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Regularity of stochastic kinetic equations

Ennio Fedrizzi * Franco Flandoli† Enrico Priola ‡ Julien Vovelle §

Abstract

We consider regularity properties of stochastic kinetic equations with multiplicative noise and drift term which belongs to a space of mixed regularity (L^p-regularity in the velocity-variable and Sobolev regularity in the space-variable). We prove that, in contrast with the deterministic case, the SPDE admits a unique weakly differentiable solution which preserves a certain degree of Sobolev regularity of the initial condition without developing discontinuities. To prove the result we also study the related degenerate Kolmogorov equation in Bessel-Sobolev spaces and construct a suitable stochastic flow.

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1 Introduction

We consider the linear Stochastic Partial Differential Equation (SPDE) of kinetic transport type

$$\mathrm{d}_t f + (v \cdot D_x f + F \cdot D_v f)\,\mathrm{d}t + D_v f \circ \mathrm{d}W_t = 0\,, \qquad f\big|_{t=0} = f_0 \tag{1.1}$$

and the associated stochastic characteristics described by the stochastic differential equation (SDE)

$$\begin{cases} \mathrm{d}X_t &= V_t\,\mathrm{d}t, \qquad \mathrm{d}V_t = F\left(X_t, V_t\right)\mathrm{d}t + \mathrm{d}W_t \\ X\left(0\right) &= x_0, \qquad\quad V\left(0\right) = v_0\,. \end{cases} \tag{1.2}$$

*Université Aix-Marseille, CNRS, UMR 7373, Centrale Marseille, I2M, 13453 Marseille, France. E-mail: fedrizzi.ecm@gmail.com.
†Dipartimento di Matematica, Largo Bruno Pontecorvo 5, Università di Pisa, 56127 Pisa, Italy. E-mail: flandoli@dma.unipi.it.
‡Dipartimento di Matematica, Università di Torino, via Carlo Alberto 10, 10123 Torino, Italy. E-mail: enrico.priola@unito.it,
§Université de Lyon, Université Claude Bernard Lyon 1, CNRS UMR 5208, Institut Camille Jordan, 43 blvd. du 11 novembre 1918, F-69622 Villeurbanne cedex, France. E-mail: vovelle@math.univ-lyon1.fr

Here $t \in [0,T]$, $(x,v) \in \mathbb{R}^d \times \mathbb{R}^d$, $f : [0,T] \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$, $f_0 : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$, $F : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$, $x_0, v_0 \in \mathbb{R}^d$ and $(W_t)_{t\geq 0}$ is a d-dimensional Brownian motion defined on a complete filtered probability space $\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P}\right)$; the operation $D_v f \circ \mathrm{d}W_t = \sum_{\alpha=1}^d \partial_{v_\alpha} f \circ \mathrm{d}W_t^\alpha$ will be understood in the Stratonovich sense, in order to preserve (a priori only formally) the relation $\mathrm{d}f\,(t, X_t, V_t) = 0$, when (X_t, V_t) is a solution of the SDE; we use Stratonovich not only for this mathematical convenience, but also because, in the spirit of the so called Wong-Zakai principle, the Stratonovich sense is the natural one from the physical viewpoint as a limit of correlated noise with small time-correlation. The physical meaning of the SPDE (1.1) is the transport of a scalar quantity described by the function $f\,(t,x,v)$ (or the evolution of a density $f\,(t,x,v)$, when $\mathrm{div}_v\, F = 0$, so that $F D_v f = \mathrm{div}_v\,(Ff)$), under the action of a fluid - or particle - motion described by the SDE (1.2), where we have two force components: a "mean" (large scale) component $F\,(x,v)$, plus a fast fluctuating perturbation given by $\frac{\mathrm{d}W_t}{\mathrm{d}t}$. Under suitable assumptions and more technical work one can consider more elaborate and flexible noise terms, space dependent, of the form $\sum_{k=1}^\infty \sigma_k\,(x)\,\mathrm{d}W_t^k$ (see [8], [11] for examples of assumptions on a noise with this structure and [13] for physical motivations), but for the purpose of this paper it is sufficient to consider the simplest noise $\mathrm{d}W_t = \sum_{k=1}^d e_k\,\mathrm{d}W_t^k$, $\{e_k\}_{k=1,\dots,d}$ being an orthonormal base of $\mathbb{R}^d$.

For physical reasons, we chose in (1.1) a specific linear form for the drift in the degenerate component, i.e., $v \cdot D_x$. It seems reasonable to expect that a possible generalization to the case of a nonlinear drift term like $G(x,v) \cdot D_x$ could be obtained under suitable Hörmander type conditions ensuring that the system is hypoelliptic. We shall mention that some results in this direction have already been obtained: two strong well-posedness results for the degenerate SDE (1.2) with nonlinear Hölder continuous drift terms are presented respectively in [6] and [38]. However, using our approach, such results are not enough to prove a well-posedness result for the SPDE (1.1) since a full hypoelliptic regularity result is yet not available at the level of the corresponding degenerate Kolmogorov equations.

Our aim is to show that noise has a regularizing effect on both the SDE (1.2) and the SPDE (1.1), in the sense that it provides results of existence, uniqueness and regularity under assumptions on F which are forbidden in the deterministic case. Results of this nature have been proved recently for other equations of transport type, see for instance [19], [16], [18], [2], but here, for the first time, we deal with the case of "degenerate" noise, because $\mathrm{d}W_t$ acts only on a component of the system. It is well known that the kinetic structure has good "propagation" properties from the v to the x component; however, for the purpose of regularization by noise one needs precise results which are investigated here for the first time and are technically quite non trivial. Let us describe more precisely the result proved here.

First Theorem 4.5 shows that weak existence and uniqueness in law holds for the SDE (1.2) only assuming $F \in L^p(\mathbb{R}^{2d}; \mathbb{R}^d)$ with $p > 4d$. To prove strong existence for (1.2) and existence of a stochastic flow we investigate the SDE (1.2) under the assumption (see below for more details) that F is in the mixed regularity space $L^p\left(\mathbb{R}^d_v; W^{s,p}\left(\mathbb{R}^d_x; \mathbb{R}^d\right)\right)$ for some $s \in (\frac{2}{3}, 1)$ and $p > 6d$; this means that we require

$$\int_{\mathbb{R}^d} \|F(\cdot, v)\|^p_{W^{s,p}}\,\mathrm{d}v < \infty\,,$$

where $W^{s,p} = W^{s,p}(\mathbb{R}^d; \mathbb{R}^d)$ is a fractional Sobolev space (cf. Hypothesis 2.1 and the comments after this assumption; see also Sections 3.1 and 3.2 for more details). Thus our drift is only L^p in the "good" v-variable in which the noise acts and has Sobolev regularity in the other x-variable. This is particularly clear in the special case of

$$F(x,v) = \varphi(v) G(x)\,, \tag{1.3}$$

where $G \in W^{s,p}(\mathbb{R}^d;\mathbb{R}^d)$, $\varphi \in L^p(\mathbb{R}^d)$ and $p > 6d$ with $s \in (\frac{2}{3}, 1)$. Just to mention in the case of full-noise action, the best known assumption to get pathwise uniqueness (cf. [24]) is that F must belong to $L^p(\mathbb{R}^N;\mathbb{R}^N)$, $p > N$ (in our case $N = 2d$).

According to a general scheme (see [36], [24], [19], [14], [18] [30], [15], [16], [2], [6], [37], [38]) to study regularity properties of the stochastic characteristics one first needs to establish precise regularity results for solutions to associated Kolmogorov equations. In our case such equations are degenerate elliptic equations of the type

$$\lambda\psi(x,v) - \frac{1}{2}\triangle_v\psi(x,v) - v\cdot D_x\psi(x,v) - F(x,v)\cdot D_v\psi(x,v) = g(x,v)\,, \qquad (1.4)$$

where $\lambda > 0$ is given (see Section 3.3). We prove a useful regularity result for (1.4) in Bessel-Sobolev spaces (see Theorem 3.7). Such result requires basic L^p-estimates proved in [4] and [5] and non-standard interpolation techniques for functions from $\mathbb{R}^d$ with values in Bessel-Sobolev spaces (see in particular the proofs of Theorem 3.4 and Lemma 3.6).

The results of Section 3 are exploited in Section 4 to prove existence of strong solutions to (1.2) and pathwise uniqueness. Moreover, we can also construct a continuous stochastic flow, injective and surjective, hence a flow of homeomorphisms. These maps are locally γ-Hölder continuous for every $\gamma \in (0,1)$. We cannot say that they are diffeomorphisms; however, we can show that for any t and $\mathbb{P}$-a.s. the random variable $Z_t = (X_t, V_t)$ admits a distributional derivative with respect to $z_0 = (x_0, v_0)$. Moreover, for any t and $p > 1$, the weak derivative $D_z Z_t \in L^p_{loc}(\Omega\times\mathbb{R}^{2d})$ (i.e., $D_z Z_t \in L^p(\Omega\times K)$, for any compact set $K \subset \mathbb{R}^{2d}$; see Theorem 4.19). These results are a generalization to the kinetic (hence degenerate noise) case of theorems in [16].

Well-posedness for kinetic SDEs (1.2) with non-Lipschitz drift has been recently investigated: strong existence and uniqueness have been recently proved in [6] and [37]. Moreover, a stochastic flow of diffeomorphisms has been obtained in [38] even with a multiplicative noise. In [37] and [38] the drift is assumed to be β-Hölder continuous in the x-variable with $\beta > \frac{2}{3}$ and Dini continuous in the v-variable. The results here are more general even concerning the regularity in the x-variable (see also Section 2.1). We stress that well-posedness is not true without noise, as the counter-examples given by Propositions 2.2 and 2.3 show.

Based on our results on the stochastic flow, we prove in Section 5 that if the initial condition f_0 is sufficiently smooth, the SPDE (1.1) admits a weakly differentiable solution and provide a representation formula (see Theorem 5.4). Moreover, the solution of equation (1.1) in the spatial variable is of class $W^{1,r}_{loc}\left(\mathbb{R}^{2d}\right)$, for every $r \geq 1$, $\mathbb{P}$-a.s., at every time $t \in [0,T]$. Such regularity result is not true without noise: Proposition 2.3 gives an example where solutions develop discontinuities from smooth initial conditions and with drift in the class considered here. Moreover, assuming in addition that $\mathrm{div}_v F \in L^\infty(\mathbb{R}^{2d})$ we prove uniqueness of weakly differentiable solutions (see Theorem 5.7).

The results presented here may also serve as a preliminary for the investigation of properties of interest in the theory of kinetic equations, where again we see a regularization by noise. In a forthcoming paper we shall investigate the mixing property

$$\|f_t\|_{L^\infty_x(L^1_v)} \leq C\,(t)\,\|f_0\|_{L^1_x(L^\infty_v)}\,,$$

with $C\,(t)$ diverging as $t \to 0$, to see if it holds when the noise is present in comparison to the deterministic case (cf. [20] and [21]). Again the theory of stochastic flows, absent without noise under our assumptions, is a basic ingredient for this analysis.

The paper is constructed as follows. We begin by introducing in the next section some necessary notation and presenting some examples that motivate our study. In Section 3 we state some well-posedness results for an associated degenerate elliptic equation (see Theorem 3.7, which contains the main result of this section). These results will be used in Section 4 to solve the stochastic equation of characteristics associated to (1.1). This is a degenerate stochastic equation, but we can prove existence and uniqueness of strong solutions (see Theorem 4.11), generating a weakly differentiable flow of homeomorphisms (see Theorems 4.18 and 4.19). Using all these tools, we can finally show in Section 5 that the stochastic kinetic equation (1.1) is well-posed in the class of weakly differentiable solutions.

Remark 1.1. After submitting the paper, we were informed of two papers [41] and [7] dealing with degenerate SDEs like (1.2) with multiplicative noise and time-dependent coefficients. In [41] the author proves strong well-posedness and existence of stochastic flow under more general assumptions than Hypothesis 2.1 (in particular s could be $2/3$ and $p > 2(2d+1)$). Applications to well-posedness of degenerate Fokker-Planck equations for measures are also given in [41]. The paper [7] considers (1.2) replacing $V_t \mathrm{d}t$ with a more general term like $G(t, X_t, V_t)\mathrm{d}t$ and assuming Hölder-continuity of the coefficients. A weak well-posedness result is then proven in [7].

2 Notation and examples

We will either use a dot or $\langle \ , \ \rangle$ to denote the scalar product in $\mathbb{R}^d$ and $|\cdot|$ for the Euclidian norm. Other norms will be denoted by $\|\cdot\|$, and for the sup norm we shall use both $\|\cdot\|_\infty$ and $\|\cdot\|_{L^\infty(\mathbb{R}^d)}$. $C_b(\mathbb{R}^d)$ denotes the Banach space of all real continuous and bounded functions $f : \mathbb{R}^d \to \mathbb{R}$ endowed with the sup norm; $C_b^1(\mathbb{R}^d) \subset C_b(\mathbb{R}^d)$ is the subspace of all functions wich are differentiable on $\mathbb{R}^d$ with bounded and continuous partial derivatives on $\mathbb{R}^d$; for $\alpha \in \mathbb{R}_+ \backslash \mathbb{N}$, $C^\alpha(\mathbb{R}^d) \subset C_b(\mathbb{R}^d)$ is the space of α-Hölder continuous functions on $\mathbb{R}^d$; $C_c^\infty(\mathbb{R}^d) \subset C_b(\mathbb{R}^d)$ is the space of all infinitely differentiable functions with compact support. C, c, K will denote different constants, and we use subscripts to indicate the parameters on which they depend.

Throughout the paper, we shall use the notation z to denote the point $(x, v) \in \mathbb{R}^{2d}$. Thus, for a scalar function $g(z) : \mathbb{R}^{2d} \to \mathbb{R}$, $D_z g$ will denote the vector in $\mathbb{R}^{2d}$ of derivatives with respect to all variables $z = (x, v)$, $D_x g \in \mathbb{R}^d$ denotes the vector of derivatives taken only with respect to the first d variables and similarly for $D_v g(z)$. We will have to work with spaces of functions of different regularity in the x and v variables: we will then use subscripts to distinguish the space and velocity variables, as in Hypothesis 2.1.

Let us state the regularity assumptions we impose on the force field F.

Hypothesis 2.1. *The function $F : \mathbb{R}^{2d} \to \mathbb{R}^d$ is a Borel function such that*

$$\int_{\mathbb{R}^d} \|F(\cdot, v)\|_{H_p^s}^p \,\mathrm{d}v < \infty \tag{2.1}$$

where $s \in (2/3, 1)$ and $p > 6d$. We write that $F \in L^p\left(\mathbb{R}_v^d; H_p^s\left(\mathbb{R}_x^d; \mathbb{R}^d\right)\right)$.

The Bessel space $H_p^s = H_p^s(\mathbb{R}^d; \mathbb{R}^d)$ is defined by the Fourier transform (see Section 3). According to Remark 3.2, condition (2.1) can also be rewritten using the related fractional Sobolev spaces $W^{s,p}(\mathbb{R}^d; \mathbb{R}^d)$ instead of $H_p^s(\mathbb{R}^d; \mathbb{R}^d)$. In the sequel we will also write $H_p^s(\mathbb{R}^d)$ instead of $H_p^s(\mathbb{R}^d; \mathbb{R}^d)$ when no confusion may arise.

2.1 Examples

Without noise, when F is only in the space $L^p\left(\mathbb{R}_v^d; H_p^s\left(\mathbb{R}_x^d\right)\right)$ for some $s > \frac{2}{3}$ and $p > 6d$, the equation for the characteristics

$$\begin{aligned} x' &= v, \qquad v' = F(x, v) \\ x(0) &= x_0, \qquad v(0) = v_0 \end{aligned} \tag{2.2}$$

and the associated kinetic transport equation

$$D_t f + v \cdot D_x f + F \cdot D_v f = 0, \qquad f|_{t=0} = f_0 \tag{2.3}$$

may have various types of pathologies. We shall mention here some of them in the very simple case of $d = 1$,

$$F(x,v) = \pm\theta(x,v)\, sign(x)\, |x|^{\alpha} \tag{2.4}$$
$$\text{for some } \alpha \in \left(\frac{1}{2}, 1\right), \quad \theta \in C_c^{\infty}\left(\mathbb{R}^2\right).$$

First, note that this function belongs to $L^p\left(\mathbb{R}_v; H_p^s(\mathbb{R}_x)\right)$ for some $s > \frac{2}{3}$ and $p > 6$. To check this fact one can first observe that $\partial_x(sign(x)|x|^{\alpha}) = \alpha|x|^{\alpha-1}$ in distributional sense, so that $F(v,\cdot) \in H_q^1(\mathbb{R}_x)$ for some appropriate value of $q > 2$, and then use Sobolev embedding theorem: $H_q^1(\mathbb{R}) \subset H_p^s(\mathbb{R})$ for $\frac{1}{p} = \frac{1}{q} - 1 + s$.

Thus F satisfies our Hypothesis 2.1. On the other hand when $\alpha \in \left(\frac{1}{2}, \frac{2}{3}\right)$, the function $sign(x)\,|x|^{\alpha}$ is not in $C_{loc}^{\gamma}(\mathbb{R})$ for any $\gamma > 2/3$ and the results of [37], [38] do not apply.

Let us come to the description of the pathologies of characteristics and kinetic equation when $F(x,v) = \pm\theta(x,v)\,|x|^{\alpha}$.

Proposition 2.2. *In $d = 1$, if $\theta \in C_c^{\infty}\left(\mathbb{R}^2\right)$, $\theta = 1$ on $B(0,R)$ for some $R > 0$, $F(x,v) = \theta(x,v)\, sign(x)\, |x|^{\alpha}$, then system (2.2) with initial condition $(x_0, 0)$ has infinitely many solutions. In particular, for small time (depending on R and α), $(x_t, v_t) = \left(x_0 + At^{\beta}, A\beta t^{\beta-1}\right)$, with (β, A) satisfying (2.5) below, and also $A = 0$, are solutions.*

Proof. Let us check that $(x_t, v_t) = \left(At^{\beta}, A\beta t^{\beta-1}\right)$ with the specified values of (β, A) and a small range of t, are solutions. We have $x_t' = v_t$,

$$\begin{aligned} v_t' - F(x, v_t) &= A\beta(\beta-1)\, t^{\beta-2} - sign(x_t)\,|x_t|^{\alpha} \\ &= A\beta(\beta-1)\, t^{\beta-2} - sign(A)\,|A|^{\alpha}\, t^{\alpha\beta} = 0 \end{aligned}$$

for $\alpha\beta = \beta - 2$ and $A\beta(\beta-1) = sign(A)\,|A|^{\alpha}$, namely

$$\beta = \frac{2}{1-\alpha}, \qquad A = \pm\left(\frac{1}{\beta(\beta-1)}\right)^{\frac{1}{1-\alpha}} = \pm\left(\frac{(1-\alpha)^2}{2(1+\alpha)}\right)^{\frac{1}{1-\alpha}}. \tag{2.5}$$

□

With a little greater effort one can show, in this specific example, that every solution (x_t, v_t) from the initial condition $(0,0)$ has, for small time, the form $(x_t, v_t) = \left(A(t-t_0)^{\beta}, A\beta(t-t_0)^{\beta-1}\right) 1_{t \geq t_0}$ for some $t_0 \geq 0$, or it is $(x_t, v_t) = (0,0)$ ((β, A) always given by (2.5)) and that existence and uniqueness holds from any other initial condition, even from points of the form $(0, v_0)$, $v_0 \neq 0$, around which F is not Lipschitz continuous. Given $T > 0$ and $R > 0$ large enough, there is thus, at every time $t \in [0,T]$, a set $\Lambda_t \subset \mathbb{R}^2$ of points “reached from $(0,0)$”, which is the set

$$\Lambda_t = \left\{\left(A(t-t_0)^{\beta}, A\beta(t-t_0)^{\beta-1}\right) \in \mathbb{R}^2 : t_0 \in [0,t]\right\}.$$

Using this family of sets one can construct examples of non uniqueness for the transport equation (2.3), because a solution $f(t,x,v)$ is not uniquely determined on Λ_t. However, these examples are not striking since the region of non-uniqueness, $\cup_{t\geq 0}\Lambda_t$, is thin and one could say that uniqueness is restored by a modification of f on a set of measure zero. But, with some additional effort, it is also possible to construct an example with $F(x,v) = \pm\theta(x,v)\,|x|^{\alpha}$. In this case, for some negative m (depending on R and α), one can construct infinitely many solutions (x_t, v_t) starting from any point in a segment $(x_0, 0)$, $x_0 \in [m, 0)$. Indeed, $(x_t, v_t) = (x_0, 0)$ is a solution, but there are also solutions leaving

$(x_0, 0)$ which will have $v_t > 0$, at least for some small time interval. Then one obtains that the solution $f(t, x, v)$ is not uniquely determined on a set of positive Lebesgue measure.

More relevant, for a simple class of drift as the one above, is the phenomenon of loss of regularity. Preliminary, notice that, when F is Lipschitz continuous, system (2.2) generates a Lipschitz continuous flow and, using it, one can show that, for every Lipschitz continuous $f_0 : \mathbb{R}^2 \to \mathbb{R}$, the transport equation (2.3) has a unique solution in the class of continuous functions $f : [0, T] \times \mathbb{R}^2 \to \mathbb{R}$ that are Lipschitz continuous in (x, v), uniformly in t. The next proposition identifies an example with non-Lipschitz F where this persistence of regularity is lost. More precisely, even starting from a smooth initial condition, unless it has special symmetry properties, there is a solution with a point of discontinuity. This pathology is removed by noise, since we will show that with sufficiently good initial condition, the unique solution $f(t, z)$ is of class $W^{1,r}_{loc}(\mathbb{R}^2)$ for every $r \geq 1$ and $t \in [0, T]$ a.s., hence in particular continuous. However, in the stochastic case, we do not know whether the solution is Lipschitz under our assumptions, whereas presumably it is under the stronger Hölder assumptions on F of [38].

Proposition 2.3. *In $d = 1$, if $\theta \in C_c^\infty\left(\mathbb{R}^2\right)$, $\theta = 1$ on $B\left(0, R\right)$ for some $R > 0$, $F\left(x, v\right) = \theta\left(x, v\right) sign\left(x\right) \left|x\right|^\alpha$, then system (2.2) has a unique local solution on any domain not containing the origin, for every initial condition. For every $t_0 > 0$ (small enough with respect to R), the two initial conditions $\left(At_0^\beta, -A\beta t_0^{\beta-1}\right)$ with (β, A) given by (2.5) produce the solution*

$$(x_t, v_t) = \left(A\left(t_0 - t\right)^\beta, -A\beta\left(t_0 - t\right)^{\beta-1}\right)$$

for $t \in [0, t_0]$, and $(x_{t_0}, v_{t_0}) = (0, 0)$. As a consequence, the transport equation (2.3) with any smooth f_0 such that $f_0\left(At_0^\beta, -A\beta t_0^{\beta-1}\right) \neq f_0\left(-At_0^\beta, A\beta t_0^{\beta-1}\right)$ for some $t_0 > 0$, has a solution with a discontinuity at time t_0 at position $(x, v) = (0, 0)$.

Proof. The proof is elementary but a full proof is lengthy. We limit ourselves to a few simple facts, without proving that system (2.2) is forward well posed (locally in time) and the transport equation (2.3) is also well posed in the set of weak solutions. We only stress that the claim $(x_{t_0}, v_{t_0}) = (0, 0)$ when the initial condition is $\left(At_0^\beta, -A\beta t_0^{\beta-1}\right)$ can be checked by direct computation (as in the previous proposition) and the discontinuity of the solution f of (2.3) is a consequence of the transport property, namely the fact that whenever f is regular we have

$$f\left(t, x_t, v_t\right) = f_0\left(x_0, v_0\right) \tag{2.6}$$

where (x_t, v_t) is the unique solution with initial condition (x_0, v_0). Hence we have this identity for points close (but not equal) to the coalescing ones mentioned above, where the forward flow is regular and a smooth initial condition f_0 gives rise to a smooth solution; but then, from identity (2.6) in nearby points, the limit

$$\lim_{(x,v)\to(0,0)} f\left(t_0, x, v\right)$$

does not exists if t_0 is as above and $f_0\left(At_0^\beta, -A\beta t_0^{\beta-1}\right) \neq f_0\left(-At_0^\beta, A\beta t_0^{\beta-1}\right)$. □

3 Well-posedness for degenerate Kolmogorov equations in Bessel-Sobolev spaces

3.1 Preliminaries on functions spaces and interpolation theory

Here we collect basic facts on Bessel and Besov spaces (see [3], [35] and [32] for more details). In the sequel if X and Y are real Banach spaces then $Y \subset X$ means that Y is continuously embedded in X.

The Bessel (potential) spaces are defined as follows (cf. [3, page 139] and [32, page 135]). For the sake of simplicity we only consider $p \in [2, \infty)$ and $s \in \mathbb{R}_+$.

First one considers the Bessel potential J^s,

$$J^s f = \mathcal{F}^{-1}[(1 + |\cdot|^2)^{s/2})\mathcal{F} f]$$

where $\mathcal{F}$ denotes the Fourier transform of a distribution $f \in \mathcal{S}'(\mathbb{R}^d)$, $d \geq 1$. Then we introduce

$$H^s_p(\mathbb{R}^d) = \{f \in \mathcal{S}'(\mathbb{R}^d) \ : \ J^s f \in L^p(\mathbb{R}^d)\}$$

(clearly $H^0_p(\mathbb{R}^d) = L^p(\mathbb{R}^d)$). This is a Banach space endowed with the norm $\|f\|_{H^s_p} = \|J^s f\|_p$, where $\|\cdot\|_p$ is the usual norm of $L^p(\mathbb{R}^d)$ (we identify functions with coincide a.e.). It can be proved that

$$H^s_p(\mathbb{R}^d) = \{f \in L^p(\mathbb{R}^d) \ : \ \mathcal{F}^{-1}[|\cdot|^s \mathcal{F} f] \in L^p(\mathbb{R}^d)\} \tag{3.1}$$

and an equivalent norm in $H^s_p(\mathbb{R}^d)$ is

$$\|f\|_{s,p} = \|f\|_p + \|\mathcal{F}^{-1}[|\cdot|^s \mathcal{F} f]\|_p \backsimeq \|\mathcal{F}^{-1}[(1 + |\cdot|^s)\, \mathcal{F} f]\|_p \,.$$

To show this characterization one can use that

$$(1 + 4\pi^2 |x|^2)^{s/2} = (1 + (2\pi|x|^s))\,[\mathcal{F}\phi(x) + 1]\,, \quad x \in \mathbb{R}^d,$$

for some $\phi \in L^1(\mathbb{R}^d)$ (see [32, page 134]), and basic properties of convolution and Fourier transform. We note that

$$H^k_p(\mathbb{R}^d) = W^{k,p}(\mathbb{R}^d) \tag{3.2}$$

if $k \geq 0$ is an integer with equivalence of norms (here $W^{k,p}(\mathbb{R}^d)$ is the usual Sobolev space; $W^{0,p}(\mathbb{R}^d) = L^p(\mathbb{R}^d)$); see [3, Theorem 6.2.3]. However if s is not an integer we only have (see [3, Theorem 6.4.4] or [32, page 155])

$$H^s_p(\mathbb{R}^d) \subset W^{s,p}(\mathbb{R}^d) \tag{3.3}$$

where $W^{s,p}(\mathbb{R}^d)$ is a fractional Sobolev space (see below). We have (cf. [3, Theorem 6.2.3])

$$H^{s_2}_p(\mathbb{R}^d) \subset H^{s_1}_p(\mathbb{R}^d)$$

if $s_2 > s_1$ and, moreover, $C^\infty_c(\mathbb{R}^d)$ is dense in any $H^s_p(\mathbb{R}^d)$.

One can compare Bessel spaces with Besov spaces $B^s_{p,q}(\mathbb{R}^d)$ (see, for instance, Theorem 6.2.5 in [3]). Let $p, q \geq 2$, $s \in (0, 2)$, to simplify notation.

If $s \in (0, 1)$ then $B^s_{p,q}(\mathbb{R}^d)$ consists of functions $f \in L^p(\mathbb{R}^d)$ such that

$$[f]_{B^s_{p,q}} = \Big(\int_{\mathbb{R}^d} \frac{\mathrm{d}h}{|h|^{d+sq}} \Big(\int_{\mathbb{R}^d} |f(x+h) - f(x)|^p \mathrm{d}x \Big)^{q/p} \Big)^{1/q} < \infty \,.$$

Thus we have

$$B^s_{p,p}(\mathbb{R}^d) = W^{s,p}(\mathbb{R}^d) \tag{3.4}$$

with equivalence of norms. However if $s = 1$, $B^1_{p,q}(\mathbb{R}^d)$ consists of all functions $f \in L^p(\mathbb{R}^d)$ such that

$$[f]_{B^1_{p,q}} = \Big(\int_{\mathbb{R}^d} \frac{\mathrm{d}h}{|h|^{d+q}} \Big(\int_{\mathbb{R}^d} |f(x+2h) - 2f(x+h) + f(x)|^p \mathrm{d}x \Big)^{q/p} \Big)^{1/q} < \infty \,.$$

Thus we only have $B^1_{p,p}(\mathbb{R}^d) \subset W^{1,p}(\mathbb{R}^d)$. Note that $B^s_{p,q}(\mathbb{R}^d)$ is a Banach space endowed with the norm: $\|\cdot\|_p + [\cdot]_{B^s_{p,q}}$. Similarly, if $s \in (1,2)$, then $B^s_{p,q}(\mathbb{R}^d)$ consists of functions $f \in W^{1,p}(\mathbb{R}^d)$ such that

$$[f]_{B^s_{p,q}} = \sum_{i=1}^d \Big(\int_{\mathbb{R}^d} \frac{\mathrm{d}h}{|h|^{d+sq}} \Big(\int_{\mathbb{R}^d} |\partial_{x_i} f(x+h) - \partial_{x_i} f(x)|^p \mathrm{d}x \Big)^{q/p} \Big)^{1/q} < \infty . \tag{3.5}$$

Moreover, $C^\infty_c(\mathbb{R}^d)$ is dense in any $B^s_{p,q}(\mathbb{R}^d)$ and

$$B^{s_2}_{p,q}(\mathbb{R}^d) \subset B^{s_1}_{p,q}(\mathbb{R}^d) , \quad 0 < s_1 < s_2 < 2 , \quad p \geq 2 . \tag{3.6}$$

We also have the following result (cf. [3, Theorem 6.4.4])

$$B^s_{p,2}(\mathbb{R}^d) \subset H^s_p(\mathbb{R}^d) \subset B^s_{p,p}(\mathbb{R}^d) , \tag{3.7}$$

$s \in (0,2)$, $p \geq 2$. Next we state a known result (see [3, Theorem 6.4.4]; for a direct proof see Appendix in [17]). This is useful to give an equivalent formulation to Hypothesis 2.1.

Proposition 3.1. *Let $p > 2$, s, s' such that $0 < s < s' < 1$. We have*

$$W^{s',p}(\mathbb{R}^d) \subset B^s_{p,2}(\mathbb{R}^d) \subset H^s_p(\mathbb{R}^d) .$$

It is important to notice that Besov spaces are real interpolation spaces (for the definition of interpolation spaces $(X,Y)_{\theta,q}$ with X and Y real Banach spaces and $Y \subset X$ see [29, Chapter 1] or [3]). As a particular case of [3, Theorem 6.2.4] we have for $0 \leq s_0 < s_1 \leq 2$, $\theta \in (0,1)$, $p \geq 2$,

$$(H^{s_0}_p(\mathbb{R}^d), H^{s_1}_p(\mathbb{R}^d))_{\theta,p} = B^s_{p,p}(\mathbb{R}^d) \tag{3.8}$$

with $s = (1-\theta)s_0 + \theta s_1$. Moreover, it holds (see [3, Theorem 6.4.5]):

$$(B^{s_0}_{p,p}(\mathbb{R}^d), B^{s_1}_{p,p}(\mathbb{R}^d))_{\theta,p} = B^s_{p,p}(\mathbb{R}^d) \tag{3.9}$$

with $0 < s_0 < s_1 < 2$, $s = (1-\theta)s_0 + \theta s_1$, $\theta \in (0,1)$.

3.2 Interpolation of functions with values in Banach spaces

We follow Section VII in [27] and [9]. Let A_0 be a real Banach space. We will consider the Banach space $L^p(\mathbb{R}^d; A_0)$, $1 \leq p < \infty$, $d \geq 1$. As usual this consists of all strongly measurable functions f from $\mathbb{R}^d$ into A_0 such that the real valued function $\|f(x)\|_{A_0}$ belongs to $L^p(\mathbb{R}^d)$. We have

$$\|f\|_{L^p(\mathbb{R}^d;A_0)} = \Big(\int_{\mathbb{R}^d} \|f(x)\|_{A_0} \mathrm{d}x \Big)^{1/p}, \quad f \in L^p(\mathbb{R}^d; A_0) .$$

If A_1 is another real Banach spaces with $A_1 \subset A_0$ we can define the Banach space

$$L^p(\mathbb{R}^d; (A_0, A_1)_{\theta,q}) ,$$

by using the interpolation space $(A_0, A_1)_{\theta,q}$, $q \in (1,\infty)$, $p \geq 1$ and $\theta \in (0,1)$. One can prove that

$$\big(L^p(\mathbb{R}^d; A_0), L^p(\mathbb{R}^d; A_1)\big)_{\theta,q} = L^p(\mathbb{R}^d; (A_0, A_1)_{\theta,q}) . \tag{3.10}$$

with equivalence of norms (see [27] and [9]). In the sequel we will often use, for $s \geq 0$, $p \geq 2$,

$$L^p(\mathbb{R}^d; H^s_p(\mathbb{R}^d)) . \tag{3.11}$$

We will often identify this space with the Banach space $L^p\left(\mathbb{R}^d_v; H^s_p\left(\mathbb{R}^d_x\right)\right)$ of all measurable functions $f(x,v)$, $f:\mathbb{R}^d\times\mathbb{R}^d\to\mathbb{R}$ such that $f(\cdot,v)\in H^s_p(\mathbb{R}^d)$, for a.e. $v\in\mathbb{R}^d$, and, moreover (see (3.1))

$$\int_{\mathbb{R}^d}\|f(\cdot,v)\|^p_{H^s_p}\,\mathrm{d}v=\int_{\mathbb{R}^d}\mathrm{d}v\int_{\mathbb{R}^d}|\mathcal{F}_x^{-1}[(1+|\cdot|^s)\mathcal{F}_xf(\cdot,v)](x)|^p\mathrm{d}x<\infty \tag{3.12}$$

(here $\mathcal{F}_x$ denotes the partial Fourier transform in the x-variable; as usual we identify functions which coincide a.e.). As a norm we consider

$$\|f\|_{L^p(\mathbb{R}^d_v;H^s_p(\mathbb{R}^d_x))}=\Big(\int_{\mathbb{R}^d}\|f(\cdot,v)\|^p_{H^s_p}\mathrm{d}v\Big)^{1/p}. \tag{3.13}$$

Also $L^p(\mathbb{R}^d_v;L^p(\mathbb{R}^d_x))$ can be identified with $L^p(\mathbb{R}^{2d})$. Similarly, we can define $L^p(\mathbb{R}^d_v;B^s_{p,p}(\mathbb{R}^d_x))$. Using (3.7) we have

$$L^p(\mathbb{R}^d;H^s_p(\mathbb{R}^d))\subset L^p(\mathbb{R}^d;B^s_{p,p}(\mathbb{R}^d)), \tag{3.14}$$

$p\geq 2$, $0<s<2$. Finally using (3.10) and (3.9) we get for $0<s_0<s_1<2$, $\theta\in(0,1)$, $p\geq 2$,

$$\big(L^p(\mathbb{R}^d;(B^{s_0}_{p,p}(\mathbb{R}^d)),L^p(\mathbb{R}^d;(B^{s_1}_{p,p}(\mathbb{R}^d))\big)_{\theta,p}=L^p(\mathbb{R}^d;B^s_{p,p}(\mathbb{R}^d))\,, \tag{3.15}$$

with $s=(1-\theta)s_0+\theta s_1$.

In the sequel when no confusion may arise, we will simply write $L^p(\mathbb{R}^d)$ instead of $L^p(\mathbb{R}^d;\mathbb{R}^k)$, $k\geq 1$, $p\in[1,\infty)$. Thus a function $U:\mathbb{R}^d\to\mathbb{R}^k$ belongs to $L^p(\mathbb{R}^d)$ if all its components $U_i\in L^p(\mathbb{R}^d)$, $i=1,\dots,k$. Moreover, $\|U\|_{L^p}=\Big(\sum_{i=1}^k\|U_i\|^p_{L^p}\Big)^{1/p}$. This convention about vector-valued functions will be used for other function spaces as well.

Remark 3.2. Proposition 3.1 and formula (3.3) show that Hypothesis 2.1 is equivalent to the following one: $F:\mathbb{R}^{2d}\to\mathbb{R}^d$ is a Borel function such that

$$\int_{\mathbb{R}^d}\|F(\cdot,v)\|^p_{W^{s,p}}\mathrm{d}v<\infty\,, \tag{3.16}$$

where $s\in(2/3,1)$ and $p>6d$.

3.3 Regularity results in Bessel-Sobolev spaces

Here $\mathbb{R}^N=\mathbb{R}^{2d}$ and $z=(x,v)\in\mathbb{R}^d\times\mathbb{R}^d$. Let also $p\in(1,\infty)$, $s\in(0,1)$ and $\lambda>0$. This section is devoted to the study of the equation

$$\begin{aligned}&\lambda\psi(z)-\frac{1}{2}\triangle_v\psi(z)-v\cdot D_x\psi(z)-F(z)\cdot D_v\psi(z)=g(z)\\&=\lambda\psi(z)-\frac{1}{2}\mathrm{Tr}\big(QD^2\psi(z)\big)-\langle Az,D\psi(z)\rangle-\langle B(z),D\psi(z)\rangle\end{aligned}$$

where $A=\begin{pmatrix}0&\mathbb{I}\\0&0\end{pmatrix}$, $Q=\begin{pmatrix}0&0\\0&\mathbb{I}\end{pmatrix}$ are $(2d\times 2d)$-matrices, $B=\begin{pmatrix}0\\F\end{pmatrix}:\mathbb{R}^{2d}\to\mathbb{R}^{2d}$. We shall start by considering the simpler equation with $B=0$, i.e.,

$$\lambda\psi(z)-\frac{1}{2}\triangle_v\psi(z)-v\cdot D_x\psi(z)=\lambda\psi(z)-\mathcal{L}\psi(z)=g(z)\,,\quad z\in\mathbb{R}^{2d}. \tag{3.17}$$

Recall that $D_v\psi$ and $D_x\psi$ denote respectively the gradient of ψ in the v-variables and in the x-variables; moreover, $D^2_v\psi$ indicates the Hessian matrix of ψ with respect to the v-variables (we have $\triangle_v\psi=\mathrm{T}r(D^2_v\psi)$).

Definition 3.3. *The space $X_{p,s}$ consists of all functions $f \in W^{1,p}(\mathbb{R}^{2d})$ such that $D_v^2 f$ and $v \cdot D_x f$ belong to $L^p(\mathbb{R}^d_v; H^s_p(\mathbb{R}^d_x))$. Recall that*

$$\|D_v^2 f\|^p_{L^p(\mathbb{R}^d_v;H^s_p(\mathbb{R}^d_x))} = \int_{\mathbb{R}^d} \sum_{i,j=1}^d \|\partial^2_{v_i v_j} f(\cdot, v)\|^p_{H^s_p(\mathbb{R}^d)} \mathrm{d}v\,.$$

It turns out that $X_{p,s}$ is a Banach space endowed with the norm:

$$\|f\|_{X_{p,s}} = \|f\|_{W^{1,p}(\mathbb{R}^{2d})} + \|D_v^2 f\|_{L^p(\mathbb{R}^d_v;H^s_p(\mathbb{R}^d_x))} + \|v \cdot D_x f\|_{L^p(\mathbb{R}^d_v;H^s_p(\mathbb{R}^d_x))}. \tag{3.18}$$

If $f \in X_{p,s}$ then $(\lambda f - \mathcal{L} f) \in L^p(\mathbb{R}^d_v; H^s_p(\mathbb{R}^d_x))$ (see (3.17)). With a slight abuse of notation, we will still write $f \in X_{p,s}$ for vector valued functions $f : \mathbb{R}^{2d} \to \mathbb{R}^{2d}$, meaning that all components $f_i : \mathbb{R}^{2d} \to \mathbb{R}$, $i = 1 \ldots 2d$ belong to $X_{p,s}$.

The following theorem improves results in [4] and [5]. In particular it shows that there exists the weak derivative $D_x \psi \in L^p(\mathbb{R}^{2d})$ so that (3.17) admits a strong solution ψ which solves equation (3.17) in distributional sense.

Theorem 3.4. *Let $\lambda > 0$, $p \geq 2$, $s \in (1/3, 1)$ and $g \in L^p(\mathbb{R}^d_v; H^s_p(\mathbb{R}^d_x))$. There exists a unique solution $\psi = \psi_\lambda \in X_{p,s}$ to equation* (3.17)*. Moreover, we have*

$$\lambda \|\psi\|_{L^p(\mathbb{R}^{2d})} + \sqrt{\lambda} \|D_v \psi\|_{L^p(\mathbb{R}^{2d})} + \|D_v^2 \psi\|_{L^p(\mathbb{R}^{2d})} + \|v \cdot D_x \psi\|_{L^p(\mathbb{R}^{2d})} \leq C \|g\|_{L^p(\mathbb{R}^{2d})} \tag{3.19}$$

with $C = C(d, p) > 0$ and

$$\|D_x \psi\|_{L^p(\mathbb{R}^{2d})} \leq C(\lambda)\, \|g\|_{L^p(\mathbb{R}^d_v;H^s_p(\mathbb{R}^d_x))}, \tag{3.20}$$

with $C(\lambda) = C(\lambda, s, p, d) > 0$ and $C(\lambda) \to 0$ as $\lambda \to \infty$. In addition there exists $c = c(s, p, d) > 0$ such that

$$\begin{aligned} &\lambda \|\psi\|_{L^p(\mathbb{R}^d_v;H^s_p(\mathbb{R}^d_x))} + \sqrt{\lambda} \|D_v \psi\|_{L^p(\mathbb{R}^d_v;H^s_p(\mathbb{R}^d_x))} + \|D_v^2 \psi\|_{L^p(\mathbb{R}^d_v;H^s_p(\mathbb{R}^{d_x}))} \\ &\qquad + \|v \cdot D_x \psi\|_{L^p(\mathbb{R}^d_v;H^s_p(\mathbb{R}^d_x))} \leq c \|g\|_{L^p(\mathbb{R}^d_v;H^s_p(\mathbb{R}^d_x))} \end{aligned} \tag{3.21}$$

Proof. Uniqueness. Let $\psi \in X_{p,s}$ be a solution. Note that $|\psi|^{p-2}\psi$ belongs to $L^q(\mathbb{R}^{2d})$, with $q = \frac{p}{p-1}$. Fix $\eta \in C_c^\infty(\mathbb{R}^{2d})$ such that $\eta = 1$ on the ball B_1 of center 0 and radius 1. Multiplying both sides of equation (3.17) by $|\psi(z)|^{p-2}\, \psi(z)\, \eta(\frac{z}{n})$, $n \geq 1$, we obtain

$$\begin{aligned} &\lambda \int_{\mathbb{R}^{2d}} |\psi(z)|^p\, \eta(\frac{z}{n}) dz - \frac{1}{2} \int_{\mathbb{R}^{2d}} |\psi(z)|^{p-2}\, \psi(z)\, \eta(\frac{z}{n})\, \triangle_v \psi(z)\, \mathrm{d}z \\ &- \int_{\mathbb{R}^{2d}} \eta(\frac{z}{n}) (v \cdot D_x \psi(z)) |\psi(z)|^{p-2} \psi\, \mathrm{d}z = \int_{\mathbb{R}^{2d}} \eta(\frac{z}{n}) g(z) |\psi(z)|^{p-2} \psi(z)\, \mathrm{d}z\,. \end{aligned} \tag{3.22}$$

Note that there exists the weak derivative $D_x(|\psi|^p) = p|\psi|^{p-2}\psi D_x \psi \in L^1(\mathbb{R}^{2d})$. Hence, for each $n \geq 1$, integrating by parts, we know that

$$\begin{aligned} \int_{\mathbb{R}^{2d}} \eta(z/n)(v \cdot D_x \psi(z)) |\psi(z)|^{p-2} \psi(z)\, \mathrm{d}z &= \frac{1}{p} \int_{\mathbb{R}^{2d}} \eta(z/n) v \cdot D_x(|\psi|^p)(z) \mathrm{d}z \\ &= \frac{1}{p} \int_{\mathbb{R}^{2d}} D_x \eta(\frac{x}{n}, \frac{v}{n}) \cdot \frac{v}{n}\, |\psi(z)|^p \mathrm{d}z \longrightarrow 0 \end{aligned}$$

as $n \to \infty$. Moreover

$$\begin{aligned} -\frac{1}{2} \int_{\mathbb{R}^{2d}} |\psi|^{p-2}\, \psi\, \eta(\frac{z}{n})\, \triangle_v \psi\, \mathrm{d}z &= \frac{(p-1)}{2} \sum_{k=1}^d \int_{\mathbb{R}^{2d}} \eta(z/n) |\psi|^{p-2} |\partial_{v_k} \psi|^2\, \mathrm{d}z \\ + \frac{1}{2n} \sum_{k=1}^d \int_{\mathbb{R}^{2d}} |\psi|^{p-2} \psi\, \partial_{v_k} \eta(\frac{z}{n}) \partial_{v_k} \psi\, \mathrm{d}z &\longrightarrow \sum_{k=1}^d \int_{\mathbb{R}^{2d}} |\psi|^{p-2} |\partial_{v_k} \psi|^2\, \mathrm{d}z \end{aligned}$$

Finally, passing to the limit as $n \to \infty$ in (3.22) we find

$$\lambda\|\psi\|^p_{L^p(\mathbb{R}^{2d})} + \frac{(p-1)}{2}\sum_{k=1}^d \int_{\mathbb{R}^{2d}} |\psi|^{p-2}|\partial_{v_k}\psi|^2 \,\mathrm{d}z = \int_{\mathbb{R}^{2d}} g|\psi|^{p-2}\psi \,\mathrm{d}z.$$

It follows easily that

$$\|\psi\|_{L^p(\mathbb{R}^{2d})} \le \frac{1}{\lambda}\|g\|_{L^p(\mathbb{R}^{2d})} \tag{3.23}$$

which implies uniqueness of solutions for the linear equation (3.17).

Existence. Step 1. We prove existence of solutions and estimates (3.19) and (3.20).

Let us first introduce the Ornstein-Uhlenbeck semigroup

$$\begin{aligned} P_t g(z) = P_t g(x,v) &= \int_{\mathbb{R}^{2d}} g(e^{tA}z + y) N(0,Q_t)\,\mathrm{d}y \qquad (3.24)\\ &= \int_{\mathbb{R}^{2d}} g(x+tv+y_1, v+y_2)N(0,Q_t)\,\mathrm{d}y\,, \quad g \in C_c^\infty(\mathbb{R}^{2d})\,, \ \ t\ge 0\,, \end{aligned}$$

where $N(0,Q_t)$ is the Gaussian measure with mean 0 and covariance matrix

$$Q_t = \int_0^t e^{sA}Qe^{sA^*}\mathrm{d}s = \int_0^t e^{sA}\begin{pmatrix} 0 & 0 \\ 0 & \mathbb{I}_{\mathbb{R}^d}\end{pmatrix} e^{sA^*}\mathrm{d}s \;= \begin{pmatrix} \frac13 t^3\mathbb{I}_{\mathbb{R}^d} & \frac12 t^2\mathbb{I}_{\mathbb{R}^d} \\ \frac12 t^2\mathbb{I}_{\mathbb{R}^d} & t\mathbb{I}_{\mathbb{R}^d}\end{pmatrix} \tag{3.25}$$

(A^* denotes the adjoint matrix). By the Young inequality (cf. the proof of [31, Lemma 13]) we know that $P_t g$ is well-defined also for any $g \in L^p(\mathbb{R}^{2d})$, z a.e.; moreover P_t : $L^p(\mathbb{R}^{2d}) \to L^p(\mathbb{R}^{2d})$, for any $t \ge 0$, and

$$\|P_t g\|_{L^p(\mathbb{R}^{2d})} \le \|g\|_{L^p(\mathbb{R}^{2d})}\,, \ \ g \in L^p(\mathbb{R}^{2d})\,,\ t\ge 0\,. \tag{3.26}$$

Let us consider, for any $\lambda > 0$, $z \in \mathbb{R}^d$, $g \in C_c^\infty(\mathbb{R}^{2d})$,

$$\psi(z) = G_\lambda g(z) = \int_0^{+\infty} e^{-\lambda t} P_t g(z)\,\mathrm{d}t\,. \tag{3.27}$$

Using the Jensen inequality, the Fubini theorem and (3.26) it is easy to prove that $G_\lambda g$ is well defined for $g \in L^p(\mathbb{R}^{2d})$, z a.e., and belongs to $L^p(\mathbb{R}^{2d})$. Moreover, for any $p \ge 1$,

$$G_\lambda : L^p(\mathbb{R}^{2d}) \to L^p(\mathbb{R}^{2d})\,, \quad \|G_\lambda g\|_p \le \frac{\|g\|_p}{\lambda}\,, \quad \lambda > 0\,, \ \ g\in L^p(\mathbb{R}^{2d})\,. \tag{3.28}$$

Note that $L^p(\mathbb{R}^d_v; H^s_p(\mathbb{R}^{d_x})) \subset L^p(\mathbb{R}^d_v; W^{s,p}(\mathbb{R}^d_x))$ (see (3.3)). Let us consider a sequence $(g_n) \in C_c^\infty(\mathbb{R}^{2d})$ such that

$$g_n \to g \text{ in } L^p(\mathbb{R}^d_v; W^{s,p}(\mathbb{R}^d_x))\,.$$

Arguing as in [31, Lemma 13] one can show that there exist classical solutions ψ_n to (3.17) with g replaced by g_n. Moreover, $\psi_n = G_\lambda g_n$. By [31, Theorem 11], which is based on results in [5], we have that

$$\|D^2_v\psi_n\|_{L^p(\mathbb{R}^{2d})} \le C\|g_n\|_{L^p(\mathbb{R}^{2d})}\,, \tag{3.29}$$

$\lambda > 0$, $n \ge 1$, $C = C(p,d)$. Using also (3.28) we deduce easily that (ψ_n) and $(D^2_v\psi_n)$ are both Cauchy sequences in $L^p(\mathbb{R}^{2d})$. Let us denote by $\psi \in L^p(\mathbb{R}^{2d})$ the limit function; it holds that $\psi = G_\lambda g$ and $D^2_v\psi \in L^p(\mathbb{R}^{2d})$.

Passing to the limit in (3.17) when ψ and g are replaced by ψ_n and g_n we obtain that ψ solves (3.17) in a weak sense ($v \cdot D_x\psi$ is intended in distributional sense). By (3.29) as $n \to \infty$ we also get

$$\begin{aligned} \|D^2_v\psi\|_{L^p(\mathbb{R}^{2d})} \le C\|g\|_{L^p(\mathbb{R}^{2d})}\,, \quad \|\psi\|_{L^p(\mathbb{R}^{2d})} \le \frac{1}{\lambda}\|g\|_{L^p(\mathbb{R}^{2d})} \qquad (3.30)\\ \text{and} \quad \|v\cdot D_x\psi\|_{L^p(\mathbb{R}^{2d})} \le C\,\|g\|_{L^p(\mathbb{R}^{2d})}\,. \end{aligned}$$

Note that the second estimate follows writing

$$v \cdot D_x \psi(z) = \lambda \psi(z) - \frac{1}{2} \triangle_v \psi(z) - g(z)\,.$$

To prove (3.19) it remains to show the estimate for $D_v \psi$. This follows from

$$\|D_v \psi\|^p_{L^p(\mathbb{R}^{2d})} = \int_{\mathbb{R}^d} \|D_v \psi(x,\cdot)\|^p_{L^p(\mathbb{R}^d)} \mathrm{d}x \le (\|\psi\|_{L^p(\mathbb{R}^{2d})})^{p/2} \, (\|D_v^2 \psi\|_{L^p(\mathbb{R}^{2d})})^{p/2}. \tag{3.31}$$

To prove that $\psi \in W^{1,p}(\mathbb{R}^{2d})$ it is enough to check that

$$\psi \in L^p\big(\mathbb{R}^d_v; W^{1,p}(\mathbb{R}^d_x)\big)\,. \tag{3.32}$$

Thus we have to prove that $\psi(\cdot, v) \in W^{1,p}(\mathbb{R}^d)$ for a.e. v and

$$\int_{\mathbb{R}^d} \mathrm{d}v \int_{\mathbb{R}^d} |D_x \psi(x,v)|^p \, \mathrm{d}x < \infty\,.$$

To this purpose we will use a result in [4] and interpolation theory. We consider $\eta \in C_c^\infty(\mathbb{R})$ such that $\mathrm{Supp}(\eta) \subset [-1,1]$ and $\int_{-1}^1 \eta(t)\mathrm{d}t > 0$.

Setting $f(t,z) = \eta(t)\psi(z)$, where ψ solves (3.17), we have that $f \in L^p(\mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d)$. In order to apply [4, Corollary 2.2] we note that, for $z = (x,v) \in \mathbb{R}^{2d}$, $t \in \mathbb{R}$,

$$\partial_t f(t,z) + v \cdot D_x f(t,z) = \eta'(t)\psi(z) - \eta(t) g(z) + \lambda \eta(t)\psi(z) - \frac{1}{2}\eta(t) \triangle_v \psi(z)\,.$$

Since $D_v^2 \psi \in L^p(\mathbb{R}^{2d})$ we deduce that $\partial_t f + v \cdot D_x f$ and $D_v^2 f$ both belong to $L^p(\mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d)$.

By [4, Corollary 2.2] and (3.30) we get easily that $\psi(\cdot, v) \in H_p^{2/3}(\mathbb{R}^d)$, for $v \in \mathbb{R}^d$ a.e., and

$$\int_{\mathbb{R}^d} \mathrm{d}v \int_{\mathbb{R}^d} |\mathcal{F}_x^{-1}[(1 + |\cdot|^{2/3})\mathcal{F}_x \psi(\cdot, v)](x)|^p \, \mathrm{d}x \le \Big(\frac{\lambda+1}{\lambda}\Big)^{2p/5} c \, \|g\|^p_{L^p(\mathbb{R}^{2d})}\,,$$

$\lambda > 0$, with $c = c(p,d)$, i.e.,

$$\psi = G_\lambda g \in L^p(\mathbb{R}^d_v; H_p^{2/3}(\mathbb{R}^d_x)) \quad \text{and} \quad \|G_\lambda g\|_{L^p(\mathbb{R}^d_v; H_p^{2/3}(\mathbb{R}^d_x))} \;\le\; \Big(\frac{\lambda+1}{\lambda}\Big)^{2/5} c_1 \, \|g\|_{L^p(\mathbb{R}^{2d})}\,. \tag{3.33}$$

By (3.10) and (3.8) with $s_0 = 0$ and $s_1 = 2/3$ we can interpolate between (3.33) and the estimate $\|G_\lambda g\|_{L^p(\mathbb{R}^d; L^p(\mathbb{R}^d))} \le \frac{1}{\lambda} \|g\|_{L^p(\mathbb{R}^{2d})}$ (see [29, Proposition 1.2.6]) and get, for $\varepsilon \in (0, 2/3)$,

$$\|G_\lambda g\|_{L^p(\mathbb{R}^d_v; W^{2/3-\varepsilon,p}(\mathbb{R}^d_x))} \le c_\varepsilon(\lambda) \|g\|_{L^p(\mathbb{R}^{2d})}\,. \tag{3.34}$$

with $c_\varepsilon(\lambda) \to 0$ as $\lambda \to \infty$.

Suppose now that $g \in L^p(\mathbb{R}^d_v; W^{1,p}(\mathbb{R}^d_x))$ and fix $k = 1, \ldots, d$. By approximating g with regular functions, it is not difficult to prove that there exists the weak derivative $\partial_{x_k} \psi \in L^p(\mathbb{R}^{2d})$, and

$$\partial_{x_k} \psi(z) = \partial_{x_k} G_\lambda g(z) = \int_0^{+\infty} e^{-\lambda t} P_t(\partial_{x_k} g)(z) \mathrm{d}t\,. \tag{3.35}$$

Arguing as in (3.34) we obtain that $\partial_{x_k} \psi \in L^p(\mathbb{R}^d_v; W^{2/3-\varepsilon,p}(\mathbb{R}^d_x))$ and

$$\|\partial_{x_k} \psi\|_{L^p(\mathbb{R}^d_v; W^{2/3-\varepsilon,p}(\mathbb{R}^d_x))} \le c_\varepsilon(\lambda) \|\partial_{x_k} g\|_{L^p(\mathbb{R}^{2d})}, \quad k = 1, \ldots, d\,,$$

so that

$$\|G_\lambda g\|_{L^p(\mathbb{R}^d_v; B^{1+2/3-\varepsilon,p}_{p,p}(\mathbb{R}^d_x))} \le c_\varepsilon(\lambda) \|g\|_{L^p(\mathbb{R}^d_v; W^{1,p}(\mathbb{R}^d_x))}. \tag{3.36}$$

Taking into account (3.8), (3.9) and (3.10) we can interpolate between (3.34) and (3.36) (see also (3.11) and (3.12)) and get

$$\begin{aligned} G_\lambda : L^p(\mathbb{R}^d_v; W^{s,p}(\mathbb{R}^d_x)) &= \big(L^p(\mathbb{R}^d; H^0_p(\mathbb{R}^d)), L^p(\mathbb{R}^d; W^{1,p}(\mathbb{R}^d))\big)_{s,p} \\ &\longrightarrow \big(L^p(\mathbb{R}^d; W^{2/3-\varepsilon,p}(\mathbb{R}^d)), L^p(\mathbb{R}^d; B^{5/3-\varepsilon}_{p,p}(\mathbb{R}^d))\big)_{s,p} = L^p(\mathbb{R}^d_v; B^{s+2/3-\varepsilon}_{p,p}(\mathbb{R}^d_x)). \end{aligned} \tag{3.37}$$

Since $L^p(\mathbb{R}^d_v; B^{s+2/3-\varepsilon}_{p,p}(\mathbb{R}^d_x)) \subset L^p(\mathbb{R}^d_v; W^{1,p}(\mathbb{R}^d_x))$ with ε small enough (recall that $s \in (1/3, 1)$) we finally obtain that

$$G_\lambda : L^p(\mathbb{R}^d_v; W^{s,p}(\mathbb{R}^d_x)) \to L^p(\mathbb{R}^d_v; W^{1,p}(\mathbb{R}^d_x)) \tag{3.38}$$

is linear and continuous. Moreover, we have with $\psi = G_\lambda g$

$$\int_{\mathbb{R}^d} \mathrm{d}v \int_{\mathbb{R}^d} |\partial_{x_k} \psi(x,v)|^p \, \mathrm{d}x \le C'(\lambda) \, \|g\|^p_{L^p(\mathbb{R}^d_v; W^{s,p}(\mathbb{R}^d_x))} \le C''(\lambda) \, \|g\|^p_{L^p(\mathbb{R}^d_v; H^s_p(\mathbb{R}^d_x))} , \tag{3.39}$$

$k = 1, \dots, d$, where $C'(\lambda)$ and $C''(\lambda)$ tend to 0 as $\lambda \to \infty$ (recall the estimates (3.34) and (3.36)). This proves (3.20) and (3.32).

Step 2. We prove the last assertion (3.21). The main problem is to show that

$$\begin{aligned} \int_{\mathbb{R}^d} \|D^2_v \psi(\cdot, v)\|^p_{H^s_p(\mathbb{R}^d)} \mathrm{d}v &= \int_{\mathbb{R}^d} \mathrm{d}v \int_{\mathbb{R}^d} |\mathcal{F}^{-1}_x [(1 + |\cdot|^s) \mathcal{F}_x D^2_v \psi(\cdot, v)](x)|^p \, \mathrm{d}x \\ &= \int_{\mathbb{R}^d} \mathrm{d}v \int_{\mathbb{R}^d} |D^2_v \left(\mathcal{F}^{-1}_x [(1 + |\cdot|^s) \mathcal{F}_x \psi(\cdot, v)]\right)(x)|^p \, \mathrm{d}x \le C \|g\|^p_{L^p(\mathbb{R}^d_v; H^s_p(\mathbb{R}^d_x))} \end{aligned}$$

($\mathcal{F}_x$ denotes the Fourier transform in the x-variable) with $\psi = G_\lambda g$. We introduce

$$h_s(x, v) = \mathcal{F}^{-1}_x [(1 + |\cdot|^s) \mathcal{F}_x g(\cdot, v)](x) ,$$

$x, v \in \mathbb{R}^d$. We know that $h_s \in L^p(\mathbb{R}^{2d})$ by our hypothesis on g. A straightforward computation based on the Fubini theorem shows that

$$\mathcal{F}^{-1}_x [(1 + |\cdot|^s) \mathcal{F}_x \psi(\cdot, v)](x) = G_\lambda h_s(x, v) .$$

By using (3.30) (with g replaced by h_s and ψ by $G_\lambda h_s$) we easily obtain that

$$\begin{aligned} \int_{\mathbb{R}^d} \|D^2_v \psi(\cdot, v)\|^p_{H^s_p(\mathbb{R}^d)} \mathrm{d}v &= \int_{\mathbb{R}^d} \mathrm{d}v \int_{\mathbb{R}^d} |D^2_v \, G_\lambda h_s(x, v)|^p \, \mathrm{d}x \\ &\le C \|h_s\|^p_{L^p(\mathbb{R}^{2d})} = C \|g\|^p_{L^p(\mathbb{R}^d_v; H^s_p(\mathbb{R}^d_x))} , \end{aligned} \tag{3.40}$$

where $C = C(d, p)$. Similarly, we have

$$\begin{aligned} \int_{\mathbb{R}^d} \|D_v \psi(\cdot, v)\|^p_{H^s_p(\mathbb{R}^d)} \mathrm{d}v &= \int_{\mathbb{R}^d} \mathrm{d}v \int_{\mathbb{R}^d} |D_v \, G_\lambda h_s(x, v)|^p \, \mathrm{d}x \\ &\le \frac{C}{(\lambda)^{p/2}} \|h_s\|^p_{L^p(\mathbb{R}^{2d})} = \frac{C}{(\lambda)^{p/2}} \|g\|^p_{L^p(\mathbb{R}^d_v; H^s_p(\mathbb{R}^d_x))} \end{aligned} \tag{3.41}$$

and $\|\psi\|_{L^p(\mathbb{R}^d_v; H^s_p(\mathbb{R}^d_x))} = \|G_\lambda h_s\|_{L^p(\mathbb{R}^{2d})} \le \frac{1}{\lambda} \|G_\lambda h_s\|_{L^p(\mathbb{R}^{2d})} = \frac{1}{\lambda} \|g\|_{L^p(\mathbb{R}^d_v; H^s_p(\mathbb{R}^d_x))}$. The proof is complete. □

Lemma 3.5. *Assume as in Theorem 3.4 that* $g \in L^p(\mathbb{R}^d_v; H^s_p(\mathbb{R}^d_x))$, $s \in (1/3, 1)$. *Moreover, suppose that* $p > d$. *Then the solution* $\psi = G_\lambda g$ *to* (3.17) *verifies also*

$$\sup_{v \in \mathbb{R}^d} \|D_v \psi(\cdot, v)\|_{H^s_p(\mathbb{R}^d)} \le C(\lambda) \|g\|_{L^p(\mathbb{R}^d_v; H^s_p(\mathbb{R}^d_x))} , \quad \lambda > 0 , \tag{3.42}$$

where $C(\lambda) \to 0$ *as* $\lambda \to \infty$.

Proof. Using the notation introduced in the previous proof, we have for any $v \in \mathbb{R}^d$, a.e.,

$$\|D_v \psi(\cdot, v)\|^p_{H^s_p(\mathbb{R}^d)} = \int_{\mathbb{R}^d} |D_v\, G_\lambda h_s(x,v)|^p \, \mathrm{d}x\,.$$

By (3.40), (3.41) and the Fubini theorem we know that

$$\int_{\mathbb{R}^d} \mathrm{d}x \int_{\mathbb{R}^d} |D_v\, G_\lambda h_s(x,v)|^p \, \mathrm{d}v + \int_{\mathbb{R}^d} \mathrm{d}x \int_{\mathbb{R}^d} |D^2_v\, G_\lambda h_s(x,v)|^p \, \mathrm{d}v < \infty\,.$$

It follows that, for $x \in \mathbb{R}^d$ a.e.,

$$\|D_v\, G_\lambda h_s(x,\cdot)\|^p_{W^{1,p}(\mathbb{R}^d)} = \int_{\mathbb{R}^d} |D_v\, G_\lambda h_s(x,v)|^p \, \mathrm{d}v + \int_{\mathbb{R}^d} |D^2_v\, G_\lambda h_s(x,v)|^p \, \mathrm{d}v < \infty\,.$$

In order to prove (3.42) with $C(\lambda) \to 0$, we consider $r \in (0,1)$ such that $rp > d$. Let us fix $x \in \mathbb{R}^d$, a.e.; by the previous estimate the mapping $v \mapsto D_v\, G_\lambda h_s(x,v)$ belongs to $W^{r,p}(\mathbb{R}^d) \subset W^{1,p}(\mathbb{R}^d)$.

We can apply the Sobolev embedding theorem (see [35, page 203]) and get that $v \mapsto D_v\, G_\lambda h_s(x,v)$ in particular is bounded and continuous on $\mathbb{R}^d$. Moreover,

$$\sup_{v \in \mathbb{R}^d} |D_v\, G_\lambda h_s(x,v)|^p \le c\, \|D_v\, G_\lambda h_s(x,\cdot)\|^p_{W^{r,p}(\mathbb{R}^d)}\,, \tag{3.43}$$

where $c = c(p,d,r)$. Integrating with respect to x we get

$$\int_{\mathbb{R}^d} \big[\sup_{v \in \mathbb{R}^d} |D_v\, G_\lambda h_s(x,v)|^p\big] \, \mathrm{d}x \le c \int_{\mathbb{R}^d} \|D_v\, G_\lambda h_s(x,\cdot)\|^p_{W^{r,p}(\mathbb{R}^d)} \mathrm{d}x\,.$$

By (3.8) we know that $\big(L^p(\mathbb{R}^d), W^{1,p}(\mathbb{R}^d)\big)_{r,p} = W^{r,p}(\mathbb{R}^d)$. Applying [29, Corollary 1.2.7] we obtain that, for any $f \in W^{1,p}(\mathbb{R}^d)$,

$$\|f\|_{W^{r,p}(\mathbb{R}^d)} \le c(r,p) \left(\|f\|_{L^p(\mathbb{R}^d)}\right)^{1-r} \cdot \left(\|f\|_{W^{1,p}(\mathbb{R}^d)}\right)^r.$$

It follows that

$$\begin{aligned}
\sup_{v \in \mathbb{R}^d} \|D_v \psi(\cdot, v)\|^p_{H^s_p(\mathbb{R}^d)} &= \sup_{v \in \mathbb{R}^d} \int_{\mathbb{R}^d} |D_v\, G_\lambda h_s(x,v)|^p \, \mathrm{d}x \\
&\le \int_{\mathbb{R}^d} \big[\sup_{v \in \mathbb{R}^d} |D_v\, G_\lambda h_s(x,v)|^p\big] \, \mathrm{d}x \le c \int_{\mathbb{R}^d} \|D_v\, G_\lambda h_s(x,\cdot)\|^p_{W^{r,p}(\mathbb{R}^d)} \mathrm{d}x \\
&\le c' \int_{\mathbb{R}^d} \|D_v\, G_\lambda h_s(x,\cdot)\|^{p(1-r)}_{L^p(\mathbb{R}^d)} \cdot \|D_v\, G_\lambda h_s(x,\cdot)\|^{pr}_{W^{1,p}(\mathbb{R}^d)} \mathrm{d}x \\
&\le c' \Big(\int_{\mathbb{R}^d} \|D_v\, G_\lambda h_s(x,\cdot)\|^p_{L^p(\mathbb{R}^d)} \mathrm{d}x\Big)^{1-r} \cdot \Big(\int_{\mathbb{R}^d} \|D_v\, G_\lambda h_s(x,\cdot)\|^p_{W^{1,p}(\mathbb{R}^d)} \mathrm{d}x\Big)^r.
\end{aligned}$$

Now we easily obtain (3.42) using (3.40) and (3.41), since

$$\int_{\mathbb{R}^d} \|D_v\, G_\lambda h_s(x,\cdot)\|^p_{L^p(\mathbb{R}^d)} \mathrm{d}x = \int_{\mathbb{R}^d} \mathrm{d}v \int_{\mathbb{R}^d} |D_v\, G_\lambda h_s(x,v)|^p \, \mathrm{d}x \le C(\lambda)\, \|g\|_{L^p(\mathbb{R}^d_v; H^s_p(\mathbb{R}^d_x))}$$

with $C(\lambda) \to 0$ as $\lambda \to \infty$. □

We complete the study of the regularity of solutions to equation (3.17) with the next result in which we strengthen the assumptions of Lemma 3.5. Note that the next assumption on p holds when $p > 6d$ as in Hypothesis 2.1.

Lemma 3.6. *Let $\lambda > 0$, $s \in (2/3, 1)$ and $g \in L^p(\mathbb{R}^d_v; H^s_p(\mathbb{R}^d_x))$. In addition assume that $p(s - \frac{1}{3}) > 2d$, then the following statements hold.*

(i) The solution $\psi = G_\lambda g$ (see (3.28)) is bounded and Lipschitz continuous on $\mathbb{R}^{2d}$. Moreover there exists the classical derivative $D_x\psi$ which is continuous and bounded on $\mathbb{R}^{2d}$ and, for $\lambda > 0$,

$$\|\psi\|_\infty + \|D_x\psi\|_\infty \le C(\lambda)\|g\|_{L^p(\mathbb{R}^d_v; H^s_p(\mathbb{R}^d_x))}, \quad \text{with } C(\lambda) \to 0 \text{ as } \lambda \to \infty. \tag{3.44}$$

(ii) $D_v\psi \in W^{1,p}(\mathbb{R}^{2d})$ (so in particular there exist weak partial derivatives $\partial_{x_i}\partial_{v_j}\psi \in L^p(\mathbb{R}^{2d})$, $i, j = 1, \dots, d$) and

$$\|D_v\psi\|_{W^{1,p}(\mathbb{R}^{2d})} \le c(\lambda)\|g\|_{L^p(\mathbb{R}^d_v; H^s_p(\mathbb{R}^d_x))}, \quad \lambda > 0, \quad \text{with } c = c(\lambda) \to 0 \text{ as } \lambda \to \infty. \tag{3.45}$$

Proof. (i) The boundedness of ψ follows easily from estimates (3.19) and (3.20) using the Sobolev embedding since in our case $p > 2d$. Let us concentrate on proving the Lipschitz continuity.

First we recall a Fubini type theorem for fractional Sobolev spaces (see [33]):

$$W^{\gamma,p}(\mathbb{R}^{2d}) = \Big\{ f \in L^p(\mathbb{R}^{2d}) : \int_{\mathbb{R}^d} \|f(x,\cdot)\|^p_{W^{\gamma,p}(\mathbb{R}^d))} \mathrm{d}x + \int_{\mathbb{R}^d} \|f(\cdot,v)\|^p_{W^{\gamma,p}(\mathbb{R}^d))} \mathrm{d}v < \infty \Big\}, \tag{3.46}$$

$\gamma \in (0, 1]$ (with equivalence of the respective norms). Let $\eta \in (0, s + 2/3 - 1)$ be such that

$$\eta p > 2d. \tag{3.47}$$

We will prove that $D_x\psi \in W^{\eta,p}(\mathbb{R}^{2d})$ so that by the Sobolev embedding $W^{\eta,p}(\mathbb{R}^{2d}) \subset C_b^{\eta - 2d/p}(\mathbb{R}^{2d})$ (see [35, page 203]) we get the assertion. According to (3.46) we check that

$$\int_{\mathbb{R}^d} \|D_x\psi(\cdot, v)\|^p_{W^{\eta,p}(\mathbb{R}^d)} \mathrm{d}v \le C(\lambda)\|g\|^p_{L^p(\mathbb{R}^d_v; H^s_p(\mathbb{R}^d_x))}, \tag{3.48}$$

and

$$\int_{\mathbb{R}^d} \|D_x\psi(x, \cdot)\|^p_{W^{\eta,p}(\mathbb{R}^d)} \mathrm{d}x \le C(\lambda)\|g\|^p_{L^p(\mathbb{R}^d_v; H^s_p(\mathbb{R}^d_x))}. \tag{3.49}$$

with $C(\lambda) \to 0$ as $\lambda \to \infty$. Estimate (3.48) follows by (3.37) which gives $\psi \in L^p(\mathbb{R}^d_v; B^{\eta+1}_{p,p}(\mathbb{R}^d_x))$ with $\eta = s - \epsilon - 1/3$.

Let us concentrate on (3.49). We still use the interpolation theory results of Section 3.2 but here in addition to (3.12) we also need to identify $L^p(\mathbb{R}^d; H^s_p(\mathbb{R}^d))$ with the Banach space $L^p(\mathbb{R}^d_x; H^s_p(\mathbb{R}^d_v))$ of all measurable functions $f(x, v)$, $f : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ such that $f(x, \cdot) \in H^s_p(\mathbb{R}^d)$, for $x \in \mathbb{R}^d$ a.e., $0 \le s \le 2$, and, moreover $\int_{\mathbb{R}^d} \|f(x,\cdot)\|^p_{H^s_p} \mathrm{d}x < \infty$. As a norm one considers

$$\|f\|_{L^p(\mathbb{R}^d_x; H^s_p(\mathbb{R}^d_v))} = \Big(\int_{\mathbb{R}^d} \|f(x,\cdot)\|^p_{H^s_p(\mathbb{R}^d)} \mathrm{d}x \Big)^{1/p}. \tag{3.50}$$

Similarly, we identify $L^p(\mathbb{R}^d; B^s_{p,p}(\mathbb{R}^d))$ with the Banach space $L^p(\mathbb{R}^d_x; B^s_{p,p}(\mathbb{R}^d_v))$ of all measurable functions $f : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ such that $f(x,\cdot) \in B^s_{p,p}(\mathbb{R}^d)$, for x a.e., and $\int_{\mathbb{R}^d} \|f(x,\cdot)\|^p_{B^s_{p,p}(\mathbb{R}^d)} \mathrm{d}x < \infty$. By (3.19) and (3.20) in Theorem 3.4 and using (3.35) we find with $\psi = G_\lambda g$

$$\begin{aligned} \int_{\mathbb{R}^d} \mathrm{d}x \int_{\mathbb{R}^d} |D_x\psi(x,v)|^p \,\mathrm{d}v &= \int_{\mathbb{R}^d} \|D_x\psi(x,\cdot)\|^p_{L^p(\mathbb{R}^d)} \mathrm{d}x \le C(\lambda)\|g\|^p_{L^p(\mathbb{R}^d_v; H^s_p(\mathbb{R}^d_x))}, \\ \int_{\mathbb{R}^d} \mathrm{d}x \int_{\mathbb{R}^d} |D^2_v(D_x\psi)(x,v)|^p \,\mathrm{d}v &= \int_{\mathbb{R}^d} \|D^2_v(D_x\psi)(x,\cdot)\|^p_{L^p(\mathbb{R}^d)} \mathrm{d}x \le c\|g\|^p_{L^p(\mathbb{R}^d_v; H^1_p(\mathbb{R}^d_x))}, \end{aligned} \tag{3.51}$$

with $c = c(d,p) > 0$. Thus we can consider the following linear maps ($s' \in (1/3,1)$ will be fixed below)

$$D_x G_\lambda : L^p\big(\mathbb{R}^d_v; H^{s'}_p(\mathbb{R}^d_x)\big) \to L^p(\mathbb{R}^{2d}) = L^p\big(\mathbb{R}^d_x; L^p(\mathbb{R}^d_v)\big)\,, \qquad (3.52)$$
$$D_x G_\lambda : L^p\big(\mathbb{R}^d_v; H^1_p(\mathbb{R}^d_x)\big) \to L^p\big(\mathbb{R}^d_x; H^2_p(\mathbb{R}^d_v)\big)\,.$$

Interpolating, choosing $s' \in (1/3,1)$ such that

$$s' < 2s - 1\,,$$

we get (see (3.8) and (3.10) with $\theta = \frac{s-s'}{1-s'} > 1/2$)

$$D_x G_\lambda : L^p\big(\mathbb{R}^d_v; W^{s,p}(\mathbb{R}^d_x)\big) = \Big(L^p\big(\mathbb{R}^d_v; H^{s'}_p(\mathbb{R}^d_x)\big)\,,\, L^p\big(\mathbb{R}^d_v; H^1_p(\mathbb{R}^d_x)\big)\Big)_{\theta,p} \qquad (3.53)$$
$$\longrightarrow \Big(L^p\big(\mathbb{R}^d_x; L^p(\mathbb{R}^d_v)\big)\,,\, L^p\big(\mathbb{R}^d_x; H^2_p(\mathbb{R}^d_v)\big)\Big)_{\theta,p} = L^p\big(\mathbb{R}^d_x; B^{2\theta}_{p,p}(\mathbb{R}^d_v)\big)$$

and by the estimates in (3.51) we find

$$\int_{\mathbb{R}^d} \|D_x(G_\lambda g)(x,\cdot)\|^p_{B^{2\theta}_{p,p}(\mathbb{R}^d)} \mathrm{d}x \le C'(\lambda)\|g\|^p_{L^p(\mathbb{R}^d_v; H^s_p(\mathbb{R}^d_x))}$$

(recall that $H^s_p(\mathbb{R}^d) \subset W^{s,p}(\mathbb{R}^d)$). Since $\eta < 2/3$ we have $B^{2\theta}_{p,p}(\mathbb{R}^d) \subset W^{\eta,p}(\mathbb{R}^d)$ (cf. (3.6)) and we finally get (3.49).

(ii) We fix $j = 1,\dots,d$ and prove the assertion with $D_v\psi$ replaced by $\partial_{v_j}\psi$.

By Theorem 3.4 we already know that there exists $D_v\partial_{v_j}\psi \in L^p(\mathbb{R}^{2d})$. Therefore to show the assertion it is enough to check that there exists the weak derivative

$$D_x(\partial_{v_j}\psi) = \partial_{v_j}(D_x\psi) \in L^p(\mathbb{R}^{2d})\,. \qquad (3.54)$$

We use again (3.53) with the same θ. Since $2\theta > 1$ we know in particular that $D_x G_\lambda g \in L^p(\mathbb{R}^d_x; W^{1,p}(\mathbb{R}^d_v))$. Thus we have that there exists the weak derivative $\partial_{v_j} D_x\psi(x,\cdot)$, for x a.e., and

$$\int_{\mathbb{R}^d}\mathrm{d}x\int_{\mathbb{R}^d}|\partial_{v_j}D_x\psi(x,v)|^p\,\mathrm{d}v = \int_{\mathbb{R}^d}\|\partial_{v_j}D_x\psi(x,\cdot)\|^p_{L^p(\mathbb{R}^d)}\mathrm{d}x \le C(\lambda)\,\|g\|^p_{L^p(\mathbb{R}^d_v;H^s_p(\mathbb{R}^d_x))}\,. \qquad (3.55)$$

This finishes the proof. □

Now we study the complete equation

$$\lambda\psi(z) - \frac{1}{2}\triangle_v\psi(z) - v\cdot D_x\psi(z) - F(z)\cdot D_v\psi(z) = g(z), \quad z = (x,v) \in \mathbb{R}^{2d}, \qquad (3.56)$$

assuming that $F \in L^p(\mathbb{R}^d_v; H^s_p(\mathbb{R}^d_x))$ (cf. (3.12) and (3.13)). From the previous results we obtain (see also Definition 3.3)

Theorem 3.7. *Let $s \in (2/3,1)$ and p be such that $p(s-\frac{1}{3}) > 2d$. Assume that*

$$g,\, F \in L^p(\mathbb{R}^d_v; H^s_p(\mathbb{R}^d_x))\,.$$

Then there exists $\lambda_0 = \lambda_0(s,p,d,\|F\|_{L^p(\mathbb{R}^d_v;H^s_p(\mathbb{R}^d_x))}) > 0$ such that for any $\lambda > \lambda_0$ there exists a unique solution $\psi = \psi_\lambda \in X_{p,s}$ to (3.56) and moreover

$$\lambda\|\psi\|_{L^p(\mathbb{R}^d_v;H^s_p(\mathbb{R}^d_x))} + \sqrt{\lambda}\|D_v\psi\|_{L^p(\mathbb{R}^d_v;H^s_p(\mathbb{R}^d_x))} + \|D^2_v\psi\|_{L^p(\mathbb{R}^d_v;H^s_p(\mathbb{R}^d_x))} \qquad (3.57)$$
$$+\|v\cdot D_x\psi\|_{L^p(\mathbb{R}^d_v;H^s_p(\mathbb{R}^d_x))} \le C\|g\|_{L^p(\mathbb{R}^d_v;H^s_p(\mathbb{R}^d_x))}$$

with $C = C(s,p,d,\|F\|_{L^p(\mathbb{R}^d_v;H^s_p(\mathbb{R}^d_x))}) > 0$*. We also have*

$$\sup_{v\in\mathbb{R}^d} \|D_v\psi(\cdot,v)\|_{H^s_p(\mathbb{R}^d)} \le C(\lambda)\|g\|_{L^p(\mathbb{R}^d_v;H^s_p(\mathbb{R}^d_x))}, \quad \textit{with } C(\lambda)\to 0 \textit{ as } \lambda\to\infty\,. \tag{3.58}$$

Moreover, $\psi \in C^1_b(\mathbb{R}^{2d})$*, i.e.,* ψ *is bounded on* $\mathbb{R}^{2d}$ *and there exist the classical derivatives* $D_x\psi$ *and* $D_v\psi$ *which are bounded and continuous on* $\mathbb{R}^{2d}$*; we also have with* $C(\lambda)\to 0$ *as* $\lambda\to\infty$

$$\|\psi\|_\infty + \|D_x\psi\|_{L^p(\mathbb{R}^{2d})} + \|D_x\psi\|_\infty + \|D_v\psi\|_\infty \le C(\lambda)\|g\|_{L^p(\mathbb{R}^d_v;H^s_p(\mathbb{R}^d_x))}\,. \tag{3.59}$$

Finally, $D_v\psi\in W^{1,p}(\mathbb{R}^{2d})$ *and*

$$\|D_v\psi\|_{W^{1,p}(\mathbb{R}^{2d})} \le c(\lambda)\|g\|_{L^p(\mathbb{R}^d_v;H^s_p(\mathbb{R}^d_x))}\,, \qquad c=c(\lambda)\to 0 \;\textit{ as }\; \lambda\to\infty\,. \tag{3.60}$$

Proof. First note that, since $p>2d$, the boundedness of ψ follows by the Sobolev embedding (recall also (3.18)). Similarly the second estimate in (3.59) follows from (3.60).

We consider the Banach space $Y = L^p(\mathbb{R}^d_v;H^s_p(\mathbb{R}^d_x))$ and use an argument similar to the one used in the proof of [10, Proposition 5]. Introduce the operator $T_\lambda : Y\to Y$,

$$T_\lambda f := F\cdot D_v(G_\lambda f)\,, \quad f\in Y,$$

where G_λ is defined in (3.27). It is not difficult to check that $T_\lambda f\in Y$ for $f\in Y$. Indeed by Lemma 3.5 we get

$$\begin{aligned}\int_{\mathbb{R}^d} \|T_\lambda f(\cdot,v)\|^p_{H^s_p(\mathbb{R}^d)}\mathrm{d}v &\le \sup_{v\in\mathbb{R}^d}\|D_v(G_\lambda f)(\cdot,v)\|^p_{H^s_p(\mathbb{R}^d)}\int_{\mathbb{R}^d}\|F(\cdot,v)\|^p_{H^s_p(\mathbb{R}^d)}\mathrm{d}v\\ &\le C(\lambda)\|f\|^p_{L^p(\mathbb{R}^d_v;H^s_p(\mathbb{R}^d_x))}\|F\|^p_{L^p(\mathbb{R}^d_v;H^s_p(\mathbb{R}^d_x))}.\end{aligned}$$

It is clear that T_λ is linear and bounded. Moreover we find easily that there exists $\lambda_0>0$ such that for any $\lambda>\lambda_0$ we have that the operator norm of T_λ is less than $1/2$.

Let us fix $\lambda>\lambda_0$. Since T_λ is a strict contraction, there exists a unique solution $f\in Y$ to

$$f - T_\lambda f = g\,. \tag{3.61}$$

We write $f = (\mathbb{I}-T_\lambda)^{-1}g\in Y$.

Uniqueness. Let ψ_1 and ψ_2 be solutions in $X_{p,s}$. Set $w=\psi_1-\psi_2$. We know that

$$\lambda w(z) - \frac{1}{2}\triangle_v w(z) - v\cdot D_x w(z) - F(z)\cdot D_v w(z) = 0\,.$$

We have $\lambda w - \frac12\triangle_v w - v\cdot D_x w = f\in Y$. By uniqueness (see Theorem 3.4) we get that $w = G_\lambda f$. Hence, for z a.e.,

$$0 = f(z) - F(z)\cdot D_v w(z) = f(z) - F(z)\cdot D_v G_\lambda f(z)\,.$$

Since T_λ is a strict contraction we obtain that $f=0$ and so $\psi_1=\psi_2$.

Existence. It is not difficult to prove that

$$\psi = \psi_\lambda = G_\lambda(\mathbb{I}-T_\lambda)^{-1}g\,, \tag{3.62}$$

is the unique solution to (3.56).

Regularity of ψ *and estimates.* All the assertions follow easily from (3.62) since $(\mathbb{I}-T_\lambda)^{-1}g\in Y$ and we can apply Theorem 3.4, Lemmas 3.5 and 3.6. □

In the Appendix we will also present a result on the stability of the PDE (3.56), see Lemma 6.1 .

4 Regularity of the characteristics

We will prove existence of a stochastic flow for the SDE (1.2) assuming Hypothesis 2.1.

We can rewrite our SDE as follows. Set $Z_t = (X_t, V_t) \in \mathbb{R}^{2d}$, $z_0 = (x_0, v_0)$ and introduce the functions $b(x,v) = A \cdot z + B(z) : \mathbb{R}^{2d} \to \mathbb{R}^{2d}$, where

$$A = \begin{pmatrix} 0 & \mathbb{I} \\ 0 & 0 \end{pmatrix}, \quad R = \begin{pmatrix} 0 \\ \mathbb{I} \end{pmatrix}, \quad R\,R^* = Q = \begin{pmatrix} 0 & 0 \\ 0 & \mathbb{I} \end{pmatrix}, \quad B = RF = \begin{pmatrix} 0 \\ F \end{pmatrix} : \mathbb{R}^{2d} \to \mathbb{R}^{2d}. \tag{4.1}$$

With this new notation, (1.2) can be rewritten as

$$\begin{cases} \mathrm{d}Z_t = b(Z_t)\,\mathrm{d}t + R \cdot \mathrm{d}W_t \\ Z_0 = z_0 \end{cases} \tag{4.2}$$

or

$$\begin{cases} \mathrm{d}Z_t = \big(A \cdot Z_t + B(Z_t)\big)\mathrm{d}t + R \cdot \mathrm{d}W_t \\ Z_0 = z_0 \end{cases}. \tag{4.3}$$

We have

$$\begin{aligned} X_t &= x_0 + \int_0^t V_s\,\mathrm{d}s = x_0 + tv_0 + \int_0^t (t-s)F(X_s,V_s)\,\mathrm{d}s + \int_0^t W_s\,\mathrm{d}s\,, \\ V_t &= v_0 + \int_0^t F(X_s,V_s)\,\mathrm{d}s + W_t\,. \end{aligned} \tag{4.4}$$

4.1 Strong well posedness

To prove strong well posedness for (4.2) we will also use solutions U with values in $\mathbb{R}^{2d}$ of

$$\begin{aligned} &\lambda U(z) - \frac{1}{2}\mathrm{Tr}\big(QD^2U(z)\big) - \langle Az, DU(z)\rangle - \langle B(z), DU(z)\rangle = B(z)\,, \\ &\text{i.e.,}\ \ \lambda U(z) - \mathcal{L}U(z) = B(z) \end{aligned} \tag{4.5}$$

(defined componentwise at least for λ large enough). Note that $U = \begin{pmatrix} 0 \\ \tilde{u} \end{pmatrix}$ where

$$\lambda \tilde{u}(z) - \mathcal{L}\tilde{u}(z) = F(z)$$

is again defined componentwise ($\tilde{u} : \mathbb{R}^{2d} \to \mathbb{R}^d$).

Remark 4.1. In the following, according to (4.1), we will say that the singular diffusion Z_t (the noise acts only on the last d coordinates $\{e_{d+1}, \dots, e_{2d}\}$) or the associated Kolmogorov operator

$$\mathcal{L}f(z) = \frac{1}{2}\triangle_v f(z) + \langle b(z), Df(z)\rangle\,,$$

$b(z) = Az + B(z)$, are hypoelliptic to refer to the fact that the vectors

$$\big\{e_{d+1}, \dots, e_{2d}, Ae_{d+1}, \dots, Ae_{2d}\big\}$$

generate $\mathbb{R}^{2d}$. Equivalently using Q given in (4.1) and the adjoint matrix A^* we have that the symmetric matrix $Q_t = \int_0^t e^{sA}Qe^{sA^*}\mathrm{d}s$ is positive definite for any $t > 0$ (cf. (3.25)). Note that

$$\det(Q_t) = c\,t^{4d}, \ \ t > 0\,.$$

We collect here some preliminary results, which we will later need. Recall the OU process

$$\begin{cases} \mathrm{d}L_t = AL_t\,\mathrm{d}t + R\,\mathrm{d}W_t \\ L_0 = z \in \mathbb{R}^{2d} \end{cases}, \quad \text{i.e.,}\quad L_t = L_t^z = e^{tA}z + \int_0^t e^{(t-s)A}R\,\mathrm{d}W_s\,. \tag{4.6}$$

Using the fact that L_t is hypoelliptic, for any $t>0$, one gets that the law of L_t is equivalent to the Lebesgue measure in $\mathbb{R}^{2d}$ (see for example the proof of the next lemma). We also have the following result.

Lemma 4.2. *Let (L_t^z) be the OU process solution of (4.6). Let $f:\mathbb{R}^{2d}\to\mathbb{R}$ belong to $L^q(\mathbb{R}^{2d})$ for $q>2d$. Then there exists a constant C depending on q,d and T such that*

$$\sup_{z\in\mathbb{R}^{2d}}\mathbb{E}\Big[\int_0^T f(L_s^z)\,\mathrm{d}s\Big]\le C\|f\|_{L^q(\mathbb{R}^{2d})}\,. \tag{4.7}$$

Proof. We need to compute

$$\mathbb{E}\Big[\int_0^T f(L_s^z)\,\mathrm{d}s\Big]=\int_0^T P_sf(z)\,\mathrm{d}s\,,$$

where P_t is the Ornstein-Uhlenbeck semigroup introduced in (3.24). By changing variable and using the Hölder inequality we find, for $t\in[0,T]$, $z\in\mathbb{R}^{2d}$,

$$\begin{aligned}|P_tf(z)|&=\Big|c_d\int_{\mathbb{R}^{2d}}f(e^{tA}z+\sqrt{Q_t}\,y)e^{-\frac{|y|^2}{2}}\,\mathrm{d}y\Big|\le c_q\Big(\int_{\mathbb{R}^{2d}}|f(e^{tA}z+\sqrt{Q_t}\,y)|^q\,\mathrm{d}y\Big)^{1/q}\\&=\frac{c_q}{(\det(Q_t))^{1/2q}}\Big(\int_{\mathbb{R}^{2d}}|f(e^{tA}z+w)|^q\,\mathrm{d}w\Big)^{1/q}=\frac{c_q}{(\det(Q_t))^{1/2q}}\|f\|_{L^q(\mathbb{R}^{2d})}\,.\end{aligned}$$

with c_q independent of z. We now have to study when

$$\int_0^t\frac{1}{(\det(Q_s))^{1/2q}}\,\mathrm{d}s<\infty\,. \tag{4.8}$$

By a direct computation for $s\to0^+$

$$(\det(Q_s))^{1/2q}\sim c(s^{4d})^{1/2q}\,,$$

hence the result follows for $q>2d$. □

We state now the classical Khas'minskii lemma for an OU process. The original version of this lemma ([23], or [34, Section 1, Lemma 2.1]) is stated for a Wiener process, but the proof only relies on the Markov property of the process, so that its extension to this setting requires no modification.

Lemma 4.3 (Khas'minskii 1959)**.** *Let (L_t^z) be our $2d-$dimensional OU process starting from z at time 0 and $f:\mathbb{R}^{2d}\to\mathbb{R}$ be a positive Borel function. Then, for any $T>0$ such that*

$$\alpha=\sup_{z\in\mathbb{R}^{2d}}\mathbb{E}\Big[\int_0^T f\big(L_t^z\big)\,\mathrm{d}t\Big]<1\,, \tag{4.9}$$

we also have

$$\sup_{z\in\mathbb{R}^{2d}}\mathbb{E}\Big[\exp\Big(\int_0^T f\big(L_t^z\big)\,\mathrm{d}t\Big)\Big]<\frac{1}{1-\alpha}\,. \tag{4.10}$$

We now introduce a generalization of the previous Khas'minskii lemma which we will use to prove the Novikov condition, allowing us to apply Girsanov's theorem.

Proposition 4.4. *Let (L_t) be the OU process solution of (4.6). Let $f:\mathbb{R}^{2d}\to\mathbb{R}$ belong to $L^q(\mathbb{R}^{2d})$ for $q>2d$. Then, there exists a constant K_f depending on d,q,T and continuously depending on $\|f\|_{L^q(\mathbb{R}^{2d})}$ such that*

$$\sup_{z\in\mathbb{R}^{2d}}\mathbb{E}\Big[\exp\Big(\int_0^T|f(L_s^z)|\mathrm{d}s\Big)\Big]=K_f<\infty\,. \tag{4.11}$$

Proof. From Lemma 4.2, for any $a > 1$ s.t. $q/a > 2d$ we get

$$\sup_{z\in\mathbb{R}^{2d}} \mathbb{E}\Big[\int_0^T |f|^a(L_s^z)\,\mathrm{d}s\Big] \le C\|f\|^a_{L^q(\mathbb{R}^{2d})}\,.$$

Setting $\varepsilon = (C\|f\|^a_{L^q})^{-1}\wedge 1$, we apply Young's inequality: $|f(z)| \le \frac{\epsilon}{a}|f(z)|^a + C_\epsilon \frac{a-1}{a}$ and Khas'minskii's Lemma 4.3 replacing f with $\frac{\varepsilon}{a}|f|^a$ to get

$$\sup_{z\in\mathbb{R}^{2d}} \mathbb{E}\Big[\exp\Big(\int_0^T |f(L_s^z)|\mathrm{d}s\Big)\Big] \le \sup_{z\in\mathbb{R}^{2d}} \mathbb{E}\Big[\exp\Big(\int_0^T \frac{\varepsilon}{a}|f(L_s^z)|^a\mathrm{d}s\Big)\Big]e^{Tc_{\varepsilon,a}} \le \frac{1}{1-\alpha}e^{CT} < \infty\,.$$

□

The next result can be proved by using the Girsanov theorem (cf. [22] and [28]).

Theorem 4.5. *Suppose that in* (4.2) *we have $F \in L^p(\mathbb{R}^{2d};\mathbb{R}^d)$ with $p > 4d$. Then the following statements hold.*

(i) Equation (4.2) *is well posed in the weak sense.*

(ii) For any $z \in \mathbb{R}^{2d}$, $T > 0$ the law in the space of continuous functions $C([0,T];\mathbb{R}^{2d})$ of the solution $Z = (Z_t) = (Z_t^z)$ to the equation (4.2) is equivalent to the law of the OU process $L = (L_t) = (L_t^z)$.

(iii) For any $t > 0$, $z \in \mathbb{R}^{2d}$, the law of Z_t is equivalent to the Lebesgue measure in $\mathbb{R}^{2d}$.

Proof. *(i) Existence.* We argue similarly to the proof of [22, Theorem IV.4.2]. Let $T > 0$. Starting from an Ornstein-Uhlenbeck process (cf. (4.6))

$$L_t = L_t^z = z + \int_0^t AL_s\,\mathrm{d}s + RW_t\,,\ \ t \ge 0$$

defined on a stochastic basis $(\Omega,\mathcal{F},(\mathcal{F}_t),\mathbb{P})$ on which it is defined an $\mathbb{R}^d$-valued Wiener process $(W_t) = W$, we can define the process

$$H_t := W_t - \int_0^t F(L_r)\,\mathrm{d}r\,,\ \ t\in[0,T]\,. \tag{4.12}$$

Since $p > 4d$, Proposition 4.4 with $f = F^2$ provides the Novikov condition ensuring that the process

$$\Phi_t = \exp\Big(\int_0^t \langle F(L_s),\mathrm{d}W_s\rangle \,-\, \frac{1}{2}\int_0^t |F(L_s)|^2\mathrm{d}s\Big)\,,\ \ t\in[0,T]\,,$$

is an $\mathcal{F}_t$-martingale. Then, by the Girsanov theorem $(H_t)_{t\in[0,T]}$ is a d-dimensional Wiener process on $(\Omega,\mathcal{F}_T,(\mathcal{F}_s)_{s\le T},\mathbb{Q})$, where $\mathbb{Q}$ is the probability measure on $(\Omega,\mathcal{F}_T)$ having density $\Phi = \Phi_T$ with respect to $\mathbb{P}$. We have that on the new probability space

$$L_t = L_t^z = z + \int_0^t AL_s\,\mathrm{d}s + \int_0^t RF(L_s)\,\mathrm{d}s + RH_t\,,\ \ t\in[0,T]$$

(cf. (4.1)). Hence $L = (L_t)$ is a solution to (4.2) on $(\Omega,\mathcal{F}_T,(\mathcal{F}_s)_{s\le T},\mathbb{Q})$.

Uniqueness. To prove weak uniqueness we use some results from [28]. First note that the process

$$V_t = v_0 + \int_0^t F(X_s,V_s)\,\mathrm{d}s + W_t \tag{4.13}$$

(cf. (4.4)) is a process of diffusion type according to [28, Definition 7, Section 4.2, page 118]. Indeed, since $X_t = x_0 + \int_0^t V_s\,\mathrm{d}s$ we have

$$V_t = v_0 + \int_0^t F\Big(x_0 + \int_0^s V_r\,\mathrm{d}r, V_s\Big)\,\mathrm{d}s + W_t$$

and the process $(b_s(V))_{s\in[0,T]} = (F(x_0 + \int_0^s V_r\,\mathrm{d}r, V_s))_{s\in[0,T]}$ is $(\mathcal{F}_t^V)$-adapted (here $\mathcal{F}_t^V$ is the σ-algebra generated by $\{V_s,\ s \in [0,t]\}$).

We can apply to $V = (V_t)$ [28, Theorem 7.5, page 257] (see also [28, Paragraph 7.2.7]): since $\int_0^T |b_s(V)|^2 \mathrm{d}s < \infty$, $\mathbb{P}$-a.s., we obtain that

$$\mu_V \sim \mu_W \ \text{ on } \ \mathcal{B}\big(C([0,T];\mathbb{R}^d)\big)\,,$$

i.e. the laws of $V = (V_t)_{t\in[0,T]}$ and $W = (W_t)_{t\in[0,T]}$ are equivalent. Moreover, by [28, Theorem 7.7], the Radon-Nykodim derivative $\frac{\mu_V}{\mu_W}(x)$, $x \in C([0,T];\mathbb{R}^d)$, verifies

$$\frac{\mu_V}{\mu_W}(W) = \exp\Big(\int_0^T \langle b_s(W), \mathrm{d}W_s\rangle \,-\, \frac{1}{2}\int_0^T |b_s(W)|^2\,\mathrm{d}s\Big)\,.$$

It follows that, for any Borel set $B \in \mathcal{B}(C([0,T];\mathbb{R}^d))$,

$$\mathbb{E}\big[1_B(V)\big] = \mathbb{E}^{\mu_W}\Big[1_B\,\frac{\mu_V}{\mu_W}\Big] = \mathbb{E}\Big[1_B(W)\,\exp\Big(\int_0^T \langle b_s(W), \mathrm{d}W_s\rangle \,-\, \frac{1}{2}\int_0^T |b_s(W)|^2\,\mathrm{d}s\Big)\Big];$$

this shows easily that uniqueness in law holds.

Clearly (iii) follows from (ii). Let us prove (ii).

(ii) The processes $L = (L_t)$ and $Z = (Z_t)$, $t \in [0,T]$, satisfy the same equation (4.2) in $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{Q}, (H_t))$ and $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P}, (W_t))$ respectively. Therefore, by weak uniqueness, the laws of L and Z on $C([0,T];\mathbb{R}^{2d})$ are the same (under the probability measures $\mathbb{Q}$ and $\mathbb{P}$ respectively). Hence, for any Borel set $J \subset C([0,T];\mathbb{R}^{2d})$, we have

$$\mathbb{E}[1_J(Z)] = \mathbb{E}[1_J(L)\,\Phi]\,.$$

Since $W_t = (\langle L_t, e_{d+1}\rangle, \ldots, \langle L_t, e_{2d}\rangle)$ we see that each W_s is measurable with respect to the σ-algebra generated by the random variable L_s, $s \le T$. By considering L as a random variable with values in $C([0,T];\mathbb{R}^{2d})$, we obtain that

$$\Phi = \exp[G(L)]$$

for some measurable function $G : M = C([0,T];\mathbb{R}^{2d}) \to \mathbb{R}$. Using the laws μ_Z of Z and μ_L of L we find

$$\int_M 1_J(\omega)\mu_Z(\mathrm{d}\omega) = \mathbb{E}[1_J(Z)] = \mathbb{E}\big[1_J(L)\,\exp[G(L)]\big] = \int_M 1_J(\omega)\ \exp[G(\omega)]\mu_L(\mathrm{d}\omega)\,.$$

Finally note that $|G(\omega)| < \infty$, for any $\omega \in M$ μ_L-a.s. (indeed $\int_M |G(\omega)|\mu_L(\mathrm{d}\omega) = E[|G(L)|] < \infty$). It follows that $\exp[G(\omega)] > 0$, for any $\omega \in M$ μ_L-a.s., and this shows that μ_L is equivalent to μ_Z. □

We can now prove that the result of Lemma 4.2 holds also when replacing the OU process L_t with Z_t.

Lemma 4.6. *Let $F \in L^p(\mathbb{R}^{2d};\mathbb{R}^d)$ for $p > 4d$ and Z_t^z be a solution of* (4.2)*. Let $f : \mathbb{R}^{2d} \to \mathbb{R}$ belong to $L^q(\mathbb{R}^{2d})$ for some $q > 2d$. Then there exists a constant C depending on q, d and T such that*

$$\sup_{z\in\mathbb{R}^{2d}} \mathbb{E}\Big[\int_0^T f(Z_s^z)\,\mathrm{d}s\Big] \le C\|f\|_{L^q(\mathbb{R}^{2d})} \tag{4.14}$$

and a constant K_f depending on q, d, T and continuously depending on $\|f\|_{L^q(\mathbb{R}^{2d})}$ for which

$$\sup_{z\in\mathbb{R}^{2d}} \mathbb{E}\Big[\exp\Big(\int_0^T f(Z_s^z)\,\mathrm{d}s\Big)\Big] \le K_f\,. \tag{4.15}$$

Proof. As seen in the previous proof, the laws of L_t and Z_t are the same under $\mathbb{Q}$ and $\mathbb{P}$ respectively. Then, applying Hölder's inequality with $1/a + 1/a' = 1$ we have

$$\mathbb{E}^{\mathbb{P}}\Big[\int_0^T f(Z_s)\,\mathrm{d}s\Big] = \mathbb{E}^{\mathbb{Q}}\Big[\int_0^T f(L_s)\,\mathrm{d}s\Big] \le \mathbb{E}^{\mathbb{P}}\Big[\int_0^T f^a(L_s)\,\mathrm{d}s\Big]^{1/a}\mathbb{E}^{\mathbb{P}}\Big[\Phi^{a'}\Big]^{1/a'}.$$

Taking $a > 1$ small enough so that $q/a > 2d$, we can apply Lemma 4.2 to $|f|^a$ and control the first expectation on the right hand side with a constant times $\|f\|_{L^q(\mathbb{R}^{2d})}$. Then we write

$$\Phi^{a'} = \exp\Big(\int_0^T \langle a'F(L_s), \mathrm{d}W_s\rangle - \frac{1}{2}\int_0^T |a'F(L_s)|^2\,\mathrm{d}s + \frac{(a')^2 - a'}{2}\int_0^T |F(L_s)|^2\,\mathrm{d}s\Big),$$

which has finite expectation due to Proposition 4.4. Both these estimates are uniform in z, so that (4.14) follows. Similarly, we have

$$\mathbb{E}^{\mathbb{P}}\Big[\exp\Big(\int_0^T f(Z_s)\,\mathrm{d}s\Big)\Big] \le \mathbb{E}^{\mathbb{P}}\Big[\exp\Big(2\int_0^T f(L_s)\,\mathrm{d}s\Big)\Big]^{1/2}\mathbb{E}^{\mathbb{P}}\Big[\Phi^2\Big]^{1/2}.$$

Both terms on the right hand side are finite due to Proposition 4.4: this proves (4.15). □

From now to the end of the paper we will assume Hypothesis 2.1.

Lemma 4.7. *Any process (Z_t) which is solution of the SDE (4.2) has finite moments of any order, uniformly in $t \in [0,T]$: for any $q \ge 2$*

$$\mathbb{E}\big[|Z_t^z|^q\big] \le C_{z,q,d,T} < \infty. \tag{4.16}$$

Proof. Recall that, setting $Z_t^z = Z_t$,

$$Z_t = z + \int_0^t F(Z_s)\,\mathrm{d}s + \int_0^t AZ_s\,\mathrm{d}s + \int_0^t R\,\mathrm{d}W_s.$$

It follows from (4.15) that for any $q \ge 1$, $\mathbb{E}\big[|\int_0^T F(Z_t)\,\mathrm{d}t|^q\big] \le C$. Using this bound, the explicit density of RW_t and the Grönwall lemma we obtain the assertion. □

In the proof of strong uniqueness of solutions of the SDE (4.2) we will have to deal with a new SDE with a Lipschitz drift coefficient, but a diffusion which only has derivatives in L^p. However, following an idea of Veretennikov [36], we can deal with increments of the diffusion coefficient on different solutions by means of the process N_t defined in (4.17). The following lemma generalizes Veretennikov's result to our degenerate kinetic setting and even provides bounds on the exponential of the process N_t. It will be a key element to prove continuity of the flow associated to (4.2) and will also be used in Subsection 4.3 to study weak derivatives of the flow.

Lemma 4.8. *Let Z_t, Y_t be two solutions of (4.2) starting from z, $y \in \mathbb{R}^{2d}$ respectively, $U : \mathbb{R}^{2d} \to \mathbb{R}^{2d}$, $U \in X_{p,s} \cap C_b^1$ (see Definition 3.3), and set*

$$N_t = \int_0^t 1_{\{Z_s \neq Y_s\}} \frac{\big\|[DU(Z_s) - DU(Y_s)]R\big\|_{HS}^2}{|Z_s - Y_s|^2}\,\mathrm{d}s, \tag{4.17}$$

where $\|\cdot\|_{HS}$ denotes the Hilbert-Schmidt norm. Then, N_t is a well-defined, real valued, continuous, adapted, increasing process such that $\mathbb{E}[N_T] < \infty$, for every $t \in [0,T]$

$$\int_0^t \Big\|\big[DU(Z_s^z) - DU(Y_s^y)\big]R\Big\|_{HS}^2\,\mathrm{d}s = \int_0^t \big|Z_s^z - Y_s^y\big|^2\,\mathrm{d}N_s \tag{4.18}$$

and for any $k \in \mathbb{R}$, uniformly with respect to the initial conditions z, y:

$$\sup_{z,y\in\mathbb{R}^{2d}} \mathbb{E}\big[e^{kN_T}\big] < \infty. \tag{4.19}$$

Proof. Recall that $B = \begin{pmatrix} 0 \\ F \end{pmatrix}$ and $\big\| \big[DU(Z_s) - DU(Y_s) \big] R \big\|_{HS}^2 = \big| D_v \tilde{u}(Z_s) - D_v \tilde{u}(Y_s) \big|^2$. We have

$$\begin{aligned} \big| D_v \tilde{u}(Z_s) - D_v \tilde{u}(Y_s) \big| &= \Big| \sum_{i=1}^{2d} (Z_s - Y_s)^i \int_0^1 D_i D_v \tilde{u}(r Z_s + (1-r) Y_s) \, \mathrm{d}r \Big| \\ &\le |Z_s - Y_s| \int_0^1 \big| D D_v \tilde{u}\big(r Z_s + (1-r) Y_s\big) \big| \, \mathrm{d}r \,. \end{aligned}$$

Set $Z_t^r = r Z_t + (1-r) Y_t$ (the process $(Z_t^r)_{t \ge 0}$ depends on $r \in [0,1]$). We will first prove that

$$\mathbb{E}\Big[\int_0^1 \mathrm{d}r \int_0^t \big| D D_v \tilde{u}(Z_s^r) \big|^2 \mathrm{d}s \Big] < \infty \,, \qquad t > 0 \,. \tag{4.20}$$

By setting $F_s^r = [r F(Z_s) + (1-r) F(Y_s)]$ and $z^r = r z + (1-r) y$, we obtain, for any $r \in [0,1]$,

$$Z_t^r = z^r + \int_0^t \begin{pmatrix} 0 \\ F_s^r \end{pmatrix} \mathrm{d}s + \begin{pmatrix} 0 \\ W_t \end{pmatrix} + \int_0^t A Z_s^r \mathrm{d}s \,.$$

Since $\int_0^T |F_s^r|^2 \, \mathrm{d}s \le C \int_0^T |F(Z_s)|^2 + |F(Y_s)|^2 \, \mathrm{d}s$, using Hölder's inequality and Lemma 4.6 we get for all $k \in \mathbb{R}$

$$\sup_{z,y} \mathbb{E}\Big[\exp\Big(k \int_0^T |F_s^r|^2 \, \mathrm{d}s \Big) \Big] \le C_k \, < \, \infty \,, \tag{4.21}$$

where the constant C_k depends on k, p, T and $\|F\|_{L^p(\mathbb{R}^{2d})}$, but is uniform in z, y and r.

We can use again the Girsanov theorem (cf. the proof of Theorem 4.5). The process

$$\widetilde{W}_t := W_t + \int_0^t F_s^r \, \mathrm{d}r \,, \qquad t \in [0,T]$$

is a d-dimensional Wiener process on $(\Omega, (\mathcal{F}_s)_{s \le T}, \mathcal{F}_T, \mathbb{Q})$, where $\mathbb{Q}$ is the probability measure on $(\Omega, \mathcal{F}_T)$ having the density ρ_r with respect to $\mathbb{P}$,

$$\rho_r = \exp\Big(\int_0^T -\langle F_s^r \,, \mathrm{d}W_s \rangle \, - \, \frac{1}{2} \int_0^T |F_s^r|^2 \, \mathrm{d}s \Big) \,.$$

Recalling the Ornstein-Uhlenbeck process L_t (starting at z^r), i.e.,

$$L_t = e^{tA} z^r + W_A(t), \qquad \text{where} \quad W_A(t) = \int_0^t e^{(t-s)A} R \, \mathrm{d}W_s \,, \tag{4.22}$$

we have:

$$Z_t^r = L_t + \int_0^t e^{(t-s)A} R \, F_s^r \, \mathrm{d}s \,.$$

Hence

$$Z_t^r = e^{tA} z^r \, + \, \int_0^t e^{(t-s)A} \, \mathrm{d}\widetilde{W}_s$$

is an OU process on $(\Omega, (\mathcal{F}_s)_{s \le T}, \mathcal{F}_T, \rho_r \mathbb{P})$.

We now find, by the Hölder inequality, for some $a > 1$ such that $1/a + 1/a' = 1$,

$$\begin{aligned} \mathbb{E}\Big[\rho_r^{-1/a} \rho_r^{1/a} \int_0^t |D D_v \tilde{u}(Z_s^r)|^2 \mathrm{d}s \Big] &\le c_T \Big(\mathbb{E}\Big[\rho_r \int_0^t |D D_v \tilde{u}(Z_s^r)|^{2a} \mathrm{d}s \Big] \Big)^{1/a} \big(\mathbb{E}[\rho_r^{-a'/a}] \big)^{1/a'} \\ &\le C_T \Big(\mathbb{E}\Big[\rho_r \int_0^t |D D_v \tilde{u}(Z_s^r)|^{2a} \, \mathrm{d}s \Big] \Big)^{1/a} \,, \end{aligned} \tag{4.23}$$

for any $t \in [0,T]$. Observe that the bound on the moments of ρ_r is uniform in the initial conditions $z, y \in \mathbb{R}^{2d}$ due to (4.21). Setting $f(z) = |DD_v\tilde{u}(z)|^{2a}$ and using the Girsanov Theorem, assertion (4.20) follows from Lemma 4.2 if we fix $a > 1$ such that $q = p/2a > 2d$.

Therefore, the process N_t is well defined and $\mathbb{E}[N_t] < \infty$ for all $t \in [0,T]$. (4.18) and the other properties of N_t follow.

To prove the exponential integrability of the process N_t we proceed in a way similar to [15, Lemma 4.5]. Using the convexity of the exponential function we get

$$\mathbb{E}\Big[e^{kN_T}\Big] \le \mathbb{E}\Big[\exp\Big(k\int_0^T\int_0^1 |DD_v\tilde{u}(Z_s^r)|^2\,\mathrm{d}r\mathrm{d}s\Big)\Big] \le \int_0^1 \mathbb{E}\Big[\exp\Big(k\int_0^T |DD_v\tilde{u}(Z_s^r)|^2\,\mathrm{d}s\Big)\Big]\mathrm{d}r$$

and we can continue as above (superscripts denote the probability measure used to take expectations)

$$\begin{aligned}\sup_{z,y}\mathbb{E}^{\mathbb{P}}\Big[e^{kN_T}\Big] &\le \sup_{z,y}\int_0^1 \mathbb{E}^{\mathbb{P}}\Big[\rho_r^{-1/a}\rho_r^{1/a}\exp\Big(k\int_0^T |DD_v\tilde{u}(Z_s^r)|^2\,\mathrm{d}s\Big)\Big]\mathrm{d}r \\ &\le C_T\sup_{z,y}\int_0^1 \mathbb{E}^{\mathbb{Q}}\Big[\exp\Big(ak\int_0^T |DD_v\tilde{u}(Z_s^r)|^2\,\mathrm{d}s\Big)\Big]^{1/a}\mathrm{d}r \\ &\le C_T\sup_{z,y}\int_0^1 \mathbb{E}^{\mathbb{P}}\Big[\exp\Big(ak\int_0^T |DD_v\tilde{u}(L_s)|^2\,\mathrm{d}s\Big)\Big]^{1/a}\mathrm{d}r\,.\end{aligned}$$

The last integral is finite due to Proposition 4.4 because $p/2 > 2d$. The proof is complete. □

Proposition 4.9 (Itô formula). *If $\varphi : \mathbb{R}^{2d} \to \mathbb{R}$ belongs to $X_{p,s} \cap C_b^1$ and Z_t is a solution of (4.2), for any $0 \le s \le t \le T$ the following Itô formula holds:*

$$\varphi(Z_t) = \varphi(Z_s) + \int_s^t \big[b(Z_r)\cdot D\varphi(Z_r) + \frac{1}{2}\Delta_v\varphi(Z_r)\big]\mathrm{d}r + \int_s^t D_v\varphi(Z_r)\,\mathrm{d}W_r\,. \tag{4.24}$$

Proof. Note that we can use (iii) in Theorem 4.5 to give a meaning to the critical term $\int_s^t \Delta_v\varphi(Z_r)\,\mathrm{d}r$. The result then follows approximating φ with regular functions and using Lemma 4.6.

Let $\varphi_\varepsilon \in C_c^\infty \to \varphi$ in $X_{p,s}$. φ_ε satisfy the assumptions of the classical Itô formula, which provides an analogue of (4.24) for $\varphi_\varepsilon(Z_t)$. For any fixed t, the random variables $\varphi_\varepsilon(Z_t) \to \varphi(Z_t)$ $\mathbb{P}$-almost surely. Using that $D\varphi$ is bounded and almost surely $F(Z_r)$ and AZ_r are in $L^1(0,T)$ (this follows by Lemma 4.6 and Lemma 4.7 respectively), the dominated convergence theorem gives the convergence of the first term in the Lebesgue integral. For the second term we use Lemma 4.6 with $f = \Delta_v\varphi_\varepsilon - \Delta_v\varphi$ (recall that $p > 6d$):

$$\mathbb{E}\Big[\int_s^t \Delta_v\varphi_\varepsilon(Z_r) - \Delta_v\varphi(Z_r)\,\mathrm{d}r\Big] \le C\|\Delta_v\varphi_\varepsilon - \Delta_v\varphi\|_{L^p(\mathbb{R}^{2d})} \to 0\,.$$

In the same way, one can show that $\mathbb{E}\big[\int_s^t |D_v\varphi_\varepsilon(Z_r) - D_v\varphi(Z_r)|^2\,\mathrm{d}r\big]$ converges to zero, which implies the convergence of the stochastic integral by the Itô isometry. □

Remark 4.10. Using the boundedness of φ, it is easy to generalize the above Itô formula (4.24) to $\varphi^a(Z_t)$ for any $a \ge 2$.

We can finally prove the well-posedness in the strong sense of the degenerate SDE (4.2). A different proof of this result in a Hölder setting is contained in [6], but no explicit control on the dependence on the initial data is given there, so that a flow cannot be constructed. See also the more recent results of [37]. We here present a different, and in some sense more constructive, proof. This approach, based on ideas introduced in [19],

[24], [15], will even allow us to obtain some regularity results on certain derivatives of the solution. We will use Theorem 3.7 from Section 3.3, which provides the regularity $X_{p,s} \cap C_b^1(\mathbb{R}^{2d})$ of solutions of (4.5).

Theorem 4.11. *Equation (4.2) is well posed in the strong sense.*

Proof. Since we have weak well posedness by (i) of Theorem 4.5, the Yamada-Watanabe principle provides strong existence as soon as strong uniqueness holds. Therefore, we only need to prove strong uniqueness. This can be done by using an appropriate change of variables which transforms equation (4.2) into an equation with more regular coefficients. This method was first introduced in [19], where it is used to prove strong uniqueness for a non degenerate SDE with a Hölder drift coefficient.

Here, the SDE is degenerate and we only need to regularize the second component of the drift coefficient, $F(\cdot)$, which is not Lipschitz continuous. We therefore introduce the auxiliary PDE (4.5) with λ large enough such that

$$\|U_\lambda\|_{L^\infty(\mathbb{R}^{2d})} + \|DU_\lambda\|_{L^\infty(\mathbb{R}^{2d})} < 1/2 \tag{4.25}$$

holds (see (3.59)). In the following we will always use this value of λ and to ease notation we shall drop the subscript for the solution U_λ of (4.5), writing $U_\lambda = U$.

Let Z_t be one solution to (4.2) starting from $z \in \mathbb{R}^{2d}$. Since

$$Z_t = z + \int_0^t B(Z_s)\,\mathrm{d}s + \int_0^t AZ_s\,\mathrm{d}s + R\,W_t\,,$$

and $U \in X_{p,s} \cap C_b^1$ (see Theorem 3.7), by the Itô formula of Proposition 4.9 we have

$$\begin{aligned} U(Z_t) &= U(z) + \int_0^t DU(Z_s)R\;\mathrm{d}W_s + \int_0^t \mathcal{L}U(Z_s)\,\mathrm{d}s \\ &= U(z) + \int_0^t DU(Z_s)R\,\mathrm{d}W_s + \lambda\int_0^t U(Z_s)\,\mathrm{d}s - \int_0^t B(Z_s)\,\mathrm{d}s\,. \end{aligned}$$

Using the SDE to rewrite the last term we find

$$U(Z_t) = U(z) + \int_0^t DU(Z_s)R\,\mathrm{d}W_s + \lambda\int_0^t U(Z_s)\,\mathrm{d}s - Z_t + z + \int_0^t AZ_s\,\mathrm{d}s + R\,W_t$$

and so

$$Z_t = U(z) - U(Z_t) + \int_0^t DU(Z_s)R\,\mathrm{d}W_s + \lambda\int_0^t U(Z_s)\mathrm{d}s + z + \int_0^t AZ_s\mathrm{d}s + R\,W_t\,. \tag{4.26}$$

Let now Y_t be another solution starting from $y \in \mathbb{R}^{2d}$ and let

$$\gamma(x) = x + U(x), \quad x \in \mathbb{R}^{2d}\,. \tag{4.27}$$

We have $\gamma(z) - \gamma(y) = z - y + U(z) - U(y)$, and so $|z-y| \le |U(z)-U(y)| + |\gamma(z)-\gamma(y)|$. Since we have chosen λ such that $\|DU\|_{L^\infty(\mathbb{R}^{2d})} < 1/2$, there exist finite constants $C, c > 0$ such that

$$c|\gamma(z) - \gamma(y)| \le |z-y| \le C|\gamma(z) - \gamma(y)|\,, \qquad \forall z, y \in \mathbb{R}^{2d}. \tag{4.28}$$

We find

$$\mathrm{d}\gamma(Z_t) = \big(\lambda U(Z_t) + AZ_t\big)\,\mathrm{d}t + \big(DU(Z_t) + \mathbb{I}\big)R\cdot\mathrm{d}W_t \tag{4.29}$$

and

$$\begin{aligned} \gamma(Z_t) - \gamma(Y_t) = {} & z - y + U(z) - U(y) + \int_0^t \big[DU(Z_s) - DU(Y_s)\big]R\cdot\mathrm{d}W_s \\ & + \lambda\int_0^t \big[U(Z_s) - U(Y_s)\big]\mathrm{d}s + \int_0^t A(Z_s - Y_s)\,\mathrm{d}s\,. \end{aligned} \tag{4.30}$$

For $a \geq 2$, let us apply Itô formula to $\big|\gamma(Z_t)-\gamma(Y_t)\big|^a = \big(\sum_{i=1}^{2d}\big[\gamma(Z_t)-\gamma(Y_t)\big]_i^2\big)^{a/2}$:

$$\begin{aligned}
\mathrm{d}\big[\big|\gamma(Z_t)-\gamma(Y_t)\big|^a\big] &= a\big|\gamma(Z_t)-\gamma(Y_t)\big|^{a-2}\big(\gamma(Z_t)-\gamma(Y_t)\big)\cdot \mathrm{d}\big(\gamma(Z_t)-\gamma(Y_t)\big)\\
&\quad + \frac{a}{2}\big|\gamma(Z_t)-\gamma(Y_t)\big|^{a-4}\sum_{i,j=1}^{2d}\sum_{k=1}^{d}\Big\{(a-2)\big(\gamma(Z_t)-\gamma(Y_t)\big)_i\big(\gamma(Z_t)-\gamma(Y_t)\big)_j\\
&\qquad\qquad + \delta_{i,j}\big|\gamma(Z_t)-\gamma(Y_t)\big|^2\Big\}\\
&\qquad \times \Big[\big(DU(Z_t)-DU(Y_t)\big)R\Big]_{k,i}\Big[\big(DU(Z_t)-DU(Y_t)\big)R\Big]_{k,j}\mathrm{d}t\\
&\leq a\big|\gamma(Z_t)-\gamma(Y_t)\big|^{a-2}\Big\{\big(\gamma(Z_t)-\gamma(Y_t)\big)\cdot\big[DU(Z_t)-DU(Y_t)\big]R\cdot \mathrm{d}W_t\\
&\qquad + \big(\gamma(Z_t)-\gamma(Y_t)\big)\cdot\Big(\lambda\big[U(Z_t)-U(Y_t)\big]+A(Z_t-Y_t)\Big)\,\mathrm{d}t\\
&\qquad + C_{a,d}\Big\|\big[DU(Z_t)-DU(Y_t)\big]R\Big\|_{HS}^2\mathrm{d}t\Big\}\,.
\end{aligned}$$

Note that Z_t has finite moments of all orders, and U is bounded, so that also the process $\gamma(Z_t)$ has finite moments of all orders. Using also that DU is a bounded function, we deduce that the stochastic integral is a martingale M_t:

$$M_t = \int_0^t a\big|\gamma(Z_s)-\gamma(Y_s)\big|^{a-2}\big(\gamma(Z_s)-\gamma(Y_s)\big)\cdot\big[DU(Z_s)-DU(Y_s)\big]R\cdot \mathrm{d}W_s$$

As in [24] and [14] we now consider the following process

$$B_t = \int_0^t 1_{\{Z_s\neq Y_s\}}\,\frac{\Big\|\big[DU(Z_s)-DU(Y_s)\big]R\Big\|_{HS}^2}{|\gamma(Z_s)-\gamma(Y_s)|^2}\;\mathrm{d}s \leq C^2 N_t\,, \tag{4.31}$$

where we have used the equivalence (4.28) between $|Z_t-Y_t|$ and $|\gamma(Z_t)-\gamma(Y_T)|$ and N_t is the process defined by (4.17) and studied in Lemma 4.8. Just as the process N_t, also B_t has finite moments, and even its exponential has finite moments. With these notations at hand we can rewrite

$$\begin{aligned}
\mathrm{d}\big[\big|\gamma(Z_t)-&\gamma(Y_t)\big|^a\big]\\
&\leq a\big|\gamma(Z_t)-\gamma(Y_t)\big|^{a-2}\big(\gamma(Z_t)-\gamma(Y_t)\big)\cdot\Big(\lambda\big[U(Z_t)-U(Y_t)\big]+A(Z_t-Y_t)\Big)\,\mathrm{d}t\\
&\quad + \mathrm{d}M_t + C_{a,d}\big|\gamma(Z_t)-\gamma(Y_t)\big|^a\,\mathrm{d}B_t\,.
\end{aligned}$$

Again by Itô formula we have

$$\begin{aligned}
\mathrm{d}\Big(e^{-C_{a,d}B_t}\big|\gamma(Z_t)-\gamma(Y_t)\big|^a\Big) &= -C_{a,d}\,e^{-C_{a,d}B_t}\big|\gamma(Z_t)-\gamma(Y_t)\big|^a\,\mathrm{d}B_t \qquad (4.32)\\
+\,e^{-C_{a,d}B_t}\Big\{a\big|\gamma(Z_t)-\gamma(Y_t)\big|^{a-2}&\big(\gamma(Z_t)-\gamma(Y_t)\big)\cdot\Big(\lambda\big[U(Z_t)-U(Y_t)\big]+A(Z_t-Y_t)\Big)\,\mathrm{d}t\\
&\quad + \mathrm{d}M_t + C_{a,d}\big|\gamma(Z_t)-\gamma(Y_t)\big|^a\,\mathrm{d}B_t\Big\}\,.
\end{aligned}$$

The term $e^{-C_{a,d}B_t}\mathrm{d}M_t$ is still the differential of a zero-mean martingale. Integrating and taking the expected value we find

$$\begin{aligned}
\mathbb{E}\Big[e^{-C_{a,d}B_t}\big|\gamma(Z_t)-\gamma(Y_t)\big|^a\Big] &= \big|\gamma(z)-\gamma(y)\big|^a + \mathbb{E}\Big[\int_0^t e^{-C_{a,d}B_s}a\big|\gamma(Z_s)-\gamma(Y_s)\big|^{a-2}\\
&\quad\times\big(\gamma(Z_s)-\gamma(Y_s)\big)\cdot\Big(\lambda\big[U(Z_s)-U(Y_s)\big]+A(Z_s-Y_s)\Big)\,\mathrm{d}s\Big].
\end{aligned}$$

Using again the equivalence (4.28) between $|Z_t - Y_t|$ and $|\gamma(Z_t) - \gamma(Y_T)|$ and the fact that U is Lipschitz continuous, this finally provides the following estimate:

$$\mathbb{E}\Big[e^{-C_{a,d}B_t}\big|Z_t - Y_t\big|^a\Big] \le C\Big\{|z-y|^a + \int_0^t \mathbb{E}\Big[e^{-C_{a,d}B_s}\big|Z_s - Y_s\big|^a\Big]\,\mathrm{d}s\Big\}\,.$$

By Grönwall's inequality, there exists a finite constant C' such that

$$\mathbb{E}\Big[e^{-C_{a,d}B_t}\big|Z_t - Y_t\big|^a\Big] \le C'\big|z-y\big|^a\,. \tag{4.33}$$

Using that B_t is increasing and a.s. $B_T < \infty$, taking $z = y$ we get for any fixed $t \in [0,T]$ that $\mathbb{P}\big(Z_t \neq Y_t\big) = 0$. Strong uniqueness follows by the continuity of trajectories. This completes the proof. □

Corollary 4.12. *Using the finite moments of the exponential of the process B_t, we can also prove that for any $a \ge 2$,*

$$\mathbb{E}\Big[\big|Z_t - Y_t\big|^a\Big] \le C\big|z-y\big|^a\,. \tag{4.34}$$

Proof. Using Hölder's inequality and for an appropriate constant c, we have

$$\begin{aligned}\mathbb{E}\Big[\big|Z_t - Y_t\big|^a\Big] &= \mathbb{E}\Big[e^{cB_t}e^{-cB_t}\big|Z_t - Y_t\big|^a\Big]\\ &\le C\Big(\mathbb{E}\Big[e^{-2cB_t}\big|Z_t - Y_t\big|^{2a}\Big]\Big)^{1/2} \le C\big|z-y\big|^a\,.\end{aligned}$$ □

4.2 Stochastic flow

The main result of this section is the existence of a stochastic flow generated by the SDE (4.2), which is presented in Theorem 4.18. This result follows in a standard way from the results of Lemma 4.13 and Corollary 4.17. One possible line of proof is to follow [25, Chapter II.2] or [26, Chapter 4.5], adapting such results to the irregular coefficients (as in [15]) and degenerate setting considered here.

Another standard result which follows from Corollary 4.12 and Lemma 4.13 is the (local) Hölder continuity of the flow, which we present in Theorem 4.15.

Lemma 4.13. *Let a be any real number. Then there is a positive constant C_a independent of $t \in [0,T]$ and $z \in \mathbb{R}^{2d}$ such that*

$$\mathbb{E}\Big[\Big(1 + \big|Z_t^z\big|^2\Big)^a\Big] \le C_{a,d}\big(1+|z|^2\big)^a.$$

Proof. Using the boundedness of the solution U of the PDE (4.5) (see (4.25)) one can show the equivalence

$$c(1+|z|^2) \le 1 + |\gamma(z)|^2 \le C(1+|z|^2)\,.$$

Set $\gamma_t = \gamma(Z_t^z)$. Then, it is enough to prove that $\mathbb{E}\big[\big(1+|\gamma_t|^2\big)^a\big] \le C_{a,d}\big(1+|\gamma(z)|^2\big)^a$. Set $f(z) := (1+|z|^2)$. The idea is to apply the Itô formula to $g(\gamma_t)$, where $g(z) = f^a(z)$. Since

$$\frac{\partial g}{\partial z_i}(z) = 2af^{a-1}(z)z_i\,, \qquad \frac{\partial^2 g}{\partial z_i\,\partial z_j}(z) = 4a(a-1)f^{a-2}(z)z_iz_j + 2af^{a-1}(z)\delta_{i,j}\,,$$

we see that

$$\begin{aligned}g(\gamma_t) - g(\gamma(z)) &= 2a\int_0^t f^{a-1}(\gamma_s)\,\gamma_s\cdot\widetilde{\sigma}(\gamma_s)\cdot\mathrm{d}W_s + 2a\int_0^t f^{a-1}(\gamma_s)\,\gamma_s\cdot\widetilde{b}(\gamma_s)\,\mathrm{d}s\\ &\quad+\sum_{i,j,k=1}^{2d}\int_0^t 2af^{a-2}(\gamma_s)\Big[(a-1)\gamma_s^i\gamma_s^j + \delta_{i,j}f(\gamma_s)\Big]\widetilde{\sigma}^{k,i}(\gamma_s)\widetilde{\sigma}^{k,j}(\gamma_s)\,\mathrm{d}s\,.\end{aligned} \tag{4.35}$$

Here we have used the relation $\mathrm{d}\langle\gamma_t,\gamma_t\rangle = \widetilde{\sigma}(\gamma_t)\,\widetilde{\sigma}^t(\gamma_t)\,\mathrm{d}t$. Since γ_t has finite moments, the first term on the right hand side of (4.35) is a martingale with zero mean. Note that $f(z)\ge 1$, so that $f^{a-1}\le f^a$ and $|z|\le f^{1/2}(z)$. Moreover, since $\widetilde{\sigma}$ is bounded and $\widetilde{b}$ is Lipschitz continuous, $|\widetilde{b}(z)|\le C(1+|z|)\le Cf^{1/2}(z)$. Using all this, we can see that the second and third term on the right hand side of (4.35) are dominated by a constant times $\int_0^t g(\gamma_s)\,\mathrm{d}s$. Therefore, taking expectations in (4.35) we have

$$\mathbb{E}\big[g(\gamma_t)\big]-g(\gamma_0)\le C_{a,d}\int_0^t \mathbb{E}\big[g(\gamma_s)\big]\mathrm{d}s\,,$$

and the result follows by Grönwall's lemma. □

Proposition 4.14. *Let Z_t^z be the unique strong solution to the SDE* (4.2) *given by Theorem 4.11 and starting from the point $z\in\mathbb{R}^{2d}$. For any $a>2$, $s,t\in[0,T]$ and $z,y\in\mathbb{R}^{2d}$ we have*

$$\mathbb{E}\Big[\big|Z_t^z-Z_s^y\big|^a\Big]\le C_{a,d,\lambda,T}\Big\{|z-y|^a+\big(1+|z|^a+|y|^a\big)|t-s|^{a/2}\Big\}\,.$$

Proof. Assume $t>s$. It suffice to show that

$$\mathbb{E}\Big[\big|Z_s^z-Z_s^y\big|^a\Big]\le C|z-y|^a\,,$$
$$\mathbb{E}\Big[\big|Z_t^z-Z_s^z\big|^a\Big]\le C\big(1+|z|^a\big)|t-s|^{a/2}\,.$$

The first inequality was obtained in Corollary 4.12. To prove the second inequality we use the equivalence (4.28) between Z_t and $\gamma(Z_t)$. We use the Itô formula (4.29) for $\gamma(Z_t)$ and $\gamma(Z_s)$: we can control the differences of the first and last term using the fact that U and DU are bounded, together with Burkholder's inequality

$$\begin{aligned}\mathbb{E}\Big[\big|Z_t^z-Z_s^z\big|^a\Big]&\le C\mathbb{E}\Big[\big|\gamma(Z_t^z)-\gamma(Z_s^z)\big|^a\Big]\\&\le C_{a,d}\Big\{\Big|\int_s^t\Big[\|\lambda U\|_\infty^2\mathrm{d}r\Big|^{a/2}+\mathbb{E}\Big[\Big|\int_s^t AZ_r^z\,\mathrm{d}r\Big|^a\Big]+\|DU\|_\infty\,\mathbb{E}\Big[\big|R(W_t-W_s)\big|^a\Big]\Big\}\end{aligned}$$

and for the linear part we use Hölder's inequality and Lemma 4.13:

$$\mathbb{E}\Big[\Big|\int_s^t AZ_r^z\,\mathrm{d}r\Big|^a\Big]\le(t-s)^{a/2}\,\mathbb{E}\Big[\int_s^t|AZ_r^z|^a\,\mathrm{d}r\Big]\le C(t-s)^{a/2}\int_s^t(1+|z^2|)^{a/2}\,\mathrm{d}r\,.$$ □

Applying Kolmogorov's regularity theorem (see [25, Theorem I.10.3]), we immediately obtain the following

Theorem 4.15. *The family of random variables (Z_t^z), $t\in[0,T]$, $z\in\mathbb{R}^d$, admits a modification which is locally α-Hölder continuous in z for any $\alpha<1$ and β-Hölder continuous in t for any $\beta<1/2$.*

From now on, we shall always use the continuous modification of Z provided by this theorem.

To obtain the injectivity of the flow, we review the computations of Proposition 4.14: we now want to allow the exponent a to be negative. The proofs of the following lemma is given in Appendix.

Lemma 4.16. *Let a be any real number and $\varepsilon>0$. Then there is a positive constant $C_{a,d}$ (independent of ε) such that for any $t\in[0,T]$ and $z,y\in\mathbb{R}^{2d}$*

$$\mathbb{E}\Big[\Big(\varepsilon+\big|Z_t^z-Z_t^y\big|^2\Big)^a\Big]\le C_{a,d}\big(\varepsilon+|z-y|^2\big)^a. \tag{4.36}$$

Corollary 4.17. *Let ε tend to zero in Lemma 4.16. Then, by monotone convergence, we have:*

$$\mathbb{E}\Big[\,\big|Z_t^z - Z_t^y\big|^a\Big] \le C_{a,d}|z-y|^a\,. \tag{4.37}$$

From the above results one can obtain the following theorem. The line of proof is quite standard, but the interested reader can find a complete proof in Section 4.2 of [17].

Theorem 4.18. *The unique strong solution $Z_t = (X_t, V_t)$ of the SDE* (1.2) *defines a stochastic flow of Hölder continuous homeomorphisms ϕ_t.*

4.3 Regularity of the derivatives

Although F is not even weakly differentiable, from the reformulation (4.26) of equation (4.2) it is reasonable to expect differentiability of the flow, since the derivatives DX_t, DV_t with respect to the initial conditions (x, v) formally solve suitable SDEs with well-defined, integrable coefficients. We have the following result.

Theorem 4.19. *Let $\phi_t(z)$ be the flow associated to* (4.2) *provided by Theorem 4.18. Then, for any $t \in [0,T]$, $\mathbb{P}$-a.s., the random variable $\phi_t(z)$ admits a weak distributional derivative with respect to z; moreover $D_z\phi_t \in L^p_{loc}(\Omega \times \mathbb{R}^{2d})$ (i.e., $D_z\phi_t \in L^p(\Omega \times K)$, for any compact set $K \subset \mathbb{R}^{2d}$), for any $p \ge 1$.*

Proof. Step 1. Bounds on difference quotients. It is sufficient to prove the existence and regularity of $D_{z_i}\phi_t$ for some fixed $i \in \{1, \dots, 2d\}$. We omit to write i and set $e = e_i$.

Introduce for every $h > 0$ the stochastic processes

$$\theta_t^h(z) = \frac{\phi_t(z+he) - \phi_t(z)}{h}\,, \qquad \xi_t^h(z) = \frac{\gamma\big(\phi_t(z+he)\big) - \gamma\big(\phi_t(z)\big)}{h}\,, \tag{4.38}$$

where $\gamma(z) = z + U(z)$ as in (4.27). It is clear that they have finite moments of all orders because ϕ and $\gamma(\phi)$ do. The two processes are also equivalent in the sense that there exist constants C_1, C_2 such that

$$C_1|\xi_t^h(z)| \le |\theta_t^h(z)| \le C_2|\xi_t^h(z)|\,. \tag{4.39}$$

This follows from (4.28). To fix the ideas, consider the case $i > d$. We have

$$\begin{aligned} \xi_t^h &= e + \frac{1}{h}\Big[U(z+he) - U(z)\Big] \\ &\quad + \int_0^t \frac{\lambda}{h}\Big[U\big(\phi_s(z+he)\big) - U\big(\phi_s(z)\big)\Big] + A\theta_t^h(z)\,\mathrm{d}s \\ &\quad + \frac{1}{h}\int_0^t \Big[DU\big(\phi_s(z+he)\big) - DU\big(\phi_s(z)\big)\Big]R\cdot \mathrm{d}W_s\,. \end{aligned} \tag{4.40}$$

Proceeding as in the proof of Theorem 4.11 above we have

$$\begin{aligned} \mathrm{d}\big|\xi_t^h\big|^p &\le p\big|\xi_t^h\big|^{p-2}\xi_t^h\cdot\Big(\frac{\lambda}{h}\big[U\big(\phi_t(z+he)\big) - U\big(\phi_t(z)\big)\big] + A\theta_t^h(z)\Big)\,\mathrm{d}t \\ &\quad + \frac{p}{h}\big|\xi_t^h\big|^{p-2}\xi_t^h\cdot\Big[DU\Big(\phi_t(z+he)\Big) - DU\Big(\phi_t(z)\Big)\Big]R\cdot\mathrm{d}W_t \\ &\quad + \frac{C_{p,d}}{h^2}\big|\xi_t^h\big|^{p-2}\Big\|DU\Big(\phi_t(z+he)\Big) - DU\Big(\phi_t(z)\Big)R\Big\|_{HS}^2\mathrm{d}t \\ &= p\big|\xi_t^h\big|^{p-2}\xi_t^h\cdot\Big(\frac{\lambda}{h}\big[U\big(\phi_t(z+he)\big) - U\big(\phi_t(z)\big)\big] + A\theta_t^h(z)\Big)\mathrm{d}t \\ &\quad + \mathrm{d}M_t^h + C_{p,d}\big|\xi_t^h\big|^{p-2}\big|\theta_t^h\big|^2\,\mathrm{d}N_t\,, \end{aligned}$$

where the process N_t is defined as in (4.17), but with $Z = \phi(z+he)$ and $Y = \phi(z)$, and for every $h > 0$, $\mathrm{d}M_t^h$ is the differential of a martingale because DU is bounded and ξ_t^h has finite moments. Setting $C_p = (C_2)^2 C_{p,d}$ we get

$$\begin{aligned} \mathrm{d}\Big(e^{-C_pN_t}\big|\xi_t^h\big|^p\Big) &= -C_p e^{-C_pN_t}\big|\xi_t^h\big|^p \mathrm{d}N_t + e^{-C_pN_t}\mathrm{d}\big(\big|\xi_t^h\big|^p\big) \qquad (4.41)\\ &\le e^{-C_pN_t} p\big|\xi_t^h\big|^{p-2}\xi_t^h \cdot \Big(\frac{\lambda}{h}\big[U\big(\phi_t(z+he)\big) - U\big(\phi_t(z)\big)\big] + A\theta_t^h(z)\Big)\mathrm{d}t + e^{-C_pN_t}\mathrm{d}M_t^h\,. \end{aligned}$$

After integrating and taking expectations we find

$$\begin{aligned} \mathbb{E}\Big[e^{-C_pN_t}\big|\xi_t^h\big|^p\Big] &\le \Big|e + \frac{1}{h}\big[U(z+he) - U(z)\big]\Big|^p \\ &\qquad + \int_0^t \mathbb{E}\Big[e^{-C_pN_s} p\big|\xi_s^h\big|^{p-2}\xi_t^h \cdot \Big(\frac{\lambda}{h}\big[U\big(\phi_s(z+he)\big) - U\big(\phi_s(z)\big)\big] + A\theta_s^h(z)\Big)\Big]\mathrm{d}s \\ &\le C\big(1 + \|DU\|^p_{L^\infty(\mathbb{R}^{2d})}\big) + \int_0^t C\big(\lambda\|DU\|_{L^\infty(\mathbb{R}^{2d})} + |A|\big)\mathbb{E}\Big[e^{-C_pN_s}p\big|\xi_s^h\big|^p\Big]\,\mathrm{d}s\,. \end{aligned}$$

A similar estimate holds for the case $i \le d$. We now apply Grönwall's inequality and proceeding as in the proof of Corollary 4.12 we finally get that

$$\mathbb{E}\Big[\big|\theta_t^h\big|^p\Big] \le C\mathbb{E}\Big[\big|\xi_t^h\big|^p\Big] \le C_{p,d,T,\lambda} < \infty\,. \qquad (4.42)$$

Step 2. Derivative of the Flow. Remark that, due to the boundedness of DU, the bound (4.42) is uniform in h and z, and we get

$$\sup_{z\in\mathbb{R}^{2d}}\sup_{h\in(0,1]}\mathbb{E}\Big[\big|\theta_t^h\big|^p\Big] \le C_{p,d,T,\lambda} < \infty\,. \qquad (4.43)$$

We can then apply [1, Corollary 3.5] and obtain the existence of the weak derivative for the flow $D\phi_t \in L^p_{loc}(\Omega\times\mathbb{R}^{2d})$. □

Remark 4.20. Since the bound (4.42) is also uniform in time, applying [1, Theorem 3.6] one would also get the existence of the weak derivative as a process $D\phi_t$ belonging to $L^p_{loc}([0,T]\times\mathbb{R}^{2d})$ with probability one, and the weak convergence $\theta_t^h \rightharpoonup D\phi_t$ in $L^p_{loc}(\Omega\times[0,T]\times\mathbb{R}^{2d})$.

It seems that $D\phi_t \in L^p_{loc}(\Omega\times[0,T]\times\mathbb{R}^{2d})$ for $p \in (2,\infty)$ could also be obtained directly from Corollary 4.12 using [39, Theorem 1.1]. However, to obtain an L^∞-in-time result using this approach one would first need to show that the estimate (4.34) holds with a $\sup_{t\in[0,T]}$ inside the expected value.

5 Stochastic kinetic equation

We present here results on the stochastic kinetic equation (1.1). The first result concerns existence of solutions with a certain Sobolev regularity (see Theorem 5.4). The second one is about uniqueness of solutions (see Theorem 5.7).

We will use the results of the previous sections together with results similar to the ones given in [16] to approximate the flow associated to the equation of characteristics. We report them in the Appendix for the sake of completeness. To prove that some degree of Sobolev regularity of the initial condition is preserved on has to deal with weakly differentiable solutions, according to the definition introduced in [16] for solutions of the stochastic transport equation.

Recall that, as observed in Section 2, by point 2 of the next definition and Sobolev embedding, weakly differentiable solutions of the stochastic kinetic equation are a.s.

continuous in the space variable, for every $t \in [0,T]$; this is in contrast with the deterministic kinetic equation, where solutions can be discontinuous (see Proposition 2.3). In the sequel, given a Banach space E we denote by $C^0\big([0,T];E\big)$ the Banach space of all continuous functions from $[0,T]$ into E endowed with the supremum norm.

Definition 5.1. *Assume that F satisfies Hypothesis 2.1. We say that f is a weakly differentiable solution of the stochastic kinetic equation* (1.1) *if*

1. $f:\Omega\times[0,T]\times\mathbb{R}^{2d}\to\mathbb{R}$ *is measurable,* $\int_{\mathbb{R}^{2d}} f(t,z)\,\varphi(z)\,\mathrm{d}z$ *(well defined by property 2 below) is progressively measurable for each* $\varphi\in C_c^\infty\left(\mathbb{R}^{2d}\right)$;
2. $\mathbb{P}\left(f(t,\cdot)\in\cap_{r\geq1}W^{1,r}_{loc}\left(\mathbb{R}^{2d}\right)\right)=1$ *for every* $t\in[0,T]$ *and both* f *and* Df *are in* $\cap_{r\geq1}C^0\big([0,T];L^r(\Omega\times\mathbb{R}^{2d})\big)$;
3. *setting* $b(z)=A\cdot z+B(z)$, $b:\mathbb{R}^{2d}\to\mathbb{R}^{2d}$*, see* (4.1)*, for every* $\varphi\in C_c^\infty\left(\mathbb{R}^{2d}\right)$ *and* $t\in[0,T]$*, with probability one, one has*

$$\begin{aligned}&\int_{\mathbb{R}^{2d}} f(t,z)\varphi(z)\,\mathrm{d}z+\int_0^t\int_{\mathbb{R}^{2d}} b(z)\cdot Df(s,z)\varphi(z)\,\mathrm{d}z\mathrm{d}s\\&=\int_{\mathbb{R}^{2d}} f_0(z)\varphi(z)\,\mathrm{d}z+\sum_{i=1}^d\int_0^t\left(\int_{\mathbb{R}^{2d}} f(s,z)\,\partial_{v_i}\varphi(z)\,\mathrm{d}z\right)\mathrm{d}W_s^i\\&+\frac{1}{2}\int_0^t\int_{\mathbb{R}^{2d}} f(s,z)\Delta_v\varphi(z)\,\mathrm{d}z\mathrm{d}s\,.\end{aligned}$$

Remark 5.2. The process $s\mapsto Y_s^i:=\int_{\mathbb{R}^{2d}} f(s,z)\,\partial_{v_i}\varphi(z)\,\mathrm{d}z$ is progressively measurable by property 1 and $\int_0^T\left|Y_s^i\right|^2\mathrm{d}s<\infty$ $\mathbb{P}$-a.s. by property 2, hence the Itô integral is well defined.

Remark 5.3. The term $\int_0^t\int_{\mathbb{R}^{2d}} b(z)\cdot Df(s,z)\,\varphi(z)\,\mathrm{d}z\mathrm{d}s$ is well defined with probability one because of the integrability properties of b (assumptions) and Df (property 2).

In the next result the inverse of ϕ_t will be denoted by ϕ_0^t.

Theorem 5.4. *If F satisfies Hypothesis 2.1 and $f_0\in\cap_{r\geq1}W^{1,r}(\mathbb{R}^{2d})$, then $f(t,z):=f_0(\phi_0^t(z))$ is a weakly differentiable solution of the stochastic kinetic equation* (1.1).

Proof. The proof follows the one of [16, Theorem 10]. We divide it into several steps.

Step 1. Preparation. The random field $(\omega,t,z)\mapsto f_0(\phi_0^t(z)(\omega))$ is jointly measurable and $(\omega,t)\mapsto\int_{\mathbb{R}^{2d}} f_0(\phi_0^t(z)(\omega))\,\varphi(z)\,\mathrm{d}z$ is progressively measurable for each $\varphi\in C_c^\infty(\mathbb{R}^{2d})$. Hence part 1 of Definition 5.1 is true. To prove part 2 and 3 we approximate $f(t,z)$ by smooth fields $f_n(t,z)$.

Let $f_{0,n}$ be a sequence of smooth functions which converges to f_0 in $W^{1,r}(\mathbb{R}^{2d})$, for any $r\geq1$, and so uniformly on $\mathbb{R}^{2d}$ by the Sobolev embedding. This can be done for instance by using standard convolution with mollifiers. Moreover suppose that F_n are smooth approximations converging to F in $L^p(\mathbb{R}^{2d})$ (p is given in Hypothesis 2.1), let $\phi_{t,n}$ be the regular stochastic flow generated by the SDE (4.3) where B is replaced by $B_n=RF_n$ and let $\phi_{0,n}^t$ be the inverse flow. Then $f_n(t,z):=f_{0,n}\left(\phi_{0,n}^t(z)\right)$ is a smooth solution of

$$\mathrm{d}f_n=-\big(v\cdot D_xf_n+F_n\cdot D_vf_n\big)\,\mathrm{d}t-D_vf_n\circ\mathrm{d}W_t$$

and thus for every $\varphi \in C_c^\infty(\mathbb{R}^{2d})$, $t \in [0,T]$ and bounded r.v. Y, it satisfies

$$\begin{aligned}
&\mathbb{E}\Big[Y \int_{\mathbb{R}^{2d}} f_n(t,z)\varphi(z)\,\mathrm{d}z\Big] + \mathbb{E}\left[Y \int_0^t \int_{\mathbb{R}^{2d}} b_n(z)\cdot Df_n(s,z)\,\varphi(z)\,\mathrm{d}z\mathrm{d}s\right] \\
&\quad = \mathbb{E}\Big[Y \int_{\mathbb{R}^{2d}} f_{0,n}(z)\,\varphi(z)\,\mathrm{d}z\Big] + \sum_{i=1}^d \mathbb{E}\left[Y \int_0^t \left(\int_{\mathbb{R}^{2d}} f_n(s,z)\,\partial_{v_i}\varphi(z)\,\mathrm{d}z\right)\mathrm{d}W_s^i\right] \\
&\qquad + \frac{1}{2}\mathbb{E}\left[Y \int_0^t \int_{\mathbb{R}^{2d}} f_n(s,z)\,\Delta_v\varphi(z)\,\mathrm{d}z\mathrm{d}s\right]. \qquad (5.1)
\end{aligned}$$

We shall pass to the limit in each one of these terms. We are forced to use this very weak convergence due to the term

$$\mathbb{E}\left[Y \int_0^t \int_{\mathbb{R}^{2d}} b_n(z)\cdot Df_n(s,z)\,\varphi(z)\,\mathrm{d}z\mathrm{d}s\right], \qquad (5.2)$$

where we may only use weak convergence of Df_n.

Step 2. Convergence of f_n to f. We claim that, uniformly in n and for every $r \geq 1$,

$$\sup_{t\in[0,T]} \int_{\mathbb{R}^{2d}} \mathbb{E}\Big[|f_n(t,z)|^r\Big]\mathrm{d}z \leq C_r\,, \qquad (5.3)$$

$$\sup_{t\in[0,T]} \int_{\mathbb{R}^{2d}} \mathbb{E}\Big[|Df_n(t,z)|^r\Big]\mathrm{d}z \leq C_r\,. \qquad (5.4)$$

Let us show how to prove the second bound; the first one can be obtained in the same way. The key estimate is the bound (6.6) on the derivative of the flow, which is proved in Appendix. We use the representation formula for f_n and the Hölder inequality to obtain

$$\left(\int_{\mathbb{R}^{2d}} \mathbb{E}\Big[|Df_n(t,z)|^r\Big]\mathrm{d}z\right)^2 \leq \sup_{z\in\mathbb{R}^{2d}} \mathbb{E}\big[|D\phi_{0,n}^t(z)|^{2r}\big] \int_{\mathbb{R}^{2d}} \mathbb{E}\Big[\big|Df_{0,n}\big(\phi_{0,n}^t(z)\big)\big|^{2r}\Big]\mathrm{d}z\,.$$

The first term on the right-hand side can be uniformly bounded using Lemma 6.3. Also the last integral can be bounded uniformly: changing variables (all functions are regular) we get

$$\int_{\mathbb{R}^{2d}} \mathbb{E}\Big[\big|Df_{0,n}\big(\phi_{0,n}^t(z)\big)\big|^{2r}\Big]\mathrm{d}z = \int_{\mathbb{R}^{2d}} \big|Df_{0,n}(y)\big|^{2r}\,\mathbb{E}\big[J_{\phi_{t,n}}(y)\big]\mathrm{d}y\,,$$

where $J_{\phi_{t,n}}(y)$ is the Jacobian determinant of $\phi_{t,n}(y)$. Then we conclude using again the Hölder inequality, (6.6) and the boundedness of $(f_{0,n})$ in $W^{1,r}(\mathbb{R}^{2d})$ (for every $r \geq 1$). Remark that all the bounds obtained are uniform in n and t.

We can now consider the convergence of f_n to f. Let us first prove that, given $t \in [0,T]$ and $\varphi \in C_c^\infty(\mathbb{R}^{2d})$,

$$\mathbb{P} - \lim_{n\to\infty} \int_{\mathbb{R}^{2d}} f_n(t,z)\,\varphi(z)\,\mathrm{d}z = \int_{\mathbb{R}^{2d}} f(t,z)\,\varphi(z)\,\mathrm{d}z \qquad (5.5)$$

(convergence in probability). Using the representation formulas $f_n = f_{0,n}(\phi_{0,n}^t)$, $f = f_0(\phi_0^t)$ and Sobolev embedding $W^{1,4d} \hookrightarrow C^{1/2}$ we have ($\mathrm{Supp}(\varphi) \subset B_R$ where B_R is the ball of radius $R > 0$ and center 0)

$$\begin{aligned}
\left|\int_{\mathbb{R}^{2d}} \big(f_n(t,z) - f(t,z)\big)\,\varphi(z)\,\mathrm{d}z\right| &\leq \|f_{0,n} - f_0\|_{L^\infty(\mathbb{R}^{2d})}\,\|\varphi\|_{L^1(\mathbb{R}^{2d})} \\
&\quad + C\,\|\varphi\|_{L^\infty(\mathbb{R}^{2d})} \int_{B_R} \big|\phi_{0,n}^t(z) - \phi_0^t(z)\big|^{1/2}\,\mathrm{d}z\,.
\end{aligned}$$

The first term converges to zero by the uniform convergence of $f_{0,n}$ to f_0. From Lemma 6.2 we get

$$\lim_{n\to\infty}\mathbb{E}\left[\int_{B_R}\left|\phi^t_{0,n}(z)-\phi^t_0(z)\right|\mathrm{d}z\right]=0\,,$$

and the convergence in probability (5.5) follows. This allows to pass to the limit in the first and in the last term of equation (5.1) using the uniform bound (5.3) and the Vitali convergence theorem. Similarly, we can show that, given $\varphi\in C_c^\infty\left(\mathbb{R}^{2d}\right)$,

$$\mathbb{P}-\lim_{n\to\infty}\int_0^T\left|\int_{\mathbb{R}^{2d}}\left(f_n(t,z)-f(t,z)\right)\varphi(z)\,\mathrm{d}z\right|^2\mathrm{d}t=0\,, \tag{5.6}$$

which allows to pass to the limit in the stochastic integral term of (5.1). Hence, one can easily show convergence of all terms in (5.1) except for the one in (5.2) which will be treated in Step 4.

Step 3. A bound for f. Let us prove property 2 of Definition 5.1. The key estimate is property (5.4) obtained in the previous step.

Recall we have already obtained the convergence (5.5) and the uniform bound (5.4) on Df_n. We can then apply [16, Lemma 16] which gives $\mathbb{P}\big(f(t,\cdot)\in W^{1,r}_{loc}(\mathbb{R}^{2d})\big)=1$ for any $r\geq 1$ and $t\in[0,T]$, and

$$\mathbb{E}\left[\int_{B_R}|Df(t,z)|^r\,\mathrm{d}z\right]\leq\limsup_{n\to\infty}\mathbb{E}\left[\int_{B_R}|Df_n(t,z)|^r\,\mathrm{d}z\right]\leq C_r\,,$$

for every $R>0$ and $t\in[0,T]$. Hence, by monotone convergence we have

$$\sup_{t\in[0,T]}\mathbb{E}\left[\int_{\mathbb{R}^{2d}}|Df(t,z)|^r\,\mathrm{d}z\right]\leq C_r\,. \tag{5.7}$$

A similar bound can be proved for f itself using (5.3), the convergence in probability (5.5) and the Vitali convergence theorem.

Step 4. Passage to the limit. Finally, we prove that we can pass to the limit in equation (5.1) and deduce that f satisfies property 3 of Definition 5.1. It remains to consider the term $\mathbb{E}\big[Y\int_0^t\int_{\mathbb{R}^{2d}}b_n(s,z)\cdot Df_n(s,z)\,\varphi(z)\,\mathrm{d}z\mathrm{d}s\big]$. Since $F_n\to F$ in $L^p(\mathbb{R}^{2d})$, it is sufficient to use a suitable weak convergence of Df_n to Df. Precisely, for $t\in[0,T]$,

$$\begin{aligned}
&\mathbb{E}\Big[Y\int_0^t\int_{\mathbb{R}^{2d}}b_n(z)\cdot Df_n(s,z)\,\varphi(z)\,\mathrm{d}z\mathrm{d}s\Big]\\
&\qquad-\mathbb{E}\left[Y\int_0^t\int_{\mathbb{R}^{2d}}b(z)\cdot Df(s,z)\,\varphi(z)\,\mathrm{d}z\mathrm{d}s\right]=I_n^{(1)}(t)+I_n^{(2)}(t)\ ;\\
&I_n^{(1)}(t)=\mathbb{E}\left[Y\int_0^t\int_{\mathbb{R}^{2d}}\big(F_n(z)-F(z)\big)\cdot D_vf_n(s,z)\,\varphi(z)\,\mathrm{d}z\mathrm{d}s\right];\\
&I_n^{(2)}(t)=\mathbb{E}\left[Y\int_0^t\int_{\mathbb{R}^{2d}}\varphi(z)\,b(z)\cdot\big(Df_n(s,z)-Df(s,z)\big)\mathrm{d}z\mathrm{d}s\right].
\end{aligned}$$

We have to prove that both $I_n^{(1)}(t)$ and $I_n^{(2)}(t)$ converge to zero as $n\to\infty$. By the Hölder inequality, for all $t\in[0,T]$

$$I_n^{(1)}(t)\leq C\,\|F_n-F\|_{L^p(\mathbb{R}^{2d})}\sup_{t\in[0,T]}\mathbb{E}\left[\big\|Df_n(t,\cdot)\big\|_{L^{p'}(\mathbb{R}^{2d})}\right]$$

where $1/p+1/p'=1$ and $C=C_{Y,T,\varphi}$. Thus, from (5.4), $I_n^{(1)}(t)$ converges to zero as $n\to\infty$. Let us treat $I_n^{(2)}(t)$. Using the integrability properties shown above we can change the order of integration. The function

$$h_n(s) := \mathbb{E}\left[\int_{\mathbb{R}^{2d}} Y\varphi(z)\, b(z)\cdot\big(Df_n(s,z) - Df(s,z)\big)\,\mathrm{d}z\right],\quad s\in[0,T],$$

converges to zero as $n\to\infty$ for almost every s and satisfies the assumptions of the Vitali convergence theorem (we shall prove these two claims in Step 5 below). Hence $I_n^{(2)}(t)$ converges to zero.

Now we may pass to the limit in equation (5.1) and from the arbitrariness of Y we obtain property 3 of Definition 5.1.

Step 5. Auxiliary facts. We have to prove the two properties of $h_n(s)$ claimed in Step 4. For every $s\in[0,T]$ [16, Lemma 16] gives

$$\mathbb{E}\left[\int_{\mathbb{R}^{2d}} \partial_{z_i} f(s,z)\,\varphi(z)\,Y\,\mathrm{d}z\right] = \lim_{n\to\infty}\mathbb{E}\left[\int_{\mathbb{R}^{2d}} \partial_{z_i} f_n(s,z)\,\varphi(z)\,Y\,\mathrm{d}z\right], \tag{5.8}$$

for every $\varphi\in C_c^\infty(\mathbb{R}^{2d})$ and bounded r.v. Y. Since the space $C_c^\infty(\mathbb{R}^{2d})$ is dense in $L^p(\mathbb{R}^{2d})$, we may extend the convergence property (5.8) to all $\varphi\in L^p(\mathbb{R}^{2d})$ by means of the bounds (5.4) and (5.7), which proves the first claim.

Moreover, for every $\varepsilon>0$ there is a constant $C_{Y,\varepsilon}$ such that ($\mathrm{Supp}(\varphi)\subset B_R$)

$$\sup_{n\geq 1}\int_0^T h_n^{1+\varepsilon}(s)\,\mathrm{d}s \leq C_{Y,\varepsilon}\,\|b\varphi\|_{L^p}^{1+\varepsilon}\left\{\left(\mathbb{E}\int_0^T\int_{B_R}\big|Df_n(s,z)\big|^r\mathrm{d}z\mathrm{d}s\right)^{\frac{1+\varepsilon}{r}} + \left(\mathbb{E}\int_0^T\int_{B_R}\big|Df(s,z)\big|^r\mathrm{d}z\mathrm{d}s\right)^{\frac{1+\varepsilon}{r}}\right\}$$

for a suitable r depending on ε (we have used Hölder inequality; cf. [16, page 1344]). The bounds (5.4) and (5.7) imply that $\int_0^T h_n^{1+\varepsilon}(s)\,\mathrm{d}s$ is uniformly bounded, and the Vitali theorem can be applied. The proof is complete. □

We now present the uniqueness result for weakly differentiable solutions. We exploit (in Step 2 of the proof of Theorem 5.7) a renormalization property of solutions, which is proved in Step 1. The proof seems to be of independent interest, see the following remark.

Remark 5.5. The main idea of our proof is to exploit the specific form of the equation using in Step 2 of the proof localizing test functions that have a different behavior in the x and v variables. We have then to perform two limits, and choosing the right order allows to deal with the problematic part of the drift coefficient.

It seems that this small trick allows to extend the possibility to apply the classical line of proof based on renormalized solutions, the DiPerna-Lions commutators lemma [12] and Grönwall's lemma to a wider class of degenerate equations.

Remark 5.6. Potentially, it seems that the proof can be also done by the maximum principle, along the lines of [40, Section 4]. This however requires a generalization of the known results since for the linear part of the drift term we only have $v/(1+|z|)\in L^\infty(\mathbb{R}^{2d})$ (allowing to obtain the renormalization property for solutions f), but $v/(1+|z|)\notin L^2(\mathbb{R}^{2d})$. Therefore, $b(z)/(1+|z|)\notin L^2(\mathbb{R}^{2d})$.

Theorem 5.7. *If F satisfies Hypothesis 2.1 and, moreover, $\mathrm{div}_v F\in L^\infty(\mathbb{R}^{2d})$ ($\mathrm{div}_v F$ is understood in distributional sense) weakly differentiable solutions are unique.*

Proof. By linearity of the equation we just have to show that the only solution starting from $f_0=0$ is the trivial one.

Step 1. f^2 is a solution. We prove that for any solution f, the function f^2 is still a weak solution of the stochastic kinetic equation. Take test functions of the form $\varphi^n_\zeta(z) = \rho_n(\zeta - z)$, where $(\rho_n)_n$ is a family of standard mollifiers (ρ_n has support in $B_{1/n}$). Let $\zeta = (\xi, \nu) \in \mathbb{R}^{2d}$, $f_n(t,\zeta) = \big(f(t,\cdot) \star \rho_n\big)(\zeta)$. By definition of solution we get that, $\mathbb{P}$-a.s.,

$$\begin{aligned} f_n(t,\zeta) + \int_0^t b(\zeta)\cdot Df_n(s,\zeta)\,\mathrm{d}s + \int_0^t D_v f_n(t,\zeta)\circ \mathrm{d}W_s &= \int_0^t R_n(s,\zeta)\,\mathrm{d}s\,, \\ R_n(s,\zeta) = \int_{\mathbb{R}^{2d}} \big(b(\zeta) - b(z)\big)\cdot D_z f(s,z)\,\rho_n(\zeta - z)\,\mathrm{d}z\,. \end{aligned}$$

The functions f_n are smooth in the space variable. For any fixed $\zeta \in \mathbb{R}^{2d}$, by the Itô formula we get

$$\mathrm{d}f_n^2 = 2f_n\,\mathrm{d}f_n = -2f_n b\cdot Df_n\,\mathrm{d}t - 2f_n D_v f_n \circ \mathrm{d}W_t + 2f_n R_n\,\mathrm{d}t\,.$$

Now we multiply by $\varphi \in C_c^\infty(\mathbb{R}^{2d})$ and integrate over $\mathbb{R}^{2d}$. Using the Itô integral we pass to the limit as $n \to \infty$ and find, $\mathbb{P}$-a.s.,

$$\begin{aligned} \int_{\mathbb{R}^{2d}} f_n^2(t,\zeta)\varphi(\zeta)\,\mathrm{d}\zeta - \frac{1}{2}\int_0^t\int_{\mathbb{R}^{2d}} f_n^2(s,\zeta)\triangle_v\varphi(\zeta)\,\mathrm{d}\zeta\mathrm{d}s + \int_0^t\int_{\mathbb{R}^{2d}}\varphi(\zeta)\,b(\zeta)\cdot Df_n^2(s,\zeta)\,\mathrm{d}\zeta\mathrm{d}s \\ - \int_0^t\int_{\mathbb{R}^{2d}} f_n^2(s,\zeta)D_v\varphi(\zeta)\,\mathrm{d}\zeta\cdot\mathrm{d}W_s = 2\int_0^t\int_{\mathbb{R}^{2d}} f_n(s,\zeta)\,R_n(s,\zeta)\varphi(\zeta)\,\mathrm{d}\zeta\mathrm{d}s\,. \end{aligned} \tag{5.9}$$

Recall that

$$b(z) = A\cdot z + \begin{pmatrix} 0 \\ F(z)\end{pmatrix} \in \mathbb{R}^{2d}.$$

Let us fix $t \in [0,T]$. By definition of weakly differentiable solution it is not difficult to pass to the limit in probability as $n \to \infty$ in all the terms in the left hand side of (5.9). Indeed, we can use that, for every $t \in [0,T]$, $r \geq 1$, $f_n(t,\cdot) \to f(t,\cdot)$ in $W^{1,r}_{loc}(\mathbb{R}^{2d})$, $\mathbb{P}$-a.s., together with the bounds

$$\sup_{t\in[0,T]}\int_{\mathbb{R}^{2d}}\mathbb{E}\Big[|f_n(t,z)|^r\Big]\mathrm{d}z \leq C_r\,, \quad \sup_{t\in[0,T]}\int_{\mathbb{R}^{2d}}\mathbb{E}\Big[|Df_n(t,z)|^r\Big]\mathrm{d}z \leq C_r\,, \tag{5.10}$$

and the Vitali theorem. For instance, if $\mathrm{Supp}(\varphi) \subset B_R$ we have

$$\begin{aligned} J_n(t) &= \mathbb{E}\int_0^t\int_{\mathbb{R}^{2d}} |f_n^2(s,\zeta) - f^2(s,\zeta)||\triangle_v\varphi(\zeta)|\,\mathrm{d}\zeta\mathrm{d}s \\ &\leq C_\varphi\int_0^T\mathbb{E}\int_{B_R}|f_n^2(s,\zeta) - f^2(s,\zeta)|\,\mathrm{d}\zeta\mathrm{d}s\,, \end{aligned}$$

and, for any $s \in [0,T]$, $\mathbb{P}$-a.e ω, $k_n(\omega,s) = \int_{B_R}|f_n^2(\omega,s,\zeta) - f^2(\omega,s,\zeta)|\,\mathrm{d}\zeta \to 0$ as $n \to \infty$. From (5.10) we deduce easily that

$$\sup_{n\geq 1}\mathbb{E}\Big[\int_0^T k_n^2(s)\,\mathrm{d}s\Big] < \infty$$

and so by the Vitali theorem we get $\int_0^T \mathbb{E}\big[\int_{B_R}|f_n^2(s,\zeta) - f^2(s,\zeta)|\,\mathrm{d}\zeta\big]\mathrm{d}s \to 0$, as $n \to \infty$, which implies $\lim_{n\to\infty} J_n(t) = 0$. In order to show that

$$\mathbb{E}\Big[\int_0^t\int_{\mathbb{R}^{2d}}|f_n(s,\zeta)\,R_n(s,\zeta)\varphi(\zeta)|\,\mathrm{d}\zeta\mathrm{d}s\Big] \leq C_\varphi\mathbb{E}\Big[\int_0^t\int_{B_R}|f_n(s,\zeta)\,R_n(s,\zeta)|\,\mathrm{d}\zeta\mathrm{d}s\Big] \to 0$$

as $n \to \infty$, it is enough to prove that for fixed ω, $\mathbb{P}$-a.s., and $s \in [0,T]$ we have

$$\int_{B_R} |f_n(s,\zeta)\, R_n(s,\zeta)|\, \mathrm{d}\zeta \to 0\,, \tag{5.11}$$

as $n \to \infty$. Indeed once (5.11) is proved, using the bounds (5.10) and the Hölder inequality we get

$$\sup_{n\geq 1} \mathbb{E}\Big[\int_0^t \Big| \int_{B_R} \big| f_n(s,\zeta)\, R_n(s,\zeta) \big|\, \mathrm{d}\zeta \Big|^2 \mathrm{d}s \Big] \leq C_R \sup_{n\geq 1} \mathbb{E}\Big[\int_0^t \int_{B_R} |f_n(s,\zeta)\, R_n(s,\zeta)|^2\, \mathrm{d}\zeta \mathrm{d}s \Big],$$

which is finite. Thus we can apply the Vitali theorem and deduce the assertion. Let us check (5.11).

By Sobolev regularity of weakly differentiable solutions we know that

$$\sup_{n\geq 1} \sup_{\zeta \in B_R} |f_n(s,\zeta)| = M < \infty\,.$$

Hence it is enough to prove that $\int_{B_R} |R_n(s,\zeta)| \mathrm{d}\zeta \to 0$. Recall that

$$R_n(s,\zeta) = b(\zeta) \cdot D f_n(s,\zeta) - [(b \cdot Df(s,\cdot)) * \rho_n](s,\zeta)\,.$$

Using the fact that $b \in L^p_{loc}(\mathbb{R}^{2d})$, with p given in Hypothesis 2.1, the Hölder inequality and basic properties of convolutions we have

$$\int_{B_R} \big| b(\zeta) \cdot D f_n(s,\zeta) - b(\zeta) \cdot Df(s,\zeta) \big|\, \mathrm{d}\zeta \;\; \to \;\; 0\,,$$

$$\int_{B_R} \Big| \big[(b \cdot Df(s,\cdot)) * \rho_n \big](s,\zeta) - b(\zeta) \cdot Df(s,\zeta) \Big|\, \mathrm{d}\zeta \;\; \to \;\; 0\,,$$

as $n \to \infty$. This shows that (5.11) holds. We have proved that also f^2 is a weakly differentiable solution of the stochastic kinetic equation.

Step 2. f is identically zero. Due to the integrability properties of f, the stochastic integral in Itô's form is a martingale; it follows that the function $g(t,z) = \mathbb{E}[f^2(t,z)]$ belongs to $C^0([0,T]; W^{1,r}(\mathbb{R}^{2d}))$ for any $r \geq 1$ and satisfies, for any $\varphi \in C_c^\infty(\mathbb{R}^{2d})$,

$$\int_{\mathbb{R}^{2d}} g(t,z)\varphi(z)\, \mathrm{d}z + \int_0^t \int_{\mathbb{R}^{2d}} b(z) \cdot Dg(s,z)\varphi(z)\, \mathrm{d}z\mathrm{d}s = \frac{1}{2} \int_0^t \int_{\mathbb{R}^{2d}} g(s,z) \Delta_v \varphi(z)\, \mathrm{d}z\mathrm{d}s\,.$$

We have, for any $s \in [0,T]$,

$$\begin{aligned} \int_{\mathbb{R}^{2d}} b(z) \cdot Dg(s,z)\varphi(z)\, \mathrm{d}z &= \int_{\mathbb{R}^{2d}} v \cdot D_x g(s,z)\varphi(z)\, \mathrm{d}z + \int_{\mathbb{R}^{2d}} F(z) \cdot D_v g(s,z)\varphi(z)\, \mathrm{d}z \\ &= -\int_{\mathbb{R}^{2d}} v \cdot D_x \varphi(z) g(s,z)\, \mathrm{d}z + \int_{\mathbb{R}^{2d}} F(z) \cdot D_v g(s,z) \varphi(z)\, \mathrm{d}z\,. \end{aligned}$$

Now we fix $\eta \in C_c^\infty(\mathbb{R}^d)$ such that $\eta = 1$ on the ball B_1 of center 0 and radius 1. By considering the test functions:

$$\varphi_{nm}(x,v) = \eta(x/n)\eta(v/m)\,, \quad (x,v) = z \in \mathbb{R}^{2d},$$

$n, m \geq 1$, we obtain

$$\begin{aligned} &\int_{\mathbb{R}^{2d}} g(t,z)\eta(x/n)\eta(v/m)\, \mathrm{d}z - \frac{1}{n} \int_0^t \int_{\mathbb{R}^{2d}} \eta(v/m) v \cdot D\eta(x/n) g(s,z)\, \mathrm{d}z\mathrm{d}s \\ &+ \int_0^t \int_{\mathbb{R}^{2d}} F(z) \cdot D_v g(s,z) \eta(x/n)\eta(v/m)\, \mathrm{d}z\mathrm{d}s = \frac{1}{2m^2} \int_0^t \int_{\mathbb{R}^{2d}} \eta(x/n) g(s,z) \Delta\eta(v/m)\, \mathrm{d}z\mathrm{d}s\,. \end{aligned}$$

Now we fix $m \geq 1$ and pass to the limit as $n \to \infty$ by the Lebesgue theorem. We infer

$$\int_{\mathbb{R}^{2d}} g(t,z)\eta(v/m)\,\mathrm{d}z + \int_0^t \int_{\mathbb{R}^{2d}} F(z)\cdot D_v g(s,z)\eta(v/m)\,\mathrm{d}z\mathrm{d}s = \frac{1}{2m^2}\int_0^t \int_{\mathbb{R}^{2d}} g(s,z)\Delta\eta(v/m)\,\mathrm{d}z\mathrm{d}s\,.$$

Passing to the limit as $m \to \infty$ we arrive at

$$\int_{\mathbb{R}^{2d}} g(t,z)\,\mathrm{d}z = -\int_0^t \int_{\mathbb{R}^{2d}} F(z)\cdot D_v g(s,z)\,\mathrm{d}z\mathrm{d}s\,.$$

Since in particular $g(t,z) \in C^0([0,T];W^{1,r}(\mathbb{R}^{2d}))$, with $r = \frac{p}{p-1}$, we obtain

$$\int_{\mathbb{R}^{2d}} g(t,z)\,\mathrm{d}z = \int_0^t \int_{\mathbb{R}^{2d}} \mathrm{div}_v F(z) g(s,z)\,\mathrm{d}z\mathrm{d}s \leq \|\mathrm{div}_v F(z)\|_\infty \int_0^t \int_{\mathbb{R}^{2d}} g(s,z)\,\mathrm{d}z\mathrm{d}s\,.$$

Applying the Grönwall lemma we get that g is identically zero and this proves uniqueness for the kinetic equation. □

6 Appendix

Proof of Lemma 4.16. Remark that, due to (4.28),

$$\Big(\varepsilon + \big|Z_t^z - Z_t^y\big|^2\Big)^a \leq C\Big(\varepsilon + \big|\gamma(Z_t^z) - \gamma(Z_t^y)\big|^2\Big)^a\,.$$

Therefore, we can prove (4.36) for $\gamma(Z_t)$ instead of Z_t.

We proceed as in [25, Lemma II.2.4] or [15, Lemma 5.4]. Fix any $t \in [0,T]$ and set for $z, y \in \mathbb{R}^{2d}$: $g(z) := f^a(z)$, $f(z) := (\varepsilon + |z|^2)$ and $\eta_t := \gamma(Z_t^z) - \gamma(Z_t^y)$. Then, applying the Itô formula we obtain as in the proof of Lemma 4.13

$$\begin{aligned} g(\eta_t) - g(\eta_0) = {} & 2a\int_0^t f^{a-1}(\eta_s)\,\eta_s\cdot\Big[\widetilde{b}\big(Z_s^z\big) - \widetilde{b}\big(Z_s^y\big)\Big]\,\mathrm{d}s \\ & + 2a\int_0^t f^{a-1}(\eta_s)\,\eta_s\cdot\Big[\widetilde{\sigma}\big(Z_r^z\big) - \widetilde{\sigma}\big(Z_s^y\big)\Big]\cdot\mathrm{d}W_s \\ & + a\sum_{i,j}\int_0^t f^{a-2}(\eta_s)\Big[f(\eta_s)\delta_{i,j} + 2(a-1)\,\eta_s^i\eta_s^j\Big] \\ & \qquad \times\Big[\Big(\widetilde{\sigma}\big(Z_s^z\big) - \widetilde{\sigma}\big(Z_s^y\big)\Big)\Big(\widetilde{\sigma}\big(Z_s^z\big) - \widetilde{\sigma}\big(Z_s^y\big)\Big)^t\Big]^{i,j}\,\mathrm{d}s\,. \end{aligned}$$

Recall that $|z| \leq f^{1/2}(z)$ and that the coefficient $\widetilde{b}$ is Lipschitz continuous:

$$|\widetilde{b}(z) - \widetilde{b}(y)| \leq L|z-y| \leq C|\gamma(z) - \gamma(y)| \leq C f^{1/2}\big(|\gamma(z) - \gamma(y)|\big)\,.$$

We can continue with the estimates and obtain

$$\begin{aligned} g(\eta_t) - g(\eta_0) \leq {} & 2C|a|\int_0^t f^a(\eta_s)\,\mathrm{d}s + 2\,a\int_0^t f^{a-1}(\eta_s)\eta_s\Big[\widetilde{\sigma}\big(Z_s^z\big) - \widetilde{\sigma}\big(Z_s^y\big)\Big]\,\mathrm{d}W_s \\ & + C_{a,d}|a|\int_0^t f^{a-1}(\eta_s)\,|\eta_s|^2\,\mathrm{d}N_s\,. \end{aligned} \tag{6.1}$$

Here, N_t is the process introduced and studied in Lemma 4.8:

$$\int_0^t \big\|\widetilde{\sigma}(Z_s^z) - \widetilde{\sigma}(Z_s^y)\big\|_{HS}^2\,\mathrm{d}s = \int_0^t \big|Z_s^z - Z_s^y\big|^2\,\mathrm{d}N_s\,.$$

The stochastic integral in (6.1) is a martingale with zero mean ($\widetilde{\sigma}$ is bounded). Proceeding as in (4.32), we get

$$\mathbb{E}\Big[e^{-N_t}g(\eta_t)\Big] - e^{-N_0}g(\eta_0) \le C_{a,d}\int_0^t \mathbb{E}\Big[e^{-N_s}g(\eta_s)\Big]\,\mathrm{d}s\,.$$

By Grönwall's inequality applied to the function $h(t) := \mathbb{E}\big[e^{-N_t}g(\eta_t)\big]$, it follows

$$\begin{aligned}\mathbb{E}\Big[e^{-N_t}\Big(\varepsilon + \big|Z_t^z - Z_t^y\big|^2\Big)^a\Big] \le C\mathbb{E}\Big[e^{-N_t}g(\eta_t)\Big] &\le C_{a,d}\,g(\eta_0) = C_{a,d}\big(\varepsilon + |\gamma(z)-\gamma(y)|^2\big)^a \\ &\le C_a\big(\varepsilon + |z-y|^2\big)^a\,. \end{aligned} \tag{6.2}$$

To complete the proof of the lemma, we manipulate (6.2) using Hölder's inequality and we conclude invoking Lemma 4.8 to bound the term $\mathbb{E}[e^{2N_T}]$:

$$\mathbb{E}\Big[\Big(\varepsilon + \big|Z_t^z - Z_t^y\big|^2\Big)^a\Big]^2 \le \mathbb{E}\Big[e^{2N_t}\Big]\mathbb{E}\Big[e^{-2N_t}g^2(\eta_t)\Big] \le C_{a,d}\big(\varepsilon + |z-y|^2\big)^{2a}. \qquad \square$$

We now present some results on the convergence and regularity of approximations $\phi_{0,n}^t$ of the inverse flow ϕ_0^t associated to the SDE (4.2). Note that $\phi_{0,n}^t$ are solutions of SDEs with regular coefficients, see the proof of Theorem 5.4. These results are adapted from [16] and based on the following lemma on the stability of the PDE (4.5), which is of independent interest.

Lemma 6.1 (Stability of the PDE (4.5))**.** *Let U_n be the unique solutions provided by Theorem 3.7 to the PDE* (4.5) *with smooth approximations $B_n(z) = (0, F_n(z))$ of $B(z) = (0, F(z))$ and some λ large enough for* (6.4) *to hold. If $F_n(z) \to F(z)$ in $L^p(\mathbb{R}_v^d; H_p^s(\mathbb{R}_x^d))$, with s, p as in Hypothesis 2.1, then U_n and D_vU_n converge pointwise and locally uniformly to the respective limits. In particular, for any $r > 0$ there exists a function $g(n) \to 0$ as $n \to \infty$ s.t.*

$$\begin{aligned}\sup_{z\in B_r}\big|U_n(z) - U(z)\big| &\le g(n)\,,\\ \sup_{z\in B_r}\big|D_vU_n(z) - D_vU(z)\big| &\le g(n)\,.\end{aligned} \tag{6.3}$$

Moreover, there exists a λ_0 s.t. for all $\lambda > \lambda_0$

$$\big\|D_vU_n\big\|_\infty \le 1/2\,. \tag{6.4}$$

Proof. Setting $V_n = (U_n - U)$ we write for λ large enough (cf. (4.5))

$$\begin{aligned}\lambda V_n(z) - \frac{1}{2}\mathrm{Tr}\big(QD^2V_n(z)\big) - \langle Az, DV_n(z)\rangle - \langle B(z), DV_n(z)\rangle \\ = B_n(z) - B(z) + \langle B_n(z) - B(z), D_vU_n(z)\rangle\,.\end{aligned}$$

By (3.57) we know that

$$\sqrt{\lambda}\|D_vU_n\|_{L^p(\mathbb{R}_v^d;H_p^s(\mathbb{R}_x^d))} \le C\|B_n\|_{L^p(\mathbb{R}_v^d;H_p^s(\mathbb{R}_x^d))} \le C\|B\|_{L^p(\mathbb{R}_v^d;H_p^s(\mathbb{R}_x^d))}\,,\quad n\ge 1\,,$$

with $C = C(s,p,d,\|F\|_{L^p(\mathbb{R}_v^d;H_p^s(\mathbb{R}_x^d))}) > 0$. Hence applying (3.58), (3.59), (3.60) and Sobolev embedding we obtain (6.3) with $g(n) = C\|B - B_n\|_{L^p(\mathbb{R}_v^d;H_p^s(\mathbb{R}_x^d))}$. On the other hand the last assertion follows from (3.60). $\square$

Lemma 6.2 ([16, Lemma 3])**.** *For every $R > 0$, $a \ge 1$ and $z \in B_R$,*

$$\lim_{n\to\infty}\sup_{t\in[0,T]}\sup_{z\in B_R}\mathbb{E}\left[\big|\phi_{0,n}^t(z) - \phi_0^t(z)\big|^a\right] = 0\,.$$

Proof. To ease notation, we shall prove the convergence result for the forward flows $\phi_{t,n} \to \phi_t$. This in enough since the backward flow solves the same equation with a drift of opposite sign. Since the flow ϕ_t is jointly continuous in (t,z), the image of $[0,T] \times B_R$ is contained in $[0,T] \times B_r$ for some $r < \infty$. Thus for $z \in B_R$, from Lemma 6.1 we get $|U_n(\phi_{t,n}) - U(\phi_t)| \le g(n) + 1/2|\phi_{t,n} - \phi_t|$ and $|D_v U_n(\phi_{t,n}) - D_v U(\phi_t)| \le g(n) + |D_v U_n(\phi_{t,n}) - D_v U_n(\phi_t)|$. Extending the definition (4.27) to $\gamma_n(z) = z + U_n(z)$ we have the approximate equivalence

$$\frac{2}{3}\Big(\big|\gamma_n(\phi_{t,n}) - \gamma(\phi_t)\big| - g(n)\Big) \le \big|\phi_{t,n} - \phi_t\big| \le 2\Big(\big|\gamma_n(\phi_{t,n}) - \gamma(\phi_t)\big| + g(n)\Big)\,.$$

Therefore, it is enough prove the convergence result for the transformed flows $\gamma_{t,n} = \gamma_n(\phi_{t,n}) \to \gamma(\phi_t) = \gamma_t$. Proceeding as in the proof of Theorem 4.11 we get, for any $a \ge 2$

$$\begin{aligned}\frac{1}{a}\mathrm{d}\big|\gamma_{t,n} - \gamma_t\big|^a \le \big|\gamma_{t,n} - \gamma_t\big|^{a-2}\Big\{&\big(\gamma_{t,n} - \gamma_t\big)\cdot\Big[\lambda\big(U_n(\phi_{t,n}) - U(\phi_t)\big) + A(\phi_{t,n} - \phi_t)\Big]\mathrm{d}t\\ &+ \big(\gamma_{t,n} - \gamma_t\big)\cdot\big(DU_n(\phi_{t,n}) - DU(\phi_t)\big)R\cdot\mathrm{d}W_t\\ &+ C_{a,d}\big\|\big(DU_n(\phi_{t,n}) - DU(\phi_t)\big)R\big\|_{HS}^2\,\mathrm{d}t\Big\}\,. \end{aligned} \tag{6.5}$$

The stochastic integral is a martingale. Since

$$\frac{|\phi_{t,n} - \phi_t|}{|\gamma_{t,n} - \gamma_t|} \le C\Big(1 + \frac{g(n)}{|\gamma_{t,n} - \gamma_t|}\Big)\,,$$

the term on the last line in (6.5) can be bounded using (6.3) by a constant times $|\gamma_{t,n} - \gamma_t|^a\,\mathrm{d}B_{t,n} + |\gamma_{t,n} - \gamma_t|^{a-2}g^2(n)(\mathrm{d}B_{t,n} + \mathrm{d}t)$, where for every n the process $B_{t,n}$ is defined as in (4.31) but with $DU_n(\phi_{t,n})$ and $DU_n(\phi_t)$ in the place of $DU(Z_t)$ and $DU(Y_t)$ respectively. One can show that $B_{t,n}$ share the same integrability properties of the process N_t studied in Lemma 4.8, uniformly in n, see [16, Lemma 14]. Computing $\mathbb{E}[e^{-B_{t,n}}|\gamma_{t,n} - \gamma_t|^a]$ using the Itô formula and taking the supremum over $t \in [0,T]$ leads to

$$\begin{aligned}\sup_{t\in[0,T]}\mathbb{E}\Big[e^{-B_{t,n}}\big|\gamma_{t,n} - \gamma_t\big|^a\Big] \le\; & C\mathbb{E}\Big[\int_0^T e^{-B_{s,n}}\big|\gamma_{s,n} - \gamma_s\big|^a\mathrm{d}s\Big]\\ &+ Cg(n)\mathbb{E}\Big[\int_0^T e^{-B_{s,n}}\big(\big|\gamma_{s,n} - \gamma_s\big|^{a-1} + g(n)\big|\gamma_{s,n} - \gamma_s\big|^{a-2}\big)\mathrm{d}s\Big]\\ &+ g^2(n)\mathbb{E}\Big[\int_0^T e^{-B_{s,n}}\big|\gamma_{s,n} - \gamma_s\big|^{a-2}\mathrm{d}B_{s,n}\Big]\,.\end{aligned}$$

Using the integrability properties of $\phi_t, \phi_{t,n}, U(\phi_t), U_n(\phi_{t,n})$ one can see that all terms are bounded, uniformly in n. To conclude the proof we can pass to the limit

$$\limsup_n \sup_{t\in[0,T]}\mathbb{E}\Big[e^{-B_{t,n}}\big|\gamma_{t,n} - \gamma_t\big|^a\Big] \le C\int_0^T \limsup_n \sup_{t\in[0,s]}\mathbb{E}\Big[e^{-B_{t,n}}\big|\gamma_{t,n} - \gamma_t\big|^a\Big]\mathrm{d}s\,,$$

apply Grönwall's lemma and proceed as in Corollary 4.12 to get rid of the exponential term. ☐

Lemma 6.3 ([16, Lemma 5])**.** *For every $a \ge 1$, there exists $C_{a,d,T} > 0$ such that*

$$\sup_{t\in[0,T]}\sup_{z\in\mathbb{R}^{2d}}\mathbb{E}\Big[\big|D\phi_{0,n}^t(z)\big|^a\Big] \le C_{a,d,T} \tag{6.6}$$

uniformly in n.

Proof. Let us show the bound for the forward flows $\phi_{t,n}$. These are regular flows: let $\theta_{t,n}$ and $\xi_{t,n}$ denote the weak derivative of $D\phi_{t,n}$ and $D\gamma_{t,n} = D\gamma_n(\phi_{t,n})$, respectively. They are equivalent in the sense of (4.39), so we shall prove the bound for $\xi_{t,n}$ instead of $\theta_{t,n}$. Proceeding as in the proof of Theorem 4.19 we obtain as in (4.41)

$$\mathrm{d}e^{-C_1 B_{t,n}}\big|\xi_{t,n}\big|^a \le e^{-C_1 B_{t,n}}\Big[C_2\big|\xi_{t,n}\big|^a \mathrm{d}t + \mathrm{d}M_t\Big] ,$$

where the process $B_{t,n}$ is simply given by $\int_0^t |DD_v U_n(\phi_{s,n})|^2 \mathrm{d}s$. We can integrate, take expected values, the supremum over $t \in [0,T]$ and apply Grönwall's inequality to get

$$\sup_{t\in[0,T]} \mathbb{E}\big[e^{-C_1 B_{t,n}}|\xi_{t,n}|^a\big] \le C_T|\xi_{0,n}|^a = C_{a,d,T} .$$

Observe that this bound is uniform in n and $z \in \mathbb{R}^{2d}$. Proceeding as in Corollary 4.12 we can get rid of the exponential term and obtain the desired uniform bound on $\xi_{t,n}$. □

References

[1] Beck L. and Flandoli F.: A regularity theorem for quasilinear parabolic systems under random perturbations. *Journal of Evolution Equations* **13**, (2013), 829–874. MR-3127027

[2] Beck L., Flandoli F., Gubinelli M. and Maurelli M.: Stochastic ODEs and stochastic linear PDEs with critical drift: regularity, duality and uniqueness. arXiv:1401.1530.

[3] Bergh J. and Lofström J.: Interpolation spaces. An introduction. Grundlehren der Mathematischen Wissenschaften, No. 223. *Springer-Verlag*, Berlin-New York, 1976. MR-0482275

[4] Bouchut F.: Hypoelliptic regularity in kinetic equations. *J. Math Pures Appl.* **81**, (2002), 1135–1159. MR-1949176

[5] Bramanti M., Cupini G., Lanconelli E. and Priola E.: Global L^p estimates for degenerate Ornstein-Uhlenbeck operators. *Math. Z.* **266**, (2010), 789–816. MR-2729292

[6] Chaudru de Raynal P.E.: Strong existence and uniqueness for degenerate SDE with Hölder drift. *Ann. Inst. H. Poincaré Probab. Statist.* **53** (1), (2017), 259–286. MR-3606742

[7] Chaudru de Raynal P.E.: Weak well posedness for hypoelliptic stochastic differential equations with singular drift: a sharp result. arXiv:1606.05458.

[8] Coghi M. and Flandoli F.: Propagation of chaos for interacting particles subject to environmental noise. *Ann. Appl. Probab.* **26** (3), (2016), 1407–1442. MR-3513594

[9] Cwikel M.: On $(L^{p_0}(A_0),\, L^{p_1}(A_1))_{\theta,\,q}$. *Proc. Amer. Math. Soc.* **44**, (1974), 286–292. MR-0358326

[10] Da Prato G., Flandoli F., Priola E. and Röckner M.: Strong uniqueness for stochastic evolution equations in Hilbert spaces perturbed by a bounded measurable drift. *Ann. Probab.* **41** (5), (2013), 3306–3344. MR-3127884

[11] Delarue F., Flandoli F. and Vincenzi D.: Noise prevents collapse of Vlasov-Poisson point charges. *Comm. Pure Appl. Math.* **67** (10), (2014), 1700–1736. MR-3251910

[12] DiPerna R. and LionsP.L.: Ordinary differential equations, transport theory and Sobolev spaces. *Invent. Math.* **98**, (1989), 511–547. MR-1022305

[13] Falkovich G., Gawedzki K. and Vergassola M.: Particles and fields in fluid turbulence. *Rev. Modern Phys.* **73** (4), (2001), 913–975. MR-1878800

[14] Fedrizzi E. and Flandoli F.: Pathwise uniqueness and continuous dependence for SDEs with non-regular drift. *Stochastics* **83** (3), (2011), 241–257. MR-2810591

[15] Fedrizzi E. and Flandoli F.: Hölder Flow and Differentiability for SDEs with Nonregular Drift. *Stochastic Analysis and Applications* **31** (4), (2013), 708–736. MR-3175794

[16] Fedrizzi E. and Flandoli F.: Noise prevents singularities in linear transport equations. *J. Funct. Anal.* **264**, (2013), 1329–1354. MR-3017266

[17] Fedrizzi E., Flandoli F., Priola E. and Vovelle J.: Regularity of stochastic kinetic equations. Preprint version arXiv:1606.01088.

[18] Flandoli F.: Random perturbation of PDEs and fluid dynamic models. Saint Flour Summer School Lectures, 2010, Lecture Notes in Math. **2015**, *Springer*, Berlin, 2011. MR-2796837

[19] Flandoli F., Gubinelli M. and Priola E.: Well–posedness of the transport equation by stochastic perturbation. *Invent. Math.* **180** (1), (2010), 1–53. MR-2593276

[20] Golse F. and Saint-Raymond L.: Velocity averaging in L^1 for the transport equation. *C. R. Acad. Sci. Paris Sér. I Math.* **334**, (2002), 557–562. MR-1903763

[21] Han-Kwan D.: L^1 averaging lemma for transport equations with Lipschitz force fields. *Kinet. Relat. Models* **3** (4), (2010), 669–683. MR-2735910

[22] Ikeda N. and Watanabe S.: Stochastic Differential Equations and Diffusion Processes. *North Holland-Kodansha*, II edition, 1989. MR-1011252

[23] Khas'minskii R.Z.: On positive solutions of the equation $Au + Vu = 0$. *Theoret. Probab. Appl.* **4**, (1959), 309–318. MR-0123373

[24] Krylov N.V. and Röckner M.: Strong solutions to stochastic equations with singular time dependent drift. *Probab. Theory Relat. Fields* **131**, (2005), 154–196. MR-2117951

[25] Kunita H.: Stochastic differential equations and stochastic flows of diffeomorphisms. XII École d'été de probabilités de Saint-Flour - 1982. Lecture Notes in Math. 1097, 143–303. *Springer*, Berlin, 1984. MR-0876080

[26] Kunita H.: Stochastic flows and stochastic differential equations. Cambridge Stud. Adv. Math., *Cambridge University Press*, 1990. MR-1070361

[27] Lions J.L. and Peetre J.: Sur une classe d'espaces d'interpolation. *Inst. Hautes Etudes Sci. Publ. Math.* **19**, (1964), 5–68. MR-0165343

[28] Liptser R. and Shiryaev A.N.: Statistics of Random Processes I. General Theory. *Springer*, 2001. MR-1800857

[29] Lunardi A.: Analytic Semigroups and Optimal Regularity in Parabolic Problems. *Birkhäuser*, 1995. MR-3012216

[30] Priola E.: Pathwise uniqueness for singular SDEs driven by stable processes. *Osaka Journal of Mathematics* **49**, (2012), 421–447. MR-2945756

[31] Priola E.: On weak uniqueness for some degenerate SDEs by global L^p estimates. *Potential Anal.* **42** (1), (2015), 247–281. MR-3297995

[32] Stein E.M.: Singular integrals and differentiability properties of functions. Princeton Mathematical Series, No. 30 *Princeton University Press*, Princeton, N.J. 1970. MR-0290095

[33] Strichartz R.S.: Fubini-type theorems. *Annali della Scuola Normale Superiore di Pisa - Classe di Scienze* **22**, (1968), 399–408. MR-0236695

[34] Sznitman A.-S.: Brownian motion, obstacles, and random media. *Springer*, 1998. MR-1717054

[35] Triebel H.: Interpolation theory, function spaces, differential operators. Amsterdam: North-Holland, 1978. MR-0503903

[36] Veretennikov A.J.: Strong solutions and explicit formulas for solutions of stochastic integral equations. *Mat. Sb.* (N.S.) **111** (153) (3), (1980), 434–452. MR-0568986

[37] Wang F.-Y. and Zhang X.: Degenerate SDEs in Hilbert Spaces with Rough Drifts. *Infin. Dimens. Anal. Quantum Probab. Relat. Top.* **18** (4), (2015). MR-3447227

[38] Wang F.-Y. and Zhang X.: Degenerate SDE with Hölder-Dini drift and non-Lipschitz noise coefficient. *SIAM J. Math. Anal.* **48** (3), (2016), 2189–2226. MR-3511355

[39] Xie L. and Zhang X.: Sobolev differentiable flows of SDEs with local Sobolev and super-linear growth coefficients. *Ann. Probab.* **44** (6), (2016), 3661–3687. MR-3572321

[40] Zhang X.: Stochastic partial differential equations with unbounded and degenerate coefficients. *J. Differential Equations* **250**, (2011), 1924–1966. MR-2763561

[41] Zhang X.: Stochastic Hamiltonian flows with singular coefficients. arXiv:1606.04360.

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