Electronic Journal of Probability

Electron. J. Probab. **22** (2017), no. 99, 1–23.
ISSN: 1083-6489 DOI: 10.1214/17-EJP71

Harmonic moments and large deviations for a supercritical branching process in a random environment

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Abstract

Let $(Z_n)_{n\geqslant 0}$ be a supercritical branching process in an independent and identically distributed random environment $\xi = (\xi_n)_{n\geqslant 0}$. We study the asymptotic behavior of the harmonic moments $\mathbb{E}\left[Z_n^{-r}|Z_0 = k\right]$ of order $r > 0$ as $n \to \infty$, when the process starts with k initial individuals. We exhibit a phase transition with the critical value $r_k > 0$ determined by the equation $\mathbb{E}p_1^k(\xi_0) = \mathbb{E}m_0^{-r_k}$, where $m_0 = \sum_{j=0}^{\infty} jp_j(\xi_0)$, $(p_j(\xi_0))_{j\geqslant 0}$ being the offspring distribution given the environnement ξ_0. Contrary to the constant environment case (the Galton-Watson case), this critical value is different from that for the existence of the harmonic moments of $W = \lim_{n\to\infty} Z_n/\mathbb{E}(Z_n|\xi)$. The aforementioned phase transition is linked to that for the rate function of the lower large deviation for Z_n. As an application, we obtain a lower large deviation result for Z_n under weaker conditions than in previous works and give a new expression of the rate function. We also improve an earlier result about the convergence rate in the central limit theorem for $W - W_n$, and find an equivalence for the large deviation probabilities of the ratio Z_{n+1}/Z_n.

Keywords: branching processes; random environment; harmonic moments; large deviations; phase transitions; central limit theorem.
AMS MSC 2010: Primary 60J80; 60K37; 60J05, Secondary 60J85; 92D25.
Submitted to EJP on August 26, 2016, final version accepted on May 26, 2017.

1 Introduction

A branching process in a random environment (BPRE) is a natural and important generalization of the Galton-Watson process, where the reproduction law varies according to a random environment indexed by time. It was introduced for the first time in Smith and Wilkinson [22] to modelize the growth of a population submitted to an environment. For background concepts and basic results concerning a BPRE we refer to Athreya

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and Karlin [4, 3]. In the critical and subcritical regimes the branching process goes out and the research interest has been mostly concentrated on the survival probability and conditional limit theorems, see e.g. Afanasyev, Böinghoff, Kersting, Vatutin [1, 2], Vatutin [24], Vatutin and Zheng [25], and the references therein. In the supercritical case, a great deal of current research has been focused on large deviations, see e.g. Bansaye and Berestycki [5], Bansaye and Böinghoff [6, 7, 8], Böinghoff and Kersting [10], Huang and Liu [16] and Nakashima [20]. In the particular case when the offspring distribution has a fractional linear generating function, precise asymptotics can be found in Böinghoff [9] and Kozlov [17]. An important closely linked issue is the asymptotic behavior of the harmonic moments $\mathbb{E}[Z_n^{-r}|Z_0 = k]$ of the process Z_n starting with $Z_0 = k$ initial individuals. For the Galton-Watson process which corresponds to the constant environment case, the question has been studied exhaustively in Ney and Vidyashankar [21]. For a BPRE, it has only been partially treated in [16, Theorem 1.3].

In the present paper, we give a complete description of the asymptotic behavior of the harmonic moments $\mathbb{E}_k[Z_n^{-r}] = \mathbb{E}[Z_n^{-r}|Z_0 = k]$ of the process Z_n starting with k individuals and assuming that each individual gives birth to at least one offspring (non-extinction case). As a consequence, we improve the lower large deviation result for the process Z_n obtained in [7, Theorem 3.1] by relaxing the hypothesis therein. In the meanwhile we give a new characterization of the rate function in the large deviation result stated in [7]. We also improve the exponential convergence rate in the central limit theorem for $W - W_n$ established in [16]. Furthermore, we investigate the large deviation behavior of the ratio $R_n = \frac{Z_{n+1}}{Z_n}$, i.e. the asymptotic of the large deviation probability $\mathbb{P}(|R_n - m_n| > a)$ for $a > 0$, where m_n is the expected value of the number of children of an individual in generation n given the environment ξ. For the Galton-Watson process, the quantity R_n is the Lotka-Nagaev estimator of the mean $\mathbb{E}Z_1$, whose large deviation probability has been studied in [21].

Let us explain briefly the findings of the paper in the special case when we start with $Z_0 = 1$ individual. Assume that $\mathbb{P}(Z_1 = 0) = 0$ and $\mathbb{P}(Z_1 = 1) > 0$. Define $r_1 \in (0, \infty)$ as the solution of the equation

$$\mathbb{E}m_0^{-r_1} = \gamma, \tag{1.1}$$

with

$$\gamma = \mathbb{P}(Z_1 = 1). \tag{1.2}$$

From Theorem 2.1 we get the following asymptotic behavior of the harmonic moments $\mathbb{E}\left[Z_n^{-r}\right]$ for $r > 0$. Assume that $\mathbb{E}m_0^{\varepsilon} < \infty$ for some $\varepsilon > 0$. Then, we have

$$\begin{cases} \dfrac{\mathbb{E}\left[Z_n^{-r}\right]}{\gamma^n} & \underset{n\to\infty}{\longrightarrow} \quad C(r) \qquad \text{if } \ r > r_1, \\[2ex] \dfrac{\mathbb{E}\left[Z_n^{-r}\right]}{n\gamma^n} & \underset{n\to\infty}{\longrightarrow} \quad C(r) \qquad \text{if } \ r = r_1, \\[2ex] \dfrac{\mathbb{E}\left[Z_n^{-r}\right]}{(\mathbb{E}m_0^{-r})^n} & \underset{n\to\infty}{\longrightarrow} \quad C(r) \qquad \text{if } \ r < r_1, \end{cases} \tag{1.3}$$

where $C(r)$ are positive constants for which we find integral expressions. This shows that there is a phase transition in the rate of convergence of the harmonic moments for the process Z_n, with the critical value r_1. It generalizes the result of [21] for the Galton-Watson process. For a BPRE, it completes and improves the result of [16], where the asymptotic equivalent of the quantity $\mathbb{E}\left[Z_n^{-r}\right]$ has been established in the particular case where $r < r_1$ and under stronger assumptions.

The proof presented here is new and straightforward compared to that for the Galton-Watson process given in [21]. Indeed, we prove (1.3) starting from the branching property

$$Z_{n+m} = \sum_{i=1}^{Z_m} Z_{n,i}^{(m)},$$

where conditionally on the environment ξ, for $i \geqslant 1$, the sequences of random variables $\{Z_{n,i}^{(m)} : n \geqslant 0\}$ are i.i.d. branching processes with the shifted environment $T^m(\xi_0, \xi_1, \ldots) = (\xi_m, \xi_{m+1}, \ldots)$, and are also independent of Z_m. This simple idea leads to the following equation which will play a key role in our arguments:

$$\mathbb{E}\left[Z_{n+1}^{-r}\right] = \gamma^{n+1} + \sum_{j=0}^{n} b_j \gamma^{n-j} c_r^j, \tag{1.4}$$

where $c_r = \mathbb{E} m_0^{-r}$ and $(b_j)_{j\geqslant 0}$ is an increasing and bounded sequence. Such a relation highlights the main role played by the quantities γ and c_r in the asymptotic study of $\mathbb{E}[Z_n^{-r}]$ whose behavior depends on whether $\gamma < c_r$, $\gamma = c_r$ or $\gamma > c_r$. Note that the complete proof of (1.3) relies on some recent and important results established in [13] concerning the critical value for the existence of the harmonic moments of the r.v. W and the asymptotic behavior of the distribution $\mathbb{P}(Z_n = j)$ as $n \to \infty$, with $j \geqslant 1$. For the Galton-Watson process, our approach based on (1.4) is much simpler than that in [21].

Our proof also gives an expression of the limit constants in (1.3). For the Galton-Watson process, it recovers the expressions of [21, Theorem 1] in the cases where $\gamma > c_r$ and $\gamma < c_r$. In the critical case where $r = r_1$, the limit constant obtained in this paper is different to that of [21, Theorem 1], which leads to an alternative expression of the constant and the following nice identity involving the well-known functions G, Q and ϕ: defining

$$\begin{aligned} G(t) &= \sum_{k=0}^{\infty} t^k \mathbb{P}(Z_1 = k), \\ Q(t) &= \lim_{n\to\infty} \gamma^{-n} G^{\circ n}(t), \\ \phi(t) &= \mathbb{E}[e^{-tW}], \end{aligned}$$

and denoting $m = \mathbb{E}[Z_1]$ and $\bar{G}(t) = G(t) - \gamma t$, we have

$$\frac{1}{\gamma}\int_0^\infty \bar{G}(\phi(u)) u^{r-1} du = \int_1^m Q(\phi(u)) u^{r-1} du. \tag{1.5}$$

For a BPRE, we will show a generalization of (1.5) in Proposition 2.2.

As a consequence of Theorem 2.1 and of a version of the Gärtner-Ellis theorem, we obtain a lower large deviation result for Z_n under conditions weaker than those in [7, Theorem 3.1]. Assume that $\mathbb{P}(Z_1 = 0) = 0$ and $\mathbb{E} m_0^{\varepsilon} < \infty$ for some $\varepsilon > 0$. Let

$$\Lambda(\lambda) = \log \mathbb{E} e^{\lambda X} \tag{1.6}$$

be the log-Laplace transform of $X = \log m_0$ and

$$\Lambda^*(\theta) = \sup_{\lambda\in\mathbb{R}} \{\lambda\theta - \Lambda(\lambda)\} = \sup_{\lambda\leqslant 0} \{\lambda\theta - \Lambda(\lambda)\} \quad \text{for} \quad \theta \in (0, \mathbb{E}[X]), \tag{1.7}$$

be the Fenchel-Legendre transform of $\Lambda(\cdot)$. Then, for any $\theta \in (0, \mathbb{E}[X])$, we have

$$\lim_{n\to\infty} -\frac{1}{n} \log \mathbb{P}\left(Z_n \leqslant e^{\theta n}\right) = \chi^*(\theta) \in (0, \infty), \tag{1.8}$$

where

$$\chi^*(\theta) = \begin{cases} -r_1\theta - \log\gamma & \text{if } 0 < \theta < \theta_1, \\ \Lambda^*(\theta) & \text{if } \theta_1 \leqslant \theta < \mathbb{E}[X], \end{cases} \tag{1.9}$$

with

$$\theta_1 = \Lambda'(-r_1) \in (0, \mathbb{E}[X]). \tag{1.10}$$

Equation (1.8) improves the result of [7, Theorem 3.1(ii)] in the case when $\mathbb{P}(Z_1 = 0) = 0$, since it is assumed in [7] that $\mathbb{E}m_0^t < \infty$ for all $t > 0$, whereas we only require that $\mathbb{E}m_0^\varepsilon < \infty$ for some $\varepsilon > 0$. Moreover, Equations (1.9) and (1.10) also give new and alternative expressions of the rate function and the critical value. In fact, it has been proved in [7] that, in the case when $\mathbb{P}(Z_1 = 0) = 0$ and $Z_0 = 1$,

$$\lim_{n\to\infty} -\frac{1}{n} \log \mathbb{P}\left(Z_n \leqslant e^{\theta n}\right) = I(\theta) \in (0, \infty), \tag{1.11}$$

with

$$I(\theta) \quad = \quad \begin{cases} \rho\left(1 - \frac{\theta}{\theta_1^*}\right) + \frac{\theta}{\theta_1^*}\Lambda^*(\theta_1^*) & \text{if } 0 < \theta < \theta_1^*, \\ \Lambda^*(\theta) & \text{if } \theta_1^* \leqslant \theta < \mathbb{E}[X], \end{cases} \tag{1.12}$$

where $\rho = -\log \gamma$ and θ_1^* is the unique solution on $(0, \mathbb{E}[X])$ of the equation

$$\frac{\rho - \Lambda^*(\theta_1^*)}{\theta_1^*} = \inf_{0 \leqslant \theta \leqslant \mathbb{E}[X]} \frac{\rho - \Lambda^*(\theta)}{\theta}. \tag{1.13}$$

It follows directly from the relations (1.9) to (1.13) that $\theta_1 = \theta_1^*$ and $\chi^*(\theta) = I(\theta)$ for all $\theta \in (0, \mathbb{E}[X])$. This fact can also be shown by using simple duality arguments between Λ and Λ^*, as will be seen in the next section.

The rest of the paper is organized as follows. In Section 2 we give the precise statements of the main theorems with applications. Section 3 is devoted to the proofs of the main results, Theorems 2.1 and 2.3. The proofs for the applications are deferred to Section 4.

Throughout the paper, we denote by C an absolute constant whose value may differ from line to line.

2 Main results

A BPRE (Z_n) can be described as follows. The random environment is represented by a sequence $\xi = (\xi_0, \xi_1, ...)$ of independent and identically distributed random variables (i.i.d. r.v.'s) taking values in an abstract space Ξ, whose realizations determine the probability generating functions

$$f_{\xi_n}(s) = f_n(s) = \sum_{i=0}^{\infty} p_i(\xi_n) s^i, \quad s \in [0, 1], \quad p_i(\xi_n) \geqslant 0, \quad \sum_{i=0}^{\infty} p_i(\xi_n) = 1. \tag{2.1}$$

The BPRE $(Z_n)_{n\geqslant 0}$ is defined by the relations

$$Z_0 = 1, \quad Z_{n+1} = \sum_{i=1}^{Z_n} N_{n,i}, \quad \text{for} \quad n \geqslant 0, \tag{2.2}$$

where the random variables $N_{n,i}$ $(i = 1, 2, \dots)$ represent the number of children of the i-th individual of the generation n. Conditionally on the environment ξ, the r.v.'s $N_{n,i}$ $(n \geqslant 0, i \geqslant 1)$ are independent of each other, and each $N_{n,i}$ $(i \geqslant 1)$ has common probability generating function f_n.

In the sequel we denote by $\mathbb{P}_\xi$ the *quenched law*, i.e. the conditional probability when the environment ξ is given, and by τ the law of the environment ξ. Then $\mathbb{P}(dx, d\xi) = \mathbb{P}_\xi(dx)\tau(d\xi)$ is the total law of the process, called *annealed law*. The corresponding quenched and annealed expectations are denoted respectively by $\mathbb{E}_\xi$ and $\mathbb{E}$. We also denote by $\mathbb{P}_k$ and $\mathbb{E}_k$ the corresponding annealed probability and expectation starting with $Z_0 = k$ individuals, with $\mathbb{P}_1 = \mathbb{P}$ and $\mathbb{E}_1 = \mathbb{E}$. From (2.2), it follows that the probability generating function of Z_n conditionally on the environment ξ is given by

$$g_n(t) = \mathbb{E}_\xi[t^{Z_n}] = f_0 \circ \ldots \circ\ f_{n-1}(t), \quad 0 \leqslant t \leqslant 1. \tag{2.3}$$

Since $\xi_0, \xi_1, \ldots$ are i.i.d. r.v.'s, we get that the annealed probability generating function $G_{k,n}$ of Z_n starting with $Z_0 = k$ individuals is given by

$$G_{k,n}(t) = \mathbb{E}_k[t^{Z_n}] = \mathbb{E}\left[g_n^k(t)\right], \quad 0 \leqslant t \leqslant 1. \tag{2.4}$$

We also define, for $n \geqslant 0$,

$$m_n(p) = m(p, \xi_n) = \sum_{i=0}^{\infty} i^p p_i(\xi_n) \quad \text{for } p > 0, \tag{2.5}$$

$$m_n = m_n(1) = \sum_{i=0}^{\infty} i p_i(\xi_n), \quad \Pi_0 = 1 \quad \text{and} \quad \Pi_n = \mathbb{E}_\xi Z_n = m_0 ... m_{n-1}, \tag{2.6}$$

where m_n represents the average number of children of an individual of generation n when the environment ξ is given. Let

$$W_n = \frac{Z_n}{\Pi_n}, \quad n \geqslant 0, \tag{2.7}$$

be the normalized population size. It is well known that under the quenched law $\mathbb{P}_\xi$, as well as under the annealed law $\mathbb{P}$, the sequence $(W_n)_{n\geqslant 0}$ is a non-negative martingale with respect to the filtration

$$\mathcal{F}_n = \sigma\left(\xi, N_{k,i}, 0 \leqslant k \leqslant n-1, i = 1, 2 \ldots\right),$$

where by convention $\mathcal{F}_0 = \sigma(\xi)$. Then the limit $W = \lim W_n$ exists $\mathbb{P}$ - a.s. and $\mathbb{E}W \leqslant 1$. We also denote the quenched and annealed Laplace transforms of W by

$$\phi_\xi(t) = \mathbb{E}_\xi\left[e^{-tW}\right] \quad \text{and} \quad \phi(t) = \mathbb{E}\left[e^{-tW}\right], \quad \text{for } t \geqslant 0, \tag{2.8}$$

while starting with $Z_0 = 1$ individual. For $k \geqslant 1$, while starting with $Z_0 = k$ individuals, we have

$$\phi_k(t) := \mathbb{E}_k\left[e^{-tW}\right] = \mathbb{E}[\phi_\xi^k(t)]. \tag{2.9}$$

We also mention the well-known functional equation

$$\phi_\xi\left(t\right) = f_0\left(\phi_{T\xi}\left(\frac{t}{m_0}\right)\right), \quad t > 0, \tag{2.10}$$

where T is the shift operator defined by $T(\xi_0, \xi_1, \ldots) = (\xi_1, \xi_2, \ldots)$.

Another important tool in the study of a BPRE is the associated random walk

$$S_n = \log \Pi_n = \sum_{i=1}^{n} X_i, \quad n \geqslant 1,$$

where the r.v.'s $X_i = \log m_{i-1}$ $(i \geqslant 1)$ are i.i.d. depending only on the environment ξ. For the sake of brevity, we set $X = \log m_0$ and

$$\mu = \mathbb{E}X.$$

We shall consider a supercritical BPRE where $\mu \in (0,\infty)$, so that under the extra condition $\mathbb{E}|\log(1-p_0(\xi_0))| < \infty$ (see [22]), the population size tends to infinity with positive probability. For our propose, in fact we will assume in the whole paper that each individual gives birth to at least one child, i.e.

$$p_0(\xi_0) = 0 \quad a.s. \tag{2.11}$$

Therefore, under the condition

$$\mathbb{E}\frac{Z_1}{m_0}\log^+ Z_1 < \infty, \tag{2.12}$$

the martingale (W_n) converges to W in $L^1(\mathbb{P})$ (see e.g. [23]) and

$$\mathbb{P}(W>0) = \mathbb{P}(Z_n \to \infty) = 1.$$

Now we can state the main result of the paper about the asymptotic behavior of the harmonic moments $\mathbb{E}_k[Z_n^{-r}]$ of the process (Z_n) for $r>0$, starting with $Z_0 = k$ for $k \geqslant 1$. Let

$$\gamma_k = \mathbb{P}_k(Z_1 = k) = \mathbb{E}[p_1^k(\xi_0)] \tag{2.13}$$

and $r_k \in (0,+\infty]$ be the solution of the equation

$$\mathbb{E}m_0^{-r_k} = \gamma_k \tag{2.14}$$

with the convention that

$$r_k = +\infty \quad \text{if} \quad \mathbb{P}(p_1(\xi_0) = 0) = 1.$$

For any $k \geqslant 1$ and $r > 0$, let $\mathbb{P}_k^{(r)}$ be the probability measure (depending on r) defined, for any $\mathcal{F}_n$-measurable r.v. T, by

$$\mathbb{E}_k^{(r)}[T] = \frac{\mathbb{E}_k\left[\Pi_n^{-r}T\right]}{\left(\mathbb{E}m_0^{-r}\right)^n}. \tag{2.15}$$

Set $\mathbb{P}^{(r)} = \mathbb{P}_1^{(r)}$ and $\mathbb{E}^{(r)} = \mathbb{E}_1^{(r)}$. It is easily seen that under $\mathbb{P}^{(r)}$, the process (Z_n) is still a supercritical branching process in a random environment with $\mathbb{P}^{(r)}(Z_1 = 0) = 0$, and (W_n) is still a non-negative martingale which converges a.s. to W. Moreover, by (2.12) and the fact that $m_0 \geqslant 1$, we have

$$\mathbb{E}^{(r)}\left[\frac{Z_1}{m_0}\log Z_1\right] = \mathbb{E}\left[\frac{Z_1}{m_0^{1+r}}\log Z_1\right]/\mathbb{E}\left[m_0^{-r}\right] \leqslant \mathbb{E}\left[\frac{Z_1}{m_0}\log Z_1\right]/\mathbb{E}\left[m_0^{-r}\right] < \infty,$$

which implies that

$$W_n \to W \quad \text{in} \quad L^1(\mathbb{P}^{(r)}). \tag{2.16}$$

Set, for $t \in [0,1)$,

$$\bar{G}_{k,1}(t) = G_{k,1}(t) - \gamma_k t^k = \sum_{j=k+1}^{\infty} t^j \mathbb{P}(Z_1 = j) \tag{2.17}$$

and

$$Q_k(t) = \lim_{n\to\infty}\frac{G_{k,n}(t)}{\gamma_k^n} = \sum_{j=k}^{\infty} q_{k,j}t^j, \tag{2.18}$$

where $G_{k,n}$ is defined by (2.4),

$$q_{k,j} = \lim_{n\to\infty}\gamma_k^{-n}\mathbb{P}_k(Z_n = j) \in (0,\infty), \tag{2.19}$$

and the limits in (2.18) and (2.19) exist according to Lemma 3.2 below.Let

$$A_{k,n}(r) = \begin{cases} \gamma_k^n & \text{if } r > r_k, \\ n\gamma_k^n & \text{if } r = r_k, \\ (\mathbb{E}m_0^{-r})^n & \text{if } r < r_k. \end{cases} \tag{2.20}$$

The next theorem gives the asymptotic behavior of the harmonic moments of Z_n.

Theorem 2.1. *Assume that* $\mathbb{E}m_0^{\varepsilon} < +\infty$ *for some* $\varepsilon > 0$. *Then, for any integer* $k \geqslant 1$ *and* $r \in (0,\infty)$, *we have*

$$\lim_{n\to\infty} \frac{\mathbb{E}_k\left[Z_n^{-r}\right]}{A_{k,n}(r)} = C(k,r) := \begin{cases} \dfrac{1}{\Gamma(r)} \displaystyle\int_0^\infty Q_k(e^{-t})t^{r-1}dt & \text{if } r > r_k, \\ \dfrac{\gamma_k^{-1}}{\Gamma(r)} \displaystyle\int_0^\infty \mathbb{E}^{(r)}\left[\bar{G}_{k,1}(\phi_\xi(t))t^{r-1}\right]dt & \text{if } r = r_k, \\ \dfrac{1}{\Gamma(r)} \displaystyle\int_0^\infty \phi_k^{(r)}(t)t^{r-1}dt & \text{if } r < r_k, \end{cases} \tag{2.21}$$

where $C(k,r) \in (0,\infty)$, $\Gamma(r) = \int_0^\infty x^{r-1}e^{-x}dx$ *is the Gamma function, and* $\phi_k^{(r)}(t) = \mathbb{E}_k^{(r)}[e^{-tW}]$ *is the Laplace transform of* W *under* $\mathbb{P}_k^{(r)}$.

This theorem shows that there is a phase transition in the rate of convergence of the harmonic moments of Z_n with the critical value $r_k > 0$ defined by (2.14). This critical value r_k is generally different from the critical value a_k for the existence of the harmonic moment of W. Indeed, as shown in [13, Theorem 2.1] (see Lemma 3.1 below), the critical value a_k is determined by

$$\mathbb{E}\left[p_1^k(\xi_0)m_0^{a_k}\right] = 1, \tag{2.22}$$

which is in general different from the critical value r_k determined by (2.14), that is

$$\mathbb{E}\left[p_1^k(\xi_0)\right] = \mathbb{E}\left[m_0^{-r_k}\right]. \tag{2.23}$$

This is in contrast with the Galton-Watson process where $r_k = a_k = r_1^k$, with r_1 the solution of the equation $p_1 m_0^{r_1} = 1$ which coincides with both equations (2.22) and (2.23). Theorem 2.1 generalizes the result of [21] for the Galton-Watson process. For a BPRE, it completes and improves Theorem 1.3 of [16], where the formula (2.21) was only proved for $k = 1$ and $r < r_1$, and under the following much stronger boundedness condition: there exist some constants $p, c_0, c_1 > 1$ such that

$$c_0 \leqslant m_0 \leqslant c_1 \quad \text{and} \quad c_0 \leqslant m_0(p) \leqslant c_1 \quad a.s.$$

with $m_0(p)$ defined by (2.5). Instead of this boundedness condition, here we only require the ordinary moment assumption $\mathbb{E}[m_0^{\varepsilon}] < \infty$ for some $\varepsilon > 0$.

For the Galton-Watson process with $k = 1$ initial individual, the expression of the limit constant in the case when $r = r_1$ (up to the constant factor $\Gamma(r)$) becomes

$$\frac{1}{\gamma}\int_0^\infty \bar{G}(\phi(u))u^{r-1}du,$$

whereas it has been proved in [21] that the limit constant is equal to

$$\int_1^m Q(\phi(u))u^{r-1}du.$$

Actually, the above two expressions coincide, as shown by the following result valid for a general BPRE.

Proposition 2.2. *Assume that $\mathbb{E}m_0^\varepsilon < \infty$ for some $\varepsilon > 0$. For $k \geqslant 1$, we have*

$$\frac{1}{\gamma_k}\mathbb{E}^{(r_k)}\left[\int_0^\infty \bar{G}_{k,1}(\phi_\xi(u))u^{r_k-1}du\right] = \mathbb{E}^{(r_k)}\left[\int_1^{m_0} Q_k(\phi_\xi(u))u^{r_k-1}du\right]. \tag{2.24}$$

As an application of Theorem 2.1 we get a large deviation result. Indeed, $\mathbb{E}[Z_n^\lambda] = \mathbb{E}[e^{\lambda \log Z_n}]$ is the Laplace transform of $\log Z_n$. From Theorem 2.1 we obtain for $\lambda \in (-\infty, 0]$,

$$\lim_{n\to\infty} \frac{1}{n}\log \mathbb{E}_k[Z_n^\lambda] = \chi_k(\lambda) = \begin{cases} \log\gamma_k & \text{if } \lambda \leqslant -r_k, \\ \Lambda(\lambda) & \text{if } \lambda \in (-r_k, 0], \end{cases} \tag{2.25}$$

with $\Lambda(\lambda)$ defined by (1.6). Note that $\chi_k(\lambda) = \Lambda(\lambda)$ for all $\lambda \in (-\infty, 0]$ if $r_k = \infty$.

Using (2.25) and a version of the Gärtner-Ellis theorem adapted to the study of tail probabilities (see [19, Theorem 6.1]), we get the following lower large deviation result for the BPRE (Z_n).

Theorem 2.3. *Assume that $\mathbb{E}m_0^\varepsilon < \infty$ for some $\varepsilon > 0$. Let $k \geqslant 1$ and r_k be the solution of the equation* (2.14)*. Then, for any $\theta \in (0, \mathbb{E}[X])$, we have*

$$\lim_{n\to\infty} -\frac{1}{n}\log \mathbb{P}_k\left(Z_n \leqslant e^{\theta n}\right) = \chi_k^*(\theta) \in (0,\infty), \tag{2.26}$$

where $\chi_k^*(\theta) = \Lambda^*(\theta)$ if $p_1(\xi_0) = 0$ a.s., otherwise

$$\chi_k^*(\theta) = \sup_{\lambda\leqslant 0}\{\lambda\theta - \chi_k(\lambda)\} = \begin{cases} -r_k\theta - \log\gamma_k & \text{if } 0 < \theta < \theta_k, \\ \Lambda^*(\theta) & \text{if } \theta_k \leqslant \theta < \mathbb{E}[X], \end{cases} \tag{2.27}$$

with $\Lambda^*(\cdot)$ defined by (1.7) *and*

$$\theta_k = \Lambda'(-r_k). \tag{2.28}$$

Notice that the transition occurs if and only if $\mathbb{P}(p_1(\xi_0) > 0) > 0$. Let us give some comments in this case. The value $\chi_k^*(\theta)$ can be interpreted geometrically as the maximum distance between the graphs of the linear function $l_\theta : \lambda \mapsto \theta\lambda$ with slope θ and the function $\chi_k : \lambda \mapsto \chi_k(\lambda)$ defined in (2.25). Taking into account the fact that $\chi(\lambda) = \Lambda(\lambda)$ for $\lambda \in (-r_k, 0]$ and $\chi(\lambda) = \log\gamma_k$ for $\lambda \leqslant -r_k$, we can easily describe the phase transitions of $\chi^*(\theta)$ depending on the value of the slope θ of the function l_θ:

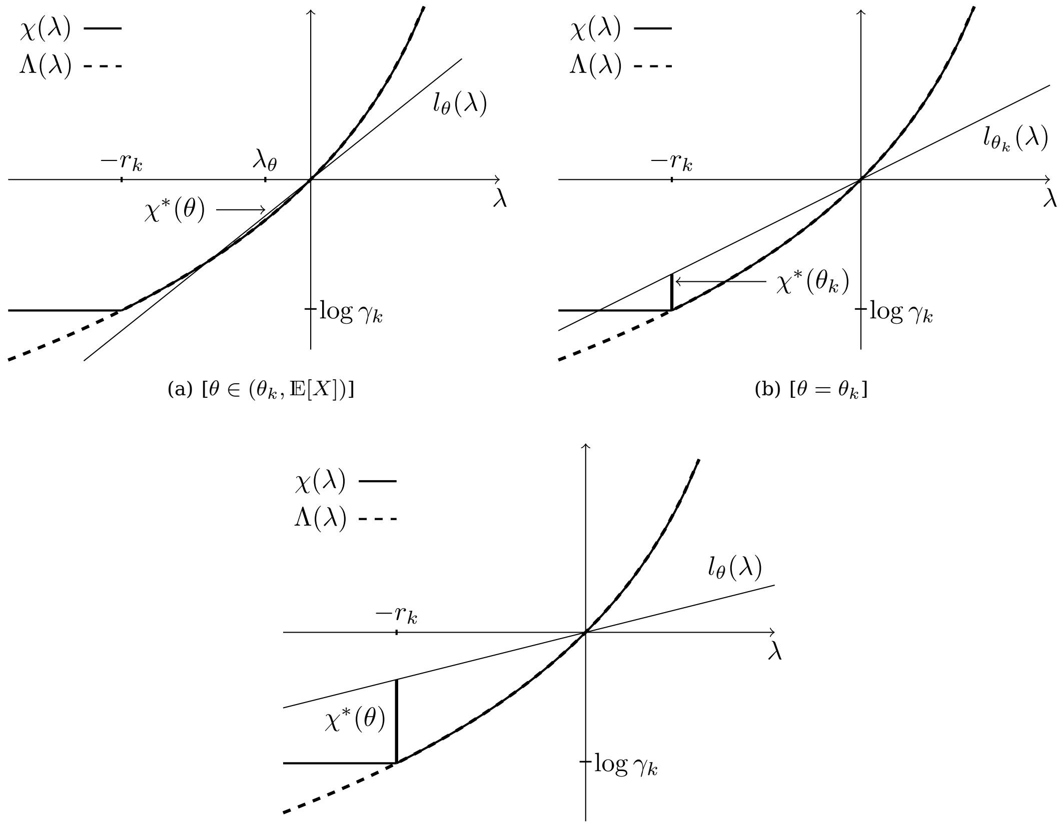
1. in the case when $\theta \in (\theta_k, \mathbb{E}[X])$, the maximum $\sup_{\lambda\leqslant 0}\{l_\theta(\lambda) - \chi(\lambda)\}$ is attained for $\lambda \in (-r_k, 0)$ such that $\chi'(\lambda_\theta) = \Lambda'(\lambda_\theta) = \theta$, whose value is $\chi^*(\theta) = \Lambda^*(\theta)$ (see Fig. 1(a));
2. the case when $\theta = \theta_k$ is the critical slope for which the equation $\Lambda'(\lambda) = \theta_k$ has a solution given by $\lambda = -r_k$ (see Fig. 1(b));
3. in the case when $\theta \in (0, \theta_k)$, the maximum $\sup_{\lambda\leqslant 0}\{l_\theta(\lambda) - \Lambda(\lambda)\}$ is attained for $\lambda = -r_k$, and then $\chi^*(\theta) = -r_k\theta - \log\gamma_k$ becomes linear in θ (see Fig. 1(c)).

Remark 2.4. Theorem 2.3 corrects and improves the result of [7, Theorem 3.1(ii)]. Moreover it gives new and alternative expressions of the rate function and the critical value. Actually it was proved in [7, Theorem 3.1(ii)] that, assuming $\mathbb{P}(Z_1 = 0) = 0$ and $\mathbb{E}m_0^t < \infty$ for all $t > 0$, we have

$$\lim_{n\to\infty} -\frac{1}{n}\log \mathbb{P}_k\left(Z_n \leqslant e^{\theta n}\right) = I_k(\theta) \in (0,\infty), \tag{2.29}$$

where $I_k(\theta) = \Lambda^*(\theta)$ if $p_1(\xi_0) = 0$ a.s., otherwise

$$I_k(\theta) = \begin{cases} \rho_k\left(1 - \frac{\theta}{\theta_k^*}\right) + \frac{\theta}{\theta_k^*}\Lambda^*(\theta_k^*) & \text{if } 0 < \theta < \theta_k^*, \\ \Lambda^*(\theta) & \text{if } \theta_k^* \leqslant \theta < \mathbb{E}[X], \end{cases} \tag{2.30}$$



(a) $[\theta \in (\theta_k, \mathbb{E}[X])]$

(b) $[\theta = \theta_k]$

(c) $[\theta \in (0, \theta_k)]$

Figure 1: Geometrical interpretation of $\chi^*(\theta)$

with

$$\rho_k = \lim_{n\to\infty} -\frac{1}{n}\log \mathbb{P}_k(Z_n = j) \tag{2.31}$$

and θ_k^* the unique solution on $(0, \mathbb{E}[X])$ of the equation

$$\frac{\rho_k - \Lambda^*(\theta_k^*)}{\theta_k^*} = \inf_{0\leqslant\theta\leqslant\mathbb{E}[X]} \frac{\rho_k - \Lambda^*(\theta)}{\theta}. \tag{2.32}$$

It has been stated mistakenly in [8] that $\rho_k = -k\log\gamma$, whereas the correct statement is

$$\rho_k = -\log\gamma_k, \tag{2.33}$$

according to [13, Theorem 2.3] (see Lemma 3.2 below). With this correction, the two critical values θ_k and θ_k^* and the two rate functions I_k and χ_k^* coincide, that is

$$\theta_k = \theta_k^* \quad \text{and} \quad \chi_k^*(\theta) = I_k(\theta) \quad \text{for all } \ \theta \in (0, \mathbb{E}[X]). \tag{2.34}$$

Indeed, by the definition of θ_k^*, the derivative of the function $\theta \mapsto \frac{\rho_k - \Lambda^*(\theta)}{\theta}$ vanishes for $\theta = \theta_k^*$. Therefore, since $(\Lambda^*)'(\theta) = \lambda_\theta$ with $\Lambda'(\lambda_\theta) = \theta$, we get, for $\theta = \theta_k^*$,

$$\Lambda^*(\theta) = \lambda_\theta\theta + \rho_k. \tag{2.35}$$

Using the identity $\Lambda^*(\theta) = \lambda_\theta\theta - \Lambda(\lambda_\theta)$, we obtain

$$\Lambda(\lambda_\theta) = -\rho_k,$$

which implies that $\lambda_\theta = -r_k$ and then $\theta_k^* = \Lambda'(-r_k) = \theta_k$.

Moreover, coming back to (2.35) and using the identities $\Lambda(-r_k) = \log \gamma_k = -\rho_k$ and $\theta_k = \Lambda'(-r_k)$, we get

$$-r_k\theta_k - \Lambda(-r_k) = -r_k\theta_k + \rho_k = \Lambda_k^*(\theta_k).$$

Therefore, for any $\theta \in [0, \theta_k]$,

$$\begin{aligned} -r_k\theta - \log\gamma_k &= -r_k\theta - \Lambda(-r_k) \\ &= \frac{\theta}{\theta_k}\left(-r_k\theta_k - \Lambda(-r_k)\right) + \frac{\theta}{\theta_k}\Lambda(-r_k) - \Lambda(-r_k) \\ &= \frac{\theta}{\theta_k}\Lambda^*(\theta_k) - \left(1 - \frac{\theta}{\theta_k}\right)\log\gamma_k, \end{aligned}$$

so that $I_k(\theta) = \chi_k^*(\theta)$, which ends the proof of (2.34). From (2.34) and Theorem 2.1, we see that (2.29) is valid assuming only $\mathbb{P}(Z_1 = 0) = 0$ and $\mathbb{E}[m_0^\varepsilon] < \infty$ for some $\varepsilon > 0$. Actually when $\mathbb{P}(Z_1 = 0) > 0$, as shown in [7, Theorem 3.1 (i)], (2.29) remains valid with $\rho_k = \lim_{n\to\infty} -\frac{1}{n}\log \mathbb{P}_k(Z_n = j) = \rho > 0$ independent of k.

Similarly, one can apply Theorem 2.1 to get the decay rate for the probability $\mathbb{P}(Z_n \leqslant k_n)$, where k_n is any sub-exponential sequence in the sense that $k_n \to \infty$ and $k_n/\exp(\theta n) \to 0$ for every $\theta > 0$, as stated in the following corollary.

Corollary 2.5. *Assume that $\mathbb{E}[m_0^\varepsilon] < \infty$ for some $\varepsilon > 0$. Let $k_n > 0$ be such that $k_n \to \infty$ and $k_n/\exp(\theta n) \to 0$ for every $\theta > 0$, as $n \to \infty$. Then for any integer $k \geqslant 1$,*

$$\lim_{n\to\infty} \frac{1}{n}\log \mathbb{P}_k\left(Z_n \leqslant k_n\right) = \log\gamma_k. \tag{2.36}$$

It was stated mistakenly in [7, Theorem 3.1(ii)] that $\lim_{n\to\infty}\frac{1}{n}\log\mathbb{P}_k\left(Z_n \leqslant k_n\right) = k\log\gamma$. To show (2.36), it suffices to note that by Markov's inequality and Theorem 2.1, we have, for any $r > r_k$,

$$\gamma_k^n = \mathbb{P}_k(Z_n = k) \leqslant \mathbb{P}_k(Z_n \leqslant k_n) \leqslant \mathbb{E}[Z_n^{-r}]k_n^r \leqslant C\min\{\gamma_k^n k_n^r, \gamma_k^n k_n^{r_k} n\},$$

where C is a constant. The above argument leads to a precise large deviation bound as stated below.

Corollary 2.6. *Assume that $\mathbb{E}[m_0^\varepsilon] < \infty$ for some $\varepsilon > 0$. Then for any integer $k \geqslant 1$,*

$$\mathbb{P}_k\left(Z_n \leqslant e^{\theta n}\right) \leqslant \inf_{r>0}\frac{\mathbb{E}_k\left[Z_n^{-r}\right]}{e^{\theta r n}} = \begin{cases} n e^{-n(-\theta r_k - \Lambda(-r_k))} & \textit{if} \quad 0 < \theta \leqslant \theta_k, \\ n e^{-n(-\theta_k r_k - \Lambda(-r_k))} & \textit{if} \quad \theta = \theta_k, \\ e^{-n\Lambda^*(\theta)} & \textit{if} \quad \theta_k \leqslant \theta < \mathbb{E}[X]. \end{cases} \tag{2.37}$$

The question of the exact decay rate of $\mathbb{P}(Z_n \leqslant e^{\theta n})$ will be treated in a forthcoming paper.

As an example, let us consider the case where the reproduction law has a fractional linear generating function, that is when

$$p_0(\xi_0) = a_0 \quad \text{and} \quad p_k(\xi_0) = \frac{(1-a_0)(1-b_0)}{b_0}b_0^k \quad \text{for all } k \geqslant 1, \tag{2.38}$$

for which the generating function is

$$f_0(t) = a_0 + \frac{(1-a_0)(1-b_0)t}{1-b_0 t},$$

where $a_0 \in [0,1)$ and $b_0 \in (0,1)$ are random variables depending on the environment ξ_0. This case has been examined by several authors (see e.g. [17, 20]). In the case where $a_0 = 0$ (non-extinction), we have $p_1(\xi_0) = (1-b_0) > 0$, $X = \log m_0 = -\log(1-b_0)$, $\log \gamma_k = \log \mathbb{E}[e^{-kX}] = \Lambda(-k)$ and $r_k = k$. Therefore, we obtain the following explicit version of Theorem 2.1:

Corollary 2.7. *Let Z_n be a fractional linear BPRE with $a_0 = 0$. Assume that $\mathbb{E}[m_0^\varepsilon] < \infty$ for some $\varepsilon > 0$. Then for any integer $k \geqslant 1$ and $\theta \in (0, \mathbb{E}[X])$, the large deviation asymptotic* (2.26) *holds with*

$$\chi_k^*(\theta) \quad = \quad \begin{cases} -k\theta - \log \mathbb{E}[e^{-kX}] & \text{if} \quad 0 < \theta < \theta_k, \\ \qquad \Lambda^*(\theta) & \text{if} \quad \theta_k \leqslant \theta < \mathbb{E}[X], \end{cases} \tag{2.39}$$

where

$$\theta_k = \mathbb{E}[X e^{-kX}] / \mathbb{E}[e^{-kX}]. \tag{2.40}$$

Corollary 2.7 recovers and completes the large deviation result in [7, Corollary 3.3] for a fractional linear BPRE with $a_0 > 0$, which states that (2.26) holds with the rate function $\chi_k^*(\theta)$ replaced by

$$I(\theta) \quad = \quad \begin{cases} -\theta - \log \mathbb{E}[e^{-X}] & \text{if} \quad 0 < \theta < \theta^*, \\ \qquad \Lambda^*(\theta) & \text{if} \quad \theta^* \leqslant \theta < \mathbb{E}[X], \end{cases} \tag{2.41}$$

where

$$\theta^* = \mathbb{E}[X e^{-X}] / \mathbb{E}[e^{-X}]. \tag{2.42}$$

In fact the result in [7, Corollary 3.3] was stated without the hypothesis $a_0 > 0$, but when $a_0 = 0$, this result is valid only for $k = 1$, as shown by Corollary 2.7 (for $k \geqslant 2$, the factor k is missing in [7, Corollary 3.3]).

As another consequence of Theorem 2.1, we improve an earlier result about the rate of convergence in the central limit theorem for $W - W_n$. Let

$$\delta_\infty^2(\xi) = \sum_{n=0}^{\infty} \frac{1}{\Pi_n} \left(\frac{m_n(2)}{m_n^2} - 1 \right). \tag{2.43}$$

The r.v. $\delta_\infty^2(\xi)$ is the variance of W under $\mathbb{P}_\xi$ (see e.g. [15]).

Theorem 2.8. *Assume essinf $\frac{m_0(2)}{m_0^2} > 1$ and $\mathbb{E} Z_1^{2+\varepsilon} < \infty$ for some $\varepsilon \in (0,1]$. Then there exists a constant $C > 0$ such that, for all $k \geqslant 1$,*

$$\sup_{x \in \mathbb{R}} \left| \mathbb{P}_k \left(\frac{\Pi_n (W - W_n)}{\sqrt{Z_n} \delta_\infty(T^n \xi)} \leqslant x \right) - \Phi(x) \right| \leqslant C A_{k,n}(\varepsilon/2), \tag{2.44}$$

where $\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2} dt$ is the standard normal distribution function.

Theorem 2.8 improves the exponential rate of convergence in [16, Theorem 1.7]. In our approach the assumption essinf $\frac{m_0(2)}{m_0^2} > 1$ is required to ensure that the quenched variance $\delta_\infty(\xi)$ of the variable W is a.s. separated from 0. This hypothesis does not seem natural and should be relaxed. One should be able to find a suitable hypothesis to ensure the existence of harmonic moments for the random variable $\delta_\infty(\xi)$, which would be enough for our objective.

As another consequence of Theorem 2.1, we give some large deviation results on the ratio

$$R_n = \frac{Z_{n+1}}{Z_n} \tag{2.45}$$

toward the conditional mean $m_n = \sum_{k=1}^{\infty} k p_k(\xi_n)$. Let

$$M_{n,j} = j^{-1} \sum_{i=1}^{j} N_{n,i} \tag{2.46}$$

be the empirical mean of m_n of size j under the environment ξ, where the r.v.'s $N_{n,i}$ ($i = 1, \ldots, j$) are i.i.d. with generating function f_n.

Theorem 2.9. *Let $k \geqslant 1$. If, for some set $D \subset \mathbb{R}$, there exist some constants $C_1 > 0$ and $r > 0$ such that, for all $j \geqslant 1$,*

$$\mathbb{P}\left(M_{0,j} - m_0 \in D\right) \leqslant \frac{C_1}{j^r}, \tag{2.47}$$

then there exists a constant $B_1 \in (0, \infty)$ such that for all $n \geqslant 1$,

$$\mathbb{P}_k\left(R_n - m_n \in D\right) \leqslant B_1 A_{k,n}(r), \tag{2.48}$$

where $A_{k,n}(r)$ is defined in (2.20)*. Similarly, if there exist some constants $C_2 > 0$ and $r > 0$ such that, for all $j \geqslant 1$,*

$$\mathbb{P}\left(M_{0,j} - m_0 \in D\right) \geqslant \frac{C_2}{j^r}, \tag{2.49}$$

then there exists a constant $B_2 \in (0, \infty)$ such that for all $n \geqslant 1$,

$$\mathbb{P}_k\left(R_n - m_n \in D\right) \geqslant B_2 A_{k,n}(r). \tag{2.50}$$

This result shows that there exist some phase transitions in the rate of convergence depending on whether the value of r is less than, equal or greater than the constant γ_k. The next result gives a bound of the large deviation probability of $R_n - m_n$ under a simple moment condition on Z_1.

Theorem 2.10. *Let $k \geqslant 1$. Assume that there exists $p > 1$ such that $\mathbb{E}|Z_1 - m_0|^p < \infty$. Then, there exists a constant $C_p > 0$ such that, for any $a > 0$,*

$$\mathbb{P}_k(|R_n - m_n| > a) \leqslant \begin{cases} C_p a^{-p} A_{k,n}(p-1) & \text{if} \quad p \in (1,2], \\ C_p a^{-p} A_{k,n}(p/2) & \text{if} \quad p \in (2,\infty). \end{cases} \tag{2.51}$$

3 Proof of main theorems

In this section we will prove the main results of this paper, Theorems 2.1 and 2.3, and the associated result, Proposition 2.2. In Section 3.1 we present some auxiliary results concerning the critical value for the existence of the harmonic moments of the r.v. W and the asymptotic behavior of the asymptotic distribution $\mathbb{P}_k(Z_n = j)$ as $n \to \infty$, with $j \geqslant k \geqslant 1$. In Sections 3.2 and 3.3 we prove respectively Theorems 2.1 and 2.3. The proof of Proposition 2.2 is given in Section 3.4 for a Galton-Watson process and in Section 3.5 for a general BPRE.

3.1 Auxiliary results

We recall some results to be used in the proofs. The first one concerns the critical value for the existence of harmonic moments of the r.v. W.

Lemma 3.1 ([13], Theorem 2.1)**.** *Assume that there exists a constant $p > 0$ such that $\mathbb{E}\left[m_0^p\right] < \infty$. Then for any $a \in (0, p)$,*

$$\mathbb{E}_k W^{-a} < \infty \quad \textit{if and only if} \quad \mathbb{E}\left[p_1^k(\xi_0) m_0^a\right] < 1.$$

The second result is about the asymptotic equivalent of the probability $\mathbb{P}_k(Z_n = j)$ as $n \to \infty$, for any $j \geqslant k \geqslant 1$.

Lemma 3.2 ([13], Theorem 2.3)**.** *Assume that $\mathbb{P}(Z_1 = 1) > 0$. For any $k \geqslant 1$ the following assertions hold.*

a) *For any accessible state $j \geqslant k$ in the sense that $\mathbb{P}_k(Z_l = j) > 0$ for some $l \geqslant 0$, we have*

$$\mathbb{P}_k(Z_n = j) \underset{n\to\infty}{\sim} \gamma_k^n q_{k,j}, \tag{3.1}$$

where $q_{k,k} = 1$ and, for $j > k$, $q_{k,j} \in (0, +\infty)$ is the solution of the recurrence relation

$$\gamma_k q_{k,j} = \sum_{i=k}^{j} p(i,j) q_{k,i}, \tag{3.2}$$

with $p(i,j) = \mathbb{P}(Z_1 = j | Z_0 = i)$ and $q_{k,i} = 0$ for any non-accessible state i (i.e. $\mathbb{P}_k(Z_l = i) = 0$ for all $l \geqslant 0$).

b) *Assume that there exists $\varepsilon > 0$ such that $\mathbb{E}[m_0^\varepsilon] < \infty$. Then, for any $r > r_k$, we have*

$$\sum_{j=k}^{\infty} j^{-r} q_{k,j} < \infty. \tag{3.3}$$

In particular, the radius of convergence of the power series

$$Q_k(t) = \sum_{j=k}^{+\infty} q_{k,j} t^j \tag{3.4}$$

is equal to 1.

c) *For all $t \in [0,1)$ and $k \geqslant 1$, we have,*

$$\frac{G_{k,n}(t)}{\gamma_k^n} \uparrow Q_k(t) \ \text{ as } \ n \to \infty, \tag{3.5}$$

where $G_{k,n}$ is the probability generating function of Z_n when $Z_0 = k$, defined in (2.4)*.*

d) *$Q_k(t)$ is the unique power series which verifies the functional equation*

$$\gamma_k Q_k(t) = \mathbb{E}\left[Q_k(f_0(t))\right], \quad t \in [0,1), \tag{3.6}$$

with the condition $Q_k^{(k)}(0) = 1$.

3.2 Proof of Theorem 2.1

In this section we give a proof of the convergence of the normalized harmonic moments $\mathbb{E}_k[Z_n^{-r}]/A_{k,n}(r)$ as $n \to \infty$, where $A_{k,n}(r)$ is defined in (2.20). For any $r > 0$, set

$$c_r = \mathbb{E} m_0^{-r}. \tag{3.7}$$

a) We first consider the case when $r < r_k$ (which corresponds to the case $\gamma_k < c_r$). By the change of measure (2.15), we obtain

$$\mathbb{E}_k\left[Z_n^{-r}\right] = \mathbb{E}_k^{(r)}[W_n^{-r}] c_r^n. \tag{3.8}$$

From (2.16) and [16, Lemma 2.1], it follows that the sequence $(\mathbb{E}_k^{(r)}[W_n^{-r}])$ is increasing and

$$\lim_{n\to\infty} \mathbb{E}_k^{(r)}[W_n^{-r}] = \sup_{n\in\mathbb{N}} \mathbb{E}_k^{(r)}[W_n^{-r}] = \mathbb{E}_k^{(r)}[W^{-r}]. \tag{3.9}$$

To prove that $\mathbb{E}_k^{(r)}[W^{-r}] < \infty$, we need to verify the conditions of Lemma 3.1 under the measure $\mathbb{E}_k^{(r)}$: by assumption, we have $\mathbb{E}^{(r)}[m_0^{r+\varepsilon}] = \mathbb{E}[m_0^{\varepsilon}]/\mathbb{E}[m_0^{-r}] < \infty$. Moreover, for any $r < r_k$, we have $\gamma_k < c_r$, which implies that $\mathbb{E}^{(r)}[p_1^k(\xi_0)m_0^r] = \gamma_k/\mathbb{E}m_0^{-r} < 1$. So, by Lemma 3.1, we get, for any $r < r_k$,

$$\mathbb{E}_k^{(r)}[W^{-r}] < \infty. \tag{3.10}$$

Therefore, coming back to (3.8) and using (3.9) and (3.10), we obtain

$$\mathbb{E}_k\left[Z_n^{-r}\right] c_r^{-n} \underset{n\to\infty}{\uparrow} \mathbb{E}_k^{(r)}\left[W^{-r}\right] \in (0,\infty). \tag{3.11}$$

To give an integral expression of the limit constant $C(k,r)$, we shall use the following expression for the inverse of a positive random variable X^r: for any $r > 0$, we have

$$\frac{1}{X^r} = \frac{1}{\Gamma(r)}\int_0^{+\infty} e^{-uX}u^{r-1}du. \tag{3.12}$$

Then, from (3.11), (3.12) and Fubini's theorem, we get

$$\mathbb{E}_k\left[Z_n^{-r}\right] c_r^{-n} \underset{n\to\infty}{\uparrow} \frac{1}{\Gamma(r)}\int_0^{\infty} \phi_k^{(r)}(u)u^{r-1}du, \tag{3.13}$$

which proves (2.21) for $r < r_k$.

b) Next we consider the case when $r > r_k$ (which corresponds to the case $\gamma_k > c_r$). Using parts a) and b) of Lemma 3.2 and the monotone convergence theorem, it follows that

$$\begin{aligned}\lim_{n\to\infty} \uparrow \frac{\mathbb{E}_k[Z_n^{-r}]}{\gamma_k^n} &= \lim_{n\to\infty} \uparrow \sum_{j=k+1}^{\infty} k^{-r}\frac{\mathbb{P}(Z_n=k)}{\gamma_k^n} \\ &= \sum_{j=k+1}^{\infty} k^{-r}q_{k,j} < \infty. \end{aligned}\tag{3.14}$$

We now give an integral expression of the limit constant in (3.14). Using the definition of Q_k in (2.18), Fubini's theorem and (3.12), we obtain

$$\frac{1}{\Gamma(r)}\int_0^1 Q_k(e^{-u})u^{r-1}du = \sum_{j=k}^{\infty} q_{k,j}\frac{1}{\Gamma(r)}\int_0^{\infty} e^{-uj}u^{r-1}du = \sum_{j=k}^{\infty} q_{k,j}j^{-r}.$$

Therefore, coming back to (3.14), we get

$$\lim_{n\to\infty} \uparrow \frac{\mathbb{E}_k[Z_n^{-r}]}{\gamma_k^n} = \frac{1}{\Gamma(r)}\int_0^1 Q_k(e^{-u})u^{r-1}du \in (0,\infty), \tag{3.15}$$

which proves (2.21) for $r > r_k$.

c) Now we consider the case when $r = r_k$ (which corresponds to $c_r = \gamma_k$). For $n \geqslant 1$ and $m \geqslant 0$, we have the following well-known branching property for Z_n:

$$Z_{n+m} = \sum_{i=1}^{Z_m} Z_{n,i}^{(m)}, \tag{3.16}$$

where the r.v.'s $Z_{n,i}^{(m)}$ $(i \geqslant 1)$ are independent of Z_m under $\mathbb{P}_\xi$ and $\mathbb{P}$. Moreover, under $\mathbb{P}_\xi$, for each $n \geqslant 0$, the r.v.'s $Z_{n,i}^{(m)}$ $(i \geqslant 1)$ are i.i.d. with the same conditional probability law $\mathbb{P}_\xi\left(Z_{n,i}^{(m)} \in \cdot\right) = \mathbb{P}_{T^m\xi}\left(Z_n \in \cdot\right)$, where T^m is the shift operator defined by $T^m(\xi_0,\xi_1,\ldots) =$

$(\xi_m, \xi_{m+1}, \ldots)$. Intuitively, relation (3.16) shows that, conditionally on $Z_m = i$, the annealed law of the process $\{Z_{n+m} : n \geqslant 0\}$ is the same as that of a process $\{Z_n : n \geqslant 0\}$ starting with i individuals. Using (3.16) with $m = 1$, we obtain

$$\mathbb{E}_k\left[Z_{n+1}^{-r}\right] = \mathbb{E}_k\left[Z_n^{-r}\right]\mathbb{P}_k(Z_1 = k) + \sum_{i=k+1}^{\infty} \mathbb{E}_i\left[Z_n^{-r}\right]\mathbb{P}_k(Z_1 = i). \tag{3.17}$$

From (3.8), we have $\mathbb{E}_i\left[Z_n^{-r}\right] = \mathbb{E}_i^{(r)}[W_n^{-r}]c_r^n$. Substituting this into (3.17) and setting

$$b_n = \sum_{i=k+1}^{\infty} \mathbb{E}_i^{(r)}[W_n^{-r}]\mathbb{P}_k(Z_1 = i),$$

we get

$$\mathbb{E}_k[Z_{n+1}^{-r}] = \mathbb{E}_k[Z_n^{-r}]\gamma_k + b_n c_r^n, \tag{3.18}$$

with $\gamma_k = \mathbb{P}_k(Z_1 = k)$. Iterating (3.18) leads to

$$\mathbb{E}_k\left[Z_{n+1}^{-r}\right] = \gamma_k^{n+1}k^{-r} + \sum_{j=0}^{n}\gamma_k^{n-j}b_j c_r^j. \tag{3.19}$$

Using the fact that $r = r_k$ (which corresponds to $\gamma_k = c_r$) and dividing (3.19) by $\gamma_k^{n+1}n$, we get

$$\frac{\mathbb{E}_k\left[Z_{n+1}^{-r}\right]}{n\gamma_k^{n+1}} = \frac{k^{-r}}{n} + \frac{\gamma_k^{-1}}{n}\sum_{j=0}^{n} b_j. \tag{3.20}$$

To prove the convergence in (3.20) we need to show that $\lim_{n\to\infty} b_n < \infty$. By (3.9) and the monotone convergence theorem, we have

$$b := \lim_{n\to\infty} \uparrow b_n = \sum_{i=k+1}^{\infty} \mathbb{E}_i^{(r)}[W^{-r}]\mathbb{P}_k(Z_1 = i). \tag{3.21}$$

Now we show that $b < \infty$. Using (3.16) for $m = 0$ and $Z_0 = i$, with $i > k$, and the fact that $Z_{n,j} > 0$ for all $1 \leqslant j \leqslant i$, we obtain

$$\mathbb{E}_i[Z_n^{-r}] = \mathbb{E}\left[(Z_{n,1} + \ldots + Z_{n,i})^{-r}\right] \leqslant \mathbb{E}\left[(Z_{n,1} + \ldots + Z_{n,k})^{-r}\right] = \mathbb{E}_k[Z_n^{-r}]. \tag{3.22}$$

By (3.22) and the change of measure (3.8), we get, for any $i \geqslant k+1$,

$$\mathbb{E}_i^{(r)}[W_n^{-r}] \leqslant \mathbb{E}_{k+1}^{(r)}[W_n^{-r}].$$

Then, as in (3.9), letting $n \to \infty$ leads to

$$\mathbb{E}_i^{(r)}[W^{-r}] \leqslant \mathbb{E}_{k+1}^{(r)}[W^{-r}]. \tag{3.23}$$

Now we shall prove that

$$\mathbb{E}_{k+1}^{(r)}[W^{-r}] < \infty. \tag{3.24}$$

For this it is enough to verify the condition of Lemma 3.1 under the measure $\mathbb{E}_k^{(r)}$ defined by (2.15): by assumption, we have $\mathbb{E}^{(r)}[m_0^{r+\varepsilon}] = \mathbb{E}[m_0^{\varepsilon}]/\mathbb{E}[m_0^{-r}] < \infty$. Moreover, since for $r = r_k$ we have $\gamma_k = c_r$, it implies that

$$\mathbb{E}^{(r)}[p_1^{k+1}(\xi_0)m_0^r] = \frac{\gamma_{k+1}}{\mathbb{E}m_0^{-r}} = \frac{\gamma_{k+1}}{\gamma_k} < 1.$$

This proves (3.24). Using (3.23) and (3.24), we obtain

$$b = \sum_{i=k+1}^{\infty} \mathbb{E}_i^{(r)}[W^{-r}]\mathbb{P}_k(Z_1 = i) \leqslant \mathbb{E}_{k+1}^{(r)}[W^{-r}] \sum_{j=k+1}^{\infty} \mathbb{P}_k(Z_1 = j) < \infty.$$

Therefore, coming back to (3.20), using (3.21) and Cesaro's lemma, we get

$$\lim_{n\to\infty} \frac{\mathbb{E}_k\left[Z_{n+1}^{-r}\right]}{n\gamma_k^{n+1}} = \frac{1}{\gamma_k} \sum_{i=k+1}^{\infty} \mathbb{E}_i^{(r)}[W^{-r}]\mathbb{P}_k(Z_1 = i) < \infty, \tag{3.25}$$

which proves (2.21) for $r = r_k$, with

$$C(k,r) = \frac{1}{\gamma_k} \sum_{i=k+1}^{\infty} \mathbb{E}_i^{(r)}[W^{-r}]\mathbb{P}_k(Z_1 = i).$$

We now show an integral expression of the constant $C(k,r)$. Using the branching property (3.16) with $m = 0$, $Z_0 = i$, dividing by Π_n and taking the limit $n \to \infty$ leads to the decomposition

$$W = \sum_{j=1}^{i} W(j), \tag{3.26}$$

where conditionally on the environment ξ, the r.v.'s $W(j)$ $(j = 1, 2, \ldots, i)$ are i.i.d. with common law $\mathbb{P}_\xi(W(j) \in \cdot) = \mathbb{P}_\xi(W \in \cdot)$. With these considerations, it can be easily seen that

$$\phi_i^{(r)}(t) = \mathbb{E}_i^{(r)}[\phi_\xi(u)] = \mathbb{E}^{(r)}[\phi_\xi(u)^i]. \tag{3.27}$$

Therefore, using (3.12) with $r = r_k$, together with (3.27) and Fubini's theorem, we obtain

$$\begin{aligned}
\frac{1}{\gamma_k} \sum_{i=k+1}^{\infty} \mathbb{E}_i^{(r)}[W^{-r}]\mathbb{P}_k(Z_1 = i) &= \frac{1}{\gamma_k\Gamma(r)} \sum_{i=k+1}^{\infty} \mathbb{E}_i^{(r)} \int_0^\infty e^{-uW} u^{r-1} du \ \mathbb{P}_k(Z_1 = i) \\
&= \frac{1}{\gamma_k\Gamma(r)} \sum_{i=k+1}^{\infty} \mathbb{E}^{(r)} \int_0^\infty \phi_\xi^i(u) u^{r-1} du \ \mathbb{P}_k(Z_1 = i) \\
&= \frac{1}{\gamma_k\Gamma(r)} \mathbb{E}^{(r)} \int_0^\infty \sum_{i=k+1}^{\infty} \phi_\xi^i(u) \ \mathbb{P}_k(Z_1 = i) u^{r-1} du \\
&= \frac{1}{\gamma_k\Gamma(r)} \mathbb{E}^{(r)} \left[\int_0^\infty \bar{G}_{k,1}(\phi_\xi(u)) u^{r-1} du\right], \qquad (3.28)
\end{aligned}$$

where $\bar{G}_{k,1}(u) = G_{k,1}(u) - \gamma_k u^k = \sum_{j=k+1}^{\infty} u^j \mathbb{P}(Z_1 = j)$. Therefore, using (3.25) and (3.28), we get

$$\lim_{n\to\infty} \frac{\mathbb{E}_k\left[Z_{n+1}^{-r}\right]}{n\gamma_k^{n+1}} = \frac{1}{\gamma_k\Gamma(r)} \mathbb{E}^{(r)} \left[\int_0^\infty \bar{G}_{k,1}(\phi_\xi(u)) u^{r-1} du\right],$$

which ends the proof of Theorem 2.1.

3.3 Proof of Theorem 2.3

In this section we prove Theorem 2.3. For convenience, let $\lambda_k = -r_k$. From Theorem 2.1, for any $k \geqslant 1$, we have

$$\lim_{n\to\infty} \frac{1}{n} \log \mathbb{E}_k[Z_n^\lambda] = \chi_k(\lambda) = \begin{cases} \log \gamma_k & \text{if } \lambda \leqslant \lambda_k, \\ \Lambda(\lambda) & \text{if } \lambda \in (\lambda_k, 0]. \end{cases} \tag{3.29}$$

Thus using a version of the Gärtner-Ellis theorem adapted to the study of tail probabilities (see [19, Theorem 6.1]) and the fact that $\chi_k(\lambda) = \log \gamma_k$ for all $\lambda \leqslant \lambda_k$, we obtain, for all $\theta \in (0, \mathbb{E}[X])$,

$$\lim_{n\to\infty} -\frac{1}{n}\log \mathbb{P}_k(Z_n \leqslant e^{\theta n}) = \chi^*(\theta), \tag{3.30}$$

with

$$\chi_k^*(\theta) = \sup_{\lambda \leqslant 0}\{\lambda\theta - \chi_k(\lambda)\} = \max\left\{\lambda_k\theta - \Lambda(\lambda_k), \sup_{\lambda_k \leqslant \lambda \leqslant 0}\{\lambda\theta - \Lambda(\lambda)\}\right\}. \tag{3.31}$$

It is well-known (see e.g. [12, Lemma 2.2.5]) that the function

$$\Lambda^*(\theta) = \sup_{\lambda \leqslant 0}\{\lambda\theta - \Lambda(\lambda)\} = \{\lambda_\theta\theta - \Lambda(\lambda_\theta)\}, \quad \text{with} \quad \Lambda'(\lambda_\theta) = \theta,$$

is non-increasing for $\theta \in (0, \mathbb{E}[X])$. Therefore, letting

$$\theta_k = \Lambda'(\lambda_k), \tag{3.32}$$

it follows that:

1. for any $\theta \in (0, \theta_k]$,

$$\lambda_k\theta - \Lambda(\lambda_k) \geqslant \lambda_k\theta_k - \Lambda(\lambda_k) = \Lambda^*(\theta_k) = \sup_{\lambda_k \leqslant \lambda \leqslant 0}\{\lambda\theta - \Lambda(\lambda)\};$$

2. for any $\theta \in [\theta_k, \mu)$,

$$\Lambda^*(\theta) = \sup_{\lambda_k \leqslant \lambda \leqslant 0}\{\lambda\theta - \Lambda(\lambda)\} \geqslant \Lambda^*(\theta_k) = \lambda_k\theta_k - \Lambda(\lambda_k) \geqslant \lambda_k\theta - \Lambda(\lambda_k).$$

With these considerations, we get from (3.31) that

$$\chi_k^*(\theta) = \begin{cases} \lambda_k\theta - \Lambda(\lambda_k) & \text{if } \theta \in (0, \theta_k], \\ \Lambda^*(\theta) & \text{if } \theta \in (\theta_k, \mathbb{E}[X]), \end{cases} \tag{3.33}$$

which ends the proof of Theorem 2.3.

3.4 Proof of Proposition 2.2 for the Galton-Watson case

In this section we assume that (Z_n) is a Galton-Watson process and prove (1.5), which is a particular but simpler case of Proposition 2.2.

Proof. First note that for the Galton-Watson case, we have

$$\gamma m_1^r = 1. \tag{3.34}$$

For convenience, we shall write $r = r_1$. Using the additive property of integration and the change of variable $u = tm^k$ for $k \geqslant 0$, together with Fubini's theorem and the fact that $\gamma m^r = 1$, we obtain

$$\begin{aligned} \frac{1}{\gamma}\int_1^\infty \bar{G}(\phi(u))u^{r-1}du &= \frac{1}{\gamma}\sum_{k=0}^\infty \int_{m^k}^{m^{k+1}} \bar{G}(\phi(u))u^{r-1}du \\ &= \frac{1}{\gamma}\sum_{k=0}^\infty \int_1^m \bar{G}(\phi(tm^k))(m^r)^k t^{r-1}dt \\ &= \frac{1}{\gamma}\int_1^m \sum_{k=0}^\infty \gamma^{-k}\bar{G}(\phi(tm^k))t^{r-1}dt. \end{aligned} \tag{3.35}$$

Since $\bar{G}(t) = G(t) - \gamma t$ and $G(\phi(t)) = \phi(tm)$ (as a particular case of (2.10)), we obtain, for any $k \geqslant 0$,

$$\begin{aligned}\gamma^{-k}\bar{G}(\phi(tm^k)) &= \gamma^{-k}G(\phi(tm^k)) - \gamma^{-k}\gamma\phi(tm^k) \\ &= \gamma^{-k}G^{\circ k+1}(\phi(t)) - \gamma^{-(k-1)}G^{\circ k}(\phi(t)). \end{aligned} \tag{3.36}$$

By (3.36), using a telescoping argument and the fact that $\lim_{k\to\infty}\gamma^{-k}G^{\circ k}(t) = Q(t)$, we get

$$\sum_{k=0}^{\infty}\gamma^{-k}\bar{G}(\phi(tm^k)) = \gamma Q(\phi(t)) - \gamma\phi(t). \tag{3.37}$$

Therefore, coming back to (3.35) and using (3.37), we have

$$\frac{1}{\gamma}\int_1^{\infty}\bar{G}(\phi(u))u^{r-1}du = \int_1^m Q(\phi(u))u^{r-1}du - \int_1^m \phi(u)u^{r-1}du. \tag{3.38}$$

Moreover, using the change of variable $u = t/m$ and the relations $G(\phi(t/m)) = \phi(t)$ and $\gamma m^r = 1$, we get

$$\frac{1}{\gamma}\int_0^1 G(\phi(u))u^{r-1}du = \frac{m^{-r}}{\gamma}\int_0^m \phi(t)t^{r-1}dt = \int_0^m \phi(t)t^{r-1}dt.$$

Therefore, since $\bar{G}(u) = G(u) - \gamma u$, we obtain

$$\begin{aligned}\frac{1}{\gamma}\int_0^1 \bar{G}(\phi(u))u^{r-1}du &= \frac{1}{\gamma}\int_0^1 G(\phi(u))u^{r-1}du - \int_0^1 \phi(u)u^{r-1}du \\ &= \int_0^m \phi(u)u^{r-1}du - \int_0^1 \phi(u)u^{r-1}du \\ &= \int_1^m \phi(u)u^{r-1}du. \end{aligned} \tag{3.39}$$

Finally, using (3.38), (3.39) and the additive property of integration, we obtain

$$\frac{1}{\gamma}\int_0^{\infty}\bar{G}(\phi(u))u^{r-1}du = \int_1^m Q(\phi(u))u^{r-1}du, \tag{3.40}$$

which ends the proof of (1.5). ☐

3.5 Proof of Proposition 2.2

Let $k \geqslant 1$. For convenience, let $r = r_k$. Using the additive property of integration, the change of variable $u = t\Pi_j^k$ for $j \geqslant 0$ and Fubini's theorem, we have

$$\begin{aligned}\frac{1}{\gamma_k}\mathbb{E}^{(r)}\left[\int_1^{\infty}\bar{G}_{k,1}(\phi_\xi(u))u^{r-1}du\right] &= \frac{1}{\gamma_k}\mathbb{E}^{(r)}\left[\sum_{j=0}^{\infty}\int_{\Pi_j}^{\Pi_{j+1}}\bar{G}_{k,1}(\phi_\xi(u))u^{r-1}du\right] \\ &= \frac{1}{\gamma_k}\mathbb{E}^{(r)}\left[\sum_{j=0}^{\infty}\int_1^{m_j}\bar{G}_{k,1}(\phi_\xi(t\Pi_j))\Pi_j^r t^{r-1}dt\right] \\ &= \frac{1}{\gamma_k}\sum_{j=0}^{\infty}\mathbb{E}^{(r)}\left[\int_1^{m_j}\bar{G}_{k,1}(\phi_\xi(t\Pi_j))\Pi_j^r t^{r-1}dt\right]. \end{aligned} \tag{3.41}$$

Using the functional equation (2.10), it can be easily seen by induction that $\phi_\xi(t\Pi_j) = g_j(\phi_{T^j\xi}(t))$, where $g_j(t) = f_0 \circ \ldots \circ f_{j-1}(t)$ is a random function depending on the environment $\xi_0, \ldots, \xi_{j-1}$ and $T^j\xi = (\xi_j, \xi_{j+1}, \ldots)$. Then, using the change of measure (2.15), the independence of the environment sequence (ξ_i) and Fubini's theorem, we see that

$$\mathbb{E}^{(r)}\left[\int_1^{m_j} \bar{G}_{k,1}(\phi_\xi(t\Pi_j))\Pi_j^r t^{r-1}dt\right] = \mathbb{E}^{(r)}\left[\int_1^{m_j} c_r^{-j}\mathbb{E}_{T^j\xi}\left[\bar{G}_{k,1}(g_j(\phi_{T^j\xi}(t))\right] t^{r-1}dt\right]. \tag{3.42}$$

Using the fact that $\bar{G}_{k,1}(t) = G_{k,1}(t) - \gamma_k t^k$ and the relations $\mathbb{E}[G_{k,n}(g_j(t))] = G_{k,n+j}(t)$ and $\mathbb{E}[g_j^k(t)] = G_{k,j}(t)$, we get

$$\begin{aligned} &\mathbb{E}^{(r)}\left[\int_1^{m_j} c_r^{-j}\mathbb{E}_{T^j\xi}\left[\bar{G}_{k,1}(g_j(\phi_{T^j\xi}(t))\right] t^{r-1}dt\right] \\ =\ &\mathbb{E}^{(r)}\left[\int_1^{m_j} c_r^{-j}\left[G_{k,j+1}(\phi_{T^j\xi}(t)) - G_{k,j}(\phi_{T^j\xi}(t))\right] t^{r-1}dt\right]. \end{aligned} \tag{3.43}$$

Moreover, since the environment sequence $(\xi_0, \xi_1, \ldots)$ is i.i.d., we obtain, for any $j \geqslant 0$,

$$\begin{aligned} &\mathbb{E}^{(r)}\left[\int_1^{m_j} c_r^{-j}\left[G_{k,j+1}(\phi_{T^j\xi}(t)) - G_{k,j}(\phi_{T^j\xi}(t))\right] t^{r-1}dt\right] \\ =\ &\mathbb{E}^{(r)}\left[\int_1^{m_0} c_r^{-j}\left[G_{k,j+1}(\phi_\xi(t)) - G_{k,j}(\phi_\xi(t))\right] t^{r-1}dt\right]. \end{aligned} \tag{3.44}$$

Therefore, using equations (3.41) to (3.44), the fact that $c_r = \gamma_k$ (for $r = r_k$) and Fubini's theorem, we obtain

$$\frac{1}{\gamma_k}\mathbb{E}^{(r)}\left[\int_1^\infty \bar{G}_{k,1}(\phi_\xi(u))u^{r-1}du\right] = \mathbb{E}^{(r)}\left[\int_1^{m_0}\sum_{j=0}^\infty\left[\frac{G_{k,j+1}(\phi_\xi(t))}{\gamma_k^{j+1}} - \frac{G_{k,j}(\phi_\xi(t))}{\gamma_k^j}\right] t^{r-1}dt\right]. \tag{3.45}$$

Assuming $\mathbb{E}[m_0^\varepsilon] < \infty$, it follows from Lemma 3.2 that $\lim_{j\to\infty}\gamma_k^j G_{k,j}(t) = Q_k(t) \in (0,\infty)$ for all $t \in [0,1)$. Then by a telescoping argument, we get

$$\begin{aligned} \sum_{j=0}^\infty\left[\frac{G_{k,j+1}(\phi_\xi(t))}{\gamma_k^{j+1}} - \frac{G_{k,j}(\phi_\xi(t))}{\gamma_k^j}\right] &= Q_k(\phi_\xi(t)) - G_{k,0}(\phi_\xi(t)) \\ &= Q_k(\phi_\xi(t)) - \phi_\xi^k(t). \end{aligned} \tag{3.46}$$

Therefore, by (3.45) and (3.46), we have

$$\begin{aligned} &\frac{1}{\gamma_k}\mathbb{E}^{(r)}\left[\int_1^\infty \bar{G}_{k,1}(\phi_\xi(u))u^{r-1}du\right] \\ =\ &\mathbb{E}^{(r)}\left[\int_1^{m_0} Q_k(\phi_\xi(t))t^{r-1}dt\right] - \mathbb{E}^{(r)}\left[\int_1^{m_0}\phi_\xi^k(t)t^{r-1}dt\right]. \end{aligned} \tag{3.47}$$

Moreover, using the identity $\phi_\xi(u) = f_0\left(\phi_{T\xi}(t/m_0)\right)$, the change of variable $t = u/m_0$, the independence between ξ_0 and $T\xi$, the relation $\gamma_k = c_r$ and Fubini's theorem, we get

$$\begin{aligned} \mathbb{E}^{(r)}\left[\int_0^{m_0}\phi_\xi^k(t)t^{r-1}dt\right] &= \mathbb{E}^{(r)}\left[\int_0^1 f_0^k\left(\phi_{T\xi}(u)\right)m_0^r u^{r-1}du\right] \\ &= \mathbb{E}^{(r)}\left[\int_0^1 \mathbb{E}_{T\xi}^{(r)}\left[f_0^k\left(\phi_{T\xi}(u)\right)m_0^r\right]u^{r-1}du\right] \\ &= \mathbb{E}^{(r)}\left[\int_0^1 c_r^{-1}G_{k,1}\left(\phi_{T\xi}(u)\right)u^{r-1}du\right] \\ &= \gamma_k^{-1}\mathbb{E}^{(r)}\left[\int_0^1 G_{k,1}\left(\phi_\xi(u)\right)u^{r-1}du\right]. \end{aligned} \tag{3.48}$$

Therefore, from the identity $\bar{G}_{k,1}(t) = G_{k,1}(t) - \gamma_k t^k$ and (3.48), it follows that

$$\begin{aligned}
\frac{1}{\gamma_k}\mathbb{E}^{(r)}\left[\int_0^1 \bar{G}_{k,1}(\phi_\xi(u))u^{r-1}du\right] &= \frac{1}{\gamma_k}\mathbb{E}^{(r)}\left[\int_0^1 G_{k,1}(\phi_\xi(t))t^{r-1}dt\right] - \mathbb{E}^{(r)}\left[\int_0^1 \phi_\xi^k(t)t^{r-1}dt\right] \\
&= \mathbb{E}^{(r)}\left[\int_0^{m_0} \phi_\xi^k(t)t^{r-1}dt\right] - \mathbb{E}^{(r)}\left[\int_0^1 \phi_\xi^k(t)t^{r-1}dt\right] \\
&= \mathbb{E}^{(r)}\left[\int_1^{m_0} \phi_\xi^k(t)t^{r-1}dt\right]. \qquad (3.49)
\end{aligned}$$

Finally, using (3.47) and (3.49), we obtain

$$\frac{1}{\gamma_k}\mathbb{E}^{(r)}\left[\int_0^\infty \bar{G}_{k,1}(\phi_\xi(u))u^{r-1}du\right] = \mathbb{E}^{(r)}\left[\int_1^{m_0} Q_k(\phi_\xi(u))u^{r-1}du\right], \qquad (3.50)$$

which ends the proof of Proposition 2.2.

4 Applications

In this section we present the proofs of Theorems 2.8, 2.9 and 2.10 as applications of Theorem 2.1. In Section 4.1 we give the rate of convergence in the central limit theorem for $W - W_n$ where we prove Theorem 2.8. In Section 4.2 we deal with the large deviation results for the ratio $R_n = Z_{n+1}/Z_n$, where we prove Theorems 2.9 and 2.10.

4.1 Central Limit Theorem for $W - W_n$

In this section we prove Theorem 2.8.

Proof of Theorem 2.8. It is well known that W admits the following decomposition:

$$\Pi_n(W - W_n) = \sum_{i=1}^{Z_n} (W(i) - 1),$$

where under $\mathbb{P}_\xi$, the random variables $W(i)$ $(i \geqslant 1)$ are independent of each other and independent of Z_n, with common distribution $\mathbb{P}_\xi(W(i) \in \cdot) = \mathbb{P}_{T^n\xi}(W \in \cdot)$. Notice that if $c_0 := \operatorname{essinf} \frac{m_0(2)}{m_0^2} > 1$, then $\delta_\infty^2(\xi) \geqslant c_0 - 1 > 0$ (recall that $m_0(2)$ and $\delta_\infty^2(\xi)$ are defined respectively in (2.5) and (2.43)). Therefore, condition $\mathbb{E}Z_1^{2+\varepsilon} < \infty$ implies that, for all $k \geqslant 1$, it holds $\mathbb{E}_k\left|\frac{W-1}{\delta_\infty}\right|^{2+\varepsilon} \leqslant \frac{C}{c_0-1}\mathbb{E}_k|W-1|^{2+\varepsilon} < \infty$ (see [14]). By the Berry-Esseen theorem (see [11, Theorem 9.1.3]), we have for all $x \in \mathbb{R}$,

$$\left|\mathbb{P}_\xi\left(\frac{\Pi_n(W-W_n)}{\sqrt{Z_n}\delta_\infty(T^n\xi)} \leqslant x\right) - \Phi(x)\right| \leqslant C\mathbb{E}_{T^n\xi}\left|\frac{W-1}{\delta_\infty}\right|^{2+\varepsilon}\mathbb{E}_\xi\left[Z_n^{-\varepsilon/2}\right].$$

Taking expectation with $Z_0 = k$ and using Theorem 2.1, we get

$$\begin{aligned}
\left|\mathbb{P}_k\left(\frac{\Pi_n(W-W_n)}{\sqrt{Z_n}\delta_\infty(T^n\xi)} \leqslant x\right) - \Phi(x)\right| &\leqslant C\,\mathbb{E}_k\left|\frac{W-1}{\delta_\infty}\right|^{2+\varepsilon}\mathbb{E}_k\left[Z_n^{-\varepsilon/2}\right] \\
&\leqslant CA_{k,n}(-\varepsilon/2). \qquad (4.1)
\end{aligned}$$

□

4.2 Large deviation rate for R_n

This section is devoted to the proof of Theorems 2.9 and 2.10. Recall that $M_{n,j}$ is defined by (2.46), where $N_{n,i}$ are i.i.d. with generating function f_n, given the environment ξ (see Section 2).

Proof of Theorem 2.9. Since for all $n \in \mathbb{N}$, $M_{n,j} - m_n$ has the same law as $M_{0,j} - m_0$, and is independent of Z_n, we obtain

$$\begin{aligned}
\mathbb{P}_k(R_n - m_n \in D) &= \sum_{j \geqslant k} \mathbb{P}(M_{n,j} - m_n \in D)\mathbb{P}_k(Z_n = j) \\
&\leqslant \sum_{j \geqslant k} \frac{C_1}{j^r} \mathbb{P}_k(Z_n = j) \\
&= C_1 \mathbb{E}_k \left[Z_n^{-r} \right].
\end{aligned}$$

The result (2.47) follows from Theorem 2.1, and (2.49) follows similarly. □

Proof of Theorem 2.10. We start with a lemma which is a direct consequence of the Marcinkiewicz-Zygmund inequality (see [11, p. 356]).

Lemma 4.1 ([18], Lemma 1.4)**.** *Let $(X_i)_{i \geqslant 1}$ be a sequence of i.i.d. centered r.v.'s. Then we have for $p \in (1, \infty)$,*

$$\mathbb{E} \left| \sum_{i=1}^{n} X_i \right|^p \leqslant \begin{cases} (B_p)^p \mathbb{E}\left(|X_i|^p\right) n, & \text{if } 1 < p \leqslant 2, \\ (B_p)^p \mathbb{E}\left(|X_i|^p\right) n^{p/2}, & \text{if } p > 2, \end{cases} \tag{4.2}$$

where $B_p = 2 \min\left\{k^{1/2} : k \in \mathbb{N}, k \geqslant p/2\right\}$ is a constant depending only on p (so that $B_p = 2$ if $1 < p \leqslant 2$).

We shall prove Theorem 2.10 in the case when $p \in (1, 2]$. Using the fact that $M_{n,j} - m_n$ has the same law as $M_{0,j} - m_0$ and is independent of Z_n, we obtain after conditioning

$$\mathbb{P}_k(|R_n - m_n| > a) = \sum_{j=k}^{\infty} \mathbb{P}(|M_{0,j} - m_0| > a)\mathbb{P}_k(Z_n = j). \tag{4.3}$$

Using (2.46) and the fact that, under $\mathbb{P}_\xi$, the r.v.'s $N_{0,i} - m_0$ $(i = 1, \dots, j)$ are i.i.d. centered and with generating function f_0, we get from Lemma 4.1 that, for $p \in (1, 2]$,

$$\begin{aligned}
\mathbb{P}_\xi(|M_{0,j} - m_0| > a) &\leqslant a^{-p} \mathbb{E}_\xi \left| M_{0,j} - m_0 \right|^p \\
&\leqslant \left(\frac{B_p}{a} \right)^p j^{1-p} \mathbb{E}_\xi |Z_1 - m_0|^p.
\end{aligned}$$

Taking expectation, we obtain

$$\mathbb{P}(|M_{n,j} - m_n| > a) \leqslant \left(\frac{B_p}{a} \right)^p j^{1-p} \mathbb{E}|Z_1 - m_0|^p.$$

Therefore, coming back to (4.3) and applying Theorem 2.1, we get

$$\begin{aligned}
\mathbb{P}_k(|R_n - m_n| > a) &\leqslant \left(\frac{B_p}{a} \right)^p \mathbb{E}|Z_1 - m_0|^p \sum_{j=k}^{\infty} j^{1-p} \mathbb{P}_k(Z_n = j) \\
&= \left(\frac{B_p}{a} \right)^p \mathbb{E}|Z_1 - m_0|^p \mathbb{E}_k \left[Z_n^{1-p} \right] \\
&= C_p a^{-p} A_{k,n}(p-1),
\end{aligned}$$

with $C_p = B_p \mathbb{E}|Z_1 - m_0|^p$. This ends the proof of Theorem 2.10 in the case when $p \in (1, 2]$. The proof in the case $p > 2$ is obtained in the same way. □

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Acknowledgments. The authors would like to thanks an anonymous referee for a carefull reading of the paper and valuable comments that led to an improvement of the exposition. The work has benefited from the support of the Centre Henri Lebesgue (CHL, ANR-11-LABX-0020-01). Q. Liu has been partially supported by the National Natural Science Foundation of China (Grants no. 11571052, no. 11401590 and no. 11731012), and by Hunan Natural Science Foundation (China, grant no. 2017JJ2271).