

Functional Erdős-Rényi law of large numbers for nonconventional sums under weak dependence

Yuri Kifer*

Abstract

We obtain a functional Erdős-Rényi law of large numbers for “nonconventional” sums of the form $\Sigma_n = \sum_{m=1}^n F(X_m, X_{2m}, \dots, X_{\ell m})$ where X_1, X_2, \dots is a sequence of exponentially fast ψ -mixing random vectors and F is a Borel vector function extending in several directions [18] where only i.i.d. random variables X_1, X_2, \dots were considered.

Keywords: laws of large numbers; large deviations; nonconventional sums; hyperbolic diffeomorphisms; Markov chains.

AMS MSC 2010: Primary 60F15, Secondary 60F10, 60F17, 37D20.

Submitted to EJP on August 5, 2016, final version accepted on February 16, 2017.

Supersedes arXiv:1608.01851.

1 Introduction

Let X_1, X_2, \dots be a sequence of independent identically distributed (i.i.d.) random variables such that $EX_1 = 0$ and the moment generating function $\phi(t) = Ee^{tX_1}$ exists. Denote by I the Legendre transform of $\ln \phi$ and set $\Sigma_n = \sum_{m=1}^n X_m$ for $n \geq 1$ and $\Sigma_0 = 0$. The Erdős-Rényi law of large numbers from [8] says that with probability one

$$I(\alpha) \lim_{n \rightarrow \infty} \max_{0 \leq m \leq n - \lfloor \frac{\ln n}{I(\alpha)} \rfloor} \frac{\Sigma_{m + \lfloor \frac{\ln n}{I(\alpha)} \rfloor} - \Sigma_m}{\ln n} = \alpha \quad (1.1)$$

for all $\alpha > 0$ such that $I(\alpha) < \infty$.

The nonconventional limit theorems initiated in [17] and partially motivated by nonconventional ergodic theorems (with the name coming from [10]) study asymptotic behaviors of sums of the form

$$\Sigma_n = \sum_{m=1}^n F(X_m, X_{2m}, \dots, X_{\ell m}) \quad (1.2)$$

(and more general ones) where F is a vector function satisfying certain conditions. The main features of such sums are nonstationarity and unboundedly long (and strong)

*Hebrew University of Jerusalem, Israel. E-mail: kifer@math.huji.ac.il

dependence of their summands. In [18] we established (1.1) for sums (1.2) where F is a bounded Borel function and X_1, X_2, \dots are independent identically distributed random variables. One of the main reasons for the independence assumption in [18] was the use of large deviations for nonconventional sums (1.2) which was established in [20] only for sums (1.2) with i.i.d. random variables X_1, X_2, \dots .

In this paper we modify our method so that only the standard (conventional) large deviations are used for sums of the form

$$T_n = \sum_{m=1}^n F(X_m^{(1)}, X_{2m}^{(2)}, \dots, X_{\ell m}^{(\ell)}) \tag{1.3}$$

where $\{X_m^{(i)}, m \geq 1\}, i = 1, 2, \dots, \ell$ are independent copies of the sequence $\{X_m, m \geq 1\}$. Now, when X_1, X_2, \dots is a stationary weakly dependent sequence then the latter sum consists of stationary weakly dependent summands with similar properties which allows applications to Markov chains satisfying the Doeblin condition and to some dynamical systems such as Axiom A diffeomorphisms, expanding transformations and topologically mixing subshifts of finite type (see [2]). We assume exponentially fast ψ -mixing of the sequence X_1, X_2, \dots which still leads to long and strongly dependent summands of Σ_n but once we justify a transition to the sums T_n we arrive at exponentially fast ψ -mixing summands there. Observe that the Erdős-Rényi law for conventional ($\ell = 1$) sums of exponentially fast ψ -mixing random variables was obtained in [6].

In fact, we derive a functional form of the Erdős-Rényi law for nonconventional sums (1.2) which was first introduced for (conventional) sums of i.i.d. random vectors in [1] and it was never considered before beyond this setup. This is a more general result and as a corollary we derive from it the standard form of the Erdős-Rényi law for nonconventional sums. Moreover, unlike the original form of this law its functional form allows to consider a multidimensional version where X_1, X_2, \dots are random vectors and F is a vector function.

The structure of this paper is as follows. In Section 2 we describe precisely our setup and results. In Section 3 we exhibit a lemma which is a version of Lemma 3.1 from [15] and which plays a crucial role here. In Sections 4 and 5 we derive the corresponding upper and lower bounds which yield the functional form of the Erdős-Rényi law for nonconventional sums. After that we show how this implies the standard form of this law. In Appendix we describe applications to Markov chains and dynamical systems and then discuss some properties of rate functions of large deviations which are relevant to our proofs but hard to find in most of the books on large deviations.

2 Preliminaries and main results

Let X_1, X_2, \dots be a \wp -dimensional stationary vector stochastic process on a probability space (Ω, \mathcal{F}, P) and let $F : \mathbb{R}^{\wp \ell} \rightarrow \mathbb{R}^d$ be a bounded Borel vector function on $\mathbb{R}^{\wp \ell}$. Our setup includes also a sequence $\mathcal{F}_{m,n} \subset \mathcal{F}, -\infty \leq m \leq n \leq \infty$ of σ -algebras such that $\mathcal{F}_{m,n} \subset \mathcal{F}_{m_1,n_1}$ whenever $m_1 \leq m$ and $n_1 \geq n$ which satisfies an exponentially fast ψ -mixing condition (see, for instance, [4]),

$$\psi(n) = \sup \left\{ \left| \frac{P(A \cap B)}{P(A)P(B)} - 1 \right| : A \in \mathcal{F}_{-\infty,k}, B \in \mathcal{F}_{k+n,\infty}, P(A)P(B) \neq 0 \right\} \leq \kappa_1^{-1} e^{-\kappa_1 n} \tag{2.1}$$

for some $\kappa_1 > 0$ and all $k, n \geq 0$.

We assume also the centering condition

$$\bar{F} = \int F(x_1, x_2, \dots, x_\ell) d\mu(x_1) \dots d\mu(x_\ell) = 0, \tag{2.2}$$

where μ is the distribution of X_1 , which is not actually a restriction since we always can take $F - \bar{F}$ in place of F . In addition, we assume that either X_n is $\mathcal{F}_{n-m, n+m}$ -measurable for some $m \in \mathbb{N}$ independent of n and then F is supposed to be only Borel measurable and bounded or F is supposed to be bounded and Hölder continuous

$$\|F\|_\infty = D < \infty, |F(x_1, \dots, x_\ell) - F(y_1, \dots, y_\ell)| \leq \kappa_2^{-1} \sum_{i=1}^\ell |x_i - y_i|^{\kappa_2}, \kappa_2 > 0 \tag{2.3}$$

and then we need only the following approximation property

$$E|X_n - X_{n,m}|^{\kappa_2} \leq \kappa_3^{-1} e^{-\kappa_3 m}, X_{n,m} = E(X_n | \mathcal{F}_{n-m, n+m}) \tag{2.4}$$

for all $n, m \in \mathbb{N}$ and some $\kappa_3 > 0$ independent of n and m .

Define two sums

$$\Sigma_n = \sum_{1 \leq k \leq n} F(X_k, X_{2k}, \dots, X_{\ell k}), \Sigma_0 = 0$$

and

$$T_n = \sum_{1 \leq k \leq n} F(X_k^{(1)}, X_{2k}^{(2)}, \dots, X_{\ell k}^{(\ell)}), T_0 = 0$$

where $\{X_k^{(i)}, k \geq 1\}, i = 1, 2, \dots, \ell$ are independent copies (in the sense of distributions) of the stationary process $\{X_k, k \geq 1\}$. Assume that for any piece-wise constant map $\gamma : [0, 1] \rightarrow \mathbb{R}^d$ the limit

$$\lim_{n \rightarrow \infty} \frac{1}{n} \ln E \exp \left(n \int_0^1 (\gamma_u, F(X_{[un]}^{(1)}, X_{2[un]}^{(2)}, \dots, X_{\ell[un]}^{(\ell)})) du \right) = \int_0^1 \Pi(\gamma_u) du \tag{2.5}$$

exists, $\Pi(\alpha), \alpha \in \mathbb{R}^d$ is a convex twice differentiable function such that $\nabla_\alpha \Pi(\alpha)|_{\alpha=0} = 0$ and the Hessian matrix $\nabla_\alpha^2 \Pi(\alpha)|_{\alpha=0}$ is positively definite (where (\cdot, \cdot) denotes the inner product).

Let

$$I(\beta) = \sup_\alpha ((\alpha, \beta) - \Pi(\alpha)) \tag{2.6}$$

and for any $\gamma : [0, 1] \rightarrow \mathbb{R}^d$ from the space $C([0, 1], \mathbb{R}^d)$ of continuous curves on \mathbb{R}^d set

$$S(\gamma) = \int_0^1 I(\dot{\gamma}_u) du \tag{2.7}$$

if γ is absolutely continuous and $S(\gamma) = \infty$, for otherwise. It follows from the existence and properties of the limit (2.5) (see, for instance, Section 7.4 in [9]) that $n^{-1}T_n$ satisfies large deviations estimates in the form that for any $a, \delta, \lambda > 0$ and every $\gamma \in C([0, 1], \mathbb{R}^d), \gamma_0 = 0$ there exists $n_0 > 0$ such that for $n \geq n_0$,

$$P\{\rho(n^{-1}T_n, \gamma) < \delta\} \geq \exp(-n(S(\gamma) + \lambda)) \text{ and} \tag{2.8}$$

$$P\{\rho(n^{-1}T_n, \Phi(a)) \geq \delta\} \leq \exp(-n(a - \lambda))$$

where $\rho(\gamma, \eta) = \sup_{u \in [0, 1]} |\gamma_u - \eta_u|$ and $\Phi(a) = \{\gamma \in C([0, 1], \mathbb{R}^d) : \gamma_0 = 0, S(\gamma) \leq a\}$.

Since S is a lower semi-continuous functional then each $\Phi(a), a < \infty$ is a closed set and, moreover, it is compact for any finite a . Indeed, $|\Pi(\alpha)| \leq D|\alpha|$ by (2.3) which implies by (2.6) that $I_t(\beta) = \infty$ provided $|\beta| > D$ (take $\alpha = a\beta/|\beta|$ in (2.6) and let $a \rightarrow \infty$). Hence, $|\dot{\gamma}_s| \leq D$ for Lebesgue almost all $s \in [0, 1]$ if $\gamma \in \Phi_t(a)$, and so the latter set is bounded and equicontinuous which by the Arzelà-Ascoli theorem implies its compactness.

For each $c > 0, u \in [0, 1]$ and integers $m \geq 0$ and $n \geq 2$ set

$$V_{m,n}^c(u) = \frac{\Sigma_{m+b_c(n,u)} - \Sigma_m}{b_c(n)}$$

where $b_c(n, u) = [ub_c(n)]$ and $b_c(n) = [c \ln n]$. Introduce also the set of random curves $W_n^c = \{V_{j,n}^c : 0 \leq j \leq n - b_c(n)\}$. The following is the main result of this paper.

Theorem 2.1. *Assume that the conditions (2.1)–(2.5) hold true. Then, for any $c > 0$ with probability one*

$$\lim_{n \rightarrow \infty} H(W_n^c, \Phi(1/c)) = 0 \tag{2.9}$$

where $H(\Gamma_1, \Gamma_2) = \inf\{\delta > 0 : \Gamma_1 \subset \Gamma_2^\delta, \Gamma_2 \subset \Gamma_1^\delta\}$ is the Hausdorff distance between sets of curves with respect to the uniform metric ρ and $\Gamma^\delta = \{\gamma : \rho(\gamma, \Gamma) < \delta\}$.

Corollary 2.2. *Let $d = 1$. Then for $c = \frac{1}{I(\beta)}$ with probability one*

$$\lim_{n \rightarrow \infty} \max_{0 \leq k \leq n - b_c(n)} V_{k,n}^c(1) = \beta \tag{2.10}$$

provided $0 < \beta < \beta_0 = \sup\{\beta : I(\beta) < \infty\}$.

Observe that $(X_k^{(1)}, X_{2k}^{(2)}, \dots, X_{\ell k}^{(\ell)})$, $k \geq 1$ is an ℓ_φ -dimensional stationary process with properties similar to the ones of the process X_k , $k \geq 1$, and so unlike Σ_n the sum T_n requires only “conventional” treatment. Our main goal here will be to show how to replace in our proofs the handling of the sums Σ_n by the sums T_n . We will mainly discuss the proof for the case where F and X_n , $n \geq 1$ satisfy the conditions (2.3) and (2.4) since the case when X_n is $\mathcal{F}_{n-m, n+m}$ -measurable and F is only a bounded Borel function is established by an obvious simplification of the proof just by eliminating the steps connected to approximations of X_n by corresponding conditional expectations $E(X_n | \mathcal{F}_{n-m, n+m})$.

Our method goes through also for more general sums $\Sigma_n = \sum_{1 \leq m \leq n} F(X_{q_1(m)}, X_{q_2(m)}, \dots, X_{q_\ell(m)})$ where $q_i(m) = im$ for $i \leq k \leq \ell$ and $q_j(m)$ for $j = k + 1, \dots, \ell$ being nonlinear indexes as in [19]. For instance, we may take $q_j(m) = m^j$ for $j > k$. In this situation, it turns out that we can replace such sums Σ_n by the sums $T_n = \sum_{1 \leq m \leq n} F(X_{q_1(m)}^{(1)}, X_{q_2(m)}^{(2)}, \dots, X_{q_\ell(m)}^{(\ell)})$ where, again, $\{X_m^{(i)}, m \geq 1\}$, $i = 1, \dots, \ell$ are independent processes, for $i = 1, \dots, k$ they are copies of X_m , $m \geq 1$ while all $X_m^{(i)}$, $m \geq 1, i > k$ are i.i.d. and have the same distribution as X_1 . Then, again, $(X_{q_1(m)}^{(1)}, X_{q_2(m)}^{(2)}, \dots, X_{q_\ell(m)}^{(\ell)})$, $m \geq 1$ is an ℓ_φ -dimensional stationary process with properties similar to the ones of the process X_m , $m \geq 1$ and we can deal with such sums T_n in the same way as in this paper.

Using dependence coefficients

$$\omega_{q,p}(\mathcal{G}, \mathcal{H}) = \sup\{\|E(g|\mathcal{G}) - Eg\|_p : g \text{ is } \mathcal{H} - \text{measurable and } \|g\|_q \leq 1\}$$

for σ -algebras $\mathcal{G}, \mathcal{H} \subset \mathcal{F}$ (see [4]) it is possible to obtain a version of Lemma 3.1 below beyond ψ -mixing (where $\psi(\mathcal{G}, \mathcal{H}) = \omega_{1,\infty}(\mathcal{G}, \mathcal{H})$), and so the proof of Theorem 2.1 can be extended assuming weaker than ψ -mixing conditions. Still, we do not give details here since our main examples (see Appendix), which should also satisfy appropriate large deviations, are anyway ψ -mixing.

Our conditions are satisfied when, for instance, X_n , $n \geq 1$ is a φ -dimensional Markov chain with transition probabilities $P(x, \Gamma)$ satisfying the strong Doeblin type condition

$$C^{-1} \leq \frac{P(x, dy)}{d\nu(y)} \leq C$$

for some probability measure ν and a constant $C > 0$ independent of x and y . Then $(X_n^{(1)}, X_{2n}^{(2)}, \dots, X_{\ell n}^{(\ell)})$ is an ℓ_φ -dimensional Markov chain satisfying similarly to X_n both exponentially fast ψ -mixing and the necessary large deviations estimates (see [4], [9] and [13]). Here we can take, for instance, the σ -algebras $\mathcal{F}_{m,n}$ generated by X_m, X_{m+1}, \dots, X_n and then F is supposed to be only bounded and Borel.

On the dynamical systems side our conditions are satisfied, for instance, when $X_n = g \circ f^n$ where g is a Hölder continuous function and f is an Axiom A diffeomorphism on a hyperbolic set, expanding transformation or a mixing subshift of finite type (see [2]). In this case $f : M \rightarrow M$ and $X_n(\omega), \omega \in M$ is a stationary sequence on the probability space (M, \mathcal{F}, P) where M is the corresponding phase space, \mathcal{F} is a Borel σ -algebra and P is a Gibbs measure constructed by a Hölder continuous function. The function F here should satisfy (2.3) and the σ -algebras $\mathcal{F}_{m,n}$ are generated by cylinder sets in the subshift case or by corresponding Markov partitions in the Axiom A and expanding case. The exponentially fast ψ -mixing for these transformations is obtained in [2] and the required large deviations results can be found in [13] and [14] and the product system (f, f^2, \dots, f^ℓ) , which plays the role of ℓ independent copies in sums T_n , has similar properties to the dynamical system f^n itself.

3 Basic estimates

We start with the following result which is a corollary of Lemma 3.1 from [15].

Lemma 3.1. *Let Y_i be \wp_i -dimensional random vectors with a distribution $\mu_i, i = 1, \dots, k$ defined on the same probability space (Ω, \mathcal{F}, P) and such that Y_i is \mathcal{F}_{m_i, n_i} -measurable where $n_{i-1} < m_i \leq n_i < m_{i+1}, i = 1, \dots, k, n_0 = -\infty, m_{k+1} = \infty$ and σ -algebras $\mathcal{F}_{n,m}$ satisfy the condition (2.1). Then, for any bounded Borel function $h = h(x_1, \dots, x_k)$ on $\mathbb{R}^{\wp_1 + \wp_2 + \dots + \wp_k}$,*

$$|Eh(Y_1, Y_2, \dots, Y_k) - \int h(x_1, x_2, \dots, x_k) d\mu_1(x_1) d\mu_2(x_2) \dots d\mu_k(x_k)| \tag{3.1}$$

$$\leq \kappa_1^{-1} \|h\|_\infty \sum_{i=2}^k e^{-\kappa_1(m_i - n_{i-1})}$$

where $\|\cdot\|$ is the L^∞ norm. In other words, if $Y_1^{(1)}, Y_2^{(2)}, \dots, Y_k^{(k)}$ are independent copies of Y_1, Y_2, \dots, Y_k , respectively, then

$$|Eh(Y_1, Y_2, \dots, Y_k) - Eh(Y_1^{(1)}, Y_2^{(2)}, \dots, Y_k^{(k)})| \leq \kappa_1^{-1} \|h\|_\infty \sum_{i=2}^k e^{-\kappa_1(m_i - n_{i-1})}. \tag{3.2}$$

Taking $h = \mathbb{I}_\Gamma$ for a Borel set $\Gamma \subset \mathbb{R}^{\wp_1 + \wp_2 + \dots + \wp_k}$ (where $\mathbb{I}_\Gamma(x) = 1$ if $x \in \Gamma$ and $\mathbb{I}_\Gamma(x) = 0$, for otherwise) it follows that

$$|P\{(Y_1, Y_2, \dots, Y_k) \in \Gamma\} - P\{(Y_1^{(1)}, Y_2^{(2)}, \dots, Y_k^{(k)}) \in \Gamma\}| \leq \kappa_1^{-1} \sum_{i=2}^k e^{-\kappa_1(m_i - n_{i-1})}. \tag{3.3}$$

Proof. If $k = 2$ then Lemma 3.1 from [15] gives that

$$|E(h(Y_1, Y_2)|\mathcal{F}_{m_1, n_1}) - g(Y_1)| \leq \psi(m_2 - n_1) \|h\|_\infty$$

where $g(y) = Eh(y, Y_2) = \int h(y, z) d\mu_2(z)$. Taking the expectation we obtain (3.1) for $k = 2$. Now let (3.1) holds true for all $k \leq j - 1$ and any bounded Borel function of the corresponding number of arguments. In order to derive (3.1) for $k = j$ we consider $(Y_1, Y_2, \dots, Y_{j-1})$ as one random vector and Y_j as another. Then we obtain from Lemma 3.1 of [15] that

$$|E(h(Y_1, Y_2, \dots, Y_j)|\mathcal{F}_{m_1, n_{j-1}}) - g(Y_1, Y_2, \dots, Y_{j-1})| \leq \psi(m_j - n_{j-1}) \|h\|_\infty$$

where $g(y_1, y_2, \dots, y_{j-1}) = Eh(y_1, y_2, \dots, y_{j-1}, Y_j)$. Now, taking the expectation and applying the induction hypothesis to g we complete the proof. \square

In the case when X_n is $\mathcal{F}_{n-m, n+m}$ -measurable for all n and a fixed m then we will be able to use Lemma 3.1 directly which will enable us to replace summands $F(X_n, X_{2n}, \dots, X_{\ell n})$ by $F(X_n^{(1)}, X_{2n}^{(2)}, \dots, X_{\ell n}^{(\ell)})$. On the other hand, under (2.3) and (2.4) we will have, first, to replace the original random vectors X_m by their approximations $X_{m,k} = E(X_m | \mathcal{F}_{m-k, m+k})$ and then using (2.3) to estimate the error.

Namely, for $\bar{k} = (k_1, k_2, \dots, k_n)$ set

$$R_{n, \bar{k}} = \sum_{1 \leq j \leq n} F(X_{j, k_j}, X_{2j, k_j}, \dots, X_{\ell j, k_j}).$$

Let $j(u) > 0, u \in [0, 1]$ be a non decreasing integer valued function. We will use that by (2.3) and (2.4),

$$\begin{aligned} & E \sup_{u \in [0, 1]} |\Sigma_{m+j(u)} - \Sigma_m - (R_{m+j(u), \bar{k}} - R_{m, \bar{k}})| \tag{3.4} \\ & \leq \sum_{m+1 \leq i \leq m+j(1)} E |F(X_i, X_{2i}, \dots, X_{\ell i}) - F(X_{i, k_i}, X_{2i, k_i}, \dots, X_{\ell i, k_i})| \\ & \leq \kappa_2^{-1} \kappa_3^{-1} \sum_{m+1 \leq i \leq m+j(1)} e^{-\kappa_3 k_i}. \end{aligned}$$

We observe that Lemma 3.1 applied to the summands of the form $F(X_{j, k_j}, \dots, X_{\ell j, k_j})$ does not yield yet the summands of the form $F(X_j^{(1)}, \dots, X_{\ell j}^{(\ell)})$ but only the summands $F(X_{j, k_j}^{(1)}, \dots, X_{\ell j, k_j}^{(\ell)})$ where $X_{ij, k_j}^{(i)}, i = 1, \dots, \ell$ are independent and have the same distributions as $X_{ij, k_j}, i = 1, \dots, \ell$, respectively. Thus an additional argument together with another use of (2.3) and (2.4) will be needed.

4 The upper bound

We will show first that with probability one,

$$\lim_{n \rightarrow \infty} \max_{0 \leq j \leq n - b_c(n)} \rho(V_{j, n}^c, \Phi(1/c)) = 0. \tag{4.1}$$

This assertion means that with probability one all limit points as $n \rightarrow \infty$ of curves from W_n^c belong to the compact set $\Phi(1/c)$.

Set $R_n = R_{n, \bar{k}}$ where $\bar{k} = (k_j, 1 \leq j \leq n)$ with $k_j = [j/3]$ and

$$\hat{V}_{m, n}^c(u) = \frac{R_{m+b_c(n, u)} - R_m}{b_c(n)}.$$

Then by (3.4) and the Chebyshev inequality

$$\begin{aligned} P\{\rho(V_{m, n}^c, \hat{V}_{m, n}^c) \geq \varepsilon\} & \leq \varepsilon^{-1} b_c^{-1}(n) \kappa_2^{-1} \kappa_3^{-1} \sum_{m+1 \leq j < \infty} e^{-\kappa_3 [j/3]} \tag{4.2} \\ & \leq \varepsilon^{-1} b_c^{-1} \kappa_2^{-1} \kappa_3^{-1} e^{-\kappa_3(m-2)/3} (1 - e^{-\frac{1}{3}\kappa_3})^{-1}. \end{aligned}$$

Observe that

$$|\Sigma_{m+b_c(n, u)} - \Sigma_m - (\Sigma_{m+k+b_c(n, u)} - \Sigma_{m+k})| \leq 2kD \quad \text{a.s.} \tag{4.3}$$

where, recall, $D = \|F\|_\infty$. Hence, a.s.,

$$0 \leq \max_{0 \leq j \leq n - b_c(n)} \rho(V_{j, n}^c, \Phi(1/c)) - \max_{\varepsilon b_c(n) \leq j \leq n - b_c(n)} \rho(V_{j, n}^c, \Phi(1/c)) \leq 2\varepsilon D \tag{4.4}$$

and, similarly,

$$0 \leq \max_{0 \leq j \leq n - b_c(n)} \rho(\hat{V}_{j, n}^c, \Phi(1/c)) - \max_{\varepsilon b_c(n) \leq j \leq n - b_c(n)} \rho(\hat{V}_{j, n}^c, \Phi(1/c)) \leq 2\varepsilon D. \tag{4.5}$$

By (4.2) we also have that

$$\begin{aligned}
 & P\{\max_{\varepsilon b_c(n) \leq j \leq n-b_c(n)} \rho(V_{j,n}^c, \hat{V}_{j,n}^c) \geq \varepsilon\} \\
 & \leq \sum_{\varepsilon b_c(n) \leq j \leq n-b_c(n)} P\{\rho(V_{j,n}^c, \hat{V}_{j,n}^c) \geq \varepsilon\} \\
 & \leq (\varepsilon b_c(n) \kappa_2 \kappa_3 (1 - e^{-\kappa_3/3})^2)^{-1} e^{2\kappa_3/3} n^{-\kappa_3 \varepsilon c/3}.
 \end{aligned}
 \tag{4.6}$$

Next, observe that the family of compact sets $\{\Phi(a), a > 0\}$ is upper semi continuous, i.e. for any $a > 0$ and $\varepsilon > 0$ there exists $\delta > 0$ such that $\Phi(a + \delta) \subset \Phi(a)^\varepsilon$ where, as before, G^ε denotes the open ε -neighborhood of a set G . Indeed, if $G_\varepsilon = \Phi(a + \delta) \setminus \Phi(a)^\varepsilon \neq \emptyset$ for some $a, \varepsilon > 0$ and all δ then the decreasing with $\delta \downarrow 0$ compact sets G_δ must have a common point $\gamma_0 \in \Phi(a)$ contradicting the fact that $\gamma_0 \notin \Phi(a)^\varepsilon$. Now, choosing such δ for ε and $a = 1/c$ we obtain that

$$P\{\rho(\hat{V}_{j,n}^c, \Phi(1/c)) > 3\varepsilon\} \leq P\{\rho(\hat{V}_{j,n}^c, \Phi(1/c + \delta)) > 2\varepsilon\}.
 \tag{4.7}$$

For each vector

$$\bar{x}(m) = (x_1^{(1)}, \dots, x_m^{(1)}; x_1^{(2)}, \dots, x_m^{(2)}; \dots; x_1^{(\ell)}, \dots, x_m^{(\ell)}) \in \mathbb{R}^{m\ell\varphi}$$

define the curve $\gamma(\bar{x}(m))$ in \mathbb{R}^d by

$$\gamma_u(\bar{x}(m)) = m^{-1} \sum_{1 \leq j \leq [um]} F(x_j^{(1)}, x_j^{(2)}, \dots, x_j^{(\ell)}), \quad u \in [0, 1]$$

where the sum over the empty set is considered to be zero. Introduce the Borel set

$$\Gamma = \{\bar{x}(b_c(n)) \in \mathbb{R}^{b_c(n)\ell\varphi} : \rho(\gamma(\bar{x}(b_c(n))), \Phi(\frac{1}{c} + \delta)) > 2\varepsilon\}.$$

Let the pairs $\{(X_{ij}^{(i)}, X_{ij,[j/3]}^{(i)}), j = 1, 2, \dots\}, i = 1, \dots, \ell$ be independent copies (in the sense of joint distributions) of pairs of processes $\{(X_{ij}, X_{ij,[j/3]}), j = 1, 2, \dots\}, i = 1, \dots, \ell$, respectively, which can be constructed on a product space. We identify the processes $\{X_{ij}^{(i)}, j = 1, 2, \dots\}, i = 1, \dots, \ell$ with the processes having the same notation in Section 2 since they have the same joint distributions which is what only matters here. Set

$$\hat{T}_n = \sum_{j=1}^n F(X_{j,[j/3]}^{(1)}, X_{2j,[j/3]}^{(2)}, \dots, X_{\ell j,[j/3]}^{(\ell)})$$

and

$$\hat{U}_{m,n}^c(u) = \frac{\hat{T}_{m+b_c(n),u} - \hat{T}_m}{b_c(n)}, \quad u \in [0, 1].$$

Next, observe that

$$\{(X_{j,[j/3]}, X_{2j,[j/3]}, \dots, X_{\ell j,[j/3]})_{m+1 \leq j \leq m+b_c(n)} \in \Gamma\} = \{\rho(\hat{V}_{m,n}^c, \Phi(\frac{1}{c} + \delta)) > 2\varepsilon\}$$

and

$$\{(X_{j,[j/3]}^{(1)}, X_{2j,[j/3]}^{(2)}, \dots, X_{\ell j,[j/3]}^{(\ell)})_{m+1 \leq j \leq m+b_c(n)} \in \Gamma\} = \{\rho(\hat{U}_{m,n}^c, \Phi(\frac{1}{c} + \delta)) > 2\varepsilon\}.$$

Taking in Lemma 3.1,

$$Y_i = (X_{ij,[j/3]}, j = m + 1, m + 2, \dots, m + b_c(n)), \quad i = 1, 2, \dots, \ell$$

and observing that

$$i(m+1) - \frac{1}{3}(m+1) - (i-1)(m+b_c(n)) - \frac{1}{3}(m+b_c(n)) \geq \frac{1}{6}m, \tag{4.8}$$

provided that $m \geq 6\ell b_c(n)$, we obtain from (3.3) that for $m \geq 6\ell b_c(n)$,

$$|P\{\rho(\hat{V}_{m,n}^c, \Phi(\frac{1}{c} + \delta)) > 2\varepsilon\} - P\{\rho(\hat{U}_{m,n}^c, \Phi(\frac{1}{c} + \delta)) > 2\varepsilon\}| \leq \kappa_1^{-1} \ell e^{-\kappa_1 m/6}. \tag{4.9}$$

Next, using the same notation for $T_n = \sum_{j=1}^n F(X_j^{(1)}, X_{2j}^{(2)}, \dots, X_{\ell j}^{(\ell)})$ as in Section 2 with $X_j^{(1)}, X_{2j}^{(2)}, \dots, X_{\ell j}^{(\ell)}$ introduced in this section we set

$$U_{m,n}^c(u) = \frac{T_{m+b_c(n,u)} - T_m}{b_c(n)}, \quad u \in [0, 1].$$

Now, recall that each pair $(X_{ij}^{(i)}, X_{ij,[j/3]}^{(i)})$ has the same joint distribution as $(X_{ij}, X_{ij,[j/3]})$, and so by (2.4),

$$E|X_{ij}^{(i)} - X_{ij,[j/3]}^{(i)}| = E|X_{ij} - E(X_{ij} | \mathcal{F}_{ij-[j/3], ij+[j/3]})| \leq \kappa_3^{-1} e^{-\kappa_3 [j/3]}. \tag{4.10}$$

Hence, similarly to (3.4) and (4.2) we conclude that

$$P\{\rho(U_{m,n}^c, \hat{U}_{m,n}^c) > \varepsilon\} \leq (\varepsilon b_c(n) \kappa_2 \kappa_3 (1 - e^{-\kappa_3/3}))^{-1} e^{-\kappa_3(m-2)/3}. \tag{4.11}$$

Now, repeating (4.4)–(4.6) with U^c in place of V^c we write also

$$0 \leq \max_{0 \leq j \leq n-b_c(n)} \rho(U_{j,n}^c, \Phi(1/c)) - \max_{\varepsilon b_c(n) \leq j \leq n-b_c(n)} \rho(U_{j,n}^c, \Phi(1/c)) \leq 2\varepsilon D, \tag{4.12}$$

$$0 \leq \max_{0 \leq j \leq n-b_c(n)} \rho(\hat{U}_{j,n}^c, \Phi(1/c)) - \max_{\varepsilon b_c(n) \leq j \leq n-b_c(n)} \rho(\hat{U}_{j,n}^c, \Phi(1/c)) \leq 2\varepsilon D, \tag{4.13}$$

$$P\{\max_{\varepsilon b_c(n) \leq j \leq n-b_c(n)} \rho(U_{j,n}^c, \hat{U}_{j,n}^c) \geq \varepsilon\} \leq \sum_{\varepsilon b_c(n) \leq j \leq n-b_c(n)} \tag{4.14}$$

$$P\{\rho(U_{j,n}^c, \hat{U}_{j,n}^c) \geq \varepsilon\} \leq (\varepsilon b_c(n) \kappa_2 \kappa_3 (1 - e^{-\kappa_3/3})^2)^{-1} e^{2\kappa_3/3} n^{-\kappa_3 \varepsilon c/3}.$$

Taking into account stationarity of the sequence $F(X_m^{(1)}, X_{2m}^{(2)}, \dots, X_{\ell m}^{(\ell)})$, $m \geq 1$ we obtain from the upper large deviations bound in (2.8) that for any $\varepsilon, \lambda > 0$ there exists $n(\varepsilon, \lambda)$ such that for all $n \geq n(\varepsilon, \lambda)$,

$$P\{\rho(U_{m,n}^c, \Phi(\frac{1}{c} + \delta)) \geq \varepsilon\} = P\{\rho(b_c^{-1}(n)T_{b_c(n)}, \Phi(\frac{1}{c} + \delta)) \geq \varepsilon\} \tag{4.15}$$

$$\leq e^{-(\frac{1}{c} + \delta - \lambda)b_c(n)} \leq e^{-\frac{1}{c}n - (1+c\sigma)}$$

where we choose $\lambda > 0$ so small that $\sigma = \delta - \lambda > 0$.

Now, by (4.2), (4.7), (4.9), (4.10) and (4.15) we obtain that for $m \geq 6\ell b_c(n)$,

$$P\{\rho(V_{m,n}^c, \Phi(1/c)) > 4\varepsilon\} \leq d_1^{-1} (e^{-d_1 m} + n^{-(1+c\sigma)}) \tag{4.16}$$

for some $d_1 = d_1(\varepsilon) > 0$ independent of m and n but dependent on $\varepsilon > 0$ which is fixed for now. Hence, we obtain from (4.2), (4.4), (4.5), (4.7), (4.12), (4.13) and (4.16) that

$$P\{\max_{0 \leq m \leq n-b_c(n)} \rho(V_{m,n}^c, \Phi(1/c)) > 8D + 4\varepsilon\} \tag{4.17}$$

$$\leq P\{\max_{\varepsilon b_c(n) \leq m \leq n-b_c(n)} \rho(V_{m,n}^c, \Phi(1/c)) > 4\varepsilon\}$$

$$\leq P\{\max_{\varepsilon b_c(n) \leq m < 6\ell b_c(n)} \rho(V_{m,n}^c, \Phi(1/c)) > 4\varepsilon\}$$

$$+ \sum_{6\ell b_c(n) \leq m \leq n-b_c(n)} P\{\rho(V_{m,n}^c, \Phi(1/c)) > 4\varepsilon\}$$

$$\leq \sum_{\varepsilon b_c(n) \leq m \leq 6\ell b_c(n)} P\{\rho(\hat{V}_{m,n}^c, \Phi(1/c)) > 3\varepsilon\} + d_2^{-1} n^{-d_2}$$

$$\leq \sum_{\varepsilon b_c(n) \leq m \leq 6\ell b_c(n)} P\{\rho(\hat{V}_{m,n}^c, \Phi(1/c + \delta)) > 2\varepsilon\} + d_2^{-1} n^{-d_2}$$

for some $d_2 > 0$ independent of n but dependent on $\varepsilon > 0$.

Next, we have to modify our approach for $\varepsilon b_c(n) \leq m < 6\ell b_c(n)$ in order to estimate the last sum in the right hand side of (4.17). Fix an integer M and define new random curves setting for $\frac{k-1}{M} \leq u < k/M$,

$$\hat{V}_{m,n,k}^{c,M}(u) = M(\hat{V}_{m,n}^c(u) - \hat{V}_{m,n}^c(\frac{k-1}{M})), k = 1, \dots, M$$

while $\hat{V}_{m,n,k}^{c,M}(u) = 0$ if $u \notin [\frac{k-1}{M}, \frac{k}{M})$. We define also for $u \in [\frac{k-1}{M}, \frac{k}{M})$,

$$\hat{U}_{m,n,k}^{c,M}(u) = M(\hat{U}_{m,n}^c(u) - \hat{U}_{m,n}^c(\frac{k-1}{M})), k = 1, \dots, M$$

with $\hat{U}_{m,n,k}^{c,M}(u) = 0$ if $u \notin [\frac{k-1}{M}, \frac{k}{M})$. Observe that for $u \in [\frac{k-1}{M}, \frac{k}{M})$,

$$\hat{V}_{m,n,k}^{c,M}(u) = Mb_c^{-1}(n) \sum_{j=m+[(k-1)M^{-1}b_c(n)]+1}^{m+b_c(n,u)} F(X_{j,[j/3]}, X_{2j,[j/3]}, \dots, X_{\ell j,[j/3]})$$

and

$$\hat{U}_{m,n,k}^{c,M}(u) = Mb_c^{-1}(n) \sum_{j=m+[(k-1)M^{-1}b_c(n)]+1}^{m+b_c(n,u)} F(X_{j,[j/3]}^{(1)}, X_{2j,[j/3]}^{(2)}, \dots, X_{\ell j,[j/3]}^{(\ell)}).$$

Taking in Lemma 3.1,

$$Y_i = (X_{ij,[j/3]}, j = m + [(k-1)M^{-1}b_c(n)] + 1, \dots, m + [kM^{-1}b_c(n)]), i = 1, 2, \dots, \ell$$

and observing that

$$(i - \frac{1}{3})(m + (k-1)M^{-1}b_c(n)) - (i - \frac{2}{3})(m + kM^{-1}b_c(n)) - \frac{1}{3} \geq (\frac{\varepsilon}{3} + \frac{k+1}{3M} - \frac{\ell}{M})b_c(n), \quad (4.18)$$

provided that $m \geq \varepsilon b_c(n)$, we see that for $M = M(\varepsilon) = 6\ell([1/\varepsilon] + 1)$ the right hand side of (4.18) is not less than $\frac{1}{6}\varepsilon b_c(n)$. Thus by (3.3) for such m in the same way as in (4.9) we obtain that

$$\begin{aligned} & |P\{\rho(\hat{V}_{m,n,k}^{c,M}, \Phi(\frac{1}{c} + \delta)) > 2\varepsilon\} - P\{\rho(\hat{U}_{m,n,k}^{c,M}, \Phi(\frac{1}{c} + \delta)) > 2\varepsilon\}| \\ & \leq \kappa_1^{-1} e^{\kappa_1(1+\varepsilon)/3} \ell e^{-\frac{1}{6}\kappa_1 \varepsilon c \ln n} = \kappa_1^{-1} e^{\kappa_1(1+\varepsilon)/3} \ell n^{-\kappa_1 \varepsilon c/6}. \end{aligned} \quad (4.19)$$

Since there are no more than $6\ell c \ln n$ numbers m for which we will need this estimate it will suit our purposes.

Next, define for $u \in [\frac{k-1}{M}, \frac{k}{M})$ and $k = 1, \dots, M$,

$$\begin{aligned} U_{m,n,k}^{c,M}(u) &= M(U_{m,n}^c(u) - U_{m,n}^c(\frac{k-1}{M})) \\ &= Mb_c^{-1}(n) \sum_{j=m+[(k-1)M^{-1}b_c(n)]+1}^{m+b_c(n,u)} F(X_j^{(1)}, X_{2j}^{(2)}, \dots, X_{\ell j}^{(\ell)}). \end{aligned}$$

Relying on (2.3) and (4.10) we estimate $P\{\rho(U_{m,n,k}^{c,M}, \hat{U}_{m,n,k}^{c,M}) > \varepsilon\}$ by the right hand side of (4.11) and using (4.19) we obtain

$$P\{\rho(\hat{V}_{m,n,k}^{c,M}, \Phi(\frac{1}{c} + \delta)) > 2\varepsilon\} \leq P\{\rho(U_{m,n,k}^{c,M}, \Phi(\frac{1}{c} + \delta)) > 2\varepsilon\} + d_3^{-1} n^{-d_3} \quad (4.20)$$

for some $d_3 > 0$ independent of m, n, k but dependent on ε .

Next, using stationarity of the sequence $F(X_k^{(1)}, X_{2k}^{(2)}, \dots, X_{\ell k}^{(\ell)})$, $k \geq 1$ we can compute the rate functional of large deviations for $U_{m,n,k}^{c,M}$ as $n \rightarrow \infty$ from (2.5)–(2.7) which will provide the upper bound similarly to (4.15) in the form

$$P\{\rho(U_{m,n,k}^{c,M}, \Phi(\frac{1}{c} + \delta)) \geq \varepsilon\} \leq d_4^{-1} e^{-d_4 \ln n} = d_4^{-1} n^{-d_4} \quad (4.21)$$

for some $d_4 > 0$ depending on ε but not on n . Now observe that

$$\hat{V}_{m,n}^c = \frac{1}{M} \sum_{k=1}^M \hat{V}_{m,n,k}^{c,M}$$

Since $\Phi(\frac{1}{c} + \delta)$ is a convex set then

$$\rho(\hat{V}_{m,n}^c, \Phi(\frac{1}{c} + \delta)) \leq \frac{1}{M} \sum_{k=1}^M \rho(\hat{V}_{m,n,k}^{c,M}, \Phi(\frac{1}{c} + \delta)).$$

Hence,

$$P\{\rho(\hat{V}_{m,n}^c, \Phi(\frac{1}{c} + \delta)) > 2\varepsilon\} \leq \frac{1}{M} \sum_{k=1}^M P\{\rho(\hat{V}_{m,n,k}^{c,M}, \Phi(\frac{1}{c} + \delta)) > 2\varepsilon\}. \tag{4.22}$$

Now, collecting the estimates (4.4)–(4.7), (4.9), (4.15) and (4.17)–(4.22) we conclude that

$$P\{\max_{0 \leq j \leq n-b_c(n)} \rho(V_{j,n}^c, \Phi(1/c)) \geq (8D + 4)\varepsilon\} \leq d_5^{-1} n^{-d_5} \tag{4.23}$$

for some $d_5 > 0$ depending on ε but not on n . Replacing n by the subsequence $k_n = n^{2/d_5}$ we obtain by the Borel-Cantelli lemma that with probability one,

$$\limsup_{n \rightarrow \infty} \max_{0 \leq j \leq k_n - b_c(k_n)} \rho(V_{j,k_n}^c, \Phi(1/c)) \leq (8D + 4)\varepsilon. \tag{4.24}$$

Now take into account that if $k_n < r \leq k_{n+1}$ then

$$b_c(k_{n+1}) - b_c(r) \leq b_c(k_{n+1}) - b_c(k_n) \leq c[2/d_4] \ln(\frac{n+1}{n}) + 1 \rightarrow 1 \text{ as } n \rightarrow \infty.$$

It follows that (4.24) remains true if k_n there is replaced by n implying (4.1) since $\varepsilon > 0$ is arbitrary.

5 The lower bound

We will prove here that with probability one

$$\lim_{n \rightarrow \infty} \sup_{\gamma \in \Phi(1/c)} \min_{0 \leq j \leq n-b_c(n)} \rho(V_{j,n}^c, \gamma) = 0 \tag{5.1}$$

which will complete the proof of (2.9). For $\gamma \in \Phi(1/c)$ introduce the events

$$\Gamma_n^{(1)} = \Gamma_n^{(1)}(\gamma, \varepsilon) = \{ \min_{0 \leq j \leq n-b_c(n)} \rho(V_{j,n}^c, \gamma) \geq 6\varepsilon \}.$$

Then

$$\Gamma_n^{(1)} \subset \Gamma_n^{(2)}(\gamma, \varepsilon) = \{ \min_{(1-1/4\ell)n \leq j \leq n-b_c(n)} \rho(V_{j,n}^c, \gamma) \geq 6\varepsilon \}. \tag{5.2}$$

Set

$$\Gamma_n^{(3)}(\gamma, \varepsilon) = \{ \min_{(1-1/4\ell)n \leq j \leq n-b_c(n)} \rho(\hat{V}_{j,n}^c, \gamma) \geq 5\varepsilon \}.$$

Then, by (4.2),

$$P(\Gamma_n^{(2)}(\gamma, \varepsilon)) \leq P(\Gamma_n^{(3)}(\gamma, \varepsilon)) + d_6^{-1} e^{-d_6 n} \tag{5.3}$$

for some constant $d_6 > 0$ independent of n and ε .

Next, we apply Lemma 3.1 to random vectors

$$Y_i = (X_{ij, [j/3]}, (1 - \frac{1}{4\ell})n \leq j \leq n), i = 1, \dots, \ell.$$

Observe that

$$(i - 1/3)(1 - 1/4\ell)n - (i - 2/3)n = n/3 - (i - 1/3)n/4\ell \geq n/12,$$

and so similarly to Section 4 we derive from (3.3) that

$$|P(\Gamma_n^{(3)}(\gamma, \varepsilon)) - P(\Gamma_n^{(4)}(\gamma, \varepsilon))| \leq d_7^{-1} e^{-d_7 n} \tag{5.4}$$

where

$$\Gamma_n^{(4)}(\gamma, \varepsilon) = \left\{ \min_{(1-1/4\ell)n \leq j \leq n-b_c(n)} \rho(\hat{U}_{j,n}^c, \gamma) \geq 5\varepsilon \right\}$$

and $d_7 > 0$ does not depend on n . Now,

$$\Gamma_n^{(4)} = \cap_{(1-1/4\ell)n \leq j \leq n-b_c(n)} \Gamma_{n,j}^{(5)} \subset \cap_{j: (1-1/4\ell)n \leq j \leq [4 \ln^2 n] \leq n-b_c(n)} \Gamma_{n,j}^{(5)} \tag{5.5}$$

where

$$\Gamma_{n,j}^{(5)}(\gamma, \varepsilon) = \{ \rho(\hat{U}_{j,n}^c, \gamma) \geq 5\varepsilon \}.$$

Next, we write

$$\Gamma_{n,j}^{(5)}(\gamma, \varepsilon) \subset \Gamma_{n,j}^{(6)}(\gamma, \varepsilon) \cup \{ \rho(U_{j,n}^c, \hat{U}_{j,n}^c) > \varepsilon \}$$

where $\Gamma_{n,j}^{(6)}(\gamma, \varepsilon) = \{ \rho(U_{j,n}^c, \gamma) \geq 4\varepsilon \}$. Hence, by (4.11) and (5.2)–(5.5),

$$P(\Gamma_n^{(1)}(\gamma, \varepsilon)) \leq P(\cap_{j: (1-1/4\ell)n \leq j \leq [4 \ln^2 n] \leq n-b_c(n)} \Gamma_{n,j}^{(6)}(\gamma, \varepsilon)) + d_8 e^{-d_8 n} \tag{5.6}$$

for some $d_8 > 0$ independent of n but depending on ε .

Next, we will use ψ -mixing and approximation properties of the product process $(X_j^{(1)}, X_{2j}^{(2)}, \dots, X_{\ell j}^{(\ell)})$, $j = 1, 2, \dots$. Consider the product probability space $(\Omega^\ell, \mathcal{F}^\ell, P^\ell) = (\omega, \mathcal{F}, P) \times \dots \times (\Omega, \mathcal{F}, P)$ (ℓ -times product) and the σ -algebras $\mathcal{F}_{m_1, n_1; m_2, n_2; \dots; m_\ell, n_\ell} = \mathcal{F}_{m_1, n_1} \times \mathcal{F}_{m_2, n_2} \times \dots \times \mathcal{F}_{m_\ell, n_\ell}$ (where by this product we mean the minimal σ -algebra containing all products of sets from the factors). Let

$$\Gamma \in \mathcal{F}_{-\infty, m_1; -\infty, m_2; \dots; -\infty, m_\ell} \text{ and } \Delta \in \mathcal{F}_{m_1+k_1, \infty; m_2+k_2, \infty; \dots; m_\ell+k_\ell, \infty}. \tag{5.7}$$

We claim that

$$|P^\ell(\Gamma \cap \Delta) - P^\ell(\Gamma)P^\ell(\Delta)| \leq P^\ell(\Gamma)P^\ell(\Delta) \sum_{i=1}^{\ell} \psi(k_i) \prod_{j=i+1}^{\ell} (1 + \psi(k_j)) \tag{5.8}$$

where $\prod_{\ell+1}^{\ell} = 1$ and ψ is defined in (2.1).

Indeed, take first $\Gamma = \Gamma_1 \times \Gamma_2 \times \dots \times \Gamma_\ell$ and $\Delta = \Delta_1 \times \Delta_2 \times \dots \times \Delta_\ell$ with $\Gamma_i \in \mathcal{F}_{-\infty, m_i}$ and $\Delta_i \in \mathcal{F}_{m_i+k_i, \infty}$, $i = 1, \dots, \ell$. We proceed by induction in ℓ . For $\ell = 1$ the inequality (5.8) follows from (2.1). Now, suppose that (5.8) holds true for $\ell - 1$ in place of ℓ . Then by (2.1),

$$\begin{aligned} |P^\ell(\Gamma \cap \Delta) - P^\ell(\Gamma)P^\ell(\Delta)| &= \left| \prod_{i=1}^{\ell} P(\Gamma_i \cap \Delta_i) - \prod_{i=1}^{\ell} P(\Gamma_i)P(\Delta_i) \right| \\ &\leq P(\Gamma_\ell \cap \Delta_\ell) \left| \prod_{i=1}^{\ell-1} P(\Gamma_i \cap \Delta_i) - \prod_{i=1}^{\ell-1} P(\Gamma_i)P(\Delta_i) \right| \\ &\quad + |P(\Gamma_\ell \cap \Delta_\ell) - P(\Gamma_\ell)P(\Delta_\ell)| \prod_{i=1}^{\ell-1} P(\Gamma_i)P(\Delta_i) \\ &\leq (1 + \psi(k_\ell)) P(\Gamma_\ell)P(\Delta_\ell) \left| \prod_{i=1}^{\ell-1} P(\Gamma_i \cap \Delta_i) - \prod_{i=1}^{\ell-1} P(\Gamma_i)P(\Delta_i) \right| \\ &\quad + \psi(k_\ell) \prod_{i=1}^{\ell} P(\Gamma_i)P(\Delta_i) \end{aligned}$$

and we arrive at (5.8) using the induction hypothesis taking into account that here

$$P^\ell(\Gamma) = \prod_{i=1}^{\ell} P(\Gamma_i) \text{ and } P^\ell(\Delta) = \prod_{i=1}^{\ell} P(\Delta_i).$$

Since (5.8) remains true under finite disjoint unions and under monotone unions and intersections then by the monotone class theorem (5.8) is valid for all Γ and Δ satisfying (5.7).

Next, we redefine $X_j^{(i)}$ on the product space Ω^ℓ setting $X_j^{(i)}(\omega, \omega_2, \dots, \omega_\ell)$ equal to $X_j^{(i)}(\omega_i)$ and using the same notation for the new processes which are, again, independent for different i 's and have the same distributions as before. Using these $X_j^{(i)}$ we redefine on Ω^ℓ the sum T_n and the random curves $U_{m,n}^c(u)$, $u \in [0, 1]$, as before. Let $\mathcal{F}_0 = \{\emptyset, \Omega\}$ be the trivial σ -algebra on Ω and set

$$\mathcal{F}^{(i)} = \mathcal{F}_0 \times \dots \times \mathcal{F}_0 \times \mathcal{F} \times \mathcal{F}_0 \times \dots \times \mathcal{F}_0 \text{ and } \mathcal{F}^{(\ell)} = \mathcal{F}_0 \times \dots \times \mathcal{F}_0 \times \mathcal{F}_{m,n} \times \mathcal{F}_0 \times \dots \times \mathcal{F}_0$$

where the nontrivial σ -algebra appears as the i -th factor (and the product σ -algebras are understood in the same sense as above). Set $X_{n,m}^{(i)} = E^\ell(X_n^{(i)} | \mathcal{F}_{n-m,n+m}^{(i)})$ where E^ℓ is the expectation with respect to the probability P^ℓ . Define

$$\tilde{T}_n = \sum_{j=1}^n F(X_{j, [\ln^2 n]}^{(1)}, X_{2j, [\ln^2 n]}^{(2)}, \dots, X_{\ell j, [\ln^2 n]}^{(\ell)})$$

and

$$\tilde{U}_{i,n}^c(u) = \frac{\tilde{T}_{j+b_c(n,u)} - \tilde{T}_j}{b_c(n)}.$$

Taking into account that the pairs $(X_j^{(i)}, X_{j, [\ln^2 n]}^{(i)})$ have the same distributions as $(X_j, X_{j, [\ln^2 n]})$ we derive from (2.3) and (2.4) similarly to (4.10) and (4.11) that

$$P^\ell\{\rho(U_{j,n}^c, \tilde{U}_{j,n}^c) > \varepsilon\} \leq d_9^{-1} \exp(-d_9 \ln^2 n) \tag{5.9}$$

for some $d_9 > 0$ independent of n but depending on ε . Hence,

$$\begin{aligned} &P(\cap_{j: (1-1/4\ell)n \leq j[4 \ln^2 n] \leq n-b_c(n)} \Gamma_{n,j[4 \ln^2 n]}^{(6)}(\gamma, \varepsilon)) \\ &\leq P(\cap_{j: (1-1/4\ell)n \leq j[4 \ln^2 n] \leq n-b_c(n)} \Gamma_{n,j[4 \ln^2 n]}^{(7)}(\gamma, \varepsilon)) + d_9^{-1} n \exp(-d_9 \ln^2 n) \end{aligned} \tag{5.10}$$

where $\Gamma_{n,j}^{(7)}(\gamma, \varepsilon) = \{\rho(\tilde{U}_{j,n}^c, \gamma) \geq 3\varepsilon\}$. Now observe that $\Gamma_{n,j[4 \ln^2 n]}^{(7)}(\gamma, \varepsilon)$ is $\mathcal{F}_{k_1(j,n), m_1(j,n); k_2(j,n), m_2(j,n); \dots; k_\ell(j,n), m_\ell(j,n)}$ -measurable, where $k_i(j, n) = ij[4 \ln^2 n] - [\ln^2 n]$ and $m_i(j, n) = ij[4 \ln^2 n] + [\ln^2 n] + b_c(n)$. Hence, $k_i(j+1, n) - m_i(j, n) \geq [\ln^2 n]$ when n is large enough. Applying (5.8) successively we obtain that

$$\begin{aligned} &P(\cap_{j: (1-1/4\ell)n \leq j[4 \ln^2 n] \leq n-b_c(n)} \Gamma_{n,j[4 \ln^2 n]}^{(7)}(\gamma, \varepsilon)) \\ &\leq (1 + \Psi([\ln^2 n]))^{\frac{n}{[4 \ln^2 n]}} \prod_{j: (1-1/4\ell)n \leq j[4 \ln^2 n] \leq n-b_c(n)} P^\ell(\Gamma_{n,j[4 \ln^2 n]}^{(7)}(\gamma, \varepsilon)) \end{aligned} \tag{5.11}$$

where

$$\Psi(k) = \ell \psi(k) (1 + \psi(k))^\ell \leq \kappa_1^{-1} \ell 2^\ell e^{-\kappa_1 k} \tag{5.12}$$

for all k large enough.

Next, we use (5.9) again in order to obtain that

$$P^\ell(\Gamma_{n,j}^{(7)}(\gamma, \varepsilon)) \leq P^\ell(\Gamma_{n,j}^{(8)}(\gamma, \varepsilon)) + d_9^{-1} \exp(-d_9 \ln^2 n) \tag{5.13}$$

where $\Gamma_{n,j}^{(8)}(\gamma, \varepsilon) = \{\rho(U_{j,n}^c, \gamma) \geq 2\varepsilon\}$. Observe that we can replace P^ℓ by P in the right hand side of (5.13) if the independent processes $\{X_{ij}^{(i)}, j = 1, 2, \dots\}$, $i = 1, \dots, \ell$ are considered on the original probability space (Ω, \mathcal{F}, P) .

Since we consider $\gamma \in \Phi(1/c)$ then $I(\dot{\gamma}_u) < \infty$ for Lebesgue almost all $u \in [0, 1]$ and (see Appendix) if $I(\beta) < \infty$, $\beta \in \mathbb{R}^d$ then $I(a\beta) < I(\beta)$ for any $0 < a < 1$. Hence, if we define

$$\eta_u = (1 - \varepsilon(\sup_{v \in [0,1]} |\gamma_u|)^{-1})\gamma_u, \quad u \in [0, 1]$$

then

$$\rho(\gamma, \eta) \leq \varepsilon \quad \text{and} \quad S(\eta) \leq S(\gamma) - a \leq \frac{1}{c} - a \tag{5.14}$$

for some $a > 0$. By (5.14) and stationarity of the summands in T_n ,

$$\begin{aligned} P^\ell(\Gamma_{n,j}^{(8)}(\gamma, \varepsilon)) &\leq P^\ell\{\rho(U_{j,n}^c, \eta) \geq \varepsilon\} \\ &= P^\ell\{\rho(b_c^{-1}(n)T_{b_c(n)}, \eta) \geq \varepsilon\} = 1 - P^\ell\{\rho(U_{j,n}^c, \eta) < \varepsilon\}. \end{aligned} \tag{5.15}$$

Now, employing the lower large deviations bound from (2.8) we obtain that for any $\varepsilon, \lambda > 0$ there exists $n_0 > 0$ such that for all $n \geq n_0$,

$$P^\ell\{\rho(b_c^{-1}(n)T_{b_c(n)}, \eta) < \varepsilon\} \geq \exp(-b_c(n)(c^{-1} - a + \lambda)) \geq n^{-1+c\sigma} \tag{5.16}$$

where we choose $\lambda > 0$ so small that $\sigma = a - \lambda > 0$.

Hence, we obtain by (5.6), (5.10)–(5.13), (5.15) and (5.16) that for all n large enough

$$\begin{aligned} P(\Gamma_n^{(1)}(\gamma, \varepsilon)) &\leq (1 + \kappa^{-1}\ell 2^\ell \exp(-\kappa_1[\ln^2 n]))^n (1 - n^{-1+c\sigma})^{\frac{n}{5\ell[4\ln^2 n]}} \\ &\quad + d_8^{-1}e^{-d_8 n} + 2d_9^{-1}n \exp(-d_9 \ln^2 n) \leq d_{10}^{-1} \exp(-d_{10} \ln^2 n) \end{aligned} \tag{5.17}$$

for some $d_{10} > 0$ independent of n but depending on ε . Employing the Borel-Cantelli lemma we conclude from (5.17) that for any $\gamma \in \Phi(1/c)$ with probability one

$$\limsup_{n \rightarrow \infty} \min_{0 \leq j \leq n - b_c(n)} \rho(V_{j,n}^c, \gamma) < 6\varepsilon. \tag{5.18}$$

Since $\Phi(1/c)$ is a compact set we can choose there an ε -net $\gamma_1, \gamma_2, \dots, \gamma_{k(\varepsilon)}$ and then with probability one (5.18) will hold true simultaneously for all $\gamma = \gamma_i$, $i = 1, \dots, k(\varepsilon)$. It follows then that with probability one,

$$\limsup_{n \rightarrow \infty} \sup_{\gamma \in \Phi(1/c)} \min_{0 \leq j \leq n - b_c(n)} \rho(V_{j,n}^c, \gamma) \leq 7\varepsilon \tag{5.19}$$

and since $\varepsilon > 0$ is arbitrary we obtain (5.1). □

6 Proof of Corollary 2.2

Observe that (2.9) implies, in particular, that for any continuous (with respect to the metric ρ) function f on the space of curves $[0, 1] \rightarrow \mathbb{R}^d$ with probability one,

$$\lim_{n \rightarrow \infty} \max_{0 \leq k \leq n - b_c(n)} f(V_{k,n}^c) = \sup_{\gamma \in \Phi(1/c)} f(\gamma). \tag{6.1}$$

Now assume that $d = 1$ and take $f(\gamma) = \gamma(1)$ where we write $\gamma(u) = \gamma_u$. Assume that $I(\beta) < \infty$ and set $c = 1/I(\beta)$. Then

$$\sup_{\gamma \in \Phi(I(\beta))} f(\gamma) = \sup\{\gamma(1) : \gamma \in \Phi(I(\beta))\} = \beta. \tag{6.2}$$

Indeed, by convexity of the rate function I for any $\gamma \in \Phi(I(\beta))$,

$$I(\beta) \geq S(\gamma) = \int_0^1 I(\dot{\gamma}(u))du \geq I\left(\int_0^1 \dot{\gamma}(u)du\right) = I(\gamma(1))$$

and by monotonicity of I (see Appendix), $\beta \geq \gamma(1)$. On the other hand, take $\gamma(u) = u\beta$, $u \in [0, 1]$. Then $S(\gamma) = I(\beta)$ and $\gamma(1) = \beta$ implying (6.2) whenever $I(\beta) < \infty$ and (2.10) follows. □

7 Appendix

7.1 Applications

The main applications in the discrete time case of Theorem 2.1 concern Markov chains and some classes of dynamical systems such as Axiom A diffeomorphisms, expanding transformations and topologically mixing subshifts of finite type. We will restrict ourselves to several main setups to which our results are applicable rather than trying to describe most general situations. First, let X_n , $n \geq 0$ be a time homogeneous Markov chain on \mathbb{R}^φ whose transition probability $P(x, \Gamma) = P\{X_1 \in \Gamma | X_0 = x\}$ satisfies

$$\kappa\nu(\Gamma) \leq P(x, \Gamma) \leq \kappa^{-1}\nu(\Gamma) \quad (7.1)$$

for some $\kappa > 0$, a probability measure ν on \mathbb{R}^φ and any Borel set $\Gamma \subset \mathbb{R}^\varphi$. Then X_n , $n \geq 0$ is exponentially fast ψ -mixing with respect to the family of σ -algebras $\mathcal{F}_{m,n} = \sigma\{X_k, m \leq k \leq n\}$ generated by the process (see, for instance, [12]). The strong Doeblin type condition (7.1) implies geometric ergodicity

$$\|P(n, x, \cdot) - \mu\| \leq \beta^{-1}e^{-\beta n}, \beta > 0$$

where $\|\cdot\|$ is the variational norm, $P(n, x, \cdot)$ is the n -step transition probability and μ is the unique invariant measure of $\{X_n, n \geq 0\}$ which makes it a stationary process. In this situation $(X_n^{(1)}, X_{2n}^{(2)}, \dots, X_{\ell n}^{(\ell)})$, $n \geq 0$ is the product Markov chain on $\mathbb{R}^{\ell\varphi}$ satisfying similar to (7.1) strong Doeblin condition. The limit (2.5) exists here (see Lemma 4.3 in Ch.7 of [9]) and $\exp(\Pi(\alpha))$ turns out to be the principal eigenvalue of the positive operator

$$Gf(x) = E_x f(X_1^{(1)}, X_2^{(2)}, \dots, X_\ell^{(\ell)}) \exp((\alpha, F(X_1^{(1)}, X_2^{(2)}, \dots, X_\ell^{(\ell)})))$$

(see [13] and references there) where E_x is the expectation conditioned to $(X_0^{(1)}, X_0^{(2)}, \dots, X_0^{(\ell)}) = x$. It is well known (see [21], [12], [11] and references there) that $\Pi(\alpha)$ is convex and differentiable in α . Furthermore, the Hessian matrix $\nabla_\alpha^2 \Pi(\alpha)|_{\alpha=0}$ is positively definite if and only if for each $\alpha \in \mathbb{R}^d$, $\alpha \neq 0$ the limiting variance

$$\sigma_\alpha^2 = \lim_{n \rightarrow \infty} n^{-1} E \left(\sum_{k=0}^n (\alpha, F(X_k^{(1)}, X_{2k}^{(2)}, \dots, X_{\ell k}^{(\ell)})) \right) \quad (7.2)$$

is positive. The latter holds true unless there exists a representation

$$(\alpha, F(X_n^{(1)}, \dots, X_{\ell n}^{(\ell)})) = g(X_n^{(1)}, \dots, X_{\ell n}^{(\ell)}) - g(X_{n-1}^{(1)}, \dots, X_{\ell(n-1)}^{(\ell)}), n = 1, 2, \dots$$

for some bounded Borel function g (see [12]).

In the discrete time dynamical systems case we consider $X_n(\omega) = g \circ f^n(\omega)$, $n \geq 0$ where g is a Hölder continuous vector function and $f : \Omega \rightarrow \Omega$ is a C^2 Axiom A diffeomorphism on a hyperbolic set or a topologically mixing subshift of finite type or a C^2 expanding transformation. Here X_n , $n \geq 0$ is considered as a stationary process on the probability space (Ω, \mathcal{F}, P) where Ω is the corresponding phase space, \mathcal{F} is the Borel σ -algebra and P is a Gibbs measure constructed by a Hölder continuous function (see [2]). Then the exponentially fast ψ -mixing holds true (see [2]) with respect to the family of (finite) σ -algebras generated by cylinder sets in the symbolic setup of subshifts of finite type or with respect to the corresponding σ -algebras constructed via Markov partitions in the Axiom A and expanding cases.

Here the process $(X_n^{(1)}, \dots, X_{\ell n}^{(\ell)})$ is generated by the product dynamical system

$$(f \times f^2 \times \dots \times f^\ell)^n(\omega_1, \omega_2, \dots, \omega_\ell) = (f^n \omega_1, f^{2n} \omega_2, \dots, f^{\ell n} \omega_\ell)$$

so that

$$(X_n^{(1)}(\omega_1), \dots, X_n^{(\ell)}(\omega_\ell)) = G \circ (f \times f^2 \times \dots \times f^\ell)^n(\omega_1, \omega_2, \dots, \omega_\ell) \tag{7.3}$$

where $G(\omega_1, \omega_2, \dots, \omega_\ell) = (g(\omega_1), g(\omega_2), \dots, g(\omega_\ell))$. The above product dynamical system has similar properties as the original dynamical system $f^n\omega$, $n \geq 0$ itself, in particular, it satisfies large deviations bounds with respect to Gibbs measures constructed by Hölder continuous functions and exponentially fast ψ -mixing holds true, as well. The existence of the limit (2.5) and its form follows from [14]. Here $\Pi_t(\alpha)$ turns out to be the topological pressure for the function $(\alpha, F) + \varphi$ where φ is the potential of the corresponding Gibbs measure (for the product system). The differentiability properties of $\Pi_t(\alpha)$ in α are well known and, again, the Hessian matrix $\nabla_\alpha^2 \Pi_t(\alpha)|_{\alpha=0}$ is positively definite if and only if for each $\alpha \in \mathbb{R}^d$, $\alpha \neq 0$ the limiting variance (7.2) is positive where the expectation should be taken with respect to the chosen Gibbs measure (see [22], [11], [13], [14] and references there). The latter holds true unless there exists a coboundary representation $(\alpha, F) = g \circ f - g$ for some bounded Borel function g .

7.2 Some properties of rate functions

We collect here few properties of rate functions of large deviations which are essentially well known but hard to find in major books on large deviations. First, observe that if $\Pi(\alpha)$, $\alpha \in \mathbb{R}^d$ is a twice differentiable function such that $\Pi(0) = 0$, $\nabla_\alpha \Pi(\alpha)|_{\alpha=0} = 0$ then $\Pi(\alpha) = o(|\alpha|)$, and so

$$I(\beta) = \sup_\alpha ((\alpha, \beta) - \Pi(\alpha)) > 0 \tag{7.4}$$

unless $\beta = 0$. Indeed, by the above

$$I(\beta) \geq \delta|\beta|^2 - \Pi(\delta\beta) > 0$$

if $\beta \neq 0$ and $\delta > 0$ is small enough. Curiously, positivity of the rate function is not studied in several books on large deviations without which upper large deviations bounds do not make much sense.

Next, assume, in addition, that Π is convex and has a positively definite at zero Hessian matrix $\nabla_\alpha^2 \Pi(\alpha)|_{\alpha=0}$. Then $\Pi(\alpha) \geq 0$ for all $\alpha \in \mathbb{R}^d$ and for some $\delta_1, \delta_2 > 0$,

$$\Pi(\alpha) \geq \delta_1|\alpha| \quad \text{provided} \quad |\alpha| > \delta_2. \tag{7.5}$$

It follows that if $|\beta| < \delta_1$ then $\alpha_\beta = \arg \sup((\alpha, \beta) - \Pi(\alpha))$ satisfies $|\alpha_\beta| \leq \delta_2$ and, in particular, $I(\beta) < \infty$, i.e. $I(\beta)$ is finite in some neighborhood of 0.

Next, under the above conditions on Π suppose that $I(\beta) < \infty$ for some $\beta \neq 0$. Then

$$I((1 + \delta)\beta) > I(\beta) \quad \text{for any} \quad \delta > 0. \tag{7.6}$$

Indeed, for any $\varepsilon > 0$ there exists $\alpha_{\beta,\varepsilon}$ such that

$$(\alpha_{\beta,\varepsilon}, \beta) - \Pi(\alpha_{\beta,\varepsilon}) \geq I(\beta) - \varepsilon.$$

Since $\Pi(\alpha_{\beta,\varepsilon}) \geq 0$ we have

$$I((1 + \delta)\beta) \geq (1 + \delta)(\alpha_{\beta,\varepsilon}, \beta) - \Pi(\alpha_{\beta,\varepsilon}) \geq I(\beta) + \delta(I(\beta) - \varepsilon) - \varepsilon > I(\beta)$$

provided $\varepsilon < \delta(1 + \delta)^{-1}I(\beta)$ yielding (7.6).

In the Erdős-Rényi law type results it is important to know where a rate function $I(\beta)$ is finite. This issue is hidden inside the functional form of Theorems 2.1 but appears explicitly in Corollary 2.2 and in its original form (1.1). The discussion on finiteness of

rate functions is hard to find in books on large deviations though without studying this issue lower bounds there do not have much sense. We start with the rate functional $J(\nu)$ of the second level of large deviations for occupational measures

$$\zeta_n = \frac{1}{n} \sum_{k=0}^{n-1} \delta_{X_k}, \tag{7.7}$$

where δ_x denotes the unit mass at x (see [13]). Explicit formulas for $J(\nu)$ are known when X_k is a Markov chain whose transition probability satisfies (5.1) and when $X_k = f^k x$ with f being an Axiom A diffeomorphism, expanding transformation or subshift of finite type. In the former case (see [7]),

$$J(\nu) = - \inf_{u>0, \text{continuous}} \int \ln\left(\frac{Pu}{u}\right) d\nu \tag{7.8}$$

and in the latter case (see [13]),

$$J(\nu) = \begin{cases} - \int \varphi d\mu - h_\nu(f) & \text{if } \nu \text{ is } f\text{-invariant,} \\ \infty, & \text{otherwise,} \end{cases} \tag{7.9}$$

where $h_\nu(f)$ is the Kolmogorov-Sinai entropy of f with respect to ν and φ is the potential of the corresponding Gibbs measure μ playing the role of probability here.

Necessary and sufficient conditions for finiteness of $J(\nu)$ in the Markov chain case are given in [7] while in the above dynamical systems cases $J(\nu) < \infty$ for any f -invariant measure ν . If

$$\Pi(\alpha) = \lim_{n \rightarrow \infty} \frac{1}{n} \ln E \exp\left(\sum_{j=0}^{n-1} (\alpha, G(X_j))\right), \tag{7.10}$$

where X_t is a stationary process as above and $G \neq 0$ is a continuous vector function with $EG(X_0) = 0$, then by the contraction principle (see, for instance, [5]) the rate function $I(\beta)$ given by (7.4) can be represented as

$$I(\beta) = \inf\{J(\nu) : \int G d\nu = \beta\} \tag{7.11}$$

where the infimum is taken over the space $\mathcal{P}(M)$ of probability measures on M .

Set

$$\Gamma = \{\beta \in \mathbb{R}^d : \exists \nu \in \mathcal{P}(M) \text{ such that } \int G d\nu = \beta \text{ and } J(\nu) < \infty\}$$

and let $Co(\Gamma)$ be the interior of the convex hull of Γ . Then

$$I(\beta) < \infty \text{ for any } \beta \in Co(\Gamma). \tag{7.12}$$

Indeed, any $\beta \in Co(\Gamma)$ can be represented as $\beta = p_1\beta_1 + p_2\beta_2$ with $\beta_1, \beta_2 \in \Gamma$, $p_1, p_2 \geq 0$ and $p_1 + p_2 = 1$. Then $\beta_1 = \int G d\nu_1$, $\beta_2 = \int G d\nu_2$, and so $\int G d\nu = \beta$ for $\nu = p_1\nu_1 + p_2\nu_2$. Since $J(\nu_1), J(\nu_2) < \infty$ then by convexity of J we have that $J(\nu) \leq p_1J(\nu_1) + p_2J(\nu_2) < \infty$, and so (7.12) holds true.

When $d = 1$, i.e. when G is a (not vector) function we can give another description of the domain where $I(\beta) < \infty$. In this case set

$$\beta_+ = \sup\{\beta : \beta \in \Gamma\} \text{ and } \beta_- = \inf\{\beta : \beta \in \Gamma\}. \tag{7.13}$$

Then by (7.12), $I(\beta) < \infty$ for any $\beta \in (\beta_-, \beta_+)$. It is possible to extract from [6] that under ψ -mixing,

$$\beta_+ = \lim_{n \rightarrow \infty} \frac{1}{n} \text{ess sup} \sum_{j=0}^{n-1} G(X_j) \text{ and } \beta_- = \lim_{n \rightarrow \infty} \frac{1}{n} \text{ess inf} \sum_{j=0}^{n-1} G(X_j). \tag{7.14}$$

References

- [1] K.A. Borovkov, *A functional form of the Erdős-Rényi law of large numbers*, Theory Probab. Appl. **35** (1991), 762–766.
- [2] R. Bowen, *Equilibrium States and the Ergodic Theory of Anosov Diffeomorphisms*, Lecture Notes in Math. 470, Springer-Verlag, Berlin, 1975. MR-0442989
- [3] R.C. Bradley, *On the φ -mixing condition for stationary random sequences*, Duke Math. J. **47** (1980), 421–433. MR-0575905
- [4] R.C. Bradley, *Introduction to Strong Mixing Conditions*, Kendrick Press, Heber City, 2007.
- [5] A. Dembo and O. Zeitouni, *Large Deviations Techniques and Applications*, Springer-Verlag, Berlin, 2010.
- [6] M. Denker and Z. Kabluchko, *An Erdős-Rényi law for mixing processes*, Probab. Math. Stat. **27** (2007), 139–149.
- [7] M.D. Donsker and S.R.S. Varadhan, *Asymptotic evaluation of certain Markov processes expectations for large time. I*, Comm. Pure Appl. Math. **28** (1975), 1–47.
- [8] P. Erdős and A. Rényi, *On a new law of large numbers*, J. Anal. Math. **23** (1970), 103–111.
- [9] M.I. Freidlin and A.D. Wentzell, *Random Perturbations of Dynamical Systems*, 3d ed., (2012), Springer-Verlag, New York. MR-2953753
- [10] H. Furstenberg, *Nonconventional ergodic averages*, Proc. Symp. Pure Math. 50 (1990), 43–56.
- [11] Y. Guivarc’h and J. Hardy, *Théorèmes limites pour une classe de chaînes de Markov et applications aux difféomorphismes d’Anosov*, Ann. Inst. H. Poincaré (Prob. Statist.) **24** (1988), 73–98. MR-0937957
- [12] I.A. Ibragimov and Yu.V. Linnik, *Independent and Stationary Sequences of Random Variables*, Wolters-Noordhoff, Groningen (1971).
- [13] Yu. Kifer, *Large deviations in dynamical systems and stochastic processes*, Trans. Amer. Math. Soc., **321** (1990), 505–524. MR-1025756
- [14] Yu. Kifer, *Averaging in dynamical systems and large deviations*, Invent. Math., **110** (1992), 337–370. MR-1185587
- [15] Yu. Kifer, *Optimal stopping and strong approximation theorems*, Stochastics **79** (2007), 253–273.
- [16] Yu. Kifer, *Large Deviations and Adiabatic Transitions for Dynamical Systems and Markov Processes in Fully Coupled Averaging*, Memoirs of AMS **944**, AMS, Providence R.I. (2009).
- [17] Yu. Kifer, *Nonconventional limit theorems*, Probab. Th. Rel. Fields **148** (2010), 71–106.
- [18] Yu. Kifer, *An Erdős-Rényi law for nonconventional sums*, Electron. Commun. Probab. **20** (2015), no.83; Erratum: **21** (2016), no.33.
- [19] Yu. Kifer and S.R.S. Varadhan, *Nonconventional limit theorems in discrete and continuous time via martingales*, Ann. Probab. **42** (2014), 649–688.
- [20] Yu. Kifer and S.R.S. Varadhan, *Nonconventional large deviations theorem*, Probab. Th. Rel. Fields, **158** (2014), 197–224.
- [21] S.V. Nagaev, *Some limit theorems for stationary Markov chains*, Theory Probab. Appl. **2** (1957), 378–406. MR-0094846
- [22] W. Parry and M. Pollicott, *Zeta functions and the periodic orbit structure of hyperbolic dynamics*, Astérisque **187–188** (1990).

Electronic Journal of Probability

Electronic Communications in Probability

Advantages of publishing in EJP-ECP

- Very high standards
- Free for authors, free for readers
- Quick publication (no backlog)
- Secure publication (LOCKSS¹)
- Easy interface (EJMS²)

Economical model of EJP-ECP

- Non profit, sponsored by IMS³, BS⁴, ProjectEuclid⁵
- Purely electronic

Help keep the journal free and vigorous

- Donate to the IMS open access fund⁶ (click here to donate!)
- Submit your best articles to EJP-ECP
- Choose EJP-ECP over for-profit journals

¹LOCKSS: Lots of Copies Keep Stuff Safe <http://www.lockss.org/>

²EJMS: Electronic Journal Management System <http://www.vtex.lt/en/ejms.html>

³IMS: Institute of Mathematical Statistics <http://www.imstat.org/>

⁴BS: Bernoulli Society <http://www.bernoulli-society.org/>

⁵Project Euclid: <https://projecteuclid.org/>

⁶IMS Open Access Fund: <http://www.imstat.org/publications/open.htm>