

Bootstrap percolation on products of cycles and complete graphs*

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Abstract

Bootstrap percolation on a graph iteratively enlarges a set of occupied sites by adjoining points with at least θ occupied neighbors. The initially occupied set is random, given by a uniform product measure, and we say that spanning occurs if every point eventually becomes occupied. The main question concerns the critical probability, that is, the minimal initial density that makes spanning likely. The graphs we consider are products of cycles of m points and complete graphs of n points. The major part of the paper focuses on the case when two factors are complete graphs and one factor is a cycle. We identify the asymptotic behavior of the critical probability and show that, when θ is odd, there are two qualitatively distinct phases: the transition from low to high probability of spanning as the initial density increases is sharp or gradual, depending on the size of m .

Keywords: bootstrap percolation; critical probability; gradual transition; sharp transition.

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1 Introduction

Given a graph $G = (V, E)$, *bootstrap percolation* with *threshold* θ is a discrete-time growth process that, starting from an initial configuration $\omega \in \{0, 1\}^V$, generates an increasing sequence of configurations $\omega = \omega_0, \omega_1, \dots$. Given ω_j , $j \geq 0$, ω_{j+1} is given by

$$\omega_{j+1}(v) = \begin{cases} 1 & \text{if } \omega_j(v) = 1 \text{ or } \sum_{w \sim v} \omega_j(w) \geq \theta \\ 0 & \text{else} \end{cases}$$

and ω_∞ is the pointwise limit of ω_j as $j \rightarrow \infty$. A vertex, v , is called *occupied* at time j if $\omega_j(v) = 1$, and is called *empty* otherwise; a set of vertices $S \subseteq V$ is called

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occupied (resp. empty) at time j if every vertex in S is occupied (resp. empty) at time j . The initial configuration ω is random; $\{\omega(v) : v \in V\}$ is a collection of i.i.d. Bernoulli random variables with parameter p . The most natural object of study is the event $\text{Span} = \{\omega_\infty \equiv 1\}$ that *spanning* occurs (in which case we also say that the initial configuration ω *spans* V). This process was introduced in [CLR], and has been widely studied since; see [AdL, Hol2] for readable surveys.

While some of the earliest results are on infinite lattices [vEn, Sch], many of the most interesting questions are formulated for graphs with finite vertex set V , whose size increases to infinity with an integer parameter n . Starting from the foundational paper [AiL], an impressive body of work addresses the most natural example of d -dimensional lattice cube \mathbb{Z}_n^d , with n^d vertices and nearest-neighbor edges [Sch, Hol1, GHM, BB2, BBM, BBDM]. More recently, some work has been done on the Hamming torus K_n^d , which, as the Cartesian product of d complete graphs of n vertices, has the same vertex set as the lattice cube, but a much larger set of edges, which makes many percolation questions fundamentally different; see [Siv, GHPS, Sli], and for a closely related model see [BBLN].

When V is finite, $\mathbb{P}_p(\text{Span})$ is a polynomial in p that increases from 0 to 1 for $p \in [0, 1]$. Therefore, for every $\alpha \in [0, 1]$ there exists a unique $p_\alpha = p_\alpha(n)$ so that $\mathbb{P}_{p_\alpha}(\text{Span}) = \alpha$. Commonly, $p_{1/2}$ is also called the *critical probability* and is denoted by p_c . In our cases, Span happens a. a. s. for any $p > 0$, which results in $p_c \rightarrow 0$. We say that a *sharp transition* (for the event Span) occurs if $p_\alpha \sim p_c$ for all $\alpha \in (0, 1)$, as $n \rightarrow \infty$, or equivalently,

$$\mathbb{P}_{ap_c}(\text{Span}) \rightarrow \begin{cases} 0 & a \in (0, 1) \\ 1 & a \in (1, \infty) \end{cases}$$

Sharp transitions results have been proved in remarkable generality [FK]. They hold for bootstrap percolation on the lattices \mathbb{Z}_n^d , where much more is proved [Hol1, BB1, BB2, BBM, BBDM]. However, general results such as those in [FK] do not apply to our cases because the critical probabilities are too small relative to the number of vertices (see Theorem 1.2).

By contrast, *gradual transition* occurs if there exists a nondecreasing continuous function ϕ on $(0, \infty)$ with $\phi(0+) = 0$ and $\phi(\infty) = 1$, so that

$$\mathbb{P}_{ap_c}(\text{Span}) \rightarrow \phi(a)$$

for all $a \in (0, \infty)$. To date, there has been no general investigation of this phenomenon, and it is rigorously established on only a few Hamming tori examples: K_n with arbitrary θ (trivially), K_n^2 with arbitrary θ [GHPS], K_n^d with arbitrary d and $\theta = 2$ [Sli], and K_n^3 with $\theta = 3$ [GHPS]. (Due to locality of nucleation events, lattice examples with gradual transition are somewhat easier to study [GG].) Clearly, sharp and gradual transitions are not the only possibilities, and indeed we exhibit examples where neither happens; see Theorem 1.4.

In this paper, we initiate a study of bootstrap percolation on graphs of the form $V = (\mathbb{Z}_m)^{d_1} \times (K_n)^{d_2}$ where $d_1, d_2 \geq 1$ are integer parameters, \mathbb{Z}_m denotes the cycle on m vertices, K_n denotes the complete graph on n vertices, and \times denotes the Cartesian product (see Figure 1 for an example). We will assume throughout that

$$\log m \sim \gamma \log n \tag{1.1}$$

for some constant $\gamma > 0$. We use the symbol $a_n \sim b_n$ to denote $a_n/b_n \rightarrow 1$. Bootstrap percolation on these graphs may be viewed as an extreme case of anisotropic bootstrap percolation, where the neighborhood in some directions (d_1 of them) is nearest-neighbor, but in other directions (d_2 of them) the neighborhood extends as far as possible. The

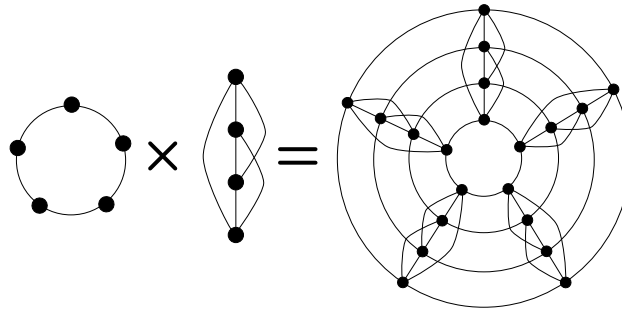


Figure 1: Cartesian product of \mathbb{Z}_5 and K_4 .

graphs under study could be viewed as limiting cases when the Holroyd-Liggett-Romik model [HLR] is combined with anisotropic graphs studied recently in [DE]. These graphs are also of great interest for the study of epidemics due to their community structure, see [LZ] for analysis of an epidemic model on the infinite graphs $\mathbb{Z}^d \times K_n$ for $d = 2, 3$. For another bootstrap percolation model on graphs that combine local and long-range connections, see [TV].

We are able to prove a general result for $V = \mathbb{Z}_m^d \times K_n$, which turns out to be mostly an application of bootstrap percolation results on lattices [Hol1, BBM, BBDM]. Denote by $\log_{(k)}$ the k th iterate of \log and let $\lambda(d, \theta)$ be the bootstrap percolation scaling constant for the lattice \mathbb{Z}_m^d defined in [BBDM].

Theorem 1.1. *Assume bootstrap percolation on $V = \mathbb{Z}_m^d \times K_n$ for any $d \geq 1$ and $\theta \geq 2$. Then, for $\theta \leq d$,*

$$p_c \sim \frac{1}{n} \left(\frac{\lambda(d, \theta)}{\log_{(\theta-1)} m} \right)^{d-\theta+1}.$$

For $\theta > d$,

$$p_c \sim \frac{d}{2^{\max(2d+1-\theta, 0)}} \cdot \frac{\log m}{n}.$$

In all cases, the transition is sharp.

The above theorem, whose proof is relegated to Section 8, demonstrates that there is no transition in behaviors for different γ , therefore this case is not of primary interest and we include it mainly for comparison. Indeed, we will see that the situation is very different when the complete graph K_n is replaced by the two-dimensional Hamming torus K_n^2 . In this paper we focus on $\mathbb{Z}_m \times K_n^2$, as the simplest case that exhibits sharp, gradual and hybrid phase transitions, depending on the relative scaling for m and n . We will address the more demanding case $\mathbb{Z}_m^{d_1} \times K_n^2$ in a subsequent paper, although some of the aforementioned phenomena appear to be limited to the case $d_1 = 1$. Higher-dimensional Hamming tori $K_n^{d_2}$, $d_2 > 2$, are much more complex [GHPS, Sli]. We now state our main results.

Theorem 1.2. *Assume bootstrap percolation on $\mathbb{Z}_m \times K_n^2$ and $\theta \geq 2$.*

- If $\theta = 2\ell + 1$ and $\gamma > 1/\ell$, then

$$p_c \sim \left(\frac{1}{2}(\gamma + 1/\ell)\ell! \right)^{1/\ell} \cdot \frac{(\log n)^{1/\ell}}{n^{1+1/\ell}}, \tag{1.2}$$

with sharp transition.

- If $\theta = 2\ell + 1$ and $\gamma < 1/\ell$, then

$$p_c \sim \left(\frac{1}{2}(\ell + 1)! \log 2\right)^{1/(\ell+1)} \cdot \frac{1}{n^{1+1/(\ell+1)} m^{1/(\ell+1)}}, \tag{1.3}$$

with gradual transition, and $\phi(a) = 1 - 2^{-a^{(\ell+1)}}$.

- If $\theta = 2\ell$, then

$$p_c \sim \begin{cases} \left(\frac{1}{4}\gamma \ell!\right)^{1/\ell} \cdot \frac{(\log n)^{1/\ell}}{n^{1+1/\ell}} & \theta \geq 4 \\ \frac{1}{2}\gamma \cdot \frac{\log n}{n^2} & \theta = 2, \end{cases} \tag{1.4}$$

with sharp transition.

Therefore, $m \approx n^{1/\ell}$ marks the boundary between sharp and gradual transition in case of odd-threshold θ , while there is no such boundary when the threshold is even. In fact, the odd threshold case has, when $\gamma > 1/\ell$, another sharp transition. Henceforth, we make the customary identification of the configuration ω_∞ with the occupied set $\{v : \omega_\infty(v) = 1\}$.

Theorem 1.3. Assume bootstrap percolation on $\mathbb{Z}_m \times K_n^2$, and suppose that $\theta = 2\ell + 1$ and $\gamma > 1/\ell$. Assume that

$$p \sim a \cdot \frac{(\log n)^{1/\ell}}{n^{1+1/\ell}}. \tag{1.5}$$

Then

$$\frac{|\omega_\infty|}{|V|} = \frac{|\omega_\infty|}{mn^2} \rightarrow \begin{cases} 0 & a^\ell/\ell! < 1/\ell \\ 1 & a^\ell/\ell! > 1/\ell \end{cases} \quad \text{in probability as } n \rightarrow \infty.$$

In other words, when the scaling constant a of (1.5) is smaller than $(\ell!/\ell)^{1/\ell}$, then a very small proportion of sites becomes occupied during the bootstrap process. When a is larger than $(\ell!/\ell)^{1/\ell}$ but smaller than $(\frac{1}{2}(\gamma + 1/\ell)\ell!)^{1/\ell}$, then most, but not all, of the sites become occupied. Finally, when $a > (\frac{1}{2}(\gamma + 1/\ell)\ell!)^{1/\ell}$, then all sites become occupied.

Finally, we give the promised example with mixed phase transition, for which we need to assume that m satisfies a particular boundary scaling with $\gamma = 1/\ell$.

Theorem 1.4. Assume bootstrap percolation on $\mathbb{Z}_m \times K_n^2$, and suppose that $\theta = 2\ell + 1$, that

$$m \sim \frac{n^{1/\ell}}{(\log n)^{1+1/\ell}}, \tag{1.6}$$

and that p satisfies (1.5). Then

$$\mathbb{P}_p(\text{Span}) \rightarrow \begin{cases} 0 & a^\ell/\ell! < 1/\ell \\ 1 - \exp(-2a^{\ell+1}/(\ell + 1)!) & a^\ell/\ell! > 1/\ell. \end{cases} \tag{1.7}$$

The proofs of Theorems 1.2–1.4 are completed in Section 7, after the auxiliary results are established in Sections 2–7. In particular, we use a result on the birthday problem in a “slightly supercritical” regime, which is proved in Section 2.

2 Preliminary results

2.1 Two simple lemmas

Lemma 2.1. In a sequence of k independent Bernoulli random variables, which are 1 with a small probability r ,

$$\mathbb{P}(\text{no two consecutive 1s}) = \exp(-kr^2 + \mathcal{O}(kr^3 + r^2)).$$

Proof. Let a_k be the probability of no two consecutive 1s in a sequence of length k . Then a_k satisfies the recursion

$$a_k = r(1 - r)a_{k-2} + (1 - r)a_{k-1} \quad \text{for } k \geq 2,$$

and $a_0 = a_1 = 1$. Let $S = \sqrt{1 + 2r - 3r^2}$. Then the solution of the above recursion is

$$a_k = \frac{1 + r + S}{2S} \left(\frac{1 - r + S}{2} \right)^k - \frac{1 + r - S}{2S} \left(\frac{1 - r - S}{2} \right)^k,$$

and the result follows by Taylor expansion. □

We will also use the following well-known Chernoff bounds.

Lemma 2.2. *For any $p, \epsilon \in (0, 1)$ and integer n ,*

$$\begin{aligned} \mathbb{P}(\text{Binomial}(n, p) \leq (1 - \epsilon)np) &\leq \exp(-np\epsilon^2/2), \\ \mathbb{P}(\text{Binomial}(n, p) \geq (1 + \epsilon)np) &\leq \exp(-np\epsilon^2/3). \end{aligned}$$

2.2 Birthday problem

In this self-contained section, we use m and n as is customary in the classic birthday problem, therefore these variables do not have the same meaning as in the rest of the paper.

The k -coincidence birthday problem asks, “What is the probability that, among m people with birthdays chosen independently and uniformly at random from $[n] = \{1, \dots, n\}$, there exists a set of k people that have the same birthday?” Let $A = A_{n,m,k}$ be the event that such a k -coincidence exists. An equivalent formulation of the birthday problem, relevant to our setting, is as follows. Given an m -by- n binary array, chosen uniformly at random from arrays that have a single 1 on every row, what is the probability that there exists a column with at least k 1’s? This formulation appears in the proof of Lemma 3.1, where the following lemma is applied.

Lemma 2.3. *Assume k is fixed, n is large, and m depends on n in such a way that $m^{k+1} \ll n^k$. Then*

$$\mathbb{P}(A^c) \sim \exp\left(-\frac{1}{k!} \frac{m^k}{n^{k-1}}\right), \tag{2.1}$$

as $n \rightarrow \infty$.

It is not difficult to show, using Poisson approximation, that (2.1) holds when m^k/n^{k-1} approaches a constant [AGG]. However, we need the formula when m is larger by a multiplicative power of $\log n$, in which case the standard upper bound for the error in Poisson approximation [BHJ] is too large. Instead, we use the following asymptotic expansion result. For a function f analytic in a neighborhood of 0, we denote by $\widehat{f}[m]$ the coefficient of z^m in its power series expansion.

Theorem 2.4 (Theorem 4 in [Gar]). *Let $f(z) = \sum_{i=0}^{\infty} a_i z^i$, and suppose that $a_0 > 0$, $a_1 > 0$, $a_i \geq 0$ for $i \geq 2$, and the series has positive radius of convergence. Suppose $m, n \rightarrow \infty$ such that $m = o(n)$. Define $\rho > 0$ to be the unique positive solution of*

$$\rho \frac{f'(\rho)}{f(\rho)} = \frac{m}{n}.$$

Then

$$\widehat{f}^n[m] = \frac{f(\rho)^n}{\rho^m \sqrt{2\pi m}} (1 + o(1)).$$

Proof of Lemma 2.3. We will first give a formula for $\mathbb{P}(A)$ using embedding into a Poisson process.

Suppose $\{\xi_s^i\}_{i=1}^n$ is a collection of n i.i.d. Poisson Processes with rate $1/n$, so ξ_s^i is the number of people with birthday i at time s , where people arrive at rate 1. Therefore, letting $e_k(z) = \sum_{i=0}^{k-1} z^i/i!$, since the distribution of $(\xi_m^1, \dots, \xi_m^n)$ conditional on $\{\sum_{i=1}^n \xi_m^i = m\}$ is multinomial we have

$$\begin{aligned} \mathbb{P}(A^c) &= \mathbb{P}\left(\max_{i \in [n]} \xi_m^i \leq k-1 \mid \sum_{i=1}^n \xi_m^i = m\right) \\ &= \frac{\mathbb{P}\left(\sum_{i=1}^n \xi_m^i = m, \max_{i \in [n]} \xi_m^i \leq k-1\right)}{\mathbb{P}\left(\sum_{i=1}^n \xi_m^i = m\right)} \\ &= \frac{e^{-m} \left(\frac{m}{n}\right)^n \cdot \widehat{e_k(z)^n}[m]}{\mathbb{P}(\text{Poisson}(m) = m)} \\ &\sim \frac{m!}{n^m} \widehat{e_k(z)^n}[m] \end{aligned} \tag{2.2}$$

as $m, n \rightarrow \infty$.

In order to apply Theorem 2.4, we need to estimate ρ and $f(\rho)$ when $f = e_k$. We observe that

$$\begin{aligned} f(\rho) &= e^\rho - \frac{1}{k!} \rho^k + \mathcal{O}(\rho^{k+1}) \\ f'(\rho) &= e^\rho - \frac{1}{(k-1)!} \rho^{k-1} + \mathcal{O}(\rho^k) \end{aligned}$$

From this, after factoring out e^ρ from both $f(\rho)$ and $f'(\rho)$ and a short computation

$$\rho - \frac{1}{(k-1)!} \rho^k + \mathcal{O}(\rho^{k+1}) = \frac{m}{n}$$

and then

$$\rho = \frac{m}{n} + \frac{1}{(k-1)!} \frac{m^k}{n^k} + \mathcal{O}\left(\frac{m^{k+1}}{n^{k+1}}\right).$$

Next we observe

$$\log\left(1 + \frac{1}{(k-1)!} \frac{m^{k-1}}{n^{k-1}} + \mathcal{O}\left(\frac{m^k}{n^k}\right)\right) = \frac{1}{(k-1)!} \frac{m^{k-1}}{n^{k-1}} + \mathcal{O}\left(\frac{m^k}{n^k}\right),$$

and therefore

$$\rho^m = \frac{m^m}{n^m} \exp\left(\frac{1}{(k-1)!} \frac{m^k}{n^{k-1}} + \mathcal{O}\left(\frac{m^{k+1}}{n^k}\right)\right). \tag{2.3}$$

Now,

$$\begin{aligned} \log f(\rho) &= \rho + \log\left(1 - \frac{1}{k!} e^{-\rho} \rho^k + \mathcal{O}(\rho^{k+1})\right) \\ &= \frac{m}{n} + \left(\frac{1}{(k-1)!} - \frac{1}{k!}\right) \frac{m^k}{n^k} + \mathcal{O}\left(\frac{m^{k+1}}{n^{k+1}}\right) \end{aligned}$$

and therefore

$$f(\rho)^n = \exp\left(m + \left(\frac{1}{(k-1)!} - \frac{1}{k!}\right) \frac{m^k}{n^{k-1}} + \mathcal{O}\left(\frac{m^{k+1}}{n^k}\right)\right). \tag{2.4}$$

By Theorem 2.4, (2.3) and (2.4), assuming $m^{k+1} \ll n^k$,

$$\widehat{e_k(z)^n}[m] = \widehat{f^n}[m] \sim \frac{e^m n^m}{m^m \sqrt{2\pi m}} \exp\left(-\frac{1}{k!} \frac{m^k}{n^{k-1}}\right),$$

and the asymptotic formula for $\mathbb{P}(A^c)$ follows from (2.2) and Stirling's approximation. \square

3 Single copies of Hamming planes

In the sequel, we will refer to a *copy* of K_n^2 , by which we mean a (deterministic) subgraph $\{i\} \times K_n^2$, for some $i \in \mathbb{Z}_m$. Its four *subsquares* are obtained by division of $\{i\} \times K_n^2$ (if n is even) or $\{i\} \times K_{n-1}^2$ (if n is odd) into four disjoint congruent squares. A *horizontal line* in a copy $\{i\} \times K_n^2$ is a subset of the form $\{(i, x, y) : x \in [n]\}$ for a fixed $y \in [n]$, and a *vertical line* is a subset of the form $\{(i, x, y) : y \in [n]\}$ for a fixed $x \in [n]$. We use the following terminology to describe the properties of the initial configuration ω_0 within a copy of K_n^2 or, in one case, within a copy and its neighboring copies. For a fixed $k \geq 1$, call a copy of K_n^2 :

- *k-viable* if it contains a horizontal or a vertical line with at least k initially occupied sites;
- *k-internally spanned (k-IS)* if the bootstrap dynamics with threshold k (rather than θ) restricted to it spans it;
- *k-internally inert (k-II)* if the bootstrap dynamics with threshold k restricted to it does not change the initial configuration;
- *k-inert* if no site in it becomes occupied at time 1 in the bootstrap dynamics with threshold k (perhaps with help of neighboring planes); and
- *k-proper* if within each of the four subsquares there are at least θ horizontal and θ vertical lines, such that each line contains at least k initially occupied sites.

In this section, we will assume that p is of the form

$$p = a \cdot \frac{(\log n)^{1/\ell}}{n^{1+1/\ell}}, \tag{3.1}$$

for some $a > 0$, where $\ell = \lceil (\theta - 1)/2 \rceil$. We will now briefly explain why this is the critical scaling for spanning, by sketching a simplified argument that does not establish a critical constant a but illustrates some of our arguments. When $\theta - 1$ is odd, the probability that a copy of K_n^2 is not ℓ -viable and the probability that it is not $(\theta - 1)$ -IS are both about $\exp(-2n(np)^\ell/\ell!)$. When $\theta - 1$ is even, the second probability goes up to about $\exp(-n(np)^\ell/\ell!)$. In either case, if a is large enough, every copy of K_n^2 is $(\theta - 1)$ -IS, and all we need for spanning is a single θ -IS copy of K_n^2 , which will appear if γ is large. On the other hand, take any pair of consecutive Hamming planes $\{i, i + 1\} \times K_n^2$, and form a new configuration on a Hamming plane \widehat{K}_n^2 in which $x \in \widehat{K}_n^2$ is initially occupied if either (i, x) or $(i + 1, x)$ is initially occupied. If such configuration does not span \widehat{K}_n^2 with threshold $\theta - 1$, which certainly happens if such configuration is not ℓ -viable, then the original configuration on $\mathbb{Z}_m \times K_n^2$ never adds an occupied point on $\{i, i + 1\} \times K_n^2$ even if all other points eventually become occupied. Existence of such a “blocking pair” is guaranteed if a is small enough.

The precise sufficient and necessary conditions for spanning, which yield the correct critical constant in (3.1), are given in Lemmas 4.1 and 5.1. To estimate the probabilities that these conditions are met, we need to carefully estimate the probabilities that a single Hamming plane contains various internal initial configurations.

Throughout the rest of the paper, we make use of the standard notation $\mathcal{O}(\cdot)$ and $o(\cdot)$ to describe asymptotic properties of quantities that depend on $n \rightarrow \infty$, with implied constants that may depend on θ and γ .

Lemma 3.1. *If p is given by (3.1) and $\ell \geq 2$, then*

$$\mathbb{P}_p (K_n^2 \text{ is not } \ell\text{-viable}) \sim n^{-2a^\ell/\ell!}.$$

Proof. The binomial probability that a fixed line contains at least ℓ initially occupied sites is

$$\frac{1}{\ell!}(np)^\ell + \mathcal{O}(n^{\ell-1}p^\ell + (np)^{\ell+1}) = \frac{1}{\ell!}(np)^\ell + \mathcal{O}((np)^{\ell+1}), \tag{3.2}$$

so the probability of the event H that at least one of n horizontal lines contains at least ℓ initially occupied sites satisfies

$$\mathbb{P}_p(H^c) = \exp\left(-\frac{1}{\ell!}n^{\ell+1}p^\ell + \mathcal{O}(n^{\ell+2}p^{\ell+1})\right) \sim n^{-a^\ell/\ell!}, \tag{3.3}$$

as $n^{\ell+2}p^{\ell+1}$ is $n^{-1/\ell}$ times a power of $\log n$. Conditioned on H^c , the configurations on horizontal lines are independent and the conditional probability that any one fixed horizontal line contains exactly one initially occupied site is bounded below by

$$q_n = np(1-p)^n = np + \mathcal{O}((np)^2).$$

By Lemma 2.2, with $\epsilon = C\sqrt{\log n}(nq_n)^{-1/2}$ for a large enough C ,

$$\begin{aligned} \mathbb{P}_p(\text{fewer than } (1-\epsilon)nq_n \text{ horizontal lines contain a single initially occupied site} \mid H^c) \\ = o(n^{-a^\ell/\ell!}). \end{aligned}$$

To connect to the birthday problem, focus on these horizontal lines with a single initially occupied site. The location of the initially occupied site is uniform on each of these lines, a ‘‘birthday.’’ We are looking for a vertical line with ℓ initially occupied sites, which will happen if ℓ of these ‘‘birthdays’’ coincide. Thus, by Lemma 2.3

$$\begin{aligned} &\mathbb{P}_p(\text{no vertical line contains at least } \ell \text{ initially occupied sites} \mid H^c) \\ &\leq (1 + o(1)) \exp\left(-\frac{1}{\ell!}(1-\epsilon)^\ell(nq_n)^\ell/n^{\ell-1}\right) + o(n^{-a^\ell/\ell!}) \\ &= (1 + o(1)) \exp\left(-\frac{1}{\ell!}(n^2p)^\ell/n^{\ell-1} + \mathcal{O}((\epsilon n^2p + n(np)^2)(n^2p)^{\ell-1})/n^{\ell-1}\right) + o(n^{-a^\ell/\ell!}) \\ &= (1 + o(1)) \exp\left(-\frac{1}{\ell!}a^\ell \log n + \mathcal{O}(\epsilon \log n + np \log n)\right) + o(n^{-a^\ell/\ell!}) \\ &= (1 + o(1))n^{-a^\ell/\ell!}. \end{aligned} \tag{3.4}$$

By the FKG inequality, we also have

$$\mathbb{P}_p(K_n^2 \text{ is not } \ell\text{-viable}) \geq P(H^c)^2,$$

which, together with (3.3) and (3.4), finishes the proof. \square

Lemma 3.2. *Assume that p is given by (3.1) and $\ell \geq 1$. Then*

$$\mathbb{P}_p(K_n^2 \text{ is not } (\ell-1)\text{-proper}) = \mathcal{O}(n^{-L}) \tag{3.5}$$

for any $L > 0$, while

$$\mathbb{P}_p(K_n^2 \text{ is not } \ell\text{-proper}) = o(1). \tag{3.6}$$

Proof. Let G be the event that K_n^2 is $(\ell-1)$ -proper, and G_1 the event that the top left (say) square contains θ horizontal lines containing at least $\ell-1$ initially occupied points. As $\mathbb{P}_p(G^c) \leq 8P(G_1^c)$, we in fact need to get the upper bound in (3.5) for $\mathbb{P}_p(G_1^c)$. Arguing as for (3.2), the expected number of horizontal lines in the top left square containing at

least $\ell - 1$ initially occupied points is $\Omega(n(np)^{\ell-1})$. Taking $\epsilon = 1/2$ in Lemma 2.2, we get a constant $c > 0$ so that

$$\mathbb{P}_p(G_1^c) \leq \exp(-cn^{1/\ell}(\log n)^{1-1/\ell}), \tag{3.7}$$

which is clearly enough for (3.5). To prove (3.6), we use analogous definitions of G and G_1 , and then (3.7) is replaced with

$$\mathbb{P}_p(G_1^c) \leq \exp(-c \log n), \tag{3.8}$$

which establishes (3.6). □

Lemma 3.3. *If p is given by (3.1) and $\ell \geq 2$, then*

$$\mathbb{P}_p(K_n^2 \text{ is not } (2\ell - 1)\text{-IS}) \sim \mathbb{P}_p(K_n^2 \text{ is } (2\ell - 1)\text{-II}) \sim n^{-2a^\ell/\ell!}.$$

Proof. Again, let G be the event that K_n^2 is $(\ell - 1)$ proper. It is easy to see that

$$\{K_n^2 \text{ is } \ell\text{-viable}\} \cap G \subset \{K_n^2 \text{ is } (2\ell - 1)\text{-IS}\} \subset \{K_n^2 \text{ is not } (2\ell - 1)\text{-II}\} \subset \{K_n^2 \text{ is } \ell\text{-viable}\},$$

and then Lemma 3.1 and (3.5) finish the proof. □

Lemma 3.4. *If p is given by (3.1) and $\ell \geq 2$, then*

$$\mathbb{P}_p(K_n^2 \text{ is not } (2\ell)\text{-IS}) \sim \mathbb{P}_p(K_n^2 \text{ is } (2\ell)\text{-II}) \sim 2n^{-a^\ell/\ell!}.$$

Proof. Let H (resp., V) be the event that at least one horizontal (resp., vertical) line contains at least ℓ initially occupied sites. Yet again, let G be the event that K_n^2 is $(\ell - 1)$ -proper. Further, let F be the event that there exists a line with at least $\ell + 1$ initially occupied sites. By the same reasoning as for (3.2), the probability that a fixed line contains at least $\ell + 1$ initially occupied sites is $\mathcal{O}((np)^{\ell+1})$, so

$$\mathbb{P}_p(F) = \mathcal{O}(n(np)^{\ell+1}) = \mathcal{O}(n^{-1/\ell}(\log n)^{1+1/\ell}).$$

Then

$$(H^c \cup V^c) \cap F^c \subset \{K_n^2 \text{ is } (2\ell)\text{-II}\},$$

so by the FKG inequality

$$\begin{aligned} \mathbb{P}_p(K_n^2 \text{ is } (2\ell)\text{-II}) &\geq \mathbb{P}_p(H^c \cup V^c) \mathbb{P}_p(F^c) \\ &= (1 - o(1)) \mathbb{P}_p(H^c \cup V^c) \\ &= (1 - o(1)) (2\mathbb{P}_p(H^c) - \mathbb{P}_p(H^c \cap V^c)) \\ &= (1 - o(1)) 2\mathbb{P}_p(H^c), \end{aligned} \tag{3.9}$$

by Lemma 3.1 and equation (3.3), since $H^c \cap V^c = \{K_n^2 \text{ is not } \ell\text{-viable}\}$.

Let \circ denote disjoint occurrence, so $H \circ V$ is the event that there exist a horizontal line and a vertical line with the property that there are ℓ initially occupied sites on one of the two lines, whose removal from the initially occupied set would leave at least ℓ initially occupied sites on the other line. Then we have

$$(H \circ V) \cap G \subset \{K_n^2 \text{ is } (2\ell)\text{-IS}\}$$

and, by FKG inequality and (3.5),

$$\mathbb{P}_p(K_n^2 \text{ is } (2\ell)\text{-IS}) \geq \mathbb{P}_p(H \circ V) \mathbb{P}_p(G) = \mathbb{P}_p(H \circ V) (1 - \mathcal{O}(n^{-L})) \tag{3.10}$$

for any constant $L > 0$. To find an appropriate lower bound for $\mathbb{P}_p(H \circ V)$, write H as a disjoint union $H = \cup_{k=1}^n H_k$, where H_k is the event that, counted from the top, the k th horizontal line is the first to contain at least ℓ initially occupied sites. Let V_k be the event that there exists a vertical line that contains at least ℓ initially occupied sites outside of the k th horizontal line. Then

$$\mathbb{P}_p(H \circ V) \geq \sum_{k=1}^n \mathbb{P}_p(H_k) \mathbb{P}_p(V_k | H_k) \tag{3.11}$$

Given H_k , the horizontal lines are independent, the first $k - 1$ have configurations conditioned on containing at most $\ell - 1$ initially occupied sites, and those after the k th horizontal line have the independent Bernoulli configuration. An identical argument as in equation (3.4), with the same values of q_n and ϵ , shows that

$$\mathbb{P}_p(V_k | H_k) \geq 1 - (1 + o(1))n^{-a^\ell/\ell!}, \tag{3.12}$$

where the $o(1)$ is uniform over all k . Then, from (3.11),

$$\mathbb{P}_p(H \circ V) \geq \left(1 - (1 + o(1))n^{-a^\ell/\ell!}\right) \mathbb{P}_p(H) \geq 1 - 2(1 + o(1))n^{-a^\ell/\ell!}, \tag{3.13}$$

which, together with (3.10), provides the matching bound to (3.9) and ends the proof. \square

Lemma 3.5. *If p is given by (3.1) and $\ell \geq 1$, then*

$$\mathbb{P}_p(K_n^2 \text{ is } (2\ell + 1)\text{-IS}) \sim \mathbb{P}_p(K_n^2 \text{ is not } (2\ell + 1)\text{-II}) \sim \frac{2a^{\ell+1}}{(\ell + 1)!} \cdot \frac{(\log n)^{1+1/\ell}}{n^{1/\ell}}.$$

Proof. Let now H (resp., V) be the event that there exists a horizontal (resp., vertical) line that contains at least $\ell + 1$ initially occupied sites. We have

$$\mathbb{P}_p(H) \sim n \cdot \frac{1}{(\ell + 1)!} n^{\ell+1} p^{\ell+1} = \frac{a^{\ell+1}}{(\ell + 1)!} \cdot \frac{(\log n)^{1+1/\ell}}{n^{1/\ell}}.$$

Then, adapting the proof of Lemma 3.3 in [GHPS], the occurrence of the event $(H \cap V) \setminus (H \circ V)$ implies that there exists an initially occupied site with 2ℓ additional initially occupied sites in its Hamming neighborhood, so

$$\mathbb{P}_p((H \cap V) \setminus (H \circ V)) \leq n^2 \cdot p \cdot n^{2\ell} p^{2\ell} = \mathcal{O}(p \cdot (\log n)^2) = o(\mathbb{P}_p(H)^2),$$

for all $\ell \geq 1$. It then follows from FKG and BK inequalities that

$$\mathbb{P}_p(H \cap V) \sim \mathbb{P}_p(H)^2.$$

Therefore,

$$\mathbb{P}_p(K_n^2 \text{ is not } (2\ell + 1)\text{-II}) \leq \mathbb{P}_p(H \cup V) \sim 2\mathbb{P}_p(H).$$

On the other hand, let G be the event that K_n^2 is ℓ -proper. By (3.6), $P(G) \rightarrow 1$. Then

$$\mathbb{P}_p(K_n^2 \text{ is } (2\ell + 1)\text{-IS}) \geq \mathbb{P}_p((H \cup V) \cap G) \geq \mathbb{P}_p(H \cup V) \mathbb{P}_p(G) \sim 2\mathbb{P}_p(H),$$

which establishes the desired asymptotics. \square

Lemma 3.6. *If p is given by (3.1) and $\ell \geq 1$, then*

$$\mathbb{P}_p(K_n^2 \text{ is not } (2\ell - 2)\text{-IS}) = \mathcal{O}(n^{-L}),$$

for any constant $L > 0$.

Proof. It is easy to see that

$$\{K_n^2 \text{ is } (\ell - 1)\text{-proper}\} \subset \{K_n^2 \text{ is } (2\ell - 2)\text{-IS}\},$$

and so (3.5) finishes the proof. \square

Lemma 3.7. *If p is given by (3.1) and $\ell \geq 1$, then the probability that any fixed copy of K_n^2 contains a site that has an initially occupied \mathbb{Z} -neighbor, and at least $\theta - 1$ initially occupied K -neighbors, is*

$$\begin{cases} \mathcal{O}(n^{-1}(\log n)^2) & \theta = 2\ell \\ \mathcal{O}(n^{-1-1/\ell}(\log n)^{2+1/\ell}) & \theta = 2\ell + 1 \end{cases} \\ = o(\mathbb{P}_p(K_n^2 \text{ is } \theta\text{-IS})).$$

Proof. The probability in question is $\mathcal{O}(n^2 p (np)^{\theta-1})$, which implies the \mathcal{O} bounds. Then Lemmas 3.4 and 3.5 finish the proof. \square

Lemma 3.8. *If p is given by (3.1) and $\ell \geq 1$, then then the probability that any fixed copy of K_n^2 has a site that that has an initially occupied \mathbb{Z} -neighbor, and at least $\theta - 2$ initially occupied K -neighbors, is*

$$\begin{cases} \mathcal{O}(n^{-1+1/\ell}(\log n)^{1-1/\ell}) & \theta = 2\ell \\ \mathcal{O}(n^{-1}(\log n)^2) & \theta = 2\ell + 1 \end{cases} \\ = o(\mathbb{P}_p(K_n^2 \text{ is } \theta\text{-IS})).$$

Proof. The probability in question is $\mathcal{O}(1/(np))$ times the one in previous lemma. \square

4 Spanning: sufficient condition

Lemma 4.1. *Assume the following three conditions are satisfied:*

- (1) *Every copy of K_n^2 is $(\theta - 2)$ -IS.*
- (2) *Between any pair (if any) of copies of K_n^2 that are not $(\theta - 1)$ -IS there is a copy of K_n^2 that is θ -IS.*
- (3) *There is at least one θ -IS copy of K_n^2 .*

Then $\mathbb{Z}_m \times K_n^2$ is spanned.

Proof. Assume that the graph is not spanned. Then, by (3), there is a contiguous interval, $\{i, i + 1, \dots, j\} \times K_n^2$, such that no Hamming plane in the interval is fully occupied in the final configuration, ω_∞ . By (1), every such interval contains at least two Hamming planes. The boundary planes of this interval, $\{i\} \times K_n^2$ and $\{j\} \times K_n^2$, are not $(\theta - 1)$ -IS. By (2), a plane in the interval must be θ -IS, and thus is fully occupied in the final configuration, a contradiction. \square

Lemma 4.2. *Suppose $\theta = 2\ell + 1$, $\ell \geq 2$, and p is given by (3.1). Assume that $\gamma > 1/\ell$ and $a^\ell/\ell! > (\gamma + 1/\ell)/2$. Then $\mathbb{P}_p(\text{Span}) \rightarrow 1$.*

Proof. Let G_1 , (resp., G_2 , G_3) be the event that condition (1) (resp., (2), (3)) above is satisfied. As $\gamma > 1/\ell$, $\mathbb{P}_p(G_3) \rightarrow 1$ by Lemma 3.5. Since $2a^\ell/\ell! > \gamma$, Lemma 3.3 implies $\mathbb{P}_p(G_1) \rightarrow 1$. To show that G_2 occurs a. a. s., let p_{good} be the probability that K_n^2 is θ -IS, and p_{bad} the probability that K_n^2 is not $(\theta - 1)$ -IS. Using Lemmas 3.4 and 3.5, the lower bound $a^\ell/\ell! > (\gamma + 1/\ell)/2 > 1/\ell$ implies that $p_{\text{bad}} \ll p_{\text{good}}$ and $mp_{\text{bad}}^2/p_{\text{good}} \rightarrow 0$.

Call a Hamming plane *exceptional* if it is either θ -IS or not $(\theta - 1)$ -IS. Let K be the number of exceptional planes. Each of these planes is not $(\theta - 1)$ -IS with probability $p_{\text{bad}}/(p_{\text{good}} + p_{\text{bad}}) \sim p_{\text{bad}}/p_{\text{good}}$. Moreover, by Lemma 2.2,

$$\mathbb{P}_p \left(\frac{1}{2}p_{\text{good}}m \leq K \leq 2p_{\text{good}}m \right) \rightarrow 1. \tag{4.1}$$

By Lemma 2.1, for $\frac{1}{2}p_{\text{good}}m \leq k \leq 2p_{\text{good}}m$ and n large enough,

$$\mathbb{P}_p (G_2 \mid K = k) \geq \exp(-2kp_{\text{bad}}^2/p_{\text{good}}^2) \geq \exp(-4mp_{\text{bad}}^2/p_{\text{good}}) \rightarrow 1. \tag{4.2}$$

By (4.1) and (4.2), $\mathbb{P}_p (G_2) \rightarrow 1$, which finishes the proof. □

Lemma 4.3. *Suppose $\theta = 2\ell$, $\ell \geq 2$, and p is given by (3.1). Assume that $a^\ell/\ell! > \gamma/4$. Then $\mathbb{P}_p (\text{Span}) \rightarrow 1$.*

Proof. Observe that condition (3) now holds a. a. s. as soon as $m \rightarrow \infty$, due to Lemma 3.4. The rest of the proof is similar to the one for the previous lemma. □

Lemma 4.4. *Suppose $\theta = 2\ell + 1$, $\ell \geq 2$, and p is given by (3.1). Assume $\gamma > 1/\ell$ and $a^\ell/\ell! > 1/\ell$. Then $|\omega_\infty|/(mn^2) \rightarrow 1$ in probability.*

Proof. Let $\gamma' \in (1/\ell, \gamma)$ be such that $a^\ell/\ell! > (\gamma' + 1/\ell)/2$, and let $m' = n^{\gamma'}$. Divide the cycle, \mathbb{Z}_m , into m/m' intervals of length m' (leaving out any leftover interval of smaller length), and identify each interval with a subgraph of $\mathbb{Z}_m \times (K_n)^2$ in the obvious way. Denote these subgraphs by $R_1, \dots, R_{m/m'}$. As in the proof of Lemma 4.2, call a Hamming plane *exceptional* if it is either θ -IS or not $(\theta - 1)$ -IS.

Observe that a subgraph, R_i , is internally spanned if it satisfies conditions (1), (2) and (3) of Lemma 4.1, in addition to

- (4) The first and last exceptional Hamming planes in R_i are θ -IS.

If G_4 is the event that condition (4) is satisfied, then Lemmas 3.4 and 3.5 imply that $\mathbb{P}_p (G_4) \rightarrow 1$. Now, it follows from the same proof as Lemma 4.2 that

$$\mathbb{P}_p (R_i \text{ is internally spanned}) \rightarrow 1.$$

Let N be the random number of subgraphs among $R_1, \dots, R_{m/m'}$ that are internally spanned, and observe that $|\omega_\infty| \geq m'n^2N$. Fix $\epsilon > 0$. We have

$$\mathbb{P}_p \left(\frac{|\omega_\infty|}{mn^2} > 1 - \epsilon \right) \geq \mathbb{P}_p (N > (1 - \epsilon)m/m') \rightarrow 1,$$

which proves the claim. □

5 Spanning: necessary condition

For $i_1, i_2 \in \mathbb{Z}_m$, $i_1 \neq i_2$, we say that $[i_1, i_2]$ is a *blocking interval* if all Hamming planes $\{i\} \times K_n^2$, $i_1 < i < i_2$, are θ -inert, all vertices on $\{i_1\} \times K_n^2$ have at most $\theta - 2$ initially occupied neighbors in $\{i_1, i_1 + 1\} \times K_n^2$, and all vertices on $\{i_2\} \times K_n^2$ have at most $\theta - 2$ initially occupied neighbors in $\{i_2 - 1, i_2\} \times K_n^2$.

Lemma 5.1. *Assume that $\mathbb{Z}_m \times K_n^2$ is spanned. Then both of the following conditions hold:*

- (1) *There is no blocking interval.*
- (2) *There is at least one copy of K_n^2 that is not θ -inert.*

Proof. The necessity of (2) is trivial. Necessity of (1) is also easy, because if (1) fails and $[i_1, i_2]$ is a blocking interval, no point in $[i_1, i_2] \times K_n^2$ gets added even if $([i_1, i_2] \times K_n^2)^c$ is completely occupied. \square

Lemma 5.2. *Suppose $\theta = 2\ell + 1$, $\ell \geq 2$, and p is given by (3.1). Assume that either $\gamma < 1/\ell$, or $(\gamma > 1/\ell$ and $a^\ell/\ell! < (\gamma + 1/\ell)/2$). Then $\mathbb{P}_p(\text{Span}) \rightarrow 0$.*

Proof. If $\gamma < 1/\ell$, then (2) fails with probability tending to 1 as $n \rightarrow \infty$ by Lemmas 3.5 and 3.7. For the remainder of the proof, we assume $\gamma > 1/\ell$ and $a^\ell/\ell! < (\gamma + 1/\ell)/2$.

This time call a Hamming plane K_n^2 *exceptional* if it is either not θ -II or it is $(\theta - 1)$ -II, and let p_{good} be the probability that K_n^2 is not θ -II, and p_{bad} the probability that K_n^2 is $(\theta - 1)$ -II.

Embed the random configuration on $\mathbb{Z}_m \times K_n^2$ into a random configuration on $\mathbb{Z}_+ \times K_n^2$. For any $i \geq 0$, let ξ_i be the random configuration on $[0, i] \times K_n^2$. Let I_1 and I_2 be the smallest (random) indices of two consecutive exceptional planes that are $(\theta - 1)$ -II. Fix $i_1 \geq 0$ and $k \geq 1$. Then, conditioned on ξ_{i_1-1} , the event $\{I_1 = i_1, I_2 - I_1 \leq k\}$ is a decreasing function of the configuration on $[i_1, \infty) \times K_n^2$.

The key to the argument that follows is the event G_{i_1, i_1+k} , that there is no neighboring-plane interference in $[i_1, i_1 + k] \times K_n^2$. To be more precise, this is the event that no vertex in $\{i_1\} \times K_n^2$ has an initially occupied neighbor in $\{i_1 + 1\} \times K_n^2$ together with at least $\theta - 2$ initially occupied neighbors in $\{i_1\} \times K_n^2$; and that no vertex in $\{j\} \times K_n^2$, $i_1 + 1 \leq j \leq i_1 + k$, has an initially occupied neighbor in $\{j - 1\} \times K_n^2$ together with at least $\theta - 2$ initially occupied neighbors in $\{j\} \times K_n^2$. Note that G_{i_1, i_1+k} is also decreasing, and independent of the configuration on ξ_{i_1-1} . Observe also that

$$G_{i_1, i_1+k} \cap \{I_1 = i_1, I_2 \leq i_1 + k\} \subset \{\text{there is blocking interval in } [i_1, i_1 + k]\}.$$

We will denote by ξ a generic realization of ξ_{i_1-1} , and let $k = \log n / (p_{\text{good}} + p_{\text{bad}})$. Conditioned on $\xi_{i_1-1} = \xi$ and $I_1 = i_1$, $I_2 - i_1$ is a geometric random variable with success probability $p_{\text{good}} + p_{\text{bad}}$, and therefore

$$\mathbb{P}_p(I_2 - i_1 > k \mid \xi_{i_1-1} = \xi, I_1 = i_1) \leq e^{-k(p_{\text{good}} + p_{\text{bad}})} = n^{-1}. \tag{5.1}$$

We proceed by a similar argument as in the proof of Lemma 4.2. Given $\gamma > 1/\ell$ and $a^\ell/\ell! < (\gamma + 1/\ell)/2$, if T is the number of exceptional planes in $[0, m/2] \times K_n^2$, then by Lemma 2.2,

$$\mathbb{P}_p(T \geq \frac{1}{4}(p_{\text{good}} + p_{\text{bad}})m) \rightarrow 1. \tag{5.2}$$

By Lemmas 2.1, 3.4 and 3.5, for large enough n ,

$$\mathbb{P}_p(I_1 > m/2 \mid T \geq \frac{1}{4}(p_{\text{good}} + p_{\text{bad}})m) \leq \exp(-\frac{1}{8}mp_{\text{bad}}^2/(p_{\text{good}} + p_{\text{bad}})) \rightarrow 0,$$

because $mp_{\text{bad}}^2/p_{\text{good}} \rightarrow \infty$ in the case $p_{\text{good}} > p_{\text{bad}}$, and $mp_{\text{bad}} \rightarrow \infty$ otherwise. Therefore,

$$\mathbb{P}_p(I_1 \leq m/2) \rightarrow 1. \tag{5.3}$$

Also, by Lemma 3.8, provided $\ell > 1$,

$$\mathbb{P}_p(G_{i_1, i_1+k}^c) = \mathcal{O}(kn^{-1}(\log n)^2) = o(1), \tag{5.4}$$

as $k = \mathcal{O}(\log n/p_{\text{good}}) = \mathcal{O}(n^{1/\ell})$. Now,

$$\begin{aligned}
 & \mathbb{P}_p(\mathbb{Z}_m \times K_n^2 \text{ is not spanned}) \\
 & \geq \mathbb{P}_p\left(\bigcup_{i_1 \leq m/2} \{I_1 = i_1, I_2 - i_1 \leq k, G_{i_1, i_1+k}\}\right) \\
 & = \sum_{i_1 \leq m/2} \sum_{\xi} \mathbb{P}_p(\xi_{i_1-1} = \xi) \cdot \mathbb{P}_p(I_1 = i_1, I_2 - i_1 \leq k, G_{i_1, i_1+k} \mid \xi_{i_1-1} = \xi) \\
 & \stackrel{\text{FKG}}{\geq} \sum_{i_1 \leq m/2} \sum_{\xi} \mathbb{P}_p(\xi_{i_1-1} = \xi) \cdot \mathbb{P}_p(I_1 = i_1, I_2 - i_1 \leq k \mid \xi_{i_1-1} = \xi) \cdot \mathbb{P}_p(G_{i_1, i_1+k}) \\
 & \stackrel{(5.4)}{=} (1 - o(1)) \sum_{i_1 \leq m/2} \sum_{\xi} \mathbb{P}_p(\xi_{i_1-1} = \xi) \cdot \mathbb{P}_p(I_1 = i_1, I_2 - i_1 \leq k \mid \xi_{i_1-1} = \xi) \\
 & = (1 - o(1)) \sum_{i_1 \leq m/2} \sum_{\xi} \mathbb{P}_p(\xi_{i_1-1} = \xi, I_1 = i_1) \cdot \mathbb{P}_p(I_2 - i_1 \leq k \mid \xi_{i_1-1} = \xi, I_1 = i_1) \\
 & \stackrel{(5.1)}{=} (1 - o(1)) \sum_{i_1 \leq m/2} \sum_{\xi} \mathbb{P}_p(\xi_{i_1-1} = \xi, I_1 = i_1) \\
 & \stackrel{(5.3)}{=} (1 - o(1)) \mathbb{P}_p(I_1 \leq m/2) = 1 - o(1).
 \end{aligned} \tag{5.5}$$

This completes the proof of the lemma. □

Lemma 5.3. *Suppose $\theta = 2\ell$, $\ell \geq 2$, and p is given by (3.1). Assume that $a^\ell/\ell < \gamma/4$. Then $\mathbb{P}_p(\text{Span}) \rightarrow 0$.*

Proof. The argument is similar to that of the previous lemma, and is somewhat simpler, so we omit the details. □

Lemma 5.4. *Suppose $\theta = 2\ell + 1$, $\ell \geq 2$, and p is given by (3.1). Assume that $\gamma \geq 1/\ell$ and $a^\ell/\ell < 1/\ell$. Then $|\omega_\infty|/mn^2 \rightarrow 0$ in probability.*

Proof. Choose constants α, β such that $a^\ell/\ell < \alpha < \beta < 1/\ell$, and let $m' = n^\beta$. As in the proof of Lemma 4.4, divide the cycle \mathbb{Z}_m into m/m' intervals of length m' , and denote the resulting subgraphs of $\mathbb{Z}_m \times K_n^2$ by $R_1, \dots, R_{m/m'}$. Call a site in R_i \mathbb{Z} -assisted if it has an initially occupied \mathbb{Z} -neighbor in R_i and at least $2\ell - 1$ initially occupied K -neighbors

Call a subgraph R_i *almost inert* if the following conditions are met.

- (1) All Hamming planes in R_i are θ -II.
- (2) No sites in R_i are \mathbb{Z} -assisted.
- (3) There is a $(\theta - 1)$ -II plane among the first n^α Hamming planes and among the last n^α Hamming planes in R_i .

If R_i is almost inert, then it contains an interval of Hamming planes of length at least $m' - 2n^\alpha = m'(1 - o(1))$, in which the initial configuration remains unchanged by the bootstrap dynamics, even if every site outside of the interval becomes occupied. The probability that R_i satisfies condition (1) converges to 1 by Lemma 3.5, since $\beta < 1/\ell$. The probability that R_i does not satisfy condition (2) is, by Lemma 3.8, $\mathcal{O}(n^{\beta-1}(\log n)^2) = o(1)$. Finally, the probability that R_i satisfies condition (3) tends to 1 by Lemma 3.4. So we have

$$\mathbb{P}_p(R_i \text{ is almost inert}) \rightarrow 1.$$

Let N be the number of subgraphs among $R_1, \dots, R_{m/m'}$ that are *not* almost inert, and observe that

$$|\omega_\infty| \leq m'n^2(N + 1) + 2n^\alpha(m/m')n^2 + |\omega_0|,$$

and that $m'n^2 = o(mn^2)$, $n^\alpha(m/m')n^2 = o(mn^2)$. Fix $\epsilon > 0$. Then, for large enough n ,

$$\mathbb{P}_p \left(\frac{|\omega_\infty|}{mn^2} > \epsilon \right) \leq \mathbb{P}_p \left(N > \frac{\epsilon}{3} \cdot \frac{m}{m'} \right) + \mathbb{P}_p \left(|\omega_0| > \frac{\epsilon}{2} \cdot mn^2 \right) \rightarrow 0,$$

where the first term goes to zero because each subgraph $R_1, \dots, R_{m/m'}$ is independently almost inert, and the second term goes to zero because $p \rightarrow 0$. This completes the proof of the lemma. \square

6 Scaling in the gradual regime

Lemma 5.2 shows that the magnitude of p of the form (3.1) is too small when $\gamma < 1/\ell$ and $\theta = 2\ell + 1$. In this case, we need to scale p so that a θ -IS plane has a chance to appear, so we let

$$p = a \cdot \frac{1}{n^{1+1/(\ell+1)} m^{1/(\ell+1)}}, \tag{6.1}$$

where $a > 0$ is a constant.

Lemma 6.1. *If p is given by (6.1), $\ell \geq 1$, and $\gamma < 1/\ell$, then*

$$\mathbb{P}_p (K_n^2 \text{ is } (2\ell + 1)\text{-IS}) \sim \mathbb{P}_p (K_n^2 \text{ is not } (2\ell + 1)\text{-II}) \sim \frac{2a^{\ell+1}}{(\ell + 1)!} \cdot \frac{1}{m}.$$

Proof. We emulate the proof of Lemma 3.5, with the same notation. Now,

$$\mathbb{P}_p (H) \sim \frac{a^{\ell+1}}{(\ell + 1)!} \cdot \frac{1}{m}$$

and

$$\mathbb{P}_p ((H \cap V) \setminus (H \circ V)) \leq n^2 \cdot p \cdot n^{2\ell} p^{2\ell} = \mathcal{O}(m^{-2}/(n^2 p)) = o(\mathbb{P}_p (H)^2),$$

as $\gamma < 1/\ell \leq 1$. The rest of the proof follows from that of Lemma 3.5; note that p of the form (6.1) is larger than that of the form (3.1) and thus (3.6) holds. \square

Lemma 6.2. *Assume that $\theta = 2\ell + 1$ and $\gamma < 1/\ell$, and that p is of the form (6.1). Then*

$$\mathbb{P}_p (\text{Span}) \rightarrow 1 - \exp(-2a^{\ell+1}/(\ell + 1)!).$$

Proof. Observe that

$$\left(\bigcap_{i=0}^{m-1} \{ \{i\} \times K_n^2 \text{ is } (2\ell)\text{-IS} \} \right) \cap \left(\bigcup_{i=0}^{m-1} \{ \{i\} \times K_n^2 \text{ is } (2\ell + 1)\text{-IS} \} \right) \subseteq \text{Span}.$$

As the size of p given by (6.1) is much larger than that given by (3.1) we can, by monotonicity in p , take the scaling constant a in Lemma 3.4 to be arbitrarily large, which implies that

$$\mathbb{P}_p \left(\bigcap_{i=0}^{m-1} \{ \{i\} \times K_n^2 \text{ is } (2\ell)\text{-IS} \} \right) \rightarrow 1.$$

So, by FKG inequality,

$$\begin{aligned} \mathbb{P}_p (\text{Span}) &\geq (1 - o(1)) \mathbb{P}_p \left(\bigcup_{i=0}^{m-1} \{ \{i\} \times K_n^2 \text{ is } (2\ell + 1)\text{-IS} \} \right) \\ &\rightarrow 1 - \exp(-2a^{\ell+1}/(\ell + 1)!), \end{aligned} \tag{6.2}$$

by Lemma 6.1 and elementary Poisson convergence.

Call a site \mathbb{Z} -assisted if one of the following holds: the site has an initially occupied \mathbb{Z} -neighbor and at least 2ℓ initially occupied K -neighbors, or the site has two initially occupied \mathbb{Z} -neighbors and at least $2\ell - 1$ initially occupied K -neighbors. The probability that there exists a \mathbb{Z} -assisted site is bounded above by a constant times

$$mn^2 [p(np)^{2\ell} + p^2(np)^{2\ell-1}] = \mathcal{O}(m^{-1}/(n^2p)) = o(1).$$

If an initially empty site becomes occupied in the first step of bootstrap percolation, then either that site is in a plane that is not $(2\ell + 1)$ -II, or that site is \mathbb{Z} -assisted. Therefore,

$$\begin{aligned} & \mathbb{P}_p(\text{there exists a site that becomes occupied in the first step}) \\ & \leq \mathbb{P}_p\left(\bigcup_{i=1}^m \{\{i\} \times K_n^2 \text{ is not } (2\ell + 1)\text{-II}\}\right) + o(1) \\ & \rightarrow 1 - \exp(-2a^{\ell+1}/(\ell + 1)!), \end{aligned} \tag{6.3}$$

again by Lemma 6.1. The two asymptotic bounds (6.2) and (6.3) establish the desired convergence. \square

7 Exceptional cases and proofs of main theorems

7.1 Threshold 2

We now show that when $\theta = 2$, the critical a is twice as large as obtained by taking $\ell = 1$ in Lemmas 4.3 and 5.3. Thus we will still assume (in this and in the next subsection) that p has the scaling given by (3.1), that is,

$$p = a \frac{\log n}{n^2}. \tag{7.1}$$

Lemma 7.1. *Assume that p is of the form (7.1). Then*

$$\mathbb{P}_p(K_n^2 \text{ is not 2-IS}) \sim \mathbb{P}_p(K_n^2 \text{ is 2-II}) \sim \frac{a \log n}{n^a}.$$

Proof. Let G_k be the event that K_n^2 contains exactly k initially occupied points. Then

$$\mathbb{P}_p(K_n^2 \text{ is 2-II}) \geq \mathbb{P}_p(G_1) \sim n^2 p e^{-n^2 p} = an^{-a} \log n. \tag{7.2}$$

Moreover,

$$\begin{aligned} & \mathbb{P}_p(K_n^2 \text{ is not 2-IS}) \\ & = \mathbb{P}_p(G_0) + \mathbb{P}_p(G_1) \\ & \quad + \sum_{k \geq 2} \mathbb{P}_p(G_k) \mathbb{P}_p(\text{all initially occupied points lie on the same line} \mid G_k) \\ & \leq (1 + o(1))an^{-a} \log n + \sum_{k \geq 2} (n^2 p)^k \exp(-(n^2 - k)p) \cdot \frac{1}{2n - 1} \cdot \frac{1}{n^{k-2}} \\ & = (1 + o(1))an^{-a} \log n + \frac{e^{-n^2 p}}{2n - 1} (n^2 p e^p)^2 (1 - n p e^p)^{-1} \\ & = (1 + o(1))an^{-a} \log n + \mathcal{O}(n^{-a-1}(\log n)^2) \end{aligned} \tag{7.3}$$

Together, (7.2) and (7.3) establish the desired asymptotics. \square

Lemma 7.2. *Suppose $\theta = 2$, and p has the form (7.1). Then $\mathbb{P}_p(\text{Span})$ converges to 0 when $a < \gamma/2$ and to 1 when $a > \gamma/2$.*

Proof. If two neighboring copies of K_n^2 are initially empty then spanning cannot occur. By Lemma 2.1, the probability of this is close to 0 (resp., 1) if $m(1-p)^{2n^2}$ goes to ∞ (resp., to 0), which happens when $a > \gamma/2$ (resp., when $a < \gamma/2$). Thus, when $a < \gamma/2$, $\mathbb{P}_p(\text{Span}) \rightarrow 0$. On the other hand, a sufficient condition for spanning is that there are no initially empty neighboring copies of K_n^2 and there is at least one 2-IS copy of K_n^2 . Therefore, Lemma 7.1 and the FKG inequality imply that, if $a > \gamma/2$, $\mathbb{P}_p(\text{Span}) \rightarrow 1$. \square

7.2 Threshold 3

We now handle the case $\theta = 3$, beginning with the restatement of Lemma 3.5 for this case.

Lemma 7.3. *Assume p has the form (7.1). Then*

$$\mathbb{P}_p(K_n^2 \text{ is 3-IS}) \sim \mathbb{P}_p(K_n^2 \text{ is not 3-II}) \sim a^2 \cdot \frac{(\log n)^2}{n}.$$

Lemma 7.4. *Assume p has the form (7.1). Then the conclusions of Lemmas 4.2, 5.2, and 6.2 hold, with $\ell = 1$.*

Proof. Lemmas 4.2 and 6.2 are proved in the same fashion, substituting Lemma 7.1 for Lemma 3.4. Clearly, Lemma 5.2 holds when $\gamma < 1/\ell$.

The rest of the proof of Lemma 5.2 needs to be slightly adapted, as now the probability in Lemma 3.8 is only $\mathcal{O}(\mathbb{P}_p(K_n^2 \text{ is 3-IS}))$, by Lemma 7.3. However, this means that $\mathbb{P}_p(G_{i_1, i_1+k}) \geq \alpha > 0$ for some fixed number $\alpha > 0$ (instead of converging to 1), and as a result the probability of occurrence of a blocking interval is at least $(1 - o(1))\alpha$. This is still true if we replace m by $m' = n^{\gamma'}$, where $\gamma' < \gamma$ and $a^\ell/\ell! < (\gamma' + 1/\ell)/2$. But this means that we have $m/m' \gg 1$ independent possibilities for a blocking interval to occur, which is sufficient. \square

7.3 A boundary case

Here, we provide an example with $\theta = 2\ell + 1$ whereby the transition is neither sharp nor gradual. This occurs at a boundary case $\gamma = 1/\ell$; more precisely, we assume that

$$m = \frac{n^{1/\ell}}{(\log n)^{1+1/\ell}}, \tag{7.4}$$

and that p is given by either (3.1) or (6.1), which now match.

Lemma 7.5. *Assume that $\theta = 2\ell + 1 \geq 3$, m is given by (7.4), and p is given by (3.1). Then (1.7) holds.*

Proof. If $a^\ell/\ell! > 1/\ell$, then by Lemma 3.4 (when $\ell > 1$) or Lemma 7.1 (when $\ell = 1$), a. a. s. every copy of K_n^2 is (2ℓ) -IS, and computations (6.2) and (6.3) apply. On the other hand, if $a^\ell/\ell! < 1/\ell$, then we apply Lemma 5.4; this lemma also holds for $\ell = 1$ because we can apply Lemma 7.1 in place of Lemma 3.4 in its proof. \square

7.4 Proofs of main theorems

Proof of Theorem 1.2. Lemmas 4.2, 5.2, and 7.4 prove (1.2). Lemmas 6.2 and 7.4 prove (1.3). Finally, Lemmas 4.3, 5.3, and 7.2 prove (1.4). \square

Proof of Theorem 1.3. Since ω_∞ is stochastically non-decreasing in p , we may assume (3.1) instead of (1.5). For $\theta > 3$, the theorem is then clearly a consequence of Lemmas 4.4 and 5.4. As the second lemma holds for $\theta = 3$, we only need to observe that the proof of Lemma 4.4 holds when $\theta = 3$ by Lemmas 7.4 and 7.2. \square

Proof of Theorem 1.4. This follows from Lemma 7.5 and monotonicity. \square

8 Proof of Theorem 1.1

The proof of Theorem 1.1, in particular the case $\theta \leq d$, is largely an application of established results for bootstrap percolation on lattices, so we only give a sketch.

Lemma 8.1. *Assume that $\theta \in [d + 1, 2d + 1]$. Suppose $A \subset \mathbb{Z}^d$ and $|A| < 2^{2d+1-\theta}$. Then there is a point in A with at least θ neighbors in A^c .*

Proof. We prove this by induction on d . For $d = 1$ the claim is obvious. Assume now the claim holds for $d - 1$. To prove it for d , first observe that it is trivially true for $\theta = 2d + 1$, or when A consists of a single point. Otherwise, let $\underline{i} = \min\{i : (\mathbb{Z}^{d-1} \times \{i\}) \cap A \neq \emptyset\}$ and $\bar{i} = \max\{i : (\mathbb{Z}^{d-1} \times \{i\}) \cap A \neq \emptyset\}$. We may, without loss of generality, assume that $\underline{i} < \bar{i}$ (otherwise we permute the coordinates), and that $A' = (\mathbb{Z}^{d-1} \times \{\underline{i}\}) \cap A$ has cardinality $|A'| < 2^{2d+1-\theta}/2 = 2^{2(d-1)+1-(\theta-1)}$. As $\theta - 1 \in [d, 2d - 1] = [(d - 1) + 1, 2(d - 1) + 1]$, we may apply the induction hypothesis to find a point in A' with at least $\theta - 1$ neighbors in $(\mathbb{Z}^{d-1} \times \{\underline{i}\}) \cap A^c$, but any point in A' also has a neighbor in $(\mathbb{Z}^{d-1} \times \{\underline{i} - 1\}) \subset A^c$. \square

Proof of Theorem 1.1 when $\theta > d$. Assume $\theta \in [d + 1, 2d + 1]$. We call a *safe box* a set of the form $\prod_{i=1}^d [a_i, b_i] \times K_n$, where $b_i = a_i + 1$ for $2d + 1 - \theta$ indices i and $b_i = a_i$ for the rest. Every vertex in a safe box has exactly $2d - (2d + 1 - \theta) = \theta - 1$ neighbors outside. Therefore, if there exists a completely empty safe box, then spanning is impossible. It follows that a. a. s. spanning does not occur when

$$(1 - p)^{n \cdot 2^{2d+1-\theta}} \gg m^{-d},$$

and this inequality is satisfied if

$$p < (1 - \epsilon) \frac{d}{2^{2d+1-\theta}} \cdot \frac{\log m}{n},$$

for some $\epsilon > 0$, implying the lower bound in Theorem 1.1. Conversely, call a point $z \in \mathbb{Z}_m^d$ *white* if $\{z\} \times K_n$ contains at most $\theta - 1$ initially occupied points. If

$$p > (1 + \epsilon) \frac{d}{2^{2d+1-\theta}} \cdot \frac{\log m}{n},$$

then the probability that there exists a white connected set of size $k \geq 1$ is bounded above by

$$C^k m^d ((np)^{\theta-1} e^{-np})^k \leq C^k m^d (\log m)^{k(\theta-1)} m^{-(1+\epsilon)dk} 2^{-2d+1+\theta},$$

where C denotes two different constants dependent on d and θ . Clearly, the above expression goes to 0 as $n \rightarrow \infty$ provided that $k \geq 2^{2d+1-\theta}$. It follows that a. a. s. there is no connected white set of size at least $2^{2d+1-\theta}$ in \mathbb{Z}_m^d . By Lemma 8.1, any connected white set of smaller size has at least one point z_0 that has θ non-white neighbors and thus the entire line $\{z_0\} \times K_n$ becomes occupied by time 2. Thus the entire V becomes occupied by time $2^{2d+1-\theta}$. The proof for $\theta > 2d + 1$ is the same as when $\theta = 2d + 1$. \square

Proof of Theorem 1.1 when $\theta \leq d$. Write $\lambda = \lambda(d, \theta)$, fix an $\epsilon > 0$, and assume p is of the form $p = an^{-1}(\log_{(\theta-1)} m)^{-(d-\theta+1)}$. Call a point $z \in \mathbb{Z}_m^d$ *grey* if the line $\{z\} \times K_n$ contains at least one initially occupied site, and *black* if the line $\{z\} \times K_n$ contains at least θ initially occupied points. It is a necessary condition for spanning that initially grey points span \mathbb{Z}_m^d under the bootstrap percolation process with threshold θ . By the main result of [BBDM], $\mathbb{P}_p(\text{Span}) \rightarrow 0$ unless

$$np \geq 1 - (1 - p)^n \geq (1 - \epsilon) \left(\frac{\lambda}{\log_{(\theta-1)} m} \right)^{d-\theta+1},$$

which proves that $\mathbb{P}_p(\text{Span}) \rightarrow 0$ when $a < \lambda^{d-\theta+1}$. To get the upper bound, observe that

$$\mathbb{P}_p(z \text{ is initially black}) = \Omega\left(\mathbb{P}_p(z \text{ is initially grey})^\theta\right). \quad (8.1)$$

Next, observe that if a point $z \in \mathbb{Z}_m^d$ has $\theta - 1$ black points in its neighborhood and an additional grey point (that could be z itself), then z eventually becomes black. The local growth of black points can then be constructed in the same way as in [BBM], and due to (8.1) has the same leading order asymptotics as the local growth probability of grey points, as the main contribution comes from a lower-dimensional process where, apart from an initial black nucleus, only grey points are used. This shows that $\mathbb{P}_p(\text{spanning}) \rightarrow 1$ when $a > \lambda^{d-\theta+1}$. \square

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