z are independent, $P_Tf=\mathbb{E}\left[P_T^{\ell,v}f(\cdot)\Big|_{\substack{\ell=Z\\ v=V}}\right]$ holds for all bounded Borel functions $f:\mathbb{R}^d\to\mathbb{R}$. Thus, the Jensen inequality yields for all $x,y\in\mathbb{R}^d$ and $f\in C_b(\mathbb{R}^d)$ with $f\geq 1$

$$\begin{aligned}
P_T\log f(y)&=\mathbb{E}\left[P_T^{\ell,v}\log f(y)\Big|_{\substack{\ell=Z\\ v=V}}\right]\\
&\leq \mathbb{E}\left[\log P_T^{\ell,v}f(x)\Big|_{\substack{\ell=Z\\ v=V}}\right]+\Theta_H\mathbb{E}\left[\frac{Z(T)^{2-2H}}{\left(\int_0^T \mathrm{e}^{-K(t)}\,\mathrm{d}Z(t)\right)^2}\right]|x-y|^2\\
&\leq \log P_Tf(x)+\Theta_H\mathbb{E}\left[\frac{Z(T)^{2-2H}}{\left(\int_0^T \mathrm{e}^{-K(t)}\,\mathrm{d}Z(t)\right)^2}\right]|x-y|^2.
\end{aligned}$$

For the power-Harnack inequality we use Hölder's inequality to find for all $x,y\in\mathbb{R}^d$ and non-negative $f\in C_b(\mathbb{R}^d)$

$$\begin{aligned}
P_Tf(y)&=\mathbb{E}\left[P_T^{\ell,v}f(y)\Big|_{\substack{\ell=Z\\ v=V}}\right]\\
&\leq \mathbb{E}\left[\left(P_T^{\ell,v}f^p(x)\right)^{\frac1p}\exp\left[\frac{\frac{\ell(T)^{2-2H}}{p-1}\Theta_H|x-y|^2}{\left(\int_0^T \mathrm{e}^{-K(t)}\,\mathrm{d}\ell(t)\right)^2}\right]\Bigg|_{\substack{\ell=Z\\ v=V}}\right]\\
&\leq \left(P_Tf^p(x)\right)^{\frac1p}\left(\mathbb{E}\exp\left[\frac{\frac{pZ(T)^{2-2H}}{(p-1)^2}\Theta_H|x-y|^2}{\left(\int_0^T \mathrm{e}^{-K(t)}\,\mathrm{d}Z(t)\right)^2}\right]\right)^{1-\frac1p}.
\end{aligned}$$

Step 2: For the general case, we use the approximation argument proposed in [21, part (c) of proof of Theorem 2.1]. Let

$$\tilde{b}_t(x):=b_t(x)-k(t)x,\quad t\geq 0,\ x\in\mathbb{R}^d.$$

($k(t)$ is the constant appearing in (H) on p. 2.) Using (H), it is not difficult to see that the mapping $\mathrm{id} - n^{-1}\tilde{b}_t : \mathbb{R}^d \to \mathbb{R}^d$ is injective for any $n \in \mathbb{N}$ and $t \geq 0$. The maps

$$b_t^{(n)}(x) := n\left[\left(\mathrm{id} - n^{-1}\tilde{b}_t\right)^{-1}(x) - x\right] + k(t)x, \quad n \in \mathbb{N},\ t \geq 0,\ x \in \mathbb{R}^d,$$

are, uniformly for t in compact intervals, globally Lipschitz continuous, see [4]. Denote by $(X_t^{(n)}(x))_{t\geq 0}$ the solution of (3.1) with b replaced by $b^{(n)}$, and define $P_t^{(n)}$ by (3.3) with $X_t(x)$ replaced by $X_t^{(n)}(x)$. Because of the first part of the proof, the statements of Theorem 3.2 hold with P_T replaced by $P_T^{(n)}$.

On the other hand, we see as in [21, part (c) of proof of Theorem 2.1], that

$$\lim_{n\to\infty} X_T^{(n)}(x) = X_T(x) \text{ a.s.,} \quad \text{hence,} \quad \lim_{n\to\infty} P_T^{(n)} f = P_T f \quad \text{for all } f \in C_b(\mathbb{R}^d).$$

Therefore, the claim follows if we let $n \to \infty$. □

3.3 Applications

Let $Z = (Z(t))_{t\geq 0}$ be a subordinator (without killing), i.e. a nondecreasing Lévy process on $[0,\infty)$ with $S_0 = 0$; its Laplace transform is of the form

$$\mathbb{E}\,\mathrm{e}^{-rZ(t)} = \mathrm{e}^{-t\phi(r)}, \quad r > 0,\ t \geq 0,$$

and the characteristic (Laplace) exponent $\phi : (0,\infty) \to (0,\infty)$ is a Bernstein function with $\phi(0+) = 0$. Recall that a Bernstein function is a smooth function $\phi \in C^\infty((0,\infty))$ such that $(-1)^{n-1}\phi^{(n)} \geq 0$ for all $n = 1, 2, \ldots$; it is well known, see e.g. [18, Theorem 3.2], that every Bernstein function enjoys a unique Lévy–Khintchine representation

$$\phi(r) = \phi(0+) + \vartheta r + \int_{(0,\infty)} \left(1 - \mathrm{e}^{-rx}\right) \nu(\mathrm{d}x), \quad r > 0,$$

where $\vartheta \geq 0$ is the drift parameter and ν is a Lévy measure, that is, a measure on $(0,\infty)$ satisfying $\int_{(0,\infty)} (1 \wedge x)\, \nu(\mathrm{d}x) < \infty$.

For the constant $k(t)$ from (H) and its primitive $K(t)$, cf. (3.4), we set

$$K^*(T) := \exp\left[2 \sup_{t\in[0,T]} K(t)\right], \quad T > 0.$$

Obviously, if $k(t) \leq 0$ for all $t \geq 0$, then $K^*(T) \leq 1$ for all $T > 0$.

Corollary 3.4. *Let Z be a subordinator whose characteristic exponent is the Bernstein function ϕ and assume that* (H) *and* (3.2) *hold. We have for all $T > 0$, $x, y \in \mathbb{R}^d$ and all bounded Borel functions $f : \mathbb{R}^d \to [0,\infty)$ the following assertions:*

i) *If $\liminf_{r\to\infty} \phi(r) r^{-\rho} > 0$ for some $\rho > 0$, then*

$$P_T \log f(y) \leq \log P_T f(x) + \frac{C_{H,\rho} K^*(T)|x-y|^2}{(T\wedge 1)^{2H/\rho}}, \quad f \geq 1,$$

$$|\nabla P_T f|^2(x) \leq \left\{P_T f^2(x) - \left(P_T f(x)\right)^2\right\} \frac{C_{H,\rho} K^*(T)}{(T\wedge 1)^{2H/\rho}}.$$

If, in addition, $\liminf_{r\downarrow 0} \phi(r) r^{-\rho} > 0$, then we can replace $T \wedge 1$ by T and get

$$P_T \log f(y) \leq \log P_T f(x) + \frac{C_{H,\rho} K^*(T)|x-y|^2}{T^{2H/\rho}}, \quad f \geq 1,$$

$$|\nabla P_T f|^2(x) \leq \left\{P_T f^2(x) - \left(P_T f(x)\right)^2\right\} \frac{C_{H,\rho} K^*(T)}{T^{2H/\rho}}.$$

ii) *If* $\liminf_{r\to\infty}\phi(r)r^{-\rho}>0$ *for some* $\rho>2H/(1+2H)$ *and* $p>1$*, then*

$$\begin{aligned}&\big(P_Tf(y)\big)^p\le P_Tf^p(x)\\&\quad\times\exp\left[\frac{C_{H,\rho}\,p}{p-1}K^*(T)\left(1+\frac{1}{T^{2H/\rho}}\right)|x-y|^2+C_{H,\rho}\frac{\big(pK^*(T)|x-y|^2\big)^{\frac{\rho}{\rho-2H(1-\rho)}}}{(p-1)^{\frac{\rho+2H(1-\rho)}{\rho-2H(1-\rho)}}T^{\frac{2H}{\rho-2H(1-\rho)}}}\right].\end{aligned}$$

If, in addition, $\liminf_{r\downarrow 0}\phi(r)r^{-\rho}>0$*, then*

$$\begin{aligned}&\big(P_Tf(y)\big)^p\le P_Tf^p(x)\\&\qquad\times\exp\left[\frac{C_{H,\rho}\,p}{p-1}\frac{K^*(T)|x-y|^2}{T^{2H/\rho}}+C_{H,\rho}\frac{\big(pK^*(T)|x-y|^2\big)^{\frac{\rho}{\rho-2H(1-\rho)}}}{(p-1)^{\frac{\rho+2H(1-\rho)}{\rho-2H(1-\rho)}}T^{\frac{2H}{\rho-2H(1-\rho)}}}\right].\end{aligned}$$

Proof. Since we have

$$Z(T)^{2-2H}\left(\int_0^T \mathrm{e}^{-K(t)}\,\mathrm{d}Z(t)\right)^{-2}\le K^*(T)Z(T)^{-2H},\quad T>0, \tag{3.15}$$

the assertion follows from Theorem 3.2 and [7, Theorem 3.8 (a) and (b)]. □

We will now assume that the subordinator S is strictly increasing, i.e. we have $\nu(0,\infty)=\infty$ or $\vartheta>0$. Define the (generalized, right-continuous) inverse of S

$$S^{-1}(t):=\inf\{s\ge 0\,:\,S(s)>t\}=\sup\{s\ge 0\,:\,S(s)\le t\},\quad t\ge 0.$$

We will call $S^{-1}=(S^{-1}(t))_{t\ge 0}$ an inverse subordinator associated with the Bernstein function ϕ. Since we assume that the subordinator S is strictly increasing, we know that almost all paths of S^{-1} are continuous and non-decreasing. We will frequently use the following identity:

$$\mathbb{P}\left(S(r)\ge t\right)=\mathbb{P}\left(S^{-1}(t)\le r\right),\quad r,t>0. \tag{3.16}$$

Corollary 3.5. *Let* Z *be an inverse subordinator associated with the Bernstein function* ϕ *and assume that* (H) *and* (3.2) *hold.*

If $\limsup_{r\downarrow 0}\phi(r)r^{-\sigma}<\infty$ *and* $\limsup_{r\to\infty}\phi(r)r^{-\sigma}<\infty$ *for some* $\sigma>0$*, then the following assertions hold.*

i) *For any* $T>0$*,* $x,y\in\mathbb{R}^d$ *and all bounded Borel functions* $f:\mathbb{R}^d\to[1,\infty)$

$$P_T\log f(y)\le\log P_Tf(x)+\frac{C_{H,\sigma}K^*(T)}{T^{2H\sigma}}|x-y|^2.$$

ii) *For any* $T>0$*,* $x\in\mathbb{R}^d$ *and all bounded Borel functions* $f:\mathbb{R}^d\to[0,\infty)$

$$|\nabla P_Tf|^2(x)\le\left\{P_Tf^2(x)-\big(P_Tf(x)\big)^2\right\}\frac{C_{H,\sigma}K^*(T)}{T^{2H\sigma}}.$$

Corollary 3.5 follows, if we combine (3.15) with Theorem 3.2 i), ii) and the next lemma.

Lemma 3.6. *Let* S^{-1} *be an inverse subordinator with Bernstein function* ϕ *satisfying the conditions of Corollary 3.5. For any* $\theta\in(0,1)$*,*

$$\mathbb{E}\left[\big(S^{-1}(t)\big)^{-\theta}\right]\le C_{\sigma,\theta}\,t^{-\sigma\theta},\quad t>0.$$

Proof. By our assumption, there exists a constant $c = c(\sigma) > 0$ such that $\phi(r) \le c\, r^\sigma$ for all $r > 0$. Combining this with

$$\mathbb{1}_{[t,\infty)}(S(s)) \le \frac{2S(s)}{t+S(s)}, \qquad \frac{x}{1+x} = \int_0^\infty \left(1-\mathrm{e}^{-xr}\right)\mathrm{e}^{-r}\,\mathrm{d}r, \quad x>0,$$

and Tonelli's theorem, we get that for all $s, t > 0$

$$\begin{aligned}
\mathbb{P}\left(S(s)\ge t\right) &= \mathbb{E}\left[\mathbb{1}_{[t,\infty)}(S(s))\right]\\
&\le 2\mathbb{E}\left[\frac{S(s)/t}{1+S(s)/t}\right]\\
&= 2\mathbb{E}\left[\int_0^\infty \left(1-\mathrm{e}^{-rS(s)/t}\right)\mathrm{e}^{-r}\,\mathrm{d}r\right]\\
&= 2\int_0^\infty \left(1-\mathrm{e}^{-s\phi(r/t)}\right)\mathrm{e}^{-r}\,\mathrm{d}r\\
&\le 2s\int_0^\infty \phi\left(\frac{r}{t}\right)\mathrm{e}^{-r}\,\mathrm{d}r\\
&\le 2c\,s\int_0^\infty \left(\frac{r}{t}\right)^\sigma \mathrm{e}^{-r}\,\mathrm{d}r\\
&= 2c\,\Gamma(\sigma+1)st^{-\sigma}.
\end{aligned}$$

This yields for all $t > 0$

$$\begin{aligned}
\mathbb{E}\big[\big(S^{-1}(t)\big)^{-\theta}\big] &= \int_0^\infty \mathbb{P}\big(S(s^{-1/\theta})\ge t\big)\,\mathrm{d}s\\
&\le \int_0^\infty \left(1\wedge\left[2c\,\Gamma(\sigma+1)s^{-1/\theta}t^{-\sigma}\right]\right)\mathrm{d}s\\
&= \frac{1}{1-\theta}\left[2c\,\Gamma(\sigma+1)\right]^\theta t^{-\sigma\theta}.
\end{aligned}$$

□

Remark 3.7. Let $\alpha \in (0,1)$. For an α-stable subordinator S, the corresponding Bernstein function $\phi(r) = r^\alpha$ satisfies the conditions of Corollary 3.5 with $\sigma = \alpha$. Because of (3.16) and the well known two-sided estimate

$$\mathbb{P}\left(S(r)\ge t\right) \asymp 1\wedge\left[rt^{-\alpha}\right], \quad r,t>0,$$

($f \asymp g$ means that $c^{-1}f(t) \le g(t) \le cf(t)$ for some $c \ge 1$ and all t) we have for any $t > 0$

$$\begin{aligned}
\mathbb{E}\big[\big(S^{-1}(t)\big)^{-\theta}\big] &= \int_0^\infty \mathbb{P}\big(\big(S^{-1}(t)\big)^{-\theta}\ge s\big)\,\mathrm{d}s\\
&= \int_0^\infty \mathbb{P}\big(S(s^{-1/\theta})\ge t\big)\,\mathrm{d}s\\
&\asymp \int_0^\infty \left(1\wedge\left[s^{-1/\theta}t^{-\alpha}\right]\right)\mathrm{d}s\\
&= \frac{1}{1-\theta}\,t^{-\alpha\theta}.
\end{aligned} \tag{3.17}$$

This shows that Lemma 3.6 is sharp for α-stable subordinators.

Remark 3.8. Let Z be an inverse α-stable subordinator, i.e. $Z(t) = S^{-1}(t)$ for $t \ge 0$, where $(S(t))_{t\ge 0}$ is an α-stable subordinator. For any $t, \theta, \delta > 0$ we have

$$\mathbb{E}\exp\left[\frac{\delta}{Z(t)^\theta}\right] = \infty.$$

The proof is similar to (3.17):

$$\begin{aligned}\mathbb{E}\exp\left[\frac{\delta}{Z(t)^\theta}\right] &\geq \int_1^\infty \mathbb{P}\left(\exp\left[\delta\left(S^{-1}(t)\right)^{-\theta}\right]\geq s\right)\,\mathrm{d}s\\ &= \int_1^\infty \mathbb{P}\left(S\left((\delta/\log s)^{1/\theta}\right)\geq t\right)\,\mathrm{d}s\\ &\asymp \int_1^\infty \left(1\wedge\left[(\delta/\log s)^{1/\theta}\,t^{-\alpha}\right]\right)\,\mathrm{d}s\\ &= \infty.\end{aligned}$$

This means that we cannot expect, in the setting of Corollary 3.5, to get a power-Harnack inequality as we did in Corollary 3.4 iii).

4 SDEs with non-Lipschitz drift and anisotropic noise

Let $W^{H_i,(i)} = (W_t^{H_i,(i)})_{t\geq 0}$, $Z^{(i)} = (Z^{(i)}(t))_{t\geq 0}$, $1\leq i\leq d$, $V = (V_t)_{t\geq 0}$ be $2d+1$ independent stochastic processes such that

$$\begin{cases} W^{H_i,(i)} \text{ are fBMs on } \mathbb{R} \text{ with Hurst parameter } H_i\in(0,1/2);\\ Z^{(i)} \text{ are non-decreasing processes on } [0,\infty) \text{ with } Z^{(i)}(0)=0;\\ V \text{ is a locally bounded measurable process on } \mathbb{R}^d \text{ with } V_0=0.\end{cases}\tag{4.1}$$

We consider the following stochastic equation on $\mathbb{R}^d$:

$$X_t(x) = x + \int_0^t b_s\big(X_s(x)\big)\,\mathrm{d}s + \big(W^{H_1,(1)}_{Z^{(1)}(t)},\dots,W^{H_d,(d)}_{Z^{(d)}(t)}\big) + V_t,\quad t\geq 0,\, x\in\mathbb{R}^d,\tag{4.2}$$

where $b=(b^{(1)},\dots,b^{(d)}) : [0,\infty)\times\mathbb{R}^d\to\mathbb{R}^d$, $b=b_t(x)$, is measurable, locally bounded in the variable $t\geq 0$ and continuous as a function of x. By $\mathscr{U}$ we denote the family of functions $u:(0,\infty)\to(0,\infty)$ which are continuous, non-decreasing, grow at most linearly as $x\to\infty$ and satisfy $\int_{0+}\frac{\mathrm{d}s}{u(s)}=\infty$. Typical examples of such functions are $u(s)=s$, $u(s)=s\log(\mathrm{e}\vee s^{-1})$, $u(s)=s\cdot\{\log(\mathrm{e}\vee s^{-1})\}\cdot\log\log(\mathrm{e}^{\mathrm{e}}\vee s^{-1})$.

In this section, we will use the ℓ^1-norm on $\mathbb{R}^d$, i.e. $\|x\|_1 := |x^{(1)}|+\dots+|x^{(d)}|$, $x\in\mathbb{R}^d$, and we replace the one-sided Lipschitz condition (H) by the following Yamada–Watanabe-type condition

$$\begin{cases}\text{There exists some } u\in\mathscr{U} \text{ and a locally bounded}\\ \text{measurable function } k:[0,\infty)\to[0,\infty) \text{ such that}\\ \|b_t(x)-b_t(y)\|_1\leq k(t)u(\|x-y\|_1),\ t\geq 0,\ x,y\in\mathbb{R}^d.\end{cases}\tag{A}$$

As in Section 3, it is easy to see that (A) guarantees the existence, uniqueness and non-explosion of the solution to (4.2). We define for bounded Borel functions $f:\mathbb{R}^d\to\mathbb{R}$ the operator

$$P_tf(x) := \mathbb{E}f(X_t(x))\quad t\geq 0,\ x\in\mathbb{R}^d.$$

Remark 4.1. Note that it is, in general, difficult to compare (A) with the condition (H) used in Section 3, since neither of them implies the other one.

4.1 Statement of the main result

Let $k(t)$ be the constant appearing in (A), and denote by $K(t)=\int_0^t k(s)\,\mathrm{d}s$ its primitive. For $i\in\{1,\dots,d\}$ we define Θ_{H_i} by (3.5) with H_i instead of H. Finally, we set for $u\in\mathscr{U}$

and $k(t)$

$$G_u(r) := \begin{cases} -\int_r^1 \frac{\mathrm{d}s}{u(s)}, & \text{if } r \in [0,1), \\ \int_1^r \frac{\mathrm{d}s}{u(s)}, & \text{if } r \in [1,\infty), \end{cases}$$

and

$$\Phi_{u,k}(t,r) := r + \int_0^t k(s)\, u \circ G_u^{-1}\big(G_u(r) + K(s)\big)\,\mathrm{d}s, \quad t, r \geq 0;$$

G_u^{-1} is the inverse function of G_u. Since $u \in \mathscr{U}$, it is easy to see that G_u is strictly increasing with $G_u(0) = -\infty$ and $\lim_{r\uparrow\infty} G_u(r) = \infty$, so that $\Phi_{u,k}$ is well-defined. If, in particular, $u(s) = cs$ for some constant $c > 0$, then

$$\Phi_{u,k}(t,r) = \left(1 + c\int_0^t k(s)\,\mathrm{e}^{cK(s)}\,\mathrm{d}s\right) r, \quad t, r \geq 0.$$

Since we use the ℓ^1-norm in this section, the local Lipschitz constant of a function f on $\mathbb{R}^d$ at the point $x \in \mathbb{R}^d$ is defined by

$$|\nabla f|(x) := \limsup_{y\to x} \frac{|f(y) - f(x)|}{\|y - x\|_1}.$$

Theorem 4.2. *Assuming* (4.1) *and* (A)*, let $X_t(x)$ denote the unique solution to the SDE* (4.2)*.*

i) *For $T > 0$, $x, y \in \mathbb{R}^d$, and any bounded Borel function $f : \mathbb{R}^d \to [1,\infty)$*

$$P_T \log f(y) \leq \log P_T f(x) + \Phi_{u,k}^2\left(T, \|x-y\|_1\right) \sum_{i=1}^d \mathbb{E}\left[\frac{\Theta_{H_i}}{(Z^{(i)}(T))^{2H_i}}\right].$$

ii) *For $T > 0$, $x, y \in \mathbb{R}^d$, $p > 1$ and any bounded Borel function $f : \mathbb{R}^d \to [0,\infty)$*

$$\big(P_T f(y)\big)^p \leq P_T f^p(x) \cdot \left(\mathbb{E}\exp\left[\frac{p}{(p-1)^2}\Phi_{u,k}^2\left(T, \|x-y\|_1\right) \sum_{i=1}^d \frac{\Theta_{H_i}}{(Z^{(i)}(T))^{2H_i}}\right]\right)^{p-1}.$$

iii) *If* (A) *holds with $u(s) = cs$ for some constant $c > 0$, then we have for $T > 0$, $x \in \mathbb{R}^d$ and any bounded Borel function $f : \mathbb{R}^d \to \mathbb{R}$*

$$|\nabla P_T f|^2(x) \leq \left\{P_T f^2(x) - \big(P_T f(x)\big)^2\right\} \times 2\left(1 + c\int_0^T k(s)\,\mathrm{e}^{cK(s)}\,\mathrm{d}s\right)^2 \sum_{i=1}^d \mathbb{E}\left[\frac{\Theta_{H_i}}{(Z^{(i)}(T))^{2H_i}}\right].$$

4.2 Deterministic time-changes

The proof of Theorem 4.2 uses the same strategy as the proof of Theorem 3.2. Because of the independence of the random time-change and the driving processes, we consider first a deterministic time-change $\ell = (\ell^{(1)}, \ldots, \ell^{(d)}) : [0,\infty) \to [0,\infty)^d$ such that for each $i \in \{1, \ldots, d\}$ the map $t \mapsto \ell^{(i)}(t)$ is non-decreasing and càdlàg with $\ell^{(i)}(0) = 0$. Let $v = (v^{(1)}, \ldots, v^{(d)}) : [0,\infty) \to \mathbb{R}^d$ be a locally bounded measurable function such that $v_0 = 0$. Under (A), the following SDE has a unique non-explosive strong solution

$$X_t^{\ell,v}(x) = x + \int_0^t b_s\big(X_s^{\ell,v}(x)\big)\,\mathrm{d}s + \big(W^{H_1,(1)}_{\ell^{(1)}(t)}, \ldots, W^{H_d,(d)}_{\ell^{(d)}(t)}\big) + v_t, \quad t \geq 0,\ x \in \mathbb{R}^d. \tag{4.3}$$

As before, we set for any bounded Borel function $f : \mathbb{R}^d \to \mathbb{R}$

$$P_t^{\ell,v} f(x) := \mathbb{E} f\big(X_t^{\ell,v}(x)\big) \quad t \geq 0,\ x \in \mathbb{R}^d.$$

Proposition 4.3. *Assuming* (4.1) *and* (A)*, let* $X^{\ell,v}(x)$ *denote the unique solution to the SDE* (4.3)*.*

i) *For* $T>0$*,* $x,y\in\mathbb{R}^d$ *and any bounded Borel function* $f:\mathbb{R}^d\to[1,\infty)$

$$P_T^{\ell,v}\log f(y)\le \log P_T^{\ell,v}f(x)+\Phi_{u,k}^2\left(T,\|x-y\|_1\right)\sum_{i=1}^d\frac{\Theta_{H_i}}{\left(\ell^{(i)}(T)\right)^{2H_i}}.$$

ii) *For* $T>0$*,* $p>1$*,* $x,y\in\mathbb{R}^d$ *and any bounded Borel function* $f:\mathbb{R}^d\to[0,\infty)$

$$\left(P_T^{\ell,v}f(y)\right)^p\le P_T^{\ell,v}f^p(x)\cdot\exp\left[\frac{p}{p-1}\Phi_{u,k}^2\left(T,\|x-y\|_1\right)\sum_{i=1}^d\frac{\Theta_{H_i}}{\left(\ell^{(i)}(T)\right)^{2H_i}}\right].$$

As in Section 3.2, we approximate $\ell^{(i)}$ by strictly increasing, absolutely continuous functions

$$\ell_\epsilon^{(i)}(t):=\frac{1}{\epsilon}\int_t^{t+\epsilon}\ell^{(i)}(s)\,\mathrm{d}s+\epsilon t=\int_0^1\ell^{(i)}(\epsilon s+t)\,\mathrm{d}s+\epsilon t,\quad \epsilon\in(0,1),\ 1\le i\le d,\ t\ge 0.$$

By construction, $\ell_\epsilon^{(i)}(t)\downarrow\ell^{(i)}(t)$ as $\epsilon\downarrow 0$. Denote by $\gamma_\epsilon^{(i)}$ the inverse function of $\ell_\epsilon^{(i)}$. We consider the following approximation of the SDE (4.3)

$$X_t^{\ell_\epsilon,v,(i)}(x)=x^{(i)}+\int_0^t b_s^{(i)}\big(X_s^{\ell_\epsilon,v}(x)\big)\,\mathrm{d}s+W^{H_i,(i)}_{\ell_\epsilon^{(i)}(t)-\ell_\epsilon^{(i)}(0)}+v_t^{(i)},\quad i=1,\dots,d,\tag{4.4}$$

where $t\ge 0$, $x\in\mathbb{R}^d$ and $X_t^{\ell_\epsilon,v}(x)=\big(X_t^{\ell_\epsilon,v,(1)}(x),\dots,X_t^{\ell_\epsilon,v,(d)}(x)\big)$. Again, for all bounded Borel functions $f:\mathbb{R}^d\to\mathbb{R}$

$$P_t^{\ell_\epsilon,v}f(x):=\mathbb{E}f\big(X_t^{\ell_\epsilon,v}(x)\big),\quad t\ge 0,\ x\in\mathbb{R}^d.$$

We will first prove the Harnack inequalities for $P_t^{\ell_\epsilon,v}$ using a modification of the arguments from Lemma 3.3, compare also [22].

Lemma 4.4. *Assuming* (4.1) *and* (A)*, let* $X_t^{\ell_\epsilon,v}(x)$*,* $\epsilon\in(0,1)$*, denote the unique solution to the SDE* (4.4)*.*

i) *For* $T>0$*,* $x,y\in\mathbb{R}^d$ *and any bounded Borel function* $f:\mathbb{R}^d\to[1,\infty)$

$$P_T^{\ell_\epsilon,v}\log f(y)\le\log P_T^{\ell_\epsilon,v}f(x)+\Phi_{u,k}^2\left(T,\|x-y\|_1\right)\sum_{i=1}^d\frac{\Theta_{H_i}}{\left(\ell_\epsilon^{(i)}(T)-\ell_\epsilon^{(i)}(0)\right)^{2H_i}}.$$

ii) *For* $T>0$*,* $p>1$*,* $x,y\in\mathbb{R}^d$ *and any bounded Borel function* $f:\mathbb{R}^d\to[0,\infty)$

$$\left(P_T^{\ell_\epsilon,v}f(y)\right)^p\le P_T^{\ell_\epsilon,v}f^p(x)\cdot\exp\left[\frac{p}{p-1}\Phi_{u,k}^2\left(T,\|x-y\|_1\right)\sum_{i=1}^d\frac{\Theta_{H_i}}{\left(\ell_\epsilon^{(i)}(T)-\ell_\epsilon^{(i)}(0)\right)^{2H_i}}\right].$$

Proof. Fix $\epsilon\in(0,1)$, $T>0$, $x,y\in\mathbb{R}^d$ and denote the coordinates by a superscript. Let $(Y_t)_{t\ge0}$ be a solution of the equation

$$\begin{aligned}Y_t^{(i)}=y^{(i)}&+\int_0^t b_s^{(i)}(Y_s)\,\mathrm{d}s+W^{H_i,(i)}_{\ell_\epsilon^{(i)}(t)-\ell_\epsilon^{(i)}(0)}+v_t^{(i)}\\&+\xi^{(i)}\int_0^t\mathbb{1}_{[0,\tau_i)}(s)\frac{X_s^{\ell_\epsilon,v,(i)}(x)-Y_s^{(i)}}{\left|X_s^{\ell_\epsilon,v,(i)}(x)-Y_s^{(i)}\right|}\,\mathrm{d}\ell_\epsilon^{(i)}(s),\end{aligned}\tag{4.5}$$

where $t \geq 0$, $i = 1, \ldots, d$ and

$$\xi^{(i)} := \frac{\Phi_{u,k}(T, \|x-y\|_1)}{\ell_\epsilon^{(i)}(\delta T) - \ell_\epsilon^{(i)}(0)}, \quad \delta \in (0,1), \quad \tau_i := \inf\big\{t \geq 0 \,:\, X_t^{\ell_\epsilon, v, (i)}(x) = Y_t^{(i)}\big\}.$$

As in the proof of Lemma 3.3, there is a unique non-explosive solution to (4.5) such that $Y_t^{(i)} = X_t^{\ell_\epsilon, v, (i)}(x)$ for $t \geq \tau_i$ on the set $\{\tau_i < \infty\}$, and we use the differential versions of the equations (4.4) and (4.5) along with the observation that $\mathrm{d}|\zeta_t| = \mathbb{1}_{\{\zeta_t \neq 0\}} |\zeta_t|^{-1} \zeta_t \, \mathrm{d}\zeta_t$, where $\zeta_t := X_t^{\ell_\epsilon, v, (i)}(x) - Y_t^{(i)}$, to get for $i = 1, \ldots, d$ and $t \geq 0$

$$\begin{aligned}
&\big|X_t^{\ell_\epsilon, v, (i)}(x) - Y_t^{(i)}\big| - |x^{(i)} - y^{(i)}| \\
&= \int_0^{t \wedge \tau_i} \frac{X_s^{\ell_\epsilon, v, (i)}(x) - Y_s^{(i)}}{\big|X_s^{\ell_\epsilon, v, (i)}(x) - Y_s^{(i)}\big|} \big(b_s^{(i)}\big(X_s^{\ell_\epsilon, v}(x)\big) - b_s^{(i)}(Y_s)\big) \, \mathrm{d}s - \xi^{(i)} \big[\ell_\epsilon^{(i)}(t \wedge \tau_i) - \ell_\epsilon^{(i)}(0)\big] \\
&\leq \int_0^{t \wedge \tau_i} \big|b_s^{(i)}\big(X_s^{\ell_\epsilon, v}(x)\big) - b_s^{(i)}(Y_s)\big| \, \mathrm{d}s - \xi^{(i)} \big[\ell_\epsilon^{(i)}(t \wedge \tau_i) - \ell_\epsilon^{(i)}(0)\big] \\
&\leq \int_0^{t} \big|b_s^{(i)}\big(X_s^{\ell_\epsilon, v}(x)\big) - b_s^{(i)}(Y_s)\big| \, \mathrm{d}s - \xi^{(i)} \big[\ell_\epsilon^{(i)}(t \wedge \tau_i) - \ell_\epsilon^{(i)}(0)\big].
\end{aligned}$$

Summing over i we obtain, using (A),

$$\begin{aligned}
&\big\|X_t^{\ell_\epsilon, v}(x) - Y_t\big\|_1 \\
&\qquad \leq \|x-y\|_1 + \int_0^t \big\|b_s\big(X_s^{\ell_\epsilon, v}(x)\big) - b_s(Y_s)\big\|_1 \, \mathrm{d}s - \sum_{i=1}^d \xi^{(i)} \big[\ell_\epsilon^{(i)}(t \wedge \tau_i) - \ell_\epsilon^{(i)}(0)\big] \\
&\qquad \leq \|x-y\|_1 + \int_0^t k(s)\, u\big(\big\|X_s^{\ell_\epsilon, v}(x) - Y_s\big\|_1\big) \, \mathrm{d}s - \sum_{i=1}^d \xi^{(i)} \big[\ell_\epsilon^{(i)}(t \wedge \tau_i) - \ell_\epsilon^{(i)}(0)\big].
\end{aligned}$$

We can now apply Bihari's inequality (cf. [3, Section 3]) to conclude

$$\big\|X_s^{\ell_\epsilon, v}(x) - Y_s\big\|_1 \leq G_u^{-1}\big(G_u(\|x-y\|_1) + K(s)\big), \quad s \geq 0.$$

Inserting this into the previous inequality yields for any $t \in [0, T]$

$$\begin{aligned}
&\sum_{i=1}^d \xi^{(i)} \big[\ell_\epsilon^{(i)}(t \wedge \tau_i) - \ell_\epsilon^{(i)}(0)\big] \\
&\qquad \leq \|x-y\|_1 + \int_0^t k(s)\, u \circ G_u^{-1}\big(G_u(\|x-y\|_1) + K(s)\big) \, \mathrm{d}s \\
&\qquad \leq \Phi_{u,k}(T, \|x-y\|_1),
\end{aligned}$$

which means that we have for each $n = 1, \ldots, d$

$$\frac{\ell_\epsilon^{(n)}(t \wedge \tau_n) - \ell_\epsilon^{(n)}(0)}{\ell_\epsilon^{(n)}(\delta T) - \ell_\epsilon^{(n)}(0)} \leq \sum_{i=1}^d \frac{\ell_\epsilon^{(i)}(t \wedge \tau_i) - \ell_\epsilon^{(i)}(0)}{\ell_\epsilon^{(i)}(\delta T) - \ell_\epsilon^{(i)}(0)} \leq 1.$$

Taking $t = T$ implies $\ell_\epsilon^{(n)}(T \wedge \tau_n) \leq \ell_\epsilon^{(n)}(\delta T)$ and this is only possible if $\tau_n < T$ as $\delta \in (0,1)$ and $\ell_\epsilon^{(n)}$ is strictly increasing.

Let

$$g_r^{(i)} := \xi^{(i)} \mathbb{1}_{[0, \ell_\epsilon^{(i)}(\tau_i) - \ell_\epsilon^{(i)}(0))}(r) \frac{X^{\ell_\epsilon, v, (i)}_{\gamma_\epsilon^{(i)}(r + \ell_\epsilon^{(i)}(0))}(x) - Y^{(i)}_{\gamma_\epsilon^{(i)}(r + \ell_\epsilon^{(i)}(0))}}{\big|X^{\ell_\epsilon, v, (i)}_{\gamma_\epsilon^{(i)}(r + \ell_\epsilon^{(i)}(0))}(x) - Y^{(i)}_{\gamma_\epsilon^{(i)}(r + \ell_\epsilon^{(i)}(0))}\big|}, \quad r \geq 0,\ 1 \leq i \leq d.$$

By (2.2), we know that $\int_0^\bullet g_r^{(i)}\,\mathrm{d}r \in I_{0+}^{H+1/2}(L^2[0,\ell_\epsilon^{(i)}(\tau_i)-\ell_\epsilon^{(i)}(0)])$. Let $W^{(i)}=(W^{(i)})_{t\ge 0}$, $1\le i\le d$, be independent one-dimensional standard Brownian motions, and define

$$\eta_s^{(i)} := \mathbb{1}_{[0,\tau_i)}(s)\mathcal{K}_{H_i}^{-1}\left(\int_0^\bullet g_r^{(i)}\,\mathrm{d}r\right)\left(\ell_\epsilon^{(i)}(s)-\ell_\epsilon^{(i)}(0)\right),\quad s\ge 0,\, 1\le i\le d,$$

$$\begin{aligned} M_t &:= -\sum_{i=1}^d \int_0^t \mathbb{1}_{[0,\tau_i)}(s)\eta_s^{(i)}\,\mathrm{d}W^{(i)}_{\ell_\epsilon^{(i)}(s)-\ell_\epsilon^{(i)}(0)} \\ &= -\sum_{i=1}^d \int_0^{\ell_\epsilon^{(i)}(t\wedge\tau_i)-\ell_\epsilon^{(i)}(0)} \eta^{(i)}_{\gamma_\epsilon^{(i)}(s+\ell_\epsilon^{(i)}(0))}\,\mathrm{d}W_s^{(i)},\quad t\ge 0, \end{aligned}$$

$$\widetilde{W}^{(i)}_{\ell_\epsilon^{(i)}(t)-\ell_\epsilon^{(i)}(0)} := W^{(i)}_{\ell_\epsilon^{(i)}(t)-\ell_\epsilon^{(i)}(0)} + \int_0^t \eta_s^{(i)}\,\mathrm{d}\ell_\epsilon^{(i)}(s),\quad t\ge 0,\ 1\le i\le d.$$

Noting $H_i\in(0,1/2)$ and using (2.1) we find for $s\in[0,\ell_\epsilon^{(i)}(\tau_i)-\ell_\epsilon^{(i)}(0))$

$$\begin{aligned} \left|\eta^{(i)}_{\gamma_\epsilon^{(i)}(s+\ell_\epsilon^{(i)}(0))}\right| &= \left|\frac{1}{\Gamma\left(\frac12-H_i\right)}s^{H_i-\frac12}\int_0^s r^{\frac12-H_i}g_r^{(i)}(s-r)^{-H_i-\frac12}\,\mathrm{d}r\right| \\ &\le \frac{1}{\Gamma\left(\frac12-H_i\right)}\,\xi^{(i)}s^{H_i-\frac12}\int_0^s r^{\frac12-H_i}(s-r)^{-H_i-\frac12}\,\mathrm{d}r \\ &= \frac{B\left(\frac32-H_i,\frac12-H_i\right)}{\Gamma\left(\frac12-H_i\right)}\,\xi^{(i)}s^{\frac12-H_i}. \end{aligned}$$

Therefore, the compensator of the martingale M satisfies

$$\begin{aligned} \langle M\rangle_\infty &= \sum_{i=1}^d\int_0^{\ell_\epsilon^{(i)}(\tau_i)-\ell_\epsilon^{(i)}(0)}\left|\eta^{(i)}_{\gamma_\epsilon^{(i)}(s+\ell_\epsilon^{(i)}(0))}\right|^2\mathrm{d}s \\ &\le \sum_{i=1}^d\left(\frac{B\left(\frac32-H_i,\frac12-H_i\right)}{\Gamma\left(\frac12-H_i\right)}\right)^2(\xi^{(i)})^2\int_0^{\ell_\epsilon^{(i)}(T)-\ell_\epsilon^{(i)}(0)} s^{1-2H_i}\,\mathrm{d}s \\ &= 2\Phi_{u,k}^2\left(T,\|x-y\|_1\right)\sum_{i=1}^d\Theta_{H_i}\frac{\left[\ell_\epsilon^{(i)}(T)-\ell_\epsilon^{(i)}(0)\right]^{2-2H_i}}{\left[\ell_\epsilon^{(i)}(\delta T)-\ell_\epsilon^{(i)}(0)\right]^2}. \end{aligned}$$

Set

$$R := \exp\left[M_\infty - \frac12\langle M\rangle_\infty\right].$$

Since $\mathbb{E}\,\mathrm{e}^{\frac12\langle M\rangle_\infty}<\infty$, we can use Novikov's criterion to obtain $\mathbb{E}R=1$, and by Girsanov's theorem we get

$$\widetilde{W}_{\ell_\epsilon(t)-\ell_\epsilon(0)} := \left(\widetilde{W}^{(1)}_{\ell_\epsilon^{(1)}(t)-\ell_\epsilon^{(1)}(0)},\dots,\widetilde{W}^{(d)}_{\ell_\epsilon^{(d)}(t)-\ell_\epsilon^{(d)}(0)}\right),\quad t\ge 0,$$

is a d-dimensional $(\mathscr{F}_t^{\ell_\epsilon})$-martingale under $R\mathbb{P}$, where $\mathscr{F}_t^{\ell_\epsilon}$ is the σ-algebra generated by $\{W^{(i)}_{\ell_\epsilon^{(i)}(s)-\ell_\epsilon^{(i)}(0)}:\ 0\le s\le t,\ 1\le i\le d\}$. For $0\le s\le t$ and $\theta=(\theta^{(1)},\dots,\theta^{(d)})\in\mathbb{R}^d$, it is easy to see that

$$\begin{aligned} &\mathbb{E}_{R\mathbb{P}}\left(\exp\left[\mathrm{i}\left\langle\theta,\widetilde{W}_{\ell_\epsilon(t)-\ell_\epsilon(0)}-\widetilde{W}_{\ell_\epsilon(s)-\ell_\epsilon(0)}\right\rangle\right]\,\middle|\,\mathscr{F}_s^{\ell_\epsilon}\right) \\ &\quad= \exp\left[\frac12\sum_{i=1}^d(\theta^{(i)})^2\left[\ell_\epsilon^{(i)}(t)-\ell_\epsilon^{(i)}(s)\right]\right] \\ &\quad= \mathbb{E}\left(\exp\left[\mathrm{i}\left\langle\theta,W_{\ell_\epsilon(t)-\ell_\epsilon(0)}-W_{\ell_\epsilon(s)-\ell_\epsilon(0)}\right\rangle\right]\,\middle|\,\mathscr{F}_s^{\ell_\epsilon}\right), \end{aligned}$$

where

$$W_{\ell_\epsilon(t)-\ell_\epsilon(0)} := \big(W^{(1)}_{\ell_\epsilon^{(1)}(t)-\ell_\epsilon^{(1)}(0)}, \dots, W^{(d)}_{\ell_\epsilon^{(d)}(t)-\ell_\epsilon^{(d)}(0)}\big), \quad t \geq 0.$$

This shows that the distribution of $(\widetilde{W}_{\ell_\epsilon(t)-\ell_\epsilon(0)})_{t\geq 0}$ under $R\mathbb{P}$ coincides with the law of $(W_{\ell_\epsilon(t)-\ell_\epsilon(0)})_{t\geq 0}$ under $\mathbb{P}$. If we rewrite (4.4) and (4.5) for $t \geq 0$ and $i = 1, \dots, d$ as

$$\begin{aligned} X_t^{\ell_\epsilon,v,(i)}(x) &= x^{(i)} + \int_0^t b_s^{(i)}\big(X_s^{\ell_\epsilon,v}(x)\big)\,\mathrm{d}s + v_t^{(i)} \\ &\quad + \int_0^t \mathcal{K}_{H_i}\left(\ell_\epsilon^{(i)}(t) - \ell_\epsilon^{(i)}(0), \ell_\epsilon^{(i)}(s) - \ell_\epsilon^{(i)}(0)\right)\mathrm{d}W^{(i)}_{\ell_\epsilon^{(i)}(s)-\ell_\epsilon^{(i)}(0)}, \end{aligned}$$

and

$$\begin{aligned} Y_t^{(i)} &= y^{(i)} + \int_0^t b_s^{(i)}(Y_s)\,\mathrm{d}s + v_t^{(i)} \\ &\quad + \int_0^t \mathcal{K}_{H_i}\left(\ell_\epsilon^{(i)}(t) - \ell_\epsilon^{(i)}(0), \ell_\epsilon^{(i)}(s) - \ell_\epsilon^{(i)}(0)\right)\mathrm{d}\widetilde{W}^{(i)}_{\ell_\epsilon^{(i)}(s)-\ell_\epsilon^{(i)}(0)}, \end{aligned}$$

respectively, we see that the distribution of $(X_t^{\ell_\epsilon,v}(y))_{t\geq 0}$ under $\mathbb{P}$ coincides with the distribution of $(Y_t)_{t\geq 0}$ under $R\mathbb{P}$.

As in the proof of Lemma 3.3 we get for any bounded Borel function $f : \mathbb{R}^d \to [1, \infty)$

$$\begin{aligned} P_T^{\ell_\epsilon,v} \log f(y) &= \mathbb{E}\big[R \log f\big(X_T^{\ell_\epsilon,v}(x)\big)\big] \\ &\leq \log \mathbb{E} f\big(X_T^{\ell_\epsilon,v}(x)\big) + \mathbb{E}\left[R \log R\right] \\ &= \log P_T^{\ell_\epsilon,v} f(x) + \mathbb{E}_{R\mathbb{P}} \log R \\ &\leq \log P_T^{\ell_\epsilon,v} f(x) + \Phi^2_{u,k}\left(T, \|x-y\|_1\right) \sum_{i=1}^d \Theta_{H_i} \frac{\left[\ell_\epsilon^{(i)}(T) - \ell_\epsilon^{(i)}(0)\right]^{2-2H_i}}{\left[\ell_\epsilon^{(i)}(\delta T) - \ell_\epsilon^{(i)}(0)\right]^2}, \end{aligned}$$

and for any non-negative bounded Borel function $f : \mathbb{R}^d \to [0, \infty)$

$$\begin{aligned} &\big(P_T^{\ell_\epsilon,v} f(y)\big)^p \\ &= \big(\mathbb{E}\big[R f\big(X_T^{\ell_\epsilon,v}(x)\big)\big]\big)^p \\ &\leq \big(\mathbb{E} f^p\big(X_T^{\ell_\epsilon,v}(x)\big)\big)\left(\mathbb{E}\left[R^{p/(p-1)}\right]\right)^{p-1} \\ &\leq P_T^{\ell_\epsilon,v} f^p(x) \cdot \exp\left[\frac{p}{p-1}\Phi^2_{u,k}\left(T, \|x-y\|_1\right) \sum_{i=1}^d \Theta_{H_i} \frac{\left[\ell_\epsilon^{(i)}(T) - \ell_\epsilon^{(i)}(0)\right]^{2-2H_i}}{\left[\ell_\epsilon^{(i)}(\delta T) - \ell_\epsilon^{(i)}(0)\right]^2}\right]. \end{aligned}$$

Letting $\delta \uparrow 1$ finishes the proof. □

The following result is easy; for the sake of completeness, we include its simple proof.

Lemma 4.5. *Assume* (A). *Then for any $x \in \mathbb{R}^d$ and $t \geq 0$,*

$$\lim_{\epsilon \downarrow 0} X_t^{\ell_\epsilon,v}(x) = X_t^{\ell,v}(x).$$

Proof. Fix $T > 0$, $\epsilon \in (0, 1)$, $x \in \mathbb{R}^d$ and observe that for $t \in [0, T]$

$$\begin{aligned} &\big\|X_t^{\ell_\epsilon,v}(x) - X_t^{\ell,v}(x)\big\|_1 \\ &\leq \int_0^t \big\|b_s\big(X_s^{\ell_\epsilon,v}(x)\big) - b_s\big(X_s^{\ell,v}(x)\big)\big\|_1\,\mathrm{d}s + \sum_{i=1}^d \big|W^{H_i,(i)}_{\ell_\epsilon^{(i)}(t)-\ell_\epsilon^{(i)}(0)} - W^{H_i,(i)}_{\ell^{(i)}(t)}\big| \\ &\leq \int_0^t k(s)\, u\big(\big\|X_s^{\ell_\epsilon,v}(x) - X_s^{\ell,v}(x)\big\|_1\big)\,\mathrm{d}s + \sum_{i=1}^d \big|W^{H_i,(i)}_{\ell_\epsilon^{(i)}(t)-\ell_\epsilon^{(i)}(0)} - W^{H_i,(i)}_{\ell^{(i)}(t)}\big|. \end{aligned}$$

Since the processes $X_t^{\ell_\epsilon,v}(x)$ and $X_t^{\ell,v}(x)$ are non-explosive, the last integral expression is finite. Applying Bihari's lemma with $g(\epsilon,t) := \sum_{i=1}^d \left|W^{H_i,(i)}_{\ell_\epsilon^{(i)}(t)-\ell_\epsilon^{(i)}(0)} - W^{H_i,(i)}_{\ell^{(i)}(t)}\right|$ yields that for any $t\in[0,T]$

$$\left\|X_t^{\ell_\epsilon,v}(x) - X_t^{\ell,v}(x)\right\|_1 \le G_u^{-1}\big(G_u(g(\epsilon,t)) + K(t)\big).$$

Since $\ell_\epsilon^{(i)}(t)\to\ell^{(i)}(t)$, one has $g(\epsilon,t)\to 0$ as $\epsilon\downarrow 0$. Combining this with $G_u(0+) = -\infty$, we find

$$\lim_{\epsilon\downarrow 0}\left\|X_t^{\ell_\epsilon,v}(x) - X_t^{\ell,v}(x)\right\|_1 = 0.$$

Hence,

$$\lim_{\epsilon\downarrow 0} X_t^{\ell_\epsilon,v}(x) = X_t^{\ell,v}(x) \quad \text{holds for all } t\in[0,T].$$

The claim follows since $T>0$ is arbitrary. □

4.3 Proof of Theorem 4.2

The proof parallels the argument which we have used for Theorem 3.2; in particular, Lemma 4.4 plays now the same role as Lemma 3.3 for the proof of Theorem 3.2.

The first step is to prove the log- and power-Harnack inequalities stated in i) and ii) for deterministic time-changes and for continuous functions $f\in C_b(\mathbb{R}^d)$. Lemma 3.2 has these inequalities for absolutely continuous time-changes and the operators $P^{\ell_\epsilon,v}$; letting $\epsilon\downarrow 0$, we get them for general time-changes and the operators $P^{\ell,v}$.

Since the processes Z and V are independent of $(W^{H_1,(1)},\dots,W^{H_d,(d)})$, we can indeed treat them like deterministic processes $Z=\ell$ and $V=v$, i.e. just as in Theorem 3.2 the deterministically time-changed inequalities combined with the Jensen and Hölder inequality prove Theorem 4.2 i) and ii).

Finally, the gradient estimate follows immediately from i) and [1, Proposition 2.3].

4.4 Two examples

As in Section 3.3, we apply our results to two typical examples of stochastic time-changes $Z^{(i)}$: subordinators and inverse subordinators.

Throughout this section we assume that $(X_t(x))_{t\ge 0}$ is the unique non-explosive solution to the SDE (4.2) and $P_tf(x) = \mathbb{E}f(X_t(x))$. Combining Theorem 4.2 and [7, Theorem 3.8 (a) and (b)], we obtain the following result.

Corollary 4.6. *Assume that* (4.1) *and* (A) *hold, and that for each $i=1,\dots,d$, $Z^{(i)}$ is a subordinator with Bernstein function ϕ_i such that $\liminf_{r\to\infty}\phi_i(r)r^{-\alpha_i}>0$ for some $\alpha_i>0$. Let*

$$\kappa_1 := 2\min_{1\le i\le d}\frac{H_i}{\alpha_i} \quad \textit{and} \quad \kappa_2 := 2\max_{1\le i\le d}\frac{H_i}{\alpha_i}.$$

Then there exists some constant $C = C_{\alpha_1,\dots,\alpha_d,H_1,\dots,H_d}>0$ such that the following assertions i)–iii) *hold.*

i) *For $T>0$, $x,y\in\mathbb{R}^d$ and all bounded Borel functions $f:\mathbb{R}^d\to[1,\infty)$*

$$P_T\log f(y) \le \log P_Tf(x) + \frac{Cd}{(T\wedge 1)^{\kappa_2}}\,\Phi_{u,k}^2\left(T,\|x-y\|_1\right).$$

If, in addition, $\liminf_{r\downarrow 0}\phi_i(r)r^{-\alpha_i}>0$ for each i, then

$$P_T\log f(y) \le \log P_Tf(x) + \frac{Cd}{T^{\kappa_1}\wedge T^{\kappa_2}}\,\Phi_{u,k}^2\left(T,\|x-y\|_1\right).$$

ii) *Assume that* $\alpha_i > 2H_i/(1+2H_i)$ *for each* $i = 1,\dots,d$. *For any* $T > 0$, $x, y \in \mathbb{R}^d$, $p > 1$ *and all bounded Borel functions* $f : \mathbb{R}^d \to [0,\infty)$

$$\begin{aligned}\big(P_T f(y)\big)^p \le & P_T f^p(x) \cdot \exp\Bigg[\frac{Cp}{p-1}\,\Phi^2_{u,k}\left(T, \|x-y\|_1\right)\left(1 + \frac{d}{T^{\kappa_1} \wedge T^{\kappa_2}}\right) \\ & + C(p-1)\sum_{i=1}^d \left(\frac{p\Phi^2_{u,k}\left(T, \|x-y\|_1\right)}{(p-1)^2}\right)^{\frac{\alpha_i}{\alpha_i - 2H_i(1-\alpha_i)}} T^{-\frac{2H_i}{\alpha_i - 2H_i(1-\alpha_i)}}\Bigg].\end{aligned}$$

If, in addition, $\liminf_{r\downarrow 0} \phi_i(r) r^{-\alpha_i} > 0$ *for each* i, *then*

$$\begin{aligned}\big(P_T f(y)\big)^p \le & P_T f^p(x) \cdot \exp\Bigg[\frac{Cdp}{p-1}\,\frac{\Phi^2_{u,k}\left(T, \|x-y\|_1\right)}{T^{\kappa_1} \wedge T^{\kappa_2}} \\ & + C(p-1)\sum_{i=1}^d \left(\frac{p\Phi^2_{u,k}\left(T, \|x-y\|_1\right)}{(p-1)^2}\right)^{\frac{\alpha_i}{\alpha_i - 2H_i(1-\alpha_i)}} T^{-\frac{2H_i}{\alpha_i - 2H_i(1-\alpha_i)}}\Bigg].\end{aligned}$$

iii) *If* (A) *holds for* $u(s) = cs$ *and some constant* $c > 0$, *then for* $T > 0$, $x \in \mathbb{R}^d$ *and all bounded Borel functions* $f : \mathbb{R}^d \to \mathbb{R}$

$$|\nabla P_T f|^2(x) \le \left\{P_T f^2(x) - \big(P_T f(x)\big)^2\right\} \frac{Cd}{(T \wedge 1)^{\kappa_2}} \left(1 + c\int_0^T k(s)\,\mathrm{e}^{cK(s)}\,\mathrm{d}s\right)^2.$$

If, in addition, $\liminf_{r\downarrow 0} \phi_i(r) r^{-\alpha_i} > 0$ *for each* i, *then*

$$|\nabla P_T f|^2(x) \le \left\{P_T f^2(x) - \big(P_T f(x)\big)^2\right\} \frac{Cd}{T^{\kappa_1} \wedge T^{\kappa_2}} \left(1 + c\int_0^T k(s)\,\mathrm{e}^{cK(s)}\,\mathrm{d}s\right)^2.$$

If the $Z^{(i)}$ are inverse subordinators, we cannot expect that a power-Harnack inequality will hold, see Remark 3.8. Combining Lemma 3.6 with Theorem 4.2 i) & iii), we still have the following corollary.

Corollary 4.7. *Assume that* (4.1) *and* (A) *hold, and that* $Z^{(i)}$ *is for each* $i = 1,\dots,d$ *an inverse subordinator with Bernstein function* ϕ_i.

Moreover, assume that $\limsup_{r\downarrow 0} \phi_i(r) r^{-\alpha_i} < \infty$ *and* $\limsup_{r\to\infty} \phi_i(r) r^{-\alpha_i} < \infty$ *for some* $\alpha_i > 0$. *Let*

$$\kappa_3 := 2 \min_{1\le i\le d} H_i\alpha_i \quad \text{and} \quad \kappa_4 := 2 \max_{1\le i\le d} H_i\alpha_i.$$

Then there exists some constant $C = C_{\alpha_1,\dots,\alpha_d,H_1,\dots,H_d} > 0$ *such that the following assertions* i), ii) *hold.*

i) *For* $T > 0$, $x, y \in \mathbb{R}^d$ *and all bounded Borel functions* $f : \mathbb{R}^d \to [1,\infty)$

$$P_T \log f(y) \le \log P_T f(x) + \frac{Cd}{T^{\kappa_3} \wedge T^{\kappa_4}}\,\Phi^2_{u,k}\left(T, \|x-y\|_1\right).$$

ii) *If* (A) *holds with* $u(s) = cs$ *for some constant* $c > 0$, *then for* $T > 0$, $x \in \mathbb{R}^d$ *and all bounded Borel functions* $f : \mathbb{R}^d \to \mathbb{R}$

$$|\nabla P_T f|^2(x) \le \left\{P_T f^2(x) - \big(P_T f(x)\big)^2\right\} \frac{Cd}{T^{\kappa_3} \wedge T^{\kappa_4}} \left(1 + c\int_0^T k(s)\,\mathrm{e}^{cK(s)}\,\mathrm{d}s\right)^2.$$

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