Perturbations of Voter model in one-dimension

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Abstract

We study the scaling limit of a large class of voter model perturbations in one dimension, including stochastic Potts models, to a universal limiting object, the continuum voter model perturbation. The perturbations can be described in terms of bulk and boundary nucleations of new colors (opinions). The discrete and continuum (space) models are obtained from their respective duals, the discrete net with killing and Brownian net with killing. These determine the color genealogy by means of *reduced graphs*. We focus our attention on models where the voter and boundary nucleation dynamics depend only on the colors of nearest neighbor sites, for which convergence of the discrete net with killing to its continuum analog was proved in an earlier paper by the authors. We use some detailed properties of the Brownian net with killing to prove voter model perturbations converge to their continuum counterparts. A crucial property of reduced graphs is that even in the continuum, they are finite almost surely. An important issue is how vertices of the continuum reduced graphs are strongly approximated by their discrete analogues.

1 Introduction

1.1 General voter model perturbations

Motivated by problems in Biology and Statistical Physics (see e.g., [MDDGL99] [NP00] [OHLN06]), Cox, Durrett, and Perkins [CDP13] considered a class of interacting particle systems on $\mathbb{Z}^d$, called voter model perturbations (or VMP), whose rates of

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transition are close to the ones of a classical voter model. Informally, they considered models whose transition rates are parametrized by ϵ with $\epsilon \to 0$, so that the transition rates $c_{x,\eta}^\epsilon(j)$ (the transition rate of site x to state j given a configuration η for the model with parameter ϵ) converges to the rate of a (possibly non-nearest neighbor) zero-drift voter model. For $d \geq 3$, and the number q of opinions or colors at each site equal to 2 (i.e. for spin systems) the authors show that, under mild technical assumptions, the properly rescaled local density of one of the colors converges to the solution of an explicit reaction diffusion equation. They are then able to combine this convergence result together with some percolation arguments to show some properties of the particle system when ϵ is small enough.

In this work, we consider a similar problem when $d = 1$. We will construct a natural continuous object — the continuum voter model perturbation (CVMP)— which will be seen to be the scaling limit of a certain type of voter model perturbation. As we shall see, such a limit can not be described in terms of a reaction-diffusion equation anymore (contrary to the case $d \geq 3$), but directly in terms of a duality relation with the Brownian net and Brownian net with killing. The former is a family of one-dimensional branching-coalescing Brownian motions, as first introduced by Sun and Swart [SS08] and then studied further by Newman, Ravishankar and Schertzer [NRS08]; the latter is an extension introduced in [NRS15]. See also [SSS15] for a recent review of those objects.

We will use some of the properties of the Brownian net to define continuous versions of the voter model perturbations, and show some properties of these continuous objects. We expect that a large class of VMP's in one dimension in both discrete and continuous time settings will converge to (the universal limiting object) CVMP. Our goal in this paper is to prove the convergence of a particular class of (discrete time) VMP to CVMP. In the spirit of [CDP13], we hope that our results will provide some insights into a large class of interacting particle systems which are close enough to the voter model in dimension one.

1.2 Decomposition. Bulk and Boundary Nucleations

In the present work, we are interested in the discrete time version of VMP considered in [CDP13]. We consider interacting particle systems on $\mathbb{Z}$ such that the color at each site x, is updated at time $t + 1$ according to some transition probability on $\{1, \cdots, q\}$, that we denote $P_{x,\eta}^\epsilon$ (transition probability at site x given a configuration η at time t). We describe below a large class of models of this type. While our results are for a more limited subclass, we believe our approach is applicable to the more general class. This would require proving convergence of the discrete net with more general jump kernels and branching mechanisms to the Brownian net. (Such improved convergence results are under active investigation by one or more groups of researchers. See also Section 1.6 for more discussion on this topic.)

In our particular context, we consider a family of such models parametrized by a number ϵ, i.e. a family of models with corresponding transition probabilities $\{P_{x,\eta}^\epsilon\}_{\epsilon>0}$ and such that $P_{x,\eta}^\epsilon$ is close to the transition probabilities of a discrete-time voter model when ϵ is small, i.e.

$$\forall i \in \{1, \cdots, q\}, \ \lim_{\epsilon \downarrow 0} P_{x,\eta}^\epsilon(i) = f_{x,\eta}(i)$$

where

$$f_{x,\eta}(i) := \sum_{y \in \mathbb{Z}} K(x - y) 1_{\eta(y)=i} \tag{1.1}$$

for a given zero mean transition kernel K. For many applications (see [CDP13] and the Appendix where we investigate the spatial Lotka-Volterra model, the stochastic Potts

model and the noisy biased voter model), one can decompose the perturbations part of our models into two parts: a first part that we call the boundary noise and a second part called the bulk noise. The bulk noise simply denotes a probability distribution on $\{1, \cdots, q\}$ — independent of x and the present configuration η of the system. The boundary noise requires a little more explanation. Informally, one can think of it as a transition probability $B_{x,\eta}$ which is only non-trivial if there is some local impurity around x — hence the terminology of boundary noise. More formally, a boundary noise can always be written under the form

$$B_{x,\eta}(i) \;=\; \mathbb{E}^Q \left(g_{\eta(x+Y_1), \cdots, \eta(x+Y_N)}(i) \right) \tag{1.2}$$

where (1) N is a fixed integer, (2) for every n-uplet of colors $(c_1, \cdots, c_N)$, the quantity $g_{c_1, \cdots, c_N}$ is a probability distribution on $\{1, \cdots, q\}$ with $g_{c, \cdots, c} = \delta_c$, (3) Q is a probability ditribution on $\mathbb{Z}^N$. In words, $B_{x,\eta}$ chooses N neighbors at random according to the probability distribution Q, and will align to its neighbors color if the coloring is uniform.

The reason for distinguishing between bulk and boundary noise is that in many applications, we shall see in the Appendix that $P^\epsilon_{x,\eta}$ can be decomposed into three parts, i.e.,

$$P^\epsilon_{x,\eta} \;=\; w_\epsilon \underbrace{f^\epsilon_{x,\eta}}_{\text{voter noise}} \;+\; b_\epsilon \underbrace{B^\epsilon_{x,\eta}}_{\text{boundary noise}} \;+\; \kappa_\epsilon \underbrace{p^\epsilon}_{\text{bulk noise}}, \tag{1.3}$$

with $w_\epsilon, b_\epsilon, \kappa_\epsilon$ being three non-negative numbers adding up to 1 and such that for every x, η (1) each part of the decomposition defines a probability measure on $\{1, \cdots, q\}$, (2) $f^\epsilon_{x,\eta} = \left(f_{x,\eta} + O(\kappa^2_\epsilon, b^3_\epsilon)\right)$ where $f_{x,\eta}$ is defined in (1.1) (3) $B^\epsilon_{x,\eta}$ is a boundary noise converging to a limiting boundary noise $B_{x,\eta}$, (4) p^ϵ is a bulk noise converging to a limiting bulk noise p. We note that the case $b_\epsilon = 0$ (no boundary nucleation) has been treated in [FINR06], and that it turns out to be considerably simpler than the general case treated in this paper.

Given the distributions described in the previous paragraph, the color transitions are then most easily understood via a two-step procedure with the first step determining which of three possible moves to implement at the particular (x, t) under consideration — a *walk* move, a *kill* move or a *branch* move with respective probabilities $w_\epsilon, \kappa_\epsilon, b_\epsilon$ — and the second step determining the possible change from the initial color to some new color. This terminology, which may at present sound rather mysterious, is taken from the Branching-Coalescing-Killing random walks model, which will be seen to be dual to the voter model perturbation. Once the type of move has been chosen, here are the rules for the second step.

- Walk: choose a color according to the probability distribution $f^\epsilon_{x,\eta}$. In other words, with high probability, choose the color of one of your neighbors chosen at random according to the kernel $K(x, \cdot)$.

- Kill: choose a color according to the probability distribution p^ϵ.

- Branch: first pick $(Y_1, \cdots, Y_N)$ according to the law Q, and then choose a color according to the distribution $g_{\eta(x+Y_1), \cdots, \eta(x+Y_N)}$.

1.3 Simple VMP and convergence to the continuum VMP

This work is a first step towards the understanding of the scaling limit of systems of type (1.3) when $\epsilon \downarrow 0$. We will restrict ourselves to the case of nearest neighbor voter model perturbation. By that, we mean that the distributions corresponding to the voter and boundary noise $f^\epsilon_{x,\eta}$ and $B^\epsilon_{x,\eta}$ are solely determined by the states $\{\eta(x-1), \eta(x+1)\}$.

Furthermore, we will assume that

$$f_{x,\eta} \;=\; f_{x,\eta}^{\epsilon} \;=\; \frac{1}{2}(\delta_{\eta(x-1)} + \delta_{\eta(x+1)}).$$

Regarding the boundary noise, we will assume the existence of a family of probability distributions $\{g_{i,j}^{\epsilon}\}_{i,j\leq q}$ on $\{1,\cdots,q\}$, each distribution being indexed by two elements on $\{1,\cdots,q\}$ such that

$$B_{x,\eta}^{\epsilon} = g_{\eta(x-1),\eta(x+1)}^{\epsilon}$$

with $g_{i,i}^{\epsilon} = \delta_i$. Furthermore, we will assume the existence of a limiting family of measures $\{g_{i,j}\}$ and p such that $g_{i,j}^{n} := g_{i,j}^{\epsilon_n} \to g_{i,j}$ and $p^{n} := p^{\epsilon_n} \to p$, as the sequence $\epsilon_n \downarrow 0$. In the following, such a model with be referred to as a simple Voter Model Perturbation (VMP) with boundary noise $\{g_{i,j}^{\epsilon}\}_{i,j}$ and bulk noise p^{ϵ}.

We note that in this particular case, the transition rates only depend on the states of the two nearest neighbors, but not on the current state of the site under consideration. As we shall see, this hypothesis is made mostly for technical reasons and is done solely to take advantage of the known convergence result of coalescing-branching discrete (time and space) random walks to the Brownian net; such a result for the continuous time or non-nearest neighbor version has still to be established. The continuous time setting, together with the study of more general voter model perturbations in one dimension (including the non-nearest neighbor case and more general boundary nucleation mechanism) will be the subject of future work.

Before stating our main theorem, we recall that the dynamics we consider are all nearest-neighbor in that the update probabilities of $\eta_t(x)$ at a particular lattice site x depend only on $(\eta_t(x-1), \eta_t(x+1))$. This property (which is also the case for the standard nearest neighbor voter model) implies that the Markov chain is reducible with complete independence between $\{\eta_t(x)\}$ on the even ($\mathbb{Z}_{even}^2 = \{(x,t) : x+t \text{ is even}\}$) and odd ($\mathbb{Z}_{odd}^2 = \{(x,t) : x+t \text{ is odd}\}$) subsets of discrete space-time. Thus one may restrict attention to one of these irreducible components.

Given a realization of the odd component of the voter model perturbation with parameter ϵ, we define $\theta^{\epsilon}(x,t)$ to be the color of site (x,t) at time t. In particular, $\theta^{\epsilon}(\cdot)$ defines a random mapping from $\mathbb{Z}_{odd}^2 \cap \{(x,t) \in \mathbb{Z}^2 : t \geq 0\}$ to $\{1,\cdots,q\}$. Analogously, we will define the CVMP as a mapping from $\mathbb{R} \times \mathbb{R}^+$ to the set of subsets of $\{1,\cdots,q\}$, allowing one space-time point to take several colors. The next theorem enumerates some of the properties of the CVMP. The precise construction and description will be carried out in Section 3.

Theorem 1.1. Let $\lambda, \{g_{i,j}\}, p$ be a family of probability distributions on $\{1,\cdots,q\}$, with $g_{i,i} = \delta_i$. Let $b, \kappa \geq 0$. There exists a random mapping $(x,t) \to \theta(x,t)$ from $\mathbb{R} \times \mathbb{R}^+$ to subsets of $\{1,\cdots,q\}$, with the following properties:

1. For every deterministic (x,t), $|\theta(x,t)| = 1$ a.s..

2. (Coarsening) For every deterministic t, $|\theta(x,t)| \leq 2$ a.s.. Moreover, the set $\{x : |\theta_{(x,t)}| = 2\}$ is locally finite and partitions the line into intervals of uniform color, i.e. the color of $x \to \theta(x,t)$ between two consecutive points of this set remains constant.

3. (Scaling Limit) Let $\{\epsilon_n\}$ be a sequence of positive real numbers converging to 0. Let $\theta^n(\cdot,\cdot) := \theta^{\epsilon_n}(\cdot,\cdot)$ be the simple voter model perturbation characterized by the boundary noise and bulk noise $(\{g_{i,j}^n\}_{1\leq i,j\leq q}, p^n)$, with branching and killing parameters (b_n, κ_n) such that

   (i) there exists $b, \kappa \geq 0$ such that
   $$b_n/\epsilon_n \to b, \text{ and } \kappa_n/\epsilon_n^2 \to \kappa \text{ as } n \to \infty,$$

(ii) there exists $(\{g_{i,j}\}_{1 \le i,j \le q}, p)$ a family of probability distributions on $\{1, \cdots, q\}$ such that

$$\forall i, j, \ g_{i,j}^n \to g_{i,j}, \ \text{ and } \ p^n \to p \ \text{ as } n \to \infty,$$

(iii) the initial distribution of the particle system is given by a product measure ν (not depending on n) with one-dimensional marginal given by λ.

For every $(x,t) \in \mathbb{R}^2$, let $S_\epsilon((x,t)) := (\epsilon x, \epsilon^2 t)$. For $i = 1, \cdots, k$, let z^i be a deterministic point in $\mathbb{R}^2$ and let $\{z_n^i\}_{n \ge 1}$ be a deterministic sequence in $S_{\epsilon_n}(\mathbb{Z}_{odd}^2) \cap \{(x,t) : t > 0\}$ such that $\lim_{n \to \infty} z_n^i = z^i$. Then

$$\mathbb{P}\left(\theta^n(z_n^1), \cdots, \theta^n(z_n^k) \in \cdot\right) \to \mathbb{P}\left(\theta(z^1), \cdots, \theta(z^k) \in \cdot\right). \tag{1.4}$$

Remark 1.2. In the Appendix, we show that the stochastic Potts model in one-dimension at low temperature – i.e., high β – is a VMP. Further, we shall see that the scaling in Theorem 1.1(3)(i) emerges naturally in this setting. See Remark 5.2 in the Appendix.

1.4 Duality to branching-coalescing-killling random walks

In this section, we start by introducing a percolation model on $\mathbb{Z}_{odd}^2$ (already introduced in [MNR13]) and we show how a random coloring algorithm of this percolation configuration is dual to simple voter model perturbations.

An oriented percolation model. Let $b, \kappa \ge 0$ with $b + \kappa \le 1$. Each site $v = (x,t) \in \mathbb{Z}_{odd}^2$ — where x is interpreted as a space coordinate and t as a time coordinate — has two nearest neighbors with higher time coordinates — $v_r = (x+1, t+1)$ and $v_l = (x-1, t+1)$ — and v is randomly (and independently for different v's) connected to a subset of its neighbors v_r and v_l according to the following distribution.

- With probability $\frac{(1-b-\kappa)}{2}$, draw the arrow $(v \to v_l)$ (i.e. starting at v and ending at v_l);
- with probability $\frac{(1-b-\kappa)}{2}$ draw the arrow $(v \to v_r)$;
- with probability b, draw the two arrows $(v \to v_r)$ and $(v \to v_l)$;
- finally do not draw any arrow with probability κ. See Fig. 1.

If we denote by $\mathcal{E}^{b,\kappa}$ the resulting random arrow configuration, the random graph $\mathcal{G}^{b,\kappa} = (\mathbb{Z}_{odd}^2, \mathcal{E}^{b,\kappa})$ defines a certain type of $1+1$ dimensional percolation model oriented forward in the t-direction. In this percolation model, the vertices with two outgoing arrows will be referred to as branching points whereas points with no arrow will be referred to as killing points.

By definition, a path π along $\mathcal{E}^{b,\kappa}$ will denote a path starting from any site of $\mathbb{Z}_{odd}^2$ and following the random arrow configuration until getting killed or reaching ∞. More precisely, a path π is the graph of a function defined on an interval in $\mathbb{R}$ of the form $[\sigma_\pi, e_\pi]$, with $\sigma_\pi, e_\pi \in \mathbb{Z} \cup \{\infty\}$ such that $e_\pi = \infty$ or $(\pi(e_\pi), e_\pi)$ is a killing point, for every $t \in [\sigma_\pi, e_\pi]$, $(\pi(t), t)$ connects to $(\pi(t+1), t+1)$ and π is linear between t and $t+1$.

Considering the set of all the paths along $\mathcal{E}^{b,\kappa}$, one generates an infinite family — denoted by $\mathcal{U}^{b,\kappa}$ — that can loosely be described as a collection of graphs of one dimensional coalescing simple random walks, that branch with probability b and get killed with probability κ. Roughly speaking, a walk at space-time site v can create two new walks (starting respectively at v_l and v_r) with probability b and can be killed with probability κ; two walks move independently when they are apart but become perfectly correlated (i.e., they coalesce) upon meeting at a space-time point. In the following, $\mathcal{U}^{b,\kappa}$ will be referred to as a system of branching-coalescing-killing random walks (or in short, BCK) with parameters (b, κ), or equivalently, and in analogy with their continuum
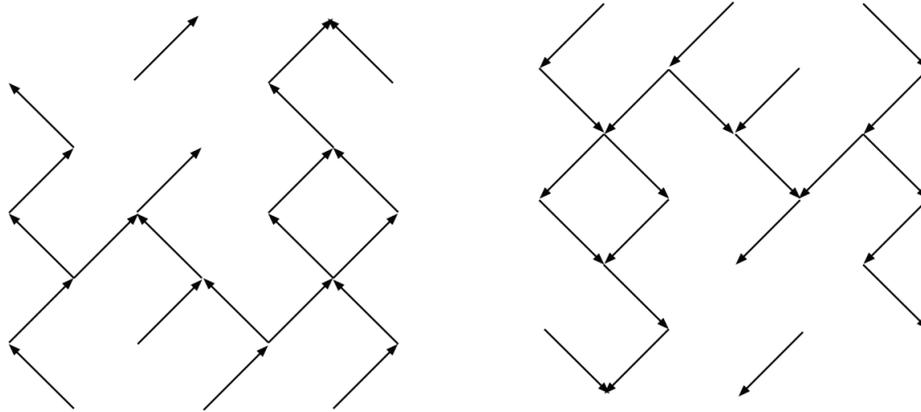
Figure 1: Forward discrete net with killing   Figure 2: Backward discrete net with killing

counterpart, as a discrete net with killing (in analogy with the Brownian net with killing as introduced in [NRS15]).

Backward discrete net. Our percolation model is oriented forward in time. We define the backward BCK — denoted by $\hat{\mathcal{U}}^{b,\kappa}$ — as the backward in time percolation model obtained from the BCK by a reflection about the x-axis. See Fig. 2.

A coloring algorithm. We now describe how simple voter model perturbations (as described in Section 1.3) are exactly dual to the backward BCK (i.e., $\hat{\mathcal{U}}^{b,\kappa}$). To describe this duality relation, we consider the subgraph of $\hat{\mathcal{G}}^{b,\kappa}$ (the backward percolation model) whose set of vertices is given by $\mathbb{Z}^2_{odd} \cap \{(x,t) \ : \ t \geq 0\}$. Then, we equip each vertex $z \in \mathbb{Z}^2_{odd} \cap \{(x,t) \ : \ t \geq 0\}$ with an independent uniform (on $[0,1]$) random variable U_z.

We start by coloring the leaves as follows (see Fig. 3). The color assigned to a point z with time coordinate equal to 0 — denoted by $\theta^\epsilon(z)$ — is the only integer in $\{1, \cdots, q\}$ such that

$$U_z \in [\sum_{j \leq \theta^\epsilon(z)-1} \lambda(j), \sum_{j \leq \theta^\epsilon(z)} \lambda(j)).$$

(With the convention that $\lambda(0) = 0$.) The color assigned to a killing point is the integer in $\{1, \cdots, q\}$ such that

$$U_z \in [\sum_{j \leq \theta^\epsilon(z)-1} p^\epsilon(j), \sum_{j \leq \theta^\epsilon(z)} p^\epsilon(j))$$

where we recall that $p^\epsilon(\cdot)$ is the probability distribution determining the transition when a bulk nucleation occurs (with the convention that $p^\epsilon(0) = 0$).

For every other vertex $z \in \mathbb{Z}^2_{odd} \cap \{(x,t) \ : \ t \geq 0\}$, we consider the component of our (backward) oriented percolation model originated from z and restricted to the upper half plane. This defines a certain acyclic directed graph $\hat{G}^z = (\hat{V}^z, \hat{E}^z)$ whose vertices have out-degree at most 2 – see Fig. 5. For every point in $\hat{V}^z$, we assign a color to each of the nodes sequentially from the leaves to the root by applying the following algorithm, where $\hat{V}^z_n$ will denote the set of color-assigned vertices at step n of the algorithm.

- **Step 1** $\hat{V}^z_1$ is the set of leaves, whose colors have already been assigned.
- **Step n**> 1 If $\hat{V}^z \setminus \hat{V}^z_n = \emptyset$ stop. Otherwise, pick a vertex z' in $\hat{V}^z$ such that z' is connected to $\hat{V}^z_{n-1}$, i.e., all its nearest (directed) neighbor(s) belong to $\hat{V}^z_{n-1}$. Next, if z' connects to a single vertex or if the color of its two neighbors match, assign to

it the color of its neighbor(s). If the colors of the two neighbors disagree, with two colors $k \neq l$, then the site is assigned a color i such that

$$U_{z'} \in [\sum_{i \leq \theta^\epsilon(z')-1} g^\epsilon_{k,l}(i), \sum_{i \leq \theta^\epsilon(z')} g^\epsilon_{k,l}(i+1)). \tag{1.5}$$

See Figs 5-8 for a concrete example in the case $q = 3$ (white, grey, black corresponding respectively to $1, 2, 3$) and p^ϵ, λ, and $g^\epsilon_{i,j}$ for $i \neq j$ are uniform distributions on $\{1, 2, 3\}$. We note that this specific choice for the bulk and boundary noises correspond to the discrete time stochastic Potts model considered in the Appendix (see Section 5.2).

Remark 1.3. At step n of the previous algorithm, there can be multiple $z' \in V^z \setminus V^z_n$, such that all of its (directed) neighbors are in $\hat{V}^z_n$. (In Fig 6, there are two such vertices – one connecting to the top white circle and one connecting to the top black circle). However, we let the reader convince herself that the final coloring of the graph is independent of the choice of z' at step n.

Remark 1.4. Our coloring is consistent in the sense that if $z' \in \hat{V}^z$, the color assigned to z' in the coloring algorithm applied to $\hat{V}^z$ is identical to $\theta^\epsilon(z')$ – i.e., the coloring algorithm applied to $\hat{V}^{z'}$.
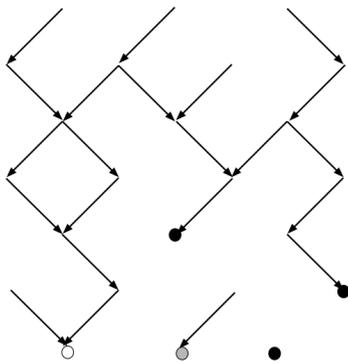

Figure 3: Realization of the coloring ($q{=}3$ colors (white, grey, black)) for the leaves of the backward percolation model restricted to the upper half plane.
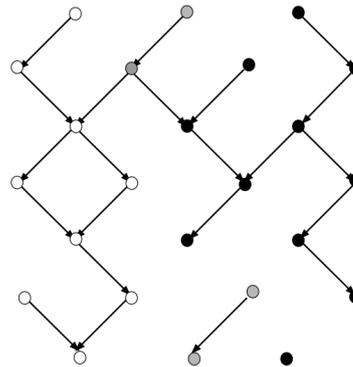
Figure 4: Final coloring of a section of $\mathbb{Z}^2_{odd} \cap \{(x,t) : t \geq 0\}$ after applying the coloring algorithm to every vertex.

Proposition 1.5. Let us consider the simple VMP with characteristic distributions $(\{\frac{1}{2}(\delta_i + \delta_j)\}, \{g^\epsilon_{i,j}\}, p^\epsilon)$, branching-killing parameters $b_\epsilon, \kappa_\epsilon \geq 0$, starting with an initial ν-coloring of the line, where ν is an i.i.d product probability measure on $\{1, \cdots, q\}^{\mathbb{Z}}$.

For every $z^1, \cdots, z^k \in \mathbb{Z}^2_{odd}$, the n-tuple $(\theta^\epsilon(z^1), \cdots, \theta^\epsilon(z^k))$, as described in the coloring algorithm above, coincides with the k-dimensional distribution of this VMP with initial distribution ν.

Proof. Since coalescing random walks are the dual of the standard voter model [HL75] [L04], it is clear that in the absence of killing and branching the color genealogy of the VMP is given by coalescing random walks. At a bulk nucleation site (killing point) a color independent of the previous color evolution is assigned to the site. This leads to the conclusion that if a path in the color genealogy evolution hits a bulk nucleation point then it should be killed at that point. The color at a boundary nucleation point is determined by the colors of the two adjacent sites. The colors of these two sites are determined by
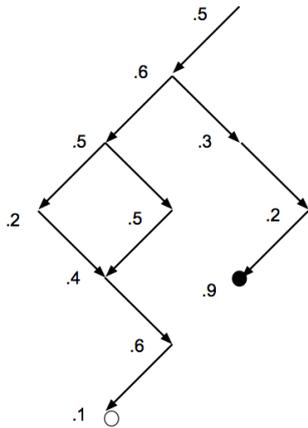
Figure 5: subgraph originated from the top-middle point of Fig. 3 decorated with fictive variables U_z.
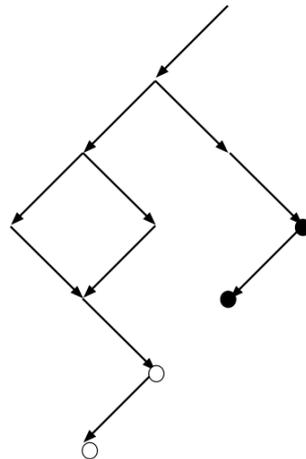

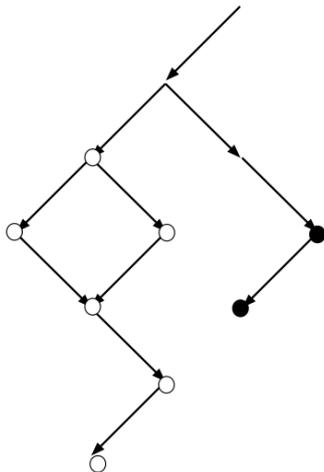Figure 6: Coloring at Step 3 of the algorithm.

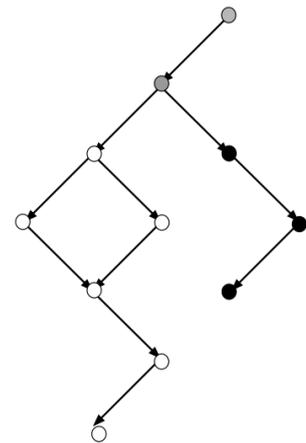Figure 7: Coloring at Step 7 of the algorithm.

Figure 8: Final coloring of the subgraph.

following the color genealogy evolution starting at these sites until reaching time zero or hitting a killing point as given by the dual percolation model starting at these two sites. The uniform random variables at the bulk and boundary nucleation sites along with the rules for determining the color at these sites given earlier ensure that the colors at these sites are chosen with the distribution p^ϵ and $g^\epsilon_{k,l}$ respectively.　□

1.5 Scaling limit

In [NRS15], we introduced the Brownian net with killing (or $\mathcal{N}^{b,\kappa}$ where b and κ are continuum branching and killing parameters) that was shown to emerge as the scaling limit of a BCK system with small branching and killing parameters (see Theorem 2.6 below for a precise statement).

In light of the previous section, it is natural to construct the CVMP as a process dual to the Brownian net with killing; the duality relation between the two models being

described in terms of the random coloring of the "continuum graphs" induced by the Brownian net.

The main difficulty in defining the CVMP lies in the fact that the set of vertices of this "continuum graph" is dense in $\mathbb{R} \times \mathbb{R}^+$. In order to deal with this extra difficulty, we now discuss some remarkable features of the coloring algorithm described in the previous section.

Definition 1.1 (Relevant Separation Point and Reduced Graph – Fig. 9-10)**.** Let $G = (V, E)$ be a finite acyclic oriented graph. We will say that an intermediate node z (i.e. a node which is neither a root nor a leaf) of the graph is relevant iff there exists two paths of G passing through z and reaching the leaves such that they do not meet between z and the leaves. Any other intermediary point will be called irrelevant.

We define the reduced version of G, denoted by $\tilde{G} = (\tilde{V}, \tilde{E})$, as the oriented graph of G obtained after "skipping" all irrelevant points in V. More precisely, the vertices of $\tilde{V}$ are obtained from V by removing the set of irrelevant separation points and placing a directed edge between two points z, z' in $\tilde{V}$ iff there exists a path $\pi = (z_1, \cdots, z_n)$ in G with $z_1 = z$ and $z_n = z'$ and such that z_i is irrelevant for $i \neq 1, n$.
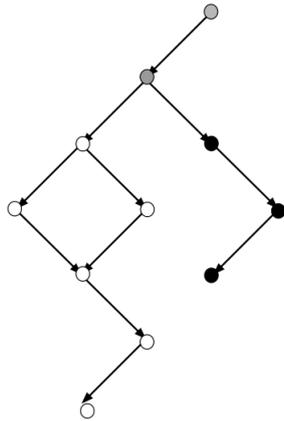

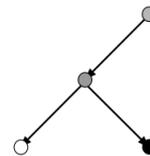Figure 9: Colored graph $\hat{G}^z$ originated from a point z.

Figure 10: Reduced version of the graph presented in the left-panel.

Proposition 1.6. Let G be a finite rooted acyclic oriented graph whose vertices have at most out degree 2 and let us assume that the vertices z of the graph are equipped with i.i.d. random variables $\{U_z\}$, uniformly distributed on $[0, 1]$. For a given coloring of the leaves, our coloring algorithm (as described in Section 1.4) applied separately to G and to the reduced graph $\tilde{G}$ induces the same coloring of the vertices $\tilde{V}$ (using the same set of i.i.d. uniform random variables for the reduced graph).

Proof. If a vertex z connects to two edges with the same color, then z must align to this color (by definition of the boundary nucleation). The same holds if z only connects to a single point. It is then easy to see that any path in the graph G that does not contain any relevant separation points is uniformly colored, and as consequence, in our coloring algorithm, irrelevant separation points can be "skipped" to deduce the color of the vertices $\tilde{V}$ (see Figs 5-8 for a concrete example). □

Significantly, even if the graph induced by the Brownian net paths starting from a point is infinite, its reduced version is finite (see Fig. 13 below). This will allow us to

apply our coloring algorithm at the continuum level and to define the CVMP by imitating the duality relation described in Section 1.4.

1.6 Perspectives

The discrete VMP's considered in Section 1.3 and Theorem 1.1 are dual to a system of coalescing-branching nearest neighbor random walks, that are known to converge (under proper rescaling) to the Brownian net with killing [NRS15].

Let us now reconsider the discrete VMP alluded to in Section 1.2, and whose transition probabilities are defined in (1.3). As for the nearest neighbor case, at least when $f_{x,\eta}^{\epsilon} = f_{x,\eta}$, it is not hard to see that such systems are dual to branching-coalescing random walks. However, the transition probabilities of such walks are more complex: (1) walks move according to the transition kernel K as described in (1.1); (2) they are killed with probability κ_ϵ; and finally (3) they branch with probability b_ϵ, and upon branching, the location of the N new particles are chosen according Q (shifted by the current location of the particle under consideration).

It is plausible that (under some restrictions on the moments of the kernels K and Q) such systems of particles also converge to the Brownian net with killing. If so, the limiting object alluded to in Theorem 1.1 – and whose construction will be carried in Section 3 – should also be the scaling limit of a large class of discrete VMP's. However, it remains unclear what should be the limiting branching and killing mechanism at the continuum. Both the limit of generalized branching-coalescing random walks, and their relation to the CVMP are challenging questions, and would certainly require techniques that go beyond the scope of this paper.

Finally, it would be natural to also consider the continuous time analog of the models alluded above (as in [CDP13] for higher dimensions). In particular, some recent work by Etheridge, Freeman and Straulino [EFS15] where they considered the paths generated by the genealogy of a spatial Fleming-Viot process in one dimension could be relevant to make progress in this direction.

1.7 Outline of the rest of the paper

The rest of the paper is organized as follows. In Section 2, we recall the definition of the scaling limit of the BCK: the Brownian net with killing introduced in [NRS15], which can be loosely described as an infinite family of one dimensional coalescing-branching-killing Brownian motions. In Section 3, this is used to construct the continuum voter model perturbation (CVMP) by mimicking at the continuum level the discrete duality relation of Section 1.4. Properties (1)-(2) of Theorem 1.1 will be shown in Proposition 3.7. Finally, we show the convergence result of Theorem 1.1 (Property (3)) in Section 4.

2 The Brownian net with killing

In Proposition 1.5, the nearest-neighbors discrete voter model perturbations were seen to be dual to an infinite family of BCK random walks. Hence, before defining a continuous analog of the voter model perturbation, it is natural to start with the construction of the BCK scaling limit: the Brownian net with killing (as introduced in [NRS15]), denoted by $\mathcal{N}^{b,\kappa}$ — where b and κ are two non-negative numbers playing a role analogous to the branching and killing parameters in the discrete setting.

2.1 The space $(\mathcal{H}, d_\mathcal{H})$

As in [FINR04], we will define the Brownian net with killing as a random compact set of paths. In this section, we briefly outline the construction of the space of compact sets. For more details, the interested reader may refer to [FINR04] and [NRS15].

First define $(\bar{\mathbb{R}}^2, \rho)$ to be the compactification of $\mathbb{R}^2$ with

$$\rho((x_1, t_1), (x_2, t_2)) = |\phi(x_1, t_1) - \phi(x_2, t_2)| \vee |\psi(t_1) - \psi(t_2)|,$$

where

$$\phi(x, t) = \frac{\tanh(x)}{1 + |t|} \quad \text{and} \quad \psi(t) = \tanh(t).$$

In particular, we note that the mapping $(x, t) \to (\phi(x, t), \psi(t))$ maps $\bar{\mathbb{R}}^2$ onto a compact subset of $\mathbb{R}^2$.

Next, let $C[t_0, t_1]$ denote the set of continuous functions from $[t_0, t_1]$ to $[-\infty, +\infty]$. From there, we define the set of continuous paths in $\mathbb{R}^2$ (with a prescribed starting and ending point) as

$$\Pi := \cup_{t_0 \leq t_1} C[t_0, t_1] \times \{t_0\} \times \{t_1\}.$$

Finally, we equip this set of paths with a metric d, defined as the maximum of the sup norm of the distance between two paths, the distance between their respective starting points and the distance between their ending points. (In particular, when no killing occurs, as in the forward Brownian web, the ending point of each path is $\{\infty\}$). More precisely, if for any path π, we denote by σ_π the starting time of π and by e_π its ending time, we have

$$d(\pi_1, \pi_2) = |\psi(\sigma_{\pi_1}) - \psi(\sigma_{\pi_2})| \vee |\psi(e_{\pi_1}) - \psi(e_{\pi_2})| \vee \max_t \rho((\bar{\pi}_1(t), t), (\bar{\pi}_2(t), t)),$$

where $\bar{\pi}$ is the extension of π into a path from $-\infty$ to $+\infty$ by setting

$$\bar{\pi}(t) = \begin{cases} \pi(\sigma_\pi) & \text{for } t < \sigma_\pi, \\ \pi(e_\pi) & \text{for } t > e_\pi. \end{cases}$$

Finally, let $\mathcal{H}$ denote the set of compact subsets of (Π, d) and let $d_{\mathcal{H}}$ denote the Hausdorff metric,

$$d_{\mathcal{H}}(K_1, K_2) = \max_{\pi_1 \in K_1} \min_{\pi_2 \in K_2} d(\pi_1, \pi_2) \vee \max_{\pi_2 \in K_2} \min_{\pi_1 \in K_1} d(\pi_1, \pi_2).$$

In [FINR04], it is proved that $(\mathcal{H}, d_{\mathcal{H}})$ is Polish. In the following, we will construct the Brownian net with killing as a random element of this space.

2.2 The Brownian web

As mentioned in the introduction, the Brownian web ([TW98] [FINR04] [SSS15]) is the scaling limit of the discrete web under diffusive space-time scaling and is defined as an element of $(\mathcal{H}, d_{\mathcal{H}})$. The next theorem, taken from [FINR04], gives some of the key properties of the BW.

Theorem 2.1. There is an $(\mathcal{H}, \mathcal{F}_{\mathcal{H}})$-valued random variable $\mathcal{W}$ whose distribution is uniquely determined by the following three properties.

(o) from any deterministic point (x, t) in $\mathbb{R}^2$, there is almost surely a unique path $B_{(x,t)}$ starting from (x, t).

(i) for any deterministic, dense countable subset $\mathcal{D}$ of $\mathbb{R}^2$, almost surely, $\mathcal{W}$ is the closure in (Π, d) of $\{B_{(x,t)} : (x, t) \in \mathcal{D}\}$.

(ii) for any deterministic n and $(x_1, t_1), \ldots, (x_n, t_n)$, the joint distribution of $B_{(x_1, t_1)}, \ldots, B_{(x_n, t_n)}$ is that of coalescing Brownian motions from those starting points (with unit diffusion constant).

Note that (i) provides a practical construction of the Brownian web. For $\mathcal{D}$ as defined above, construct coalescing Brownian motion paths starting from $\mathcal{D}$. This defines a *skeleton* for the Brownian web that is denoted by $\mathcal{W}(\mathcal{D})$. $\mathcal{W}$ is simply defined as the closure of this precompact set of paths.

2.3 The standard Brownian net

2.3.1 Idea of the construction

When $\kappa = 0$, Sun and Swart give a construction of the Brownian net which is based on the construction of two coupled drifted Brownian webs $(\mathcal{W}_l, \mathcal{W}_r)$ that interact in a sticky way. For instance, for every point (x, t), there is a unique pair $(l, r) \in (\mathcal{W}_l, \mathcal{W}_r)$ starting from (x, t), with $l \leq r$, and where where l is distributed as the path of a Brownian motion with drift $-b$, and r is a Brownian motion with drift $+b$. The difference between the two paths is a Brownian motion with sticky reflection at 0 (see [SS08] and a recent review paper [SSS15] for more details).

The Brownian net $\mathcal{N}^b$ is then constructed by concatenating paths of the right and left webs. More precisely, given two paths $\pi_1, \pi_2 \in \Pi$, let σ_{π_1} and σ_{π_2} be the starting times of those paths. For $t > \sigma_{\pi_1} \vee \sigma_{\pi_2}$ (note the strict inequality), t is called an intersection time of π_1 and π_2 if $\pi_1(t) = \pi_2(t)$. By hopping from π_1 to π_2, we mean the construction of a new path by concatenating together the piece of π_1 before and the piece of π_2 after an intersection time. Given the left-right Brownian web $(\mathcal{W}_l, \mathcal{W}_r)$, let $H(\mathcal{W}_l \cup \mathcal{W}_r)$ denote the set of paths constructed by hopping a finite number of times between paths in $\mathcal{W}_l \bigcup \mathcal{W}_r$. $\mathcal{N}^b$ is then defined as the closure of $H(\mathcal{W}_l \bigcup \mathcal{W}_r)$. After taking this limit, $\mathcal{W}_l$ and $\mathcal{W}_r$ delimit the net paths from the left and from the right, in the sense that paths in $\mathcal{N}^b$ can not cross any element of $\mathcal{W}_l$ (resp., $\mathcal{W}_r$) from the left (resp., from the right).

Finally, we cite from [SS08] a crucial property of the Brownian net that will be useful for the rest of this paper.

Proposition 2.2. Let $S < T$ and define the branching-coalescing point set

$$\xi^S(T) := \{x \in \mathbb{R} : \exists \pi \in \mathcal{N}^b \text{ s.t. } \sigma_\pi = S, \ \pi(T) = x\}.$$

For almost every realization of the Brownian net, the set $\xi^S(T)$ is locally finite.

2.3.2 Special points of the standard Brownian net

In this section, we recall the classification of the special points of the Brownian net, as described in [SSS09]. As in the Brownian web, the classification of special points will be based on the local geometry of the Brownian net. Of special interest to us will be the points with a deterministic time coordinate.

Definition 2.1 (Equivalent Ingoing and Outgoing Paths)**.** Two paths $\pi, \pi' \in (\Pi, d)$ are said to be equivalent paths entering a point $(x, t) = z \in \mathbb{R}^2$, or in short $\pi \sim_{in}^z \pi'$, iff there exists a sequence $\{t_n\}$ converging to t such that $t_n < t$ and $\pi(t_n) = \pi'(t_n)$ for every n. Equivalent paths exiting a point z, denoted by $\pi \sim_{out}^z \pi'$, are defined analogously by finding a sequence $t_n > t$ with $\{t_n\}$ converging to t and $\pi(t_n) = \pi'(t_n)$.

Despite the notation, these are not in general equivalence relations on the spaces of all paths entering resp. leaving a point. However, in [SSS09], it is shown that that if $(\mathcal{W}_l, \mathcal{W}_r)$ is a left-right Brownian web, then a.s. for all $z \in \mathbb{R}^2$, the relations $\sim_{in}^z$ and $\sim_{out}^z$ actually define equivalence relations on the set of paths in $\mathcal{W}_l \cup \mathcal{W}_r$ entering (resp., leaving) z, and the equivalence classes of paths in $\mathcal{W}_l \cup \mathcal{W}_r$ entering (resp., leaving) z are naturally ordered from left to right. Moreover, the authors gave a complete classification of points $z \in \mathbb{R}^2$ according to the structure of the equivalence classes in $\mathcal{W}_l \cup \mathcal{W}_r$ entering (resp., leaving) z.

In general, such an equivalence class may be of three types. If it contains only paths in $\mathcal{W}_l$ then we say it is of type l, if it contains only paths in $\mathcal{W}_r$ then we say it is of type r, and if it contains both paths in $\mathcal{W}_l$ and $\mathcal{W}_r$ then we say it is of type p, standing for pair. To denote the type of a point $z \in \mathbb{R}^2$ in a Brownian net $\mathcal{N}^b$, we list the incoming equivalence
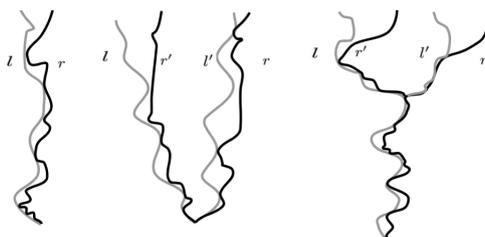
Figure 11: Structure of the left (grey) and right (black) webs $\mathcal{W}_l, \mathcal{W}_r$ at (o, p), (o, pp) and (p, pp) points of the net $\mathcal{N}^b$. In particular, for (o, pp) and (p, pp) points, there are two equivalent outgoing pairs (l, r') and (l', r) (ordered from left to right) exiting the point z.

classes in $\mathcal{W}_l \cup \mathcal{W}_r$ from left to right (see Fig. 11), and then, separated by a comma, the outgoing equivalence classes of paths from left to right. For example, a point of type (p, pp) (as in Fig. 11) has exactly one equivalent class of paths entering the point, and two distinct equivalent classes exiting the point. Further, all of those classes contain at least a path in $\mathcal{W}_r$ and path in $\mathcal{W}_l$. For a point of type (o, p), there is no path entering the point, and one equivalent class exiting the point. We cite the following results from Theorems 1.7 and 1.12 in [SSS09].

Proposition 2.3 (Geometry of the Net at Deterministic Times). Let $\mathcal{N}^b$ be a Brownian net and let $(\mathcal{W}_l, \mathcal{W}_r)$ be the left-right Brownian web associated with $\mathcal{N}^b$. Let t be a deterministic time.

1. For every deterministic point x, the point (x, t) is of type (o, p).
2. Each point (x, t) (with x deterministic or random) is either of type (o, p), (p, p) or (o, pp), and all of these types occur.
3. Every path $\pi \in \mathcal{N}^b$ starting from the line $\mathbb{R} \times \{t\}$, is squeezed between an equivalent pair of right-most and left-most paths, i.e. there exists $l \in \mathcal{W}_l$ and $r \in \mathcal{W}_r$ so that $l \sim_{out}^z r$ such that $l \leq \pi \leq r$ on $[t, \infty)$.
4. Any point (x, t) entered by a path $\pi \in \mathcal{N}^b$ with $\sigma_\pi < t$ is of type (p, p). Moreover, π is squeezed between an equivalent pair of right-most and left-most paths, i.e. there exist $l \in \mathcal{W}_l, r \in \mathcal{W}_r$ with $r \sim_{in}^z l$ and $\epsilon > 0$ and such that $l \leq \pi \leq r$ on $[t - \epsilon, t]$.

In the Brownian net, a separation point (or branching point) will refer to a point $z = (x, t)$ such that there exist two paths π_r and $\pi_l \in \mathcal{N}^b$ with $\sigma_{\pi_l}, \sigma_{\pi_r} < t$ so that (1) $\pi_l(t) = \pi_r(t) = x$ and (2) there exists $\epsilon > 0$ so that $\pi_l(s) < \pi_r(s)$ for every $s \in (t, t + \epsilon]$.

Proposition 2.4 (Geometry of the Net at a Separation Point). If $z = (x, t)$ is a separation point in the standard Brownian net $\mathcal{N}^b$ then

1. z is of the type (p,pp).
2. Every path π starting from z is squeezed between some pair of equivalent right-most and left-most paths — i.e. there exists $l \in \mathcal{W}_l$ and $r \in \mathcal{W}_r$ (which will depend on π) with $l \sim_{out}^z r$ such that $l \leq \pi \leq r$ on $[t, \infty)$.
3. Every path π entering the point z is squeezed between a pair of equivalent right-most and left-most paths — i.e., there exist $l \in \mathcal{W}_l, r \in \mathcal{W}_r$, $l \sim_{in}^z r$, entering the point z and $\epsilon > 0$ such that $l \leq \pi \leq r$ on $[t - \epsilon, t]$.

2.4 The Brownian net with killing

Construction. Having the standard Brownian net at hand, we now need to turn on the killing mechanism. To do that, we recall the definition of the time length measure of the Brownian net as introduced in [NRS15].

Definition 2.2 (Time Length Measure of the Brownian Net). For almost every realization of the standard Brownian net, $\mathcal{T}$ is the unique σ-finite measure such that for every Borel set $E \subset \mathbb{R}^2$,

$$\mathcal{T}(E) = \int_{\mathbb{R}} |\{x : (x,t) \in E \cap (p,p)\}| dt. \tag{2.1}$$

where (p,p) refers to the set of (p,p) points for the net $\mathcal{N}^b$.

Given a realization of the standard Brownian net $\mathcal{N}^b$, we define the set of the killing marks $\mathcal{M}^\kappa$ as a Poisson point process with intensity measure $\kappa\mathcal{T}$. Finally, we define the Brownian net with killing as the union of (1) all the paths $\pi \in \mathcal{N}^b$ killed at

$$e_\pi := \inf\{t > \sigma_\pi : (\pi(t),t) \in \mathcal{M}^\kappa\} \tag{2.2}$$

and (2) for every $z \in \mathbb{R}^2$, the trivial path whose starting point and ending point coincide with z. In the following, this construction will be denoted as $\mathcal{N}^{b,\kappa}$.

Let $E \subseteq \mathbb{R}^2$. We will denote by $\mathcal{N}^{b,\kappa}(E)$ the subset of paths in $\mathcal{N}^{b,\kappa}(E)$ with starting points in E. Finally, $\mathcal{M}^\kappa(S)$ will denote the set of killing points that are attained by $\pi \in \mathcal{N}^{b,\kappa}$ with $\sigma_\pi = S$. We cite the following result from [NRS15] (Proposition 2.14 therein).

Proposition 2.5. For every $S \in \mathbb{R}$, $\mathcal{M}^\kappa(S)$ is locally finite a.s..

Coupling between the net and the net with killing. Our construction of $\mathcal{N}^{b,\kappa}$ induces a natural coupling between $\mathcal{N}^b$ and $\mathcal{N}^{b,\kappa}$, and in the following, $\mathcal{N}^b$ will refer to the net used to construct the net with killing $\mathcal{N}^{b,\kappa}$. Let $\mathcal{U}^{b_\epsilon,\kappa_\epsilon}$ denote the discrete BCK (or discrete net with killing) with branching and killing parameters $(b_\epsilon, \kappa_\epsilon)$, and $\mathcal{U}^{b_\epsilon} := \mathcal{U}^{b_\epsilon,0}$. Like in the continuum, there is a natural and analogous coupling between the discrete objects $\mathcal{U}^b$ and $\mathcal{U}^{b,\kappa}$, and again, $\mathcal{U}^b$ will always refer to the discrete net coupled with $\mathcal{U}^{b,\kappa}$. (This coupling is obtained by starting from $\mathcal{U}^b$ and by removing all the outgoing arrows independently at every vertex with probability κ).

Convergence. In [NRS15], we show that the killed Brownian net is the scaling limit of the BCK system for small values of the branching and killing parameters. In the following, for every element $U \in \mathcal{H}$, $S_\epsilon(U)$ is the set paths obtained after scaling the x-axis by ϵ and the t-axis by ϵ^2.

Theorem 2.6. (Invariance Principle) Let $b, \kappa \geq 0$ and let $\{b_n\}_{n>0}$ and $\{\kappa_n\}_{n>0}$ two sequences of non-negative numbers such that $b_n + \kappa_n \leq 1$ and such that $\lim_{n\to\infty} b_n \epsilon_n^{-1} = b$ and $\lim_{n\to\infty} \kappa_n \epsilon_n^{-2} = \kappa$. Then, as $n \to \infty$,

$$(S_{\epsilon_n}(\mathcal{U}^{b_n}), S_{\epsilon_n}(\mathcal{U}^{b_n,\kappa_n})) \to (\mathcal{N}^b, \mathcal{N}^{b,\kappa}) \quad \text{in law.}$$

Proof. In [NRS15], we showed that $S_{\epsilon_n}(\mathcal{U}^{b_n,\kappa_n})$ converges to $\mathcal{N}^{b,\kappa}$ in law. However, a closer look at the proof shows that the result was shown by constructing a coupling between $\{(S_{\epsilon_n}(\mathcal{U}^{b_n}), S_{\epsilon_n}(\mathcal{U}^{b_n,\kappa_n}))\}_{n>0}$ and $(\mathcal{N}^b, \mathcal{N}^{b,\kappa})$ such that $S_{\epsilon_n}(\mathcal{U}^{b_n})$ converges to $\mathcal{N}^b$ a.s., and such that under this coupling, the convergence of the second discrete coordinate to its continuum counterpart holds in probability. This allows to extend the invariance principle derived in [NRS15] to Theorem 2.6. □

3 The continuum voter model perturbation (CVMP)

As described in the introduction, we aim at constructing the CVMP as a map from $\mathbb{R} \times \mathbb{R}^+$ onto finite subsets of $\{1, ..., q\}$. This "quasi-coloring" will be defined in terms of the duality relation given in Section 1.4, i.e, we see the CVMP as dual to the Brownian net with killing. The key observation is that even if the continuous graph starting from a "generic" point z is infinite (what we mean by generic will be discussed below in more

details), the reduced version of this graph is locally finite. (Recall that a reduced graph is constructed after removal of all the non-relevant separation points in the original graph — see Definition 1.1).

We now give an outline of the present section. In Section 3.1, we introduce the notion of relevant separation points and of reduced graph for the Brownian net with killling and prove some of their properties. Those results are simple extensions of the results already known in the case $\kappa = 0$ [SSS09, SSS14]. In Section 3.2, we construct the CVMP. In Section 3.3, we prove properties 1 and 2 listed in Theorem 1.1.

3.1 The reduced graph

We first define a notion of graph isomorphism adapted to our problem.

Definition 3.1. Let $G = (V, E)$ and $G' = (V', E')$ be two finite directed acyclic graphs. We will say that G and G' are isomorphic if there exists a bijection ψ from V to V' such that for every $z_1, z_2 \in V$

$$(z_1, z_2) \in E \ \text{ iff } (\psi(z_1), \psi(z_2)) \in E'. \tag{3.1}$$

In other words, two graphs will be isomorphic if they coincide modulo some relabelling of the nodes.

Let us consider the set of all finite directed acyclic graphs whose vertices are labelled by points in $\mathbb{R}^2$. The graph isomorphism property is an equivalence relation partitioning this set of graphs into equivalence classes. In the following, we will denote by $(\Gamma, \mathcal{G})$ the set of all equivalence classes endowed with its natural σ field. In this section, we construct a natural reduced graph generated by the Brownian net with killing paths (starting from a point z until a certain time horizon T) as an element of $(\Gamma, \mathcal{G})$.

As in the standard Brownian net, in the Brownian net with killing, a separation point (or branching point) will refer to a point $z = (x, t)$ such that there exist two paths π_r and $\pi_l \in \mathcal{N}^{b,\kappa}$ with $\sigma_{\pi_l}, \sigma_{\pi_r} < t$ and $e_{\pi_l}, e_{\pi_r} > t$ so that (1) $\pi_l(t) = \pi_r(t) = x$ and (2) there exists $\epsilon > 0$ so that $\pi_l(s) < \pi_r(s)$ for every $s \in (t, \min(t + \epsilon, e_r, e_l)]$. We note that in the coupled pair $(\mathcal{N}^b, \mathcal{N}^{b,\kappa})$ the separation points of the killed Brownian net and the ones of the standard Brownian net coincide a.s..

Definition 3.2 (Relevant Separation Point)**.** A point $z = (x, t) \in \mathbb{R}^2$ with $t \in (S, T)$ is said to be an (S, T)-relevant separation point of the Brownian net $\mathcal{N}^{b,\kappa}$, if and only if there exist two paths $\pi_1, \pi_2 \in \mathcal{N}^{b,\kappa}$ with starting time $\sigma_{\pi_1} = \sigma_{\pi_2} = S$ and ending times $e_{\pi_1}, e_{\pi_2} > t$, and such that

$$\pi_1(t) = \pi_2(t) \ \text{ but } \ \pi_1(u) \neq \pi_2(u) \ \text{ for } \ u \in (t, \inf(T, e_1, e_2)). \tag{3.2}$$

In the following, we will denote by $R^\kappa(S, T)$ the set of (S, T)-relevant separation points in the Brownian net with killing $\mathcal{N}^{b,\kappa}$ (with no b superscript in $R^\kappa(S, T)$ to ease the notation). Finally, $R(S, T)$ will denote the same set when $\kappa = 0$ (Proposition 6.1. in [SSS14]).

Proposition 3.1. For every $S < T$, the set $R^\kappa(S, T)$ is locally finite a.s.

Proof. As already mentioned, this property for $\kappa = 0$ has been established in [SSS14]. It remains to extend the property for $\kappa \neq 0$. Let $E \subset \mathbb{R}^2$ be a bounded set and let us show that $R^\kappa(S, T) \cap E$ is finite a.s..

Let $L > 0$ and define B_L to be the set of realizations such that

$$\{\pi \in \mathcal{N}^b : \sigma_\pi = S, \ trace(\pi) \cap E \neq \emptyset\} \subset \{\pi \in \mathcal{N}^b : \sigma_\pi = S, \ \forall t \in [S, T], \ -L < \pi(t) < L\}.$$

In particular, the equicontinuity of the paths in $\mathcal{N}^b$ implies that $\mathbb{P}(B_L)$ goes to 1 as $L \to \infty$.

Next, we inductively define $\theta^0 \equiv \theta^{0,L} = S$ and a sequence of killing times $\{\theta^i \equiv \theta^{i,L}\}_{i \geq 1}$ as follows:

$$\forall i \geq 0, \ \theta^{i+1} \ = \ \inf\{t > \theta^i : \exists (x,t) \in \mathcal{M}^\kappa\left(\theta^i\right) \text{ and } |x| < L\}. \tag{3.3}$$

where $\mathcal{M}^\kappa(t)$ is the set of killing points attained by the set of paths $\pi \in \mathcal{N}^{b,\kappa}$ with $\sigma_\pi = t$. From the very definitions of the θ^i's, it is not hard to see that

$$\text{conditionally on } B_L, \ R^\kappa(S,T) \cap E \ \subset \ \cup_{i \geq 0} R(\theta^i \wedge T, \ \theta^{i+1} \wedge T). \tag{3.4}$$
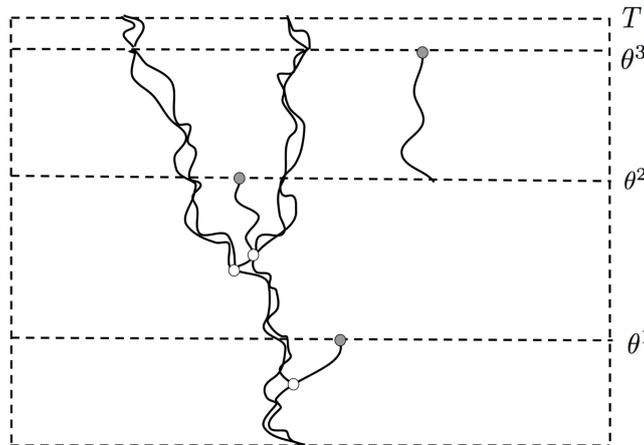
(See also Fig. 12.)

Figure 12: Grey circles represent points of the form $(x, \theta^i) \in \mathcal{M}^\kappa$. The (S,T)-relevant separation points (white circles) hit by a path starting from a given point in $(-L, L) \times \{S\}$ and staying in the box $(-L, L) \times [S, T]$ belong to $\cup_{i \geq 0} R(\theta^i \wedge T, \ \theta^{i+1} \wedge T)$.

We now claim that

$$\text{for every } i, \ R(\theta^i \wedge T, \theta^{i+1} \wedge T) \text{ is locally finite.} \tag{3.5}$$

In order to prove (3.5), it is enough to prove that (a) when $\theta^i < T$, $R(\theta^i, T)$ is locally finite, and (b) $R(\theta^i, \theta^{i+1})$ is also locally finite. For property (a), we first note that $R(U, T)$ is a.s. locally finite for every deterministic U with $U < T$, implying that the set of times U for which $R(U, T)$ is locally finite has full Lebesgue measure. Combining this with the definition of the random intensity measure used in our Poisson marking (see Definition 2.2) and using Fubini's theorem show that property (a) must hold.

For Property (b), the strong Markov Property and stationarity in time show that it is enough to prove the result for the special value $i = 0$. For any deterministic times U, we know that $R(S, U)$ is locally finite a.s.. Reasoning as in the previous paragraph completes the proof of (b), and thus (3.5).

We are now ready to complete the proof of Proposition 3.1. First of all, $\mathcal{M}^\kappa(S)$ is locally finite by Proposition 2.5. Thus we get that $\theta^1 - \theta^0 > 0$ a.s.. Furthermore, using the strong Markov property of the standard Brownian net and its stationarity in time, the sequence $(\theta^{i+1} - \theta^i)$ is a sequence of i.i.d. strictly positive random variables. As a consequence, for every realization of $\mathcal{N}^{b,\kappa}$, the sequence $\{R(\theta^i \wedge T, \theta^{i+1} \wedge T)\}_{i \geq 0}$ is empty after some finite value of i. Our proposition then immediately follows from (3.4) and (3.5) and the fact that $\mathbb{P}(B_L)$ goes to 1 as L goes to ∞. $\qquad \square$

Define

$$
\begin{aligned}
L^{\kappa}(S,T) &= \{(x,T) \ : \ \exists \pi \in \mathcal{N}^{b,\kappa}, \ \text{s.t.} \ \sigma_{\pi} = S, \ e_{\pi} \geq T, \ \pi(T) = x\}, \\
K^{\kappa}(S,T) &= \mathcal{M}^{\kappa}(S) \bigcap \mathbb{R} \times [S,T].
\end{aligned}
\tag{3.6}
$$

Definition 3.3 (Continuous Reduced Graph). Let $S < T$ be deterministic or random. Let $z = (x,S)$ and let $V^{z}(T)$ be the union of the singleton z with the subset of vertices $L^{\kappa}(S,T) \bigcup K^{\kappa}(S,T) \bigcup R^{\kappa}(S,T)$ reached by a path in $\mathcal{N}^{b,\kappa}$ starting from z.

1. A point $z = (x,S)$ will be said to be a T-finite point iff $|V^{z}(T)| < \infty$.
2. For every T-finite point $z = (x,S)$, the reduced graph starting from z on the time interval $[S,T]$ — denoted by $G^{z}(T)$ — is the random directed graph whose vertices are given by $V^{z}(T)$ and whose edges are generated by the Brownian net with killing $\mathcal{N}^{b,\kappa}$: two points $z' = (x',t'), z'' = (x'',t'') \in V^{z}(T)$, will be connected by a (directed) edge iff $t' < t''$ and there exists $\pi \in \mathcal{N}^{b,\kappa}$ such that $\sigma_{\pi} = t', \pi(t') = x', \pi(t'') = x''$ and
$$
\forall u \in (t',t''), \ (\pi(u),u) \notin L^{\kappa}(S,T) \bigcup K^{\kappa}(S,T) \bigcup R^{\kappa}(S,T).
$$

Proposition 3.2. Let $S < T$ be two deterministic times. Almost surely for every realization of the Brownian net with killing, every point (deterministic or random) in $\mathbb{R} \times \{S\}$ is T-finite.

Proof. By equicontinuity of the paths in $\mathcal{N}^{b,\kappa}$, we only need to show that

$$
L^{\kappa}(S,T) \bigcup K^{\kappa}(S,T) \bigcup R^{\kappa}(S,T)
$$

is locally finite a.s.. First, $R^{\kappa}(S,T)$ is locally finite using Proposition 3.1, and $\mathcal{M}^{\kappa}(S)$ is also locally finite by Proposition 2.5. Finally, when $\kappa = 0$, $L^{0}(S,T)$ is locally finite by Proposition 2.2. Since $L^{\kappa}(S,T)$ is stochastically dominated by $L^{0}(S,T)$, this ends the proof of Proposition 3.2. □

3.2 Construction of the continuum voter model perturbation

In this section, we construct the CVMP alluded to in Theorem 1.1. It will depend on following quantities (1) the branching and killing parameters b, κ, (2) the boundary nucleation mechanism $\{g_{i,j}\}_{i,j \leq q}$, (3) two probability distributions p and λ on $\{1, \cdots, q\}$, where p is the bulk nucleation mechanism and λ is the analog of the one-dimensional marginal of the initial distribution for the discrete voter model perturbation.

Recall that in order to determine the color of a vertex (x,t) in our discrete voter model perturbation $\{\theta^{\epsilon}(x,t)\}_{(x,t) \in \mathbb{Z}^{2}_{odd}}$, we start with a random coloring of the leaves of the graph induced by the backward BCK, starting from (x,t) as a root. The color of (x,t) is then determined by applying the algorithm described in Section 1.4 to the reduced graph rooted at (x,t) (see Proposition 1.6).

The backward Brownian net with killing. As in the discrete case, the backward Brownian net with killing $\hat{\mathcal{N}}^{b,\kappa}$ is obtained from the Brownian net with killing by a reflection about the x-axis. In the following, we will use identical notation for the backward Brownian net with killing, adding only a hat sign to indicate that we are dealing with the backward object. For instance $\hat{R}^{\kappa}(S,T)$ will refer to the (S,T)–relevant separation points for the backward net between times S and T.

Step 0. Assignment of i.i.d. random variables. In the continuum, we consider a realization of the backward Brownian net with killling $\hat{\mathcal{N}}^{b,k}$ and consider the random set of points consisting of the union of

1. all the separation points in $\hat{\mathcal{N}}^{b,\kappa}$,
2. all the killing points in $\hat{\mathcal{N}}^{b,\kappa}$,
3. the set $\{(x,0) \; : \; \exists \hat{\pi} \in \hat{\mathcal{N}}^{b,\kappa} \text{ s.t. } \sigma_{\hat{\pi}} > 0 \text{ and } \hat{\pi}(0) = x\}$

Those three sets are countable for almost every realization of the Brownian net with killing for the following reasons:

1. First, for any separation point z, there exists $p < q \in \mathbb{Q}$ such that $z \in \hat{R}^{\kappa}_{p,q}$. From Proposition 3.1, $\hat{R}^{\kappa}_{p,q}$ is locally finite a.s., which implies that the set of separation points is countable a.s.
2. Since, conditional on $\mathcal{N}^b$, the set of killing points is defined as a Poisson Point Process, this set is also countable a.s..
3. Finally, for the third set, we note that

$$\{(x,0) \; : \; \exists \hat{\pi} \in \hat{\mathcal{N}}^{b,\kappa} \text{ s.t. } \sigma_{\hat{\pi}} > 0 \text{ and } \hat{\pi}(0) = x\} \; = \; \cup_{s>0, s \in \mathbb{Q}} \; \hat{\xi}^s(0)$$

where $\hat{\xi}^s(0)$ is the backward version of the branching-coalescing point set as defined in Proposition 2.2. Since $\hat{\xi}^s(0)$ is locally finite for every deterministic s, the third item is also countable.

Finally, since the random set consisting of the union of the sets 1, 2 and 3 above is a.s. countable, given a realization of $\hat{\mathcal{N}}^{b,\kappa}$, i.i.d. uniformly distributed random variables U_z in $[0,1]$ can be assigned to each of the points z in the three sets described above.

Step 1. Pre-coloring of the leaves. We consider the set of leaves, consisting of the killing points and the intersection points of $\hat{\mathcal{N}}^{b,\kappa}$ with $\{t = 0\}$. For each such point z, we deduce the pre-coloring of z, denoted by $\bar{\theta}(z)$, from the variable U_z according to the following rule.

- If the leaf z is an intersection point between a path from $\hat{\mathcal{N}}^{b,\kappa}$ and $\{t = 0\}$ (i.e. a point from set 3), $\bar{\theta}(z)$ is the unique integer in $\{1, \cdots, q\}$ such that

$$U_z \in [\sum_{i \le \bar{\theta}(z)-1} \lambda(i), \sum_{i \le \bar{\theta}(z)} \lambda(i))$$

with the convention that $\lambda(0) = 0$.
- If the leaf z is a killing point in $\hat{\mathcal{N}}^{b,\kappa}$ (i.e. from set 2), $\bar{\theta}(z)$ is the unique integer in $\{1, \cdots, q\}$ such that

$$U_z \in [\sum_{i \le \bar{\theta}(z)-1} p(i), \sum_{i \le \bar{\theta}(z)} p(i))$$

with the convention that $p(0) = 0$.

Step 2. Pre-coloring of the "nice" points. Given a point (x,S) and a time $T > S$, recall the definition of the reduced graph $G^{(x,S)}(T)$ in the forward Brownian net with killing $\mathcal{N}^{b,\kappa}$ (see Definition 3.3). Similarly, for any $S > 0$, one can define a reduced graph $\hat{G}^{(x,S)}(0)$ whenever the point (x,S) is 0-finite in the backward Brownian net with killing.

Definition 3.4. A point is "nice" if it is a 0-finite point for the backward Brownian net $\hat{\mathcal{N}}^{b,\kappa}$.

In particular, if we consider $\mathcal{D}$, a countable deterministic dense set of $\mathbb{R} \times \mathbb{R}^+$, Proposition 3.2 (in its backward formulation) ensures that all the points in $\mathcal{D}$ are nice for almost every realization of the Brownian net with killing.

Definition 3.5. A graph is said to be simply rooted if the out-degree of the root is 1.

For nice points with a simply rooted graph, we assign the color determined by the coloring algorithm described in Section 1.4 using the family of distributions $\{g_{i,j}\}$ and the U_z's at separation points (as in (1.5)). For every such point, this defines a color that we denote by $\bar{\theta}(z)$.

For graphs with more complicated root structure (i.e. with the root connected to more than one point), we decompose the finite graph into simply rooted graphs. (For instance, if the root connects to two distinct points z' and z'', the first component is the subgraph whose set of vertices is the union of the root with the vertices that can be accessed from z'; the second component is defined analogously.) $\bar{\theta}(z)$ is then defined as the union of the coloring induced by each of the simply rooted subgraphs in the decomposition.

Remark 3.3. By construction, $|\bar{\theta}(z)|$ is less or equal to the out-degree of the root.

Step 3. Horizontal Limits. The coloring of the half-plane $\mathbb{R} \times \mathbb{R}^+$ is generated by taking horizontal limits. Our procedure is based on the following lemma.

Lemma 3.1. Almost surely for every realization of the backward Brownian net with killing, for every time t (deterministic or not) the set $\{x \; : \; (x,t)$ is nice$\}$ is dense in $\mathbb{R}$.

Proof. Let us consider a dense countable deterministic set $\mathcal{D} \subset \mathbb{R} \times \mathbb{R}^+$. By Proposition 3.2 in its backward formulation, every point in $\mathcal{D}$ is nice a.s.. Furthermore, any point z' belonging to the trace of $\hat{\mathcal{N}}^{b,k}(\mathcal{D})$ (the backward Brownian net with killing paths starting from $\mathcal{D}$), must also have a finite graph representation, since every relevant separation point for z' is also relevant for some $z \in \mathcal{D}$, and every leaf for z' is also a leaf for z. Hence, for every every realization of the Brownian net, for every t (random or deterministic) it is sufficient to prove that $\hat{\mathcal{N}}^{b,\kappa}(\mathcal{D})$ intersects the set $\mathbb{R} \times \{t\}$ at a dense set of points.

For $\kappa = 0$, the property directly follows from a property of the Brownian web: the so-called skeleton $\hat{\mathcal{W}}_r(\mathcal{D})$ (and also $\hat{\mathcal{W}}_l(\mathcal{D})$) intersect every horizontal time $\mathbb{R} \times \{t\}$ at a dense set of point. (The contrary would easily contradict the equicontinuity of the web paths.) A fortiori, the net with no killing $\hat{\mathcal{N}}^b$ enjoys the same property. Finally, this extends to the case $\kappa > 0$, by noting that our Poisson construction of the killing points can not charge any horizontal line $\mathbb{R} \times \{t\}$ with more than one point. $\qquad\square$

For every point $(x,t) \in \mathbb{R} \times \mathbb{R}^+$ (including the nice points described previously), we define the colors at point (x,t) as

$$\theta(x,t) \; = \; \{c \in \{1,\cdots,q\} \; : \; \exists\{(x_n,t)\} \text{ "nice" s.t. } x_n \to x \text{ and } \forall n, \; c \in \bar{\theta}(x_n,t)\} \quad (3.7)$$

From Lemma 3.1, $\theta(x,t)$ is well defined. Further, for every nice point $z \in \mathbb{R} \times \mathbb{R}^+$, we must have $\bar{\theta}(z) \subseteq \theta(z)$.

Remark 3.4. It is natural to ask whether one could define the CVMP by taking limits in any directions. It is not hard to show that for every deterministic (x,t), and every color c, there exists a sequence (x,t_n) with $t_n \to t$ such that $\bar{\theta}(x,t_n) = c$. As a consequence, if we allowed for vertical limits, every deterministic point will be trivially colored with $\{1,\cdots,q\}$. This property was established in the case where the boundary noise is absent (Theorem 1.1.(7) in [FINR06])

3.3 Properties of the CVMP

We will now prove some basic properties of our mapping (Theorem 1.1(1)–(2)). In order to do so, we first elaborate on the properties of the reduced graph at deterministic times.

3.3.1 Reduced graphs at deterministic times

In this subsection, we derive some properties of the reduced graph at deterministic times. We formulate these results "forward in time" – i.e., for the original Brownian net, and not its backward version. When needed (in the forthcoming sections), these results will be used in their backward formulation.

Before stating the main proposition of this section (see Proposition 3.5 below), we start with some definitions. Let z_0 and z_1 be two T-finite points. We will say that $G^{z_0}(T)$ and $G^{z_1}(T)$ only differ by their root iff they are identical up to relabeling of their root, i.e., when the mapping ψ, defined by

$$\forall z \in V^{z_0}(T), \quad \psi(z) = \left\{ \begin{array}{ll} z & \text{if } z \neq z_0 \\ z_1 & \text{otherwise} \end{array} \right. ,$$

is a graph isomorphism from $G^{z_0}(T)$ to $G^{z_1}(T)$ (as in Definition 3.1).

Before stating our next result, we recall from Proposition 2.3 that for every deterministic S, a.s. for every realization of the Brownian net with killing, every point (x, S) is either of type $(o, p), (p, p)$ or (o, pp).

Proposition 3.5. (Reduced Graph at Deterministic Times) Let S, T be a pair of deterministic times with $S < T$. For almost every realization of the Brownian net with killing, every $z_0 = (x_0, S)$ is T-finite (with x_0 random or deterministic) and the following properties hold:

1. If $z_0 = (x_0, S)$ is of type (o, p) or (p, p) then $G^{z_0}(T)$ is a simply rooted graph. In particular, for any deterministic point z_0, $G^{z_0}(T)$ is simply rooted.
2. If $z_0 = (x_0, S)$ is of type (o, pp), then the root z_0 of the graph $G^{z_0}(T)$ is either of out-degree 1 or 2.
3. Every intermediate node in $G^{z_0}(T)$ has out-degree 2.
4. The set
$$S_2(S, T) := \{x_0 \ : \ G^{(x_0, S)}(T) \text{ is not simply rooted } \} \tag{3.8}$$
   is a locally finite set and is made of (o, pp) points. Further, between two consecutive points of $S_2(S, T)$, the graphs only differ by their root. I.e., if x, y are such that
$$x_1 < x \leq y < y_1,$$
   where x_1 and y_1 are two consecutive points in $S_2(S, T)$, then $G^{(x,S)}(T)$ and $G^{(y,S)}(T)$ only differ by their root.

The proof of this result relies on some properties of the special point of the Brownian net. Let $(r, l) \in (\mathcal{W}_r, \mathcal{W}_l)$ and let z, z' be two vertices in the graph $G^{z_0}(T)$. We will write

$$z := (x, t) \rightarrow_{l,r} z' := (x', t')$$

iff $t < t'$, $l \sim_{out}^{z} r$ and $l \sim_{in}^{z'} r$, and z and z' are connected by the paths l and r, in the sense that l and r do not encounter any element of $V^{z_0}(T)$ in the time interval (t, t'). Among other things, the following technical lemma will allow to relate the out-degree of a root to its type (see Corollary 3.1).

Lemma 3.2. Let $S < T$ be two deterministic times. For almost every realization of the Brownian net with killing, for every $z_0 = (x_0, S)$ (with x_0 random or deterministic), the following properties hold.

1. Suppose $z \in V^{z_0}(T)$ is not a leaf. For every equivalent outgoing pair (l, r) from z, there exists a unique $z' \in V^{z_0}(T)$ such that $z \rightarrow_{l,r} z'$.

2. For every $z, z' \in V^{z_0}(T)$ such that $z \to z'$, there exists a unique pair $(l, r) \in (\mathcal{W}_r, \mathcal{W}_l)$ such that $z = (x, t) \to_{l,r} z' = (x', t')$.

3. Let $z_- = (x_-, S)$ and $z_+ = (x_+, S)$ with $x_- \leq x_+$ such that there exist two distinct equivalent outgoing paths (l, r') and (l', r) starting respectively from z_- and z_+, with the convention that when $x_- = x_+$, (l, r') and (l', r) are ordered from left to right. Finally, let z_l and z_r such that

$$z_- \to_{l,r'} z_l \quad \text{and} \quad z_+ \to_{l',r} z_r.$$

Then $z_l \neq z_r$ if and only if l and r do not meet on $(S, U_{l,r'} \wedge U_{l',r}]$, where

$$U_{l,r'} := T \wedge \inf\{u \geq S \ : \ \exists \pi \in \mathcal{N}^{b,\kappa} \ \text{s.t.} \ \sigma_\pi = S, \ e_\pi = u, \ \text{and} \ l \leq \pi \leq r'\} \quad (3.9)$$

and $U_{l',r}$ is defined analogously using l' and r.

Proof. We start with the proof of item 1. Let $z := (x, t) \in V^{z_0}(T)$, for every equivalent outgoing pair (l, r) at z, define $U_{l,r}$ analogously to (3.9), i.e.,

$$U_{l,r} = T \wedge \inf\{u \geq t \ : \ \exists \pi \in \mathcal{N}^{b,k} \ \text{s.t.} \ \sigma_\pi = t, \ e_\pi = u, \ \text{and} \ l \leq \pi \leq r\}.$$

Finally, define

$$\tau_{l,r} := \sup\{u \in [t, U_{l,r}] : l(u) = r(u)\} \quad (3.10)$$

to be the last time l and r separate before time $U_{l,r}$. We first claim that the point

$$z' := (l(\tau_{l,r}), \tau_{l,r}) = (r(\tau_{l,r}), \tau_{l,r}) \quad (3.11)$$

belongs to $V^{z_0}(T)$ and that $r \sim_{in}^{z'} l$. In order to prove this claim, we distinguish two cases.

Let us first assume that $\tau_{l,r} = U_{l,r}$. There are only two possibilities: If $U_{l,r} = T$, then z' is a leaf and must be of type (p, p) by Proposition 2.3(4); if $U_{l,r} \neq T$, the point z' must be a killing point (ands thus a leaf) which is also of type (p, p) (this follows directly from the structure of the intensity measure (2.1) used in the Poissonian construction of the killing points). Hence, when $\tau_{l,r} = U_{l,r}$, the point z' (as defined in (3.11)) is always a leaf of $V^{z_0}(T)$ and is always of type (p, p), which implies that $l \sim_{in}^{z'} r$.

Let us now assume that $\tau_{l,r} < U_{l,r}$. It is not hard to show that z' is an (S,T)-relevant separation point in $G^{z_0}(T)$. Since separation points are of type (p, pp) (by Proposition 2.4(1)), z' is a point of $V^{z_0}(T)$ (a separation point) with the property $l \sim_{in}^{z'} r$.

In the two previous paragraphs, we showed that the point $z' = (l(\tau_{l,r}), \tau_{l,r})$ belongs to $V^{z_0}(T)$ and that $l \sim_{in}^{z'} r$. Furthermore, the definition of $\tau_{l,r}$ directly implies that r and l do not encounter any relevant separation point or any killing point before entering the point z'. Combining those two facts implies that $z \to_{l,r} (l(\tau_{l,r}), \tau_{l,r})$. This ends the proof of item 1.

We now proceed with the proof of the second item of Lemma 3.2. Let $z, z' \in V^{z_0}(T)$ be such that $z \to z'$. Let $\pi \in \mathcal{N}^b$ be a path connecting z and z'. z is not a leaf and thus it can either be (1) a separation point, or (2) a root. In case (1), Proposition 2.4 implies that π is squeezed between a unique equivalent pair (l, r) such that $l \sim_{out}^z r$. In case (2), since we are considering the case of a reduced graph rooted at a deterministic time, the same property must hold by Proposition 2.3. Let $\tau_{l,r}$ be defined as in (3.10). By definition of $\tau_{l,r}$, π can not access any point of $V^{z_0}(T)$ on the interval $(t, \tau_{l,r})$. Since we proved that $(l(\tau_{l,r}), \tau_{l,r}) \in V^{z_0}(T)$, z' must coincide with $(l(\tau_{l,r}), \tau_{l,r})$. Finally, since we showed that $z \to_{l,r} (l(\tau_{l,r}), \tau_{l,r})$, this ends the proof of item 2.

We now proceed with the proof of the third item of Lemma 3.2. The existence of z_l and z_r is guaranteed by item 1 of Lemma 3.2. Further, from what we showed earlier, $z_l = (l(\tau_{l,r'}), \tau_{l,r'})$ and $z_r = (r(\tau_{l',r}), \tau_{l',r})$ and thus

$$z_l = z_r \iff \tau_{l,r'} = \tau_{l',r}, \ \text{and} \ l(\tau_{l,r'}) = r(\tau_{l',r}).$$
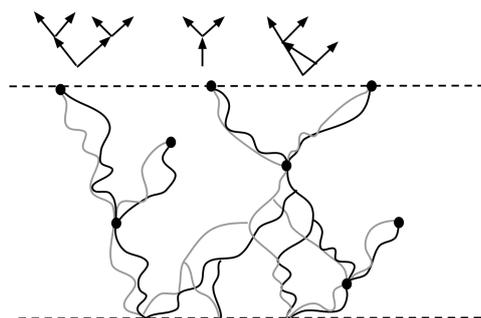
Figure 13: Three reduced graphs generated from the net paths starting from points on the line $\mathbb{R} \times \{S\}$ (with time horizon T). The leftmost and rightmost graphs are generated from two consecutive points in $S_2(S, T)$. Grey (resp., black) paths represent paths from $\mathcal{W}_l$ (resp., $\mathcal{W}_r$).

Using the non-crossing property of the Brownian web paths (in $\mathcal{W}_l$ or $\mathcal{W}_r$), if l and r meet before time $U_{l,r'} \wedge U_{l',r}$, then the condition on the RHS of the former equivalence is satisfied. Conversely, if the condition on the RHS of the former equivalence is satisfied, it is clear that l and r meet before time $U_{l,r'} \wedge U_{l',r}$. This ends the proof of Lemma 3.2. □

As a direct corollary of Lemma 3.2(1) and (2), we deduce the following result.

Corollary 3.1. Let $S < T$ be two deterministic times. For almost every realization of the Brownian net with killing, for every $z_0 = (x_0, S)$ (with x_0 random or deterministic), the out-degree of a point $z \in V^{z_0}(T)$ is at most equal to the number of pairs $(l, r) \in (\mathcal{W}_l, \mathcal{W}_r)$ such that $l \sim_{out}^z r$.

Proof of Proposition 3.5. The fact that z_0 is T-finite is the content of Proposition 3.2. We now turn to the proof of item 1. By Proposition 2.3, for any point of type (o, p) or (p, p), there exists a unique pair l, r such that $l \sim_{out}^{z_0} r$. By Corollary 3.1, this proves that the degree of z_0 is at most 1. Since z_0 is a.s. not a killing point, the out-degree is at least 1, thus showing that the finite graph representation at any point of type (o, p) or (p, p) is simply rooted. Finally, since every deterministic point is a.s. of type (o, p) (by Proposition 2.3), the same property must hold at any deterministic point a.s.. This ends the proof of item 1.

We now turn to the proof of items 2 and 3. We first claim that any node of type $(*, pp)$ — where * can be either be o (for roots of type (o,pp)) or 1 (corresponding to separation point by Proposition 2.3 and Proposition 2.4) — connects to one or two points. By definition of a point of type $(*, pp)$, there exists exactly two distinct pairs (l, r') and (l', r) in $(\mathcal{W}_l, \mathcal{W}_r)$ starting from z such that $l \sim_{out}^z r'$ and $r' \sim_{out}^z l$. By Corollary 3.1, the out-degree is at most two, thus proving item 2. Further, by definition of a relevant separation point, the out-degree for such a point is at least two, and thus the out-degree of a relevant separation point is exactly equal to 2. This ends the proof of item 3.

It remains to show item 4. The fact that $S_2(S, T)$ is a set of (o, pp) points follows from Corollary 3.1 and the fact that at deterministic times, points are either of type $(o, p), (p, p)$ or (o, pp). Next, we show that $S_2(S, T)$ is locally finite. Lemma 3.2(3) (in the special case $x_- = x_+$) easily implies that if x is in $S_2(S, T)$ then there exists a time t_x — either a killing time for a path starting from time S, or $t_x = T$ — such that the rightmost and leftmost paths l, r starting from (x, S) do not meet before time t_x.

Let $L > 0$. Since $\mathcal{M}^\kappa(S)$ is locally finite (by Proposition 2.5), the previous argument and the equicontinuity of the net paths imply that there exists a rational number $q > 0$

such that $[-L, L] \cap S_2(S, T)$ is a subset of

$$\{x : \text{the leftmost and rightmost path in } \mathcal{N}^b \text{ starting from } (x, S) \text{ do not meet on } (S, S+q)\};$$
(3.12)

e.g., take $q < \inf\{s > S : (y, s) \in \mathcal{M}^\kappa \text{ and } \exists \pi \in \mathcal{N}^b, \sigma_\pi = S, \pi(S) \in [-L, L], \pi(s) = y\}$. Since a.s. for every realization of the net, the set (3.12) is locally finite for every deterministic q, this proves that $[-L, L] \cap S_2(S, T)$ is finite a.s. and thus, $S_2(S, T)$ is locally finite. (This can easily be seen as a direct consequence from the fact that the dual branching-coalescing point set (the dual branching-coalescing point set being defined from the dual Brownian net as defined in [SS08]) starting from $t + q$ is finite at time t by Proposition 2.2.)

To conclude the proof of item 4, it remains to show that $G^{z_-}(T)$ and $G^{z_+}(T)$ are identical (up to some relabeling of the roots) if $z_- = (x_-, S)$ and $z_+ = (x_+, S)$ are located (strictly) between two consecutive elements of $S_2(S, T)$. We give an argument by contradiction and assume that the roots of $G^{z_-}(T)$ and $G^{z_+}(T)$ do not connect to the same point (i.e., $z_l \neq z_r$). We now show that this implies the existence of a point $z_0 = (x_0, S)$ between z_- and z_+ (i.e. $x_- \leq x_0 \leq x_+$) that connects to two distinct points.

Since $[x_-, x_+] \cap S_2(S, T) = \emptyset$, we can partition the segment $[x_-, x_+]$ into equivalence classes, where x_1, x_2 are equivalent iff they connect to the same vertex (i.e., to the same element of $L^\kappa(S, T) \cup R^\kappa(S, T) \cup K^\kappa(S, T)$ as defined in (3.6)). Since $L^\kappa(S, T) \cup R^\kappa(S, T) \cup K^\kappa(S, T)$ is locally finite, the equicontinuity of the net paths implies that the number of classes is finite. From there, it is straightforward to check that there exists $x_0 \in [x_-, x_+]$, z_1 and z_2 two elements in $L^\kappa(S, T) \cup R^\kappa(S, T) \cup K^\kappa(S, T)$, and two sequences of net paths $\{\pi_1^n\}$ and $\{\pi_2^n\}$ with $\sigma_{\pi_i^n} = S, i = 1, 2$ such that

$$(\pi_1^n(S), S) \text{ connects to } z_1, \ (\pi_2^n(S), S) \text{ connects to } z_2 \ \text{ and } \ \lim_{n \to \infty} \pi_1^n(S), \pi_2^n(S) = x_0.$$

For $i = 1, 2$, going to a subsequence if necessary, π_i^n converges to a net path π_i^∞ starting from (x_0, S) and passing trough the point z_i. We will now show that π_i^∞ connects z_0 to z_i (in order to get the desired contradiction). In order to do so, it is sufficient to prove that π_i^∞ does not encounter any element in $L^\kappa(S, T) \cup R^\kappa(S, T) \cup K^\kappa(S, T)$ before entering z_i.

Again, we argue by contradiction. Let us assume that π_i^∞ connects (x_0, S) to a point $\bar{z} = (\bar{x}, \bar{t})$ before entering z_i. Since $z_0 \to \bar{z}$, by the second item of Lemma 3.2, there must exist a left-right pair (l, r) such that $l \leq \pi_i^\infty \leq r$ on $[0, \bar{t}]$. Let q be any rational number in $(S, \bar{t})$. By Proposition 2.2, the set $\xi^S(q)$ is locally finite a.s., and since π_1^n converges to π_1^∞, $\pi_1^n(q)$ and $\pi_1^\infty(q)$ coincide for n large enough. This implies that π_1^n is "trapped" between l and r after a finite rank, i.e., that $l \leq \pi_i^n \leq r$ on $[q, \bar{t}]$. (Here we used the property that a net path can not cross a left-most path from the right, and a rightmost path from the left – see [SS08]). This implies that π_i^n enters the point $\bar{z}$ for n large enough, which is the desired contradiction. This completes the proof of item 4 of Proposition 3.5. $\square$

Let us take a point x in $S_2(S, T)$. By Proposition 3.5, (x, S) must be of type (o, pp), and connects to two distinct points. By Lemma 3.2(3), there exist two points z', z'' with $z' \neq z''$, such that if (l, r') and (l', r) are the two distinct outgoing equivalent paths in $(\mathcal{W}_l, \mathcal{W}_r)$ exiting the point (x, S), ordered from left to right, then

$$(x, S) \to_{l,r'} z' = (x', t') \quad (x, S) \to_{l',r} z'' = (x'', t'').$$

One can always decompose $G^{(x, S)}$ into two simply rooted graphs, a left and a right component $G_l^{(x, S)}(T)$ and $G_r^{(x, S)}(T)$: We define $G_l^{(x, S)}(T)$ as the subgraph whose set of vertices is the union of (x, S) with the vertices of the (directed) subgraph originated from z'. $G_r^{(x, S)}(T)$ is defined analogously using z''. (Note that the two sub-graphs can share a lot of common vertices, and that this decomposition is not a partition of the vertices). For a pictorial representation of the next proposition, we refer the reader to Fig. 13.

Proposition 3.6. Let x_l, x, x_r be three consecutive points in $S_2(S, T)$ and let $G_l^{(x,S)}(T)$ and $G_r^{(x,S)}(T)$ be respectively the left and right component of the graph $G^{(x,S)}(T)$.

$$\forall y \in (x_l, x), \quad G^{(y,S)}(T) \text{ and } G_l^{(x,S)}(T) \text{ only differ by their root,}$$

$$\forall y \in (x, x_r), \quad G^{(y,S)}(T) \text{ and } G_r^{(x,S)}(T) \text{ only differ by their root.}$$

Proof. We only prove the first relation. The second relation can be proved along the same lines. In the following, as before, (l, r') and (l', r) will denote the two outgoing equivalent paths in $(\mathcal{W}_l, \mathcal{W}_r)$ exiting the point (x, S), ordered from left to right. In particular, it is not hard to see from Proposition 2.3, that l (resp., r) is the leftmost (resp., rightmost) path starting from (x, S).

Take $x_n \uparrow x$, and for every n, let (l_n, r_n) be an equivalent pair of outgoing paths starting from (x_n, S). By Proposition 3.5(4), all the points in (x_l, x) have the same finite graph representation, up to relabeling of their root. As a consequence, we need to prove that if $x_n \uparrow x$ then $G^{(x_n,S)}(T)$ and $G_l^{(x,S)}(T)$ coincide up to their root, after some rank, i.e., we need to show that $(x_n, S) \to_{(l_n, r_n)} (x', t')$ (after some rank), where $z' = (x', t')$ is the only point in $G_l^{(x,S)}(T)$ connected to (x, S).

According to Lemma 3.2(3), and since $U_{l_n, r'} \leq U_{l_n, r_n} \wedge U_{l, r'}$ (as defined in (3.9)), we need to prove that l_n and r' meet before time

$$U_{l_n, r'} = T \wedge \inf\{u \geq t : \exists \pi \in \mathcal{N}^{b,k} \text{ s.t. } \sigma_\pi = t, \ e_\pi = u, \text{ and } l_n \leq \pi \leq r'\}.$$

Let us first show that l_n coalesces with l at some time $\tau_n \to S$. By compactness of the Brownian web, l_n converges to some path $\bar{l} \in \mathcal{W}_l$ starting from (x, S). Since l_n does not cross the path l, we have $l_n \leq l$, implying that $\bar{l} \leq l$. Since l is the leftmost path starting from (x, S), this yields $\bar{l} = l$, and thus l_n converges to l. Using an argument analogous to the one given in the last paragraph of the proof of Proposition 3.5, it is not hard to show that l_n coalesces with l at some $\tau_n \to S$, as claimed earlier. Next, since r_n can not cross r', we have $r_n \leq r'$. This implies that for any $s \geq \tau_n$,

$$l(s) = l_n(s) \leq r_n(s) \leq r'(s).$$

On the other hand, since $l \sim_{out}^z r'$, we can choose a time $\tilde{t}$ arbitrarily close to S, such that $l(\tilde{t}) = r'(\tilde{t})$. In particular, for n large enough such that $\tau_n \leq \tilde{t}$, the previous inequality implies that l_n and r' must have met before time $\tilde{t}$. Finally, since the set of killing points $\mathcal{M}^\kappa(S)$ is locally finite, and since $\tilde{t}$ can be chosen arbitrarily close to S, we can choose n large enough such that l_n and r' meet before any path squeezed between those two paths encounters a killing point, i.e., before time $U_{l_n, r'}$. As explained earlier, by Lemma 3.2(3), this implies that for n large enough, the graphs $G^{(x_n,S)}(T)$ and $G_l^{(x,S)}(T)$ coincide modulo their root. This ends the proof of Proposition 3.6. □

3.3.2 Proof of Theorem 1.1 (1)–(2)

Let t be a deterministic time. Proposition 3.2 implies that for any point (x, t) (with x deterministic or random) is "nice" and has a well defined pre-coloring $\bar{\theta}(x, t)$.

Lemma 3.3. At any deterministic time t, for almost every realization of the CVMP, for every x (deterministic or random), the pre-coloring $\bar{\theta}(x, t)$ coincides with the coloring $\theta(x, t)$.

Proof. The key point of our proof is that if two graphs only differ by their root, they must have the same pre-coloring (by the very definition of the coloring algorithm as

defined in Section 1.4). We denote by $\hat{S}_2(0, t)$ the backward analog of $S_2(0, t)$ as defined in Proposition 3.5(4), i.e.

$$\hat{S}_2(0, t) = \{x : \hat{G}^{(x,t)}(0) \text{ is not simply rooted.}\}$$

We first show that Lemma 3.3 holds for $z = (x, t) \notin \hat{S}_2(0, T)$. Since $\hat{S}_2(0, T)$ is a.s. locally finite, Proposition 3.5(4) — in its backward formulation — implies that there exists $\delta > 0$ small enough s.t.

$$\forall x', \text{ s.t. } |x' - x| < \delta, \quad \hat{G}^{(x,t)}(0) \text{ and } \hat{G}^{(x',t)}(0) \text{ only differ by their root}$$

As a consequence, for any $x_n \to x$, $\bar{\theta}(x_n, t)$ is stationary after a certain rank, and thus

$$\theta(x, t) = \lim_n \bar{\theta}(x_n, t) = \bar{\theta}(x, t).$$

Let us now consider $z \in \hat{S}_2(0, T)$. First, by definition of the precoloring $\bar{\theta}(\cdot)$, we must have $\bar{\theta}(z) = \{c_l\} \cup \{c_r\}$, with c_l (resp., c_r) being the color given by our coloring algorithm applied on the graph $\hat{G}_l^{x,t}(0)$ (resp., $\hat{G}_r^{x,t}(0)$) — those two graphs being defined as in Proposition 3.6 in the backward context. (Note that those two colors are potentially equal.) Let us now consider a subsequence $x_n \to x$, with $x_n \neq x$. We can decompose this sequence into two subsequences $x_n^l < x$ and $x_n^r > x$. For the sequence x_n^l, Proposition 3.6, implies that for n large enough, $\bar{\theta}(x_n^l, t) = c_l$, while for the sequence x_n^r, we must have $\bar{\theta}(x_n^r, t) = c_r$. As a consequence, we get $\theta(z) = \bar{\theta}(z)$. $\qquad \square$

Theorem 1.1 (1)–(2) is then a direct consequence of the following proposition.

Proposition 3.7. At any deterministic time t, for almost every realization of the CVMP:

1. $|\theta(x, t)| = 1$ a.s., for any deterministic point (x, t). Furthermore, the coloring $\theta(x, t)$ is obtained by applying the coloring algorithm to the reduced graph starting from (x, t) (see Section 1.4 for a definition of the coloring algorithm).

2. For every x, $|\theta(x, t)| \leq 2$.

3. The set of points such that $|\theta(x, t)| = 2$ is locally finite. Moreover, the color between two consecutive points of this set remains constant.

Proof. The first item of Proposition 3.7 is a direct consequence of Proposition 3.5(1) and Lemma 3.3. For the second item, we first note that Proposition 3.5 implies that for any point (x, t) (with x deterministic or random), the out-degree of the root of $\hat{G}^{(x,t)}(0)$ is either 1 or 2. Furthermore, the definition of our pre-coloring implies $|\bar{\theta}(x, t)|$ is less or equal to the out-degree of the reduced $\hat{G}^{(x,t)}(0)$. The second item then follows by a direct application of Lemma 3.3.

We now turn to the third item of Proposition 3.7 Since the $\hat{G}^{(x,t)}(T)$'s for the x's between two consecutive point of $\hat{S}_2(0, T)$ only differ by their root, $\bar{\theta}(x, t)$ (and hence $\theta(x, t)$) must remain constant between two elements of $\hat{S}_2(0, t)$. Since $\hat{S}_2(0, t)$ is locally finite (Proposition 3.5 (4)), this implies that the set of points such that $|\theta(x, t)| = 2$ is locally finite.

Finally, $|\bar{\theta}(x, t)| = 2$ (and hence $\theta(x, t)$) iff $(x, t) \in \hat{S}_2(0, t)$ and the colors induced by the graphs $\hat{G}_l^{(x,t)}(0)$ and $\hat{G}_r^{(x,t)}(0)$ differ (where $\hat{G}_l^{(x,t)}(0)$ and $\hat{G}_r^{(x,t)}(0)$ are defined as in Proposition 3.6). By Proposition 3.6, for every $x_n^l \uparrow x$ (resp., $x_n^r \downarrow x$) the color $\bar{\theta}(x_n^l, S)$ (resp., $\bar{\theta}(x_n^r, S)$) must coincide with the color induced by $\hat{G}_l^{(x,t)}(0)$ (resp., $\hat{G}_r^{(x,t)}(0)$) after a certain rank. This implies that $|\theta(x, t)| = 2$ iff x delimits two color intervals. $\qquad \square$

4 Convergence to the continuum model. Proof of Theorem 1.1 (3)

Let $\{\epsilon_n\}$ be a sequence of positive numbers such that $\epsilon_n \to 0$. As in Theorem 1.1, we assume that there exists $b, \kappa \geq 0$, and two positive sequences $\{b_n\}$ and $\{\kappa_n\}$ such that $b_n/\epsilon_n \to b$ and $\kappa_n/\epsilon_n^2 \to \kappa$. We also assume that the boundary and bulk mechanisms $\{g_{i,j}^n\}_{i,j\leq q}$ and p^n converge to a sequence of limiting probability distributions $\{g_{i,j}\}_{i,j\leq q}$ and p when $n \to \infty$.

Let $z_n^1, \cdots, z_n^k \in S_{\epsilon_n}(\mathbb{Z}_{odd}^2)$ be such that for every $i = 1, \cdots, k$, $\lim_{n\to\infty} z_n^i = z^i$ for some $z^i \in \mathbb{R}^2$. Our goal is to show that the distribution of the coloring of the points $(z_n^1, \cdots, z_n^k)$ for the discrete model labelled by n converges to the distribution of the coloring of $(z^1, \cdots, z^k)$ at the continuum.

By Proposition 3.7(1), the coloring of a deterministic point z in the CVMP is obtained by applying the coloring algorithm to the reduced graph $\hat{G}^z(0)$ (the backward of the object introduced in Definition 3.3). By Proposition 1.6, the color of a point in the discrete model can be recovered by applying the same procedure to the discrete backward net with killing.

From the previous observation, and since the same property holds at the discrete level (see Proposition 1.6), our convergence result easily follows from the convergence (in law) of $\cup_{i=1}^k \hat{G}_n^{z_n^i}(0)$ to $\cup_{i=1}^k \hat{G}^{z^i}(0)$, where $\hat{G}_n^{z_n^i}(0)$ is the backward reduced graph starting from z_n^i with time horizon 0 and constructed from the (backward) discrete net with killing $S_{\epsilon_n}(\hat{\mathcal{U}}^{b_n,\kappa_n})$ restricted to the upper half-plane. In this section, and for the sake of clarity, we will only prove the result for $k = 1$ (i.e., the convergence of one dimensional marginals of the coloring). The general case involves more cumbersome notation, but can be treated along exactly the same lines.

Let $z_n \in S_{\epsilon_n}(\mathbb{Z}_{odd}^2)$ such that z_n converges to $z = (x, T)$. We need to show that $\hat{G}_n^{z_n}(0)$ converges to $\hat{G}^z(0)$. By reversing the direction of time and using translation invariance (in the x and t directions), it is equivalent to show the following result in the forward-in-time setting:

Proposition 4.1. Let $z_n \in S_{\epsilon_n}(\mathbb{Z}_{odd}^2)$ such that $z_n \to (0,0)$. For every $T > 0$,

$$G_n := G_n^{z_n}(T) \ \to \ G := G^{(0,0)}(T) \ \text{ in law},$$

where $G_n^{z_n}(T)$ is the reduced graph at z_n with time horizon T for the (rescaled) forward discrete net with killing $S_{\epsilon_n}(\mathcal{U}^{b_n,\kappa_n})$.

4.1 Proof of Proposition 4.1

We start by introducing some notation. Define

$$\mathcal{N}^{b,\kappa}(\Sigma_0) \ := \ \{\pi \in \mathcal{N}^{b,\kappa} \ : \ \sigma_\pi = 0\}$$

and for any path $\pi \in \mathcal{N}^{b,\kappa}(\Sigma_0)$, let $r(\pi)$ be the (a.s. finite) sequence of points in

$$L^\kappa(0,T) \bigcup K^\kappa(0,T) \bigcup R^\kappa(0,T) \tag{4.1}$$

(as defined in (3.6)) visited by the path π, the points in the sequence being listed in their order of visit. (In particular, the final point of $r(\pi)$ must be a leaf, and every other point is a relevant separation point). In the following, $r(\pi)$ will be referred to as the finite graph representation of π.

Analogously to the continuum level, we assume that the discrete net with killing $\mathcal{U}^{b_n,\kappa_n}$ is coupled with a discrete net $\mathcal{U}^{b_n}$ (where $\kappa_n = 0$). (See Section 2.4 for more details). $S_{\epsilon_n}(\mathcal{U}^{b_n,\kappa_n}(\Sigma_0))$ will denote the discrete analog of $\mathcal{N}^{b,\kappa}(\Sigma_0)$, and for any path

$\pi_n \in S_{\epsilon_n}(\mathcal{U}^{b_n,\kappa_n}(\Sigma_0))$, $r_n(\pi_n)$ will denote the finite graph representation at the discrete level.

For any point (x,t) and $s \leq t$, define

$$d^s((x,t)) := \inf\{|y-x| \; : \; y \neq x \text{ and } y \in \xi^s(t)\}.$$

and from the Brownian net (with no killing), define the point process

$$\bar{R}(s,t) := \sum_{z \in R(s,t)} \delta_{z,d^s(z)}. \tag{4.2}$$

In words, $\bar{R}(s,t)$ records the location of the (s,t)–relevant separation points (in the standard Brownian net (with no killing)), together with a measure of "isolation" around each of those points. Analogously, we define $\bar{R}_n(s,t)$ for the rescaled discrete net $S_{\epsilon_n}(\mathcal{U}^{b_n})$.

Let $L > 0$. Recall the definition of θ^i given in (3.3) for $S = 0$, i.e. $\theta^0 = 0$ and for every $i \geq 0$

$$\theta^{i+1} := \inf\{t > \theta^i \; : \; \exists x \in (-L,L), \; (x,t) \in \mathcal{M}^\kappa(\theta^i)\}.$$

where we recall that $\mathcal{M}^\kappa(S)$ is the set of killing points attained by some path in $\mathcal{N}^{b,\kappa}$ starting at time S. Define

$$N = \inf\{i \geq 1 : \theta^i \geq T\}. \tag{4.3}$$

For every $s < t$, define

$$\square_{s,t}^L := (-L,L) \times [s,t],$$

and for every $i \geq 0$,

$$\mathcal{R}^{i+1} := R(\theta^i \wedge T, \theta^{i+1} \wedge T) \cap \square_{0,T}^L$$

$$\mathcal{L}^{i+1} := \begin{cases} \{(x,\theta^{i+1}) \in \mathcal{M}^\kappa(\theta^i) \; : \; x \in \xi^0(\theta^{i+1}) \cap (-L,L)\} & \text{if } \theta^{i+1} < T \\ \{(x,T) \; : \; x \in \xi^0(T) \cap (-L,L)\} & \text{otherwise.} \end{cases}$$

Note that when $\theta^{i+1} < T$, $\mathcal{L}^{i+1}$ is a single killing point a.s.. Analogously to (4.2), for every $i \geq 1$, we also define

$$\bar{\mathcal{R}}^i := \sum_{z \in \mathcal{R}^i} \delta_{z,d^0(z)}, \quad \bar{\mathcal{L}}^i := \sum_{z \in \mathcal{L}^i} \delta_{z,d^0(z)}.$$

(To ease the presentation, L and T are not explicit in the notation). Finally, $\theta_n^i, N_n, \mathcal{R}_n^i, \bar{\mathcal{R}}_n^i$ and $\mathcal{L}_n^i, \bar{\mathcal{L}}_n^i$ will denote the analogous quantities for the rescaled process $S_{\epsilon_n}(\mathcal{U}^{b_n}, \mathcal{U}^{b_n,\kappa_n})$. As in the proof of Proposition 3.1, the previous definitions are motivated by the fact that if the paths starting from $(0,0)$ of the Brownian net (with no killing) between time 0 and T are all contained in the box $\square_{0,T}^L$, i.e.,

$$\{(\pi(s),s) \; : \; s \in [0,T], \; \pi \in \mathcal{N}^b((0,0))\} \subset \square_{0,T}^L,$$

then any intermediary point (resp., leaf) in G must be a point in $\cup_i \mathcal{R}^i$ (resp., $\cup_i \mathcal{L}^i$). See Fig. 12. The analogous property holds at the discrete level.

The next two results will be instrumental in the proof of Proposition 4.1. Informally, Proposition 4.2 states that relevant separation points are isolated from the left and from the right. Proposition 4.3 implies that the same holds at the discrete level; furthermore it shows that each continuum separation point is the limit of a single discrete separation point, which is essential for the proof of Proposition 4.1, the convergence of reduced graphs.

Proposition 4.2. For almost every realization of $\mathcal{N}^{b,\kappa}$, $d^0(z) > 0$ for every z in $\cup_{i=1}^N (\mathcal{R}^i \cup \mathcal{L}^i)$.

Proof. By Proposition 2.3 and 2.4 and the construction of the killing points, z is of type $(p, *)$ – with $*$ being equal either to p and pp. Proposition 4.2 is then the content of Proposition 3.11(c) in [SSS09]. □

Proposition 4.3. Let $\bar{\mathcal{P}}$ be the space of Radon measures on $\mathbb{R} \times \mathbb{R}^+ \times \mathbb{R}^+$ equipped with the vague topology. For every $L > 0$ and $T > 0$, the random variable N is finite and for every $1 \leq i \leq N$, $\bar{\mathcal{R}}^i$ and $\bar{\mathcal{L}}^i$ are finite measures a.s.. Further,

$$\left\{ \left(S_{\epsilon_n}(\mathcal{U}^{b_n,\kappa_n}), S_{\epsilon_n}(\mathcal{U}^{b_n,\kappa_n}(\Sigma_0)), \sum_{i=1}^{N_n} \bar{\mathcal{R}}_n^i, \sum_{i=1}^{N_n} \bar{\mathcal{L}}_n^i \right) \right\}_{n \geq 1}$$

$$\rightarrow \left(\mathcal{N}^{b,\kappa}, \mathcal{N}^{b,\kappa}(\Sigma_0), \sum_{i=1}^N \bar{\mathcal{R}}^i, \sum_{i=1}^N \bar{\mathcal{L}}^i \right) \text{ in law,}$$

where the convergence is meant in the product topology $\mathcal{H} \times \mathcal{H} \times \bar{\mathcal{P}} \times \bar{\mathcal{P}}$.

Proposition 4.4. Let $z_n = (x_n, t_n) \in S_{\epsilon_n}(\mathbb{Z}_{odd}^2)$ such that $z_n \rightarrow (0,0)$. As $n \rightarrow \infty$,

$$\mathbb{P}\left(G_n^{z_n}(T) \text{ and } G_n^{(0,0)}(T) \text{ are isomorphic} \right) \rightarrow 1$$

The proofs of Propositions 4.3 and Proposition 4.4 are rather technical, and for the sake of clarity they are postponed until Sections 4.3 and 4.4.

Proof of Proposition 4.1 (assuming Propositions 4.3 and 4.4). By Proposition 4.4, it is enough to show Proposition 4.1 for $z_n = (0,0)$.

Let A_L be the set of realizations such that $\{(\pi(s), s) \ : \ s \in [0, T], \ \pi \in \mathcal{N}^b((0,0))\} \subseteq \square_{0,T}^L$. By equicontinuity of the net paths, the probability of A_L goes to 1 as $L \rightarrow \infty$. Thus, to prove Proposition 4.1, it is enough to show that for every $L > 0$, there exists a coupling such that for almost every realization in A_L, G_n and G are isomorphic for n large enough.

By the Skorohod Representation Theorem and Proposition 4.3, for every $L > 0$, there exists a coupling between the discrete and continuum levels such that

$$\left\{ \left(S_{\epsilon_n}(\mathcal{U}^{b_n,\kappa_n}(\Sigma_0)), \sum_{i=1}^{N_n} \bar{\mathcal{R}}_n^i, \sum_{i=1}^{N_n} \bar{\mathcal{L}}_n^i \right) \right\}_{n \geq 1} \rightarrow \left(\mathcal{N}^{b,\kappa}(\Sigma_0), \sum_{i=1}^N \bar{\mathcal{R}}^i, \sum_{i=1}^N \bar{\mathcal{L}}^i \right) \text{ a.s.}$$

Until the end of the proof, we work under this coupling. For any $n \in \mathbb{N}$ and any realization in A_L, for any discrete point $z_n \in \cup_{i \geq 1} (\mathcal{R}_n^i \cup \mathcal{L}_n^i)$, define $\psi_n(z_n)$ to be the closest point in $\cup_{i \geq 1} (\mathcal{R}^i \cup \mathcal{L}^i)$ to z_n (it is easy to see that $\psi_n(z_n)$ is well defined a.s.). As argued above, we need to show that for almost every realization in A_L, for n large enough, the graphs G_n and G are isomorphic under this mapping.

We claim that conditional on A_L, for any subsequence of discrete paths $\{\pi_n\}_{n \geq 1}$ with $\pi_n \in S_{\epsilon_n}(\mathcal{U}^{b_n,\kappa_n}((0,0)))$ converging to $\pi \in \mathcal{N}^{b,\kappa}((0,0))$, the discrete and continuum paths have the same finite graph representation (up to ψ_n) after a finite rank. Once this claim is obtained, the proof of Proposition 4.1 goes by contradiction: it is rather easy to see that if Proposition 4.1 was not satisfied, this would contradict the latter claim. Thus, the rest of the proof is dedicated to the proof of this statement.

Let us now consider a sequence $\pi_n \in S_{\epsilon_n}(\mathcal{U}^{b_n,\kappa_n}((0,0)))$ converging to π, and let us show that for n large enough, for any $z_n = (x_n, t_n) \in \cup_{i \geq 1} (\mathcal{R}_n^i \cup \mathcal{L}_n^i)$ then π_n hits z_n if

and only if π hits the point $(x,t) := \psi_n(z_n)$. First,

$$\forall (x_n, t_n) \in \cup_{i \geq 1} \left(\mathcal{R}_n^i \cup \mathcal{L}_n^i \right),\ (x,t) = \psi_n(z_n):$$
$$|\pi(t) - x| \leq |\pi(t) - \pi_n(t_n)| + |\pi_n(t_n) - x_n| + |x - x_n|,$$
$$|\pi_n(t_n) - x_n| \leq |\pi_n(t_n) - \pi(t)| + |\pi(t) - x| + |x - x_n|.$$

Define

$$\gamma_n := \min_{z \in \cup_{i=1}^N (\mathcal{R}^i \cup \mathcal{L}^i)} d^0(z)\ \wedge\ \min_{z_n \in \cup_{i=1}^{N_n} (\mathcal{R}_n^i \cup \mathcal{L}_n^i)} d_n^0(z_n)$$

On the one hand, since N (as defined in (4.3)) is finite and the $\mathcal{R}^i$'s and $\mathcal{L}^i$'s are made of finitely many atoms, Proposition 4.2 implies that under our coupling $\liminf \gamma_n = \min_{z \in \cup_{i=1}^N (\mathcal{R}^i \cup \mathcal{L}^i)} d^0(z) > 0$ a.s..

On the other hand, $\{\pi_n\}$ converges to π and $\max_{z_n} \rho(z_n, \psi_n(z_n)) \to 0$ by Proposition 4.3 (where the max is taken over $\cup_{i=1}^{N_n} (\mathcal{R}_n^i \cup \mathcal{L}_n^i)$). Using the equicontinuity of the paths in the sequence $\{\pi_n\}$, we can choose n large enough such that the first and last terms of the two latter inequalities are less than $\gamma_n/3$, i.e., such that

$$\forall (x_n, t_n) \in \cup_{i \geq 1} \left(\mathcal{R}_n^i \cup \mathcal{L}_n^i \right),\ (x,t) = \psi_n(z_n):$$
$$|\pi(t) - x| \leq |\pi_n(t_n) - x_n| + \frac{2\gamma_n}{3}$$
$$\leq |\pi_n(t_n) - x_n| + \frac{2}{3} d^0(z)$$
$$|\pi_n(t_n) - x_n| \leq |\pi(t) - x| + \frac{2}{3} d_n^0(z_n).$$

From the definition of d^0 and d_n^0, this easily implies that for n large enough (so that the previous inequalities are satisfied), the following statement holds:

for any $z_n \in \cup_{i \geq 1} \left(\mathcal{R}_n^i \cup \mathcal{L}_n^i \right)$, π_n hits z_n if and only π hits the point $\psi_n(z_n)$,

as claimed earlier. Finally, for every realization in A_L, and for n large enough, we already argued that the set of vertices of G (resp., G_n) is a subset of $\cup_{i \geq 1} (\mathcal{R}^i \cup \mathcal{L}^i)$ (resp., $\cup_{i \geq 1} (\mathcal{R}_n^i \cup \mathcal{L}_n^i)$). Combining this with the previous statement implies that n large enough, the finite graph representation of π_n and π coincides (up to ψ_n).

This completes the proof of the claim and and, as argued above, the proof of Proposition 4.1 $\hfill\square$

It remains to prove Proposition 4.3 and Proposition 4.4.

4.2 Preliminary work for the proof of Proposition 4.3

For the time being, we only consider the Brownian net $\mathcal{N}^b$ (with no killing). For every $t > 0$, let $\xi_n^0(t)$ and $R_n(0,t)$ be the discrete counterpart (in the rescaled net $S_{\epsilon_n}(\mathcal{U}^{b_n})$) of $\xi^0(t)$ and $R(0,t)$. Finally, $\bar{R}_n(0,t)$ is the discrete analog of $\bar{R}(0,t)$. The rest of this section is dedicated to proving the following result, which is closely related to Proposition 4.3.

Proposition 4.5. For every $T > 0$, $(S_{\epsilon_n}(\mathcal{U}^{b_n}), \bar{R}_n(0,T)) \to (\mathcal{N}^b, \bar{R}(0,T))$ in law, where the convergence is in the product topology $\mathcal{H} \times \bar{\mathcal{P}}$.

In order to prove Proposition 4.5, we will need the following lemma.

Lemma 4.1. For almost every realization, $d^0(z) > 0$ for every $z \in V := R(0,T) \cup \{(x,T)\ :\ x \in \xi^0(T)\}$.

Proof. This a special case of Proposition 4.2 when $\kappa = 0$. $\hfill\square$

Corollary 4.1. For every $(z^1, \cdots, z^\kappa) \in (\mathbb{R}^2)^n$, let

$$
\begin{aligned}
A(z^1, \cdots, z^\kappa) &:= \{\pi \in \mathcal{N}^b(\Sigma_0) : r(\pi) = (z^1, \cdots, z^\kappa)\} \\
B(z^1, \cdots, z^\kappa) &:= \mathcal{N}^b(\Sigma_0) \setminus A(z^1, \cdots, z^\kappa).
\end{aligned}
$$

If $A(z^1, \cdots, z^\kappa) \neq \emptyset$, then $A(z^1, \cdots, z^\kappa)$ is a compact set and $d_{\mathcal{H}}\big(A(z^1, \cdots, z^\kappa),$ $B(z^1, \cdots, z^\kappa)\big) > 0$.

Proof. We first show that $A(z^1, \cdots, z^\kappa)$ is a compact set of paths. Since $\mathcal{N}^b$ is compact, it is is sufficient to prove that for any $\{p_l\}_{l \geq 1}$ in $A(z^1, \cdots, z^\kappa)$ converging to π in $\mathcal{N}^b(\Sigma_0)$, the finite graph representation $r(\pi)$ is given by $z^1, \cdots, z^\kappa$. On the one hand, it is clear that π must hit the points $z^1, \cdots, z^\kappa$. On the other hand, if π touches some $\bar{z} = (\bar{x}, \bar{t}) \in V$ distinct from $z^1, \cdots, z^\kappa$, we would have $p_l(\bar{t}) \neq \bar{x}$, but $p_l(\bar{t}) \to \bar{x}$ as $l \to \infty$, which would imply that $d^0(\bar{z}) = 0$. However, by Lemma 4.1, $d^0(\bar{z}) > 0$ a.s.. This shows that $A(z^1, \cdots, z^\kappa)$ is a compact set of paths a.s.

Next, since sets of the form $A(\bar{z}^1, \cdots, \bar{z}^\kappa)$ with $(\bar{z}^1, \cdots, \bar{z}^\kappa) \in \cup_{l \in \mathbb{N}^*}(\mathbb{R}^2)^l$ are pairwise disjoint, the distance between $A(z^1, \cdots, z^\kappa)$ and any finite union of sets of the form $A(\bar{z}^1, \cdots, \bar{z}^m) \neq \emptyset$ with $(\bar{z}^1, \cdots, \bar{z}^m) \neq (z^1, \cdots, z^\kappa)$ is strictly positive. Furthermore, by equicontinuity of the paths in $\mathcal{N}^b$ and the local finiteness of V, the same property holds for any infinite union of those sets since one can restrict attention to a finite space-time box. Combining this with the fact that $B(z^1, \cdots, z^k)$ can be written as the union of all the non-empty $A(\bar{z}^1, \cdots, \bar{z}^m)$'s with $(\bar{z}^1, \cdots, \bar{z}^m) \neq (z^1, \cdots, z^\kappa)$ completes the proof of Corollary 4.1. $\square$

Let $\mathcal{P}$ be the space of Radon measures on $\mathbb{R} \times \mathbb{R}^+$ equipped with the vague topology. From Proposition 6.14 in [SSS14], it is already known that

$$
(S_{\epsilon_n}(\mathcal{U}^{b_n}), R_n(0, T)) \to (\mathcal{N}^b, R(0, T)) \text{ in law}, \tag{4.4}
$$

where the convergence is meant in the product topology $\mathcal{H} \times \mathcal{P}$.

Lemma 4.2. For every deterministic $t > 0$,

$$
(S_{\epsilon_n}(\mathcal{U}^{b_n}), \xi_n^0(t)) \to (\mathcal{N}^b, \xi^0(t)) \text{ in law}. \tag{4.5}
$$

Proof. From the Skorohod representation theorem, there exists a coupling between the discrete and continuum levels such that

$$
S_{\epsilon_n}(\mathcal{U}^{b_n}) \to \mathcal{N}^b \quad \text{a.s.}.
$$

Let $a < b \in \mathbb{R}$. The previous convergence statement entails that under our coupling:

$$
\liminf_{n \to \infty} |\xi_n(t) \cap [a, b]| \geq |\xi(t) \cap [a, b]| \text{ a.s.}
$$

On the other hand, using the same technic as in Proposition 4.3. in [NRS15], one can show that

$$
\limsup_{n \to \infty} \mathbb{E}(|\xi_n(t) \cap [a, b]|) = \mathbb{E}(|\xi(t) \cap [a, b]|).
$$

Combing the two previous results, for every interval $[a, b]$, we get

$$
\lim_{n \to \infty} |\xi_n(t) \cap [a, b]| = |\xi(t) \cap [a, b]| \text{ a.s.},
$$

which ends the proof of the lemma. $\square$

We will need one more ingredient that is a simple extension of Lemma 6.7 in [SSS14], whose proof is left as an exercise to the reader.

Lemma 4.3. Let E be a Polish space. Let $\{F_i\}_{i \in I}$ and $\{G_i\}_{i \in I}$ be two finite or countable collections of Polish spaces and for each $i \in I$, let $f_i : E \to F_i$ and $g_i : E \to G_i$ be measurable functions. Let X, X_k, Y, Y_k and $U_{k,i}, V_{k,i}$ be random variables $\{k \geq 1, i \in I\}$ such that X, X_k, Y, Y_k take values in E and $U_{k,i}, V_{k,i}$ takes values in F_i and G_i respectively; for every k, $(X_k, Y_k, U_{k,i}, V_{k,i}, i \in I)$ are defined on the same probability space; (X, Y) are defined on the same probability space. Then

$$(X_k, U_{k,i}) \xrightarrow{(d)} (X, f_i(X)), \ (Y_k, V_{k,i}) \xrightarrow{(d)} (Y, g_i(Y)), \ \forall i \in I$$
$$\text{and } (X_k, Y_k) \xrightarrow{(d)} (X, Y)$$
$$\text{implies} \quad (X_k, Y_k, U_{k,i}, (V_{k,i})_{i \in I}) \xrightarrow{(d)} (X, Y, f_i(X), (g_i(Y))_{i \in I})$$

where the convergence is meant in the product topology.

By Lemma 6.12 in [SSS14], $(S_{\epsilon_n}(\mathcal{U}^{b_n}), S_{\epsilon_n}(\mathcal{U}^{b_n}(\Sigma_0)))$ converges to $(\mathcal{N}^b, \mathcal{N}^b(\Sigma_0))$ in $(\mathcal{H}, d_{\mathcal{H}})$. Using Lemma 4.3 (in the special case $X = Y = \mathcal{N}^b$, and $U_{k,i} = V_{k,i}$) and the Skorohod Representation Theorem, (4.4) and (4.5) imply that there exists a coupling between the discrete and the continuum levels such that

$$\{\left(S_{\epsilon_n}(\mathcal{U}^{b_n}(\Sigma_0)), \ R_n(0,T), \ \{\xi_n^0(q)\}_{q \in \mathbb{Q}^+}, \ \xi_n^0(T)\right)\}_{n \geq 1}$$
$$\to \ \left(\mathcal{N}^b(\Sigma_0), R(0,T), \{\xi^0(q)\}_{q \in \mathbb{Q}^+}, \xi^0(T)\right) \quad a.s..$$

From now until the end of this subsection, we will work under this coupling. In particular, if V_n denotes the analog of V (as defined in Lemma 4.1), then $V_n \Longrightarrow V$ a.s. under our coupling.

In the next lemma, $A_n(z^1, \cdots, z^k)$ will refer to the discrete analog of $A(z^1, \cdots, z^k)$ (i.e., the set of paths in the rescaled net $S_{\epsilon_n}(\mathcal{U}^{b_n}(\Sigma_0))$ with discrete finite graph representation $(z^1, \cdots, z^k)$).

Lemma 4.4. Let $(z^1, \cdots, z^k)$ be such that $A(z^1, \cdots, z^k) \neq \emptyset$. For almost every realization of our coupling, for large enough n, there exists $(z_n^1, \cdots, z_n^k)$ such that

$$\forall i \in \{1, \cdots, k-1\}, \ z_n^i \in R_n(0,T) \to z^i$$

and $x_n^k \in \xi_n^0(T)$ such that $z_n^k = (x_n^k, T) \to z^k$. Further, the set of paths $A_n(z_n^1, \cdots, z_n^k)$ converges to $A(z^1, \cdots, z^k)$ in $(\mathcal{H}, d_{\mathcal{H}})$ a.s..

Proof. The existence of the z_n^i's is a direct consequence of the fact that $V_n \Longrightarrow V$ a.s.. The rest of the proof (i.e., the convergence $A_n(z_n^1, \cdots, z_n^k)$ to $A(z^1, \cdots, z^k)$) is decomposed into two steps.

Step 1. Let $\delta(z^1, \cdots, z^k) := d_{\mathcal{H}}\left(A(z^1, \cdots, z^k), B(z^1, \cdots, z^k)\right)$. Define

$$\bar{A}_n(z^1, \cdots, z^k) \ = \ \{\pi_n \in S_{\epsilon_n}(\mathcal{U}^{b_n}(\Sigma_0)) \ : \ \min_{\pi \in A(z^1, \cdots, z^k)} d(\pi_n, \pi) \ \leq \ \frac{1}{3}\delta(z^1, \cdots, z^k)\},$$

and

$$\bar{B}_n(z^1, \cdots, z^k) \ = \ \{\pi_n \in S_{\epsilon_n}(\mathcal{U}^{b_n}(\Sigma_0)) \ : \ \min_{\pi \in B(z^1, \cdots, z^k)} d(\pi_n, \pi) \ \leq \ \frac{1}{3}\delta(z^1, \cdots, z^k)\}.$$

In this step, we will show that after a finite rank, for any $\pi_n \in \bar{A}_n(z^1, \cdots, z^k)$, the finite graph representation of π_n is a subsequence of $(z_n^1, \cdots, z_n^k)$, and that $A_n(z_n^1, \cdots, z_n^k) \subseteq \bar{A}_n(z^1, \cdots, z^k)$.

By Corollary 4.1, $\delta(z^1, \cdots, z^k) > 0$ a.s.. From there, it is straightforward to check from the definition of $\delta(z^1, \cdots, z^k)$ and the fact that under our coupling $S_{\epsilon_n}(\mathcal{U}^{b_n}(\Sigma_0))$ converges to $\mathcal{N}^b(\Sigma_0)$ a.s. that:

$$d_{\mathcal{H}}(\bar{A}_n(z^1, \cdots, z^k), A(z^1, \cdots, z^k)) \to 0, \quad d_{\mathcal{H}}(\bar{B}_n(z^1, \cdots, z^k), B(z^1, \cdots, z^k)) \to 0 \text{ a.s. } (4.6)$$

as $n \to \infty$, and that $\bar{A}_n(z^1, \cdots, z^k)$ and $\bar{B}_n(z^1, \cdots, z^k)$ form a partition of the set $S_{\epsilon_n}(\mathcal{U}^{b_n}(\Sigma_0))$ for large enough n. Let us now derive two consequences of this result.

First, Corollary 4.1 easily implies that

$$\min_{\bar{z} \in V, \ \bar{z} \notin \{z^1, \cdots, z^k\}} \quad \min_{\pi \in A(z^1, \cdots, z^k), s \geq 0} \rho\left((\pi(s), s), \bar{z}\right) > 0.$$

Further, under our coupling, $V_n \Longrightarrow V$ a.s., and since $\bar{A}_n(z^1, \cdots, z^k) \to A(z^1, \cdots, z^k)$ (in $(\mathcal{H}, d_{\mathcal{H}})$), the equicontinuity of the paths in $\cup_n S_{\epsilon_n}(\mathcal{U}^{b_n})$ combined with the previous limit implies that

$$\liminf_{n \to \infty} \quad \min_{\bar{z}_n \in V_n, \ \bar{z}_n \notin \{z_n^1, \cdots, z_n^k\}} \quad \min_{\pi_n \in \bar{A}_n(z^1, \cdots, z^k), s \geq 0} \rho\left((\pi_n(s), s), \bar{z}_n\right) > 0.$$

As a consequence, after a finite rank, for any $\pi_n \in \bar{A}_n(z^1, \cdots, z^k)$, the (discrete) finite graph representation of π_n is a subsequence of $(z_n^1, \cdots, z_n^k)$. (In words, some z_n^i's can be "missed", but any point in the finite graph representation of π_n must be in $(z_n^1, \cdots, z_n^k)$).

Secondly, by reasoning along the same lines, one can deduce that after a finite rank, the finite graph representation of any path $\pi_n \in \bar{B}_n(z^1, \cdots, z^k)$ must be distinct from $(z_n^1, \cdots, z_n^k)$. Since $\bar{A}_n(z^1, \cdots, z^k)$ and $\bar{B}_n(z^1, \cdots, z^k)$ form a partition of the set of discrete net paths starting at time 0 (for large enough n), it follows that $A_n(z_n^1, \cdots, z_n^k) \subseteq \bar{A}_n(z^1, \cdots, z^k)$ after a finite rank.

Step 2. We now show that $\bar{A}_n(z^1, \cdots, z^k) = A_n(z_n^1, \cdots, z_n^k)$ for n large enough. Since we showed in Step 1 that after a finite rank, for any $\pi_n \in \bar{A}_n(z^1, \cdots, z^k)$, the finite graph representation of π_n is a subsequence of $(z_n^1, \cdots, z_n^k)$, and since $A_n(z_n^1, \cdots, z_n^k) \subseteq \bar{A}_n(z^1, \cdots, z^k)$, it is sufficient to show that for n large enough, any $\pi_n \in \bar{A}_n(z^1, \cdots, z^k)$ must pass through $z_n^1, \cdots, z_n^k$.

First, for n large enough, is is easy to see that any $\pi_n \in \bar{A}_n(z^1, \cdots, z^k)$ must go through z_n^k. This is a direct consequence of the fact that $\xi^0(T)$ is locally finite, $\xi_n^0(T) \Longrightarrow \xi^0(T)$ a.s. under our coupling, and the fact $\bar{A}_n(z^1, \cdots, z^k)$ converges to $A(z^1, \cdots, z^k)$ a.s. (see Step 1).

Let us now argue that after some rank, any $\pi_n \in \bar{A}_n(z^1, \cdots, z^k)$ must also go through the point z_n^i for $i < k$. In fact, we will show a little more: We will show by induction on $i \in \mathbb{N}$, that for any $m > i$ and any $\bar{z}^1, \cdots \bar{z}^m$ such that $A(\bar{z}^1, \cdots \bar{z}^m) \neq \emptyset$, for n large enough, any $\pi_n \in \bar{A}_n(\bar{z}^1, \cdots, \bar{z}^m)$ must go through $\bar{z}_n^1, \cdots, \bar{z}_n^i$, where $\{\bar{z}_n^i\}$ is defined analogously to $\{z_n^i\}$ (i.e., as a sequence of points in V_n approximating $\bar{z}^i$).

The case i=1: Let us first show the property for $i = 1$. We argue by contradiction. Going to a subsequence if necessary, let us assume that there exists an infinite sequence $\{\pi_n\}_n$ with $\pi_n \in \bar{A}_n(\bar{z}_1, \cdots, \bar{z}^m)$ (with $m > 1$) and such that π_n does not go through the point $\bar{z}_n^1$ (i.e., if $\bar{z}_n^1 = (\bar{x}_n^1, \bar{t}_n^1)$ then $\pi_n^1(\bar{t}_n^1) \neq \bar{x}_n^1$). Going to a further subsequence, we can assume w.l.o.g. that $\{\pi_n\}$ converges to $\pi \in A(\bar{z}^1, \cdots, \bar{z}^m)$. (The fact that $\pi \in A(\bar{z}^1, \cdots, \bar{z}^m)$ easily follows from Corollary 4.1.) Since $m > 1$, the point $\bar{z}^1 = (\bar{x}^1, \bar{t}^1)$ is a relevant separation point. From this observation, one can construct a path $\pi' \in \mathcal{N}^b(\Sigma_0)$ such that $\pi = \pi'$ up to $\bar{t}^1$ and the paths π and π' separate after $\bar{t}^1$. (This is done by concatenating π with one of the paths in $(\mathcal{W}_r, \mathcal{W}_l)$ starting from $\bar{z}^1$: for instance, following the notation of Proposition 2.4 and Fig. 11, if π is squeezed between l and r', we concatenate π with r. The constructed path belongs to the net using the stability of the net under hopping – See [SS08].)

Since $S_{\epsilon_n}(\mathcal{U}^{b_n}(\Sigma_0))$ converges to $\mathcal{N}^b(\Sigma_0)$, there is a sequence $\{\pi'_n\}_n$ in $S_{\epsilon_n}(\mathcal{U}^{b_n}(\Sigma_0))$ converging to π'. Let $q \in \mathbb{Q}$, with $q < \bar{t}^1$. $\xi^0(q)$ is a.s. locally finite and under our coupling $\xi_n^0(q) \Longrightarrow \xi^0(q)$ a.s.. Since

$$\lim_{n\to\infty} |\pi_n(q) - \pi'_n(q)| \;=\; |\pi(q) - \pi'(q)| = 0,$$

it follows that $\pi_n(q) = \pi'_n(q)$ for n large enough. Finally, since π and π' separate at $\bar{z}^1$, π'_n and π_n must separate at some point (x_n, t_n) with $\limsup t_n \leq \bar{t}^1$. However, in Step 1, we showed that after a finite rank, the finite graph representation of π_n is a subset of $\{\bar{z}_n^1, \cdots, \bar{z}_n^k\}$. Since $\limsup t_n \leq \bar{t}^1$, it follows that for large n, $(x_n, t_n) = \bar{z}_n^1$ and so π_n must go through $\bar{z}_n^1$ (for n large enough), which yields a contradiction. This shows the induction hypothesis for $i = 1$.

The induction step from i-1 to i: Let us now assume that our property holds at rank $i - 1$ with $i \geq 2$. Again we argue by contradiction and we assume that there exists a subsequence $\pi_n \in \bar{A}_n(\bar{z}^1, \cdots, \bar{z}^m)$ (with $m > i$) such that $\pi_n(\bar{t}_n^i) \neq \bar{x}_n^i$ (where $\bar{z}_n^i = (\bar{x}_n^i, \bar{t}_n^i)$). By arguing as in the case $i = 1$, we can assume w.l.o.g. that π_n converges to $\pi \in A(\bar{z}^1, \cdots, \bar{z}^m)$, and that there exists $\pi' \in \mathcal{N}^b$ such that π and π' coincide up to $\bar{t}^i$ and separate afterwards. Let $(\tilde{z}^1, \cdots, \tilde{z}^l)$ be the finite graph representation of π'. By construction, $l > i$ and $\tilde{z}^j = \bar{z}_j$ for $j \leq i$. Since $\bar{A}_n(\tilde{z}^1, \cdots, \tilde{z}^l) \to A(\tilde{z}^1, \cdots, \tilde{z}^l)$ (by Step 1), there exists $\pi'_n \in \bar{A}_n(\tilde{z}^1, \cdots, \tilde{z}^l)$ such that $\{\pi'_n\}$ converges to π'.

By using our induction hypothesis, π_n and π'_n must go through the point $\bar{z}_n^{i-1} = \tilde{z}_n^{i-1}$ after a finite rank. Further, since π and π' separate at $\bar{z}^i$, the two paths π_n and π'_n must separate at some time t_n with $\limsup t_n \leq \bar{t}^i$. However, for n large enough, the finite graph representation of π_n is a subset of $(\bar{z}_n^1, \cdots, \bar{z}_n^k)$, and since π_n and π'_n coincide at $\bar{t}_n^{i-1}$, π_n and π'_n must separate at $\bar{z}_n^i$. As a consequence, π_n goes through the point $\bar{z}_n^i$, which yields the desired contradiction. This ends the proof of Lemma 4.4. □

Proof of Proposition 4.5. Under our coupling, we already know that $R_n(0, T) \Longrightarrow R(0, T)$ a.s. It is enough to show that for every $z = (x, t) \in V$, and for every $z_n = (x_n, t_n) \in V_n$ with $z_n \to z$, we have $\lim_{n\to\infty} d_n^0(z_n) = d^0(z)$ a.s.. Note that if $z = (x, t) \in R(0, T)$ then

$$d^0(z) \;=\; \inf\{|x - y| \;:\; \exists(z^1, \cdots, z^k) \text{ and } \pi \in A(z^1, \cdots, z^k), \text{ s.t. } y = \pi(t) \text{ and } \forall i \leq k,\ z \neq z^i\}.$$

Thus, if we write

$$m\big((z^1, \cdots, z^k), z\big) \;=\; \inf\{|x - y| \;:\; \exists \pi \in A(z^1, \cdots, z^k) \text{ s.t. } y = \pi(t)\}$$

$d^0(z)$ is the infimum of all the $m\big((z^1, \cdots, z^k), z\big)$'s such that $\forall i \leq k,\ z \neq z^i$. The analogous statement holds at the discrete level.

Let m_n denote the discrete analog of m, and let $z_n, z_n^1, \cdots, z_n^k \in V_n$ converging to $z, z^1, \cdots, z^k \in V$ such that $A(z^1, \cdots, z^k) \neq \emptyset$. Lemma 4.4 implies that $m_n\big((z_n^1, \cdots, z_n^k), z_n\big)$ converges to $m\big((z^1, \cdots, z^k), z\big)$. The latter convergence statement extends by taking the infimum over finitely many $m\big((z^1, \cdots, z^k), z\big)$'s and $m_n\big((z_n^1, \cdots, z_n^k), z_n\big)$. Proposition 4.5 is then a direct consequence of the equicontinuity of the paths in $\mathcal{N}^b$ and $\cup_n S_{\epsilon_n}(\mathcal{U}^{b_n})$. □

4.3 Proof of Proposition 4.3

In the course of the proof of Proposition 3.1, we already proved that $N < \infty$ (where N is defined as in (4.3)) and for every $1 \leq i \leq N$, the random point measure $\bar{\mathcal{R}}^i$ is finite a.s.. When $\theta^i < T$, $\bar{\mathcal{L}}^i$ is a singleton a.s., and when $\theta^i > T$, the number of atoms in $\bar{\mathcal{L}}^i$ coincides with the cardinality of $\{x \;:\; x \in \xi^0(T) \cap (-L, L)\}$. By Proposition 2.2, this shows that that $\bar{\mathcal{L}}^i$ is also a finite measure a.s.. In this section, to prove Proposition 4.3, our main task will be to prove the following two lemmas:

Lemma 4.5. $(S_{\epsilon_n}(\mathcal{U}^{b_n, \kappa_n}), \bar{\mathcal{L}}_n^1) \to (\mathcal{N}^{b, \kappa}, \bar{\mathcal{L}}^1)$ in law.

Lemma 4.6. $(S_{\epsilon_n}(\mathcal{U}^{b_n,\kappa_n}), \bar{\mathcal{R}}_n^1) \rightarrow (\mathcal{N}^{b,\kappa}, \bar{\mathcal{R}}^1)$ in law.

We note that in both lemmas, the convergence is meant in the product topology $\mathcal{H} \times \mathcal{P}$.

By using the strong Markov property and the stationarity in time of the killed net (both at the discrete and continuum level), Lemmas 4.5 and 4.6 will imply that

$$\{(S_{\epsilon_n}(\mathcal{U}^{b_n,\kappa_n}), \{\bar{\mathcal{R}}_n^i\}_{i\geq 1})\}_{n\geq 1} \rightarrow (\mathcal{N}^{b,\kappa}, \{\bar{\mathcal{R}}^i\}_{i\geq 1}) \text{ and}$$
$$\{(S_{\epsilon_n}(\mathcal{U}^{b_n,\kappa_n}), \{\bar{\mathcal{L}}_n^i\}_{i\geq 1})\}_{n\geq 1} \rightarrow (\mathcal{N}^{b,\kappa}, \{\bar{\mathcal{L}}^i\}_{i\geq 1}) \text{ in law,}$$

where the convergence is again meant in the product topology. Since N (resp., N_n) coincides with the first i such that $\bar{\mathcal{R}}^i$ (resp., $\bar{\mathcal{R}}_n^i$) is empty, this implies

$$\left\{\left(S_{\epsilon_n}(\mathcal{U}^{b_n,\kappa_n}), \sum_{i=1}^{N_n} \bar{\mathcal{R}}_n^i\right)\right\}_{n\geq 1} \rightarrow \left(\mathcal{N}^{b,\kappa}, \sum_{i=1}^{N} \bar{\mathcal{R}}^i\right)$$

and

$$\left\{\left(S_{\epsilon_n}(\mathcal{U}^{b_n,\kappa_n}), \sum_{i=1}^{N_n} \bar{\mathcal{L}}_n^i\right)\right\}_{n\geq 1} \rightarrow \left(\mathcal{N}^{b,\kappa}, \sum_{i=1}^{N} \bar{\mathcal{L}}^i\right) \text{ in law.}$$

Using Theorem 2.6, it is not hard to prove that $S_{\epsilon_n}(\mathcal{U}^{b_n,\kappa_n}, \mathcal{U}^{b_n,\kappa_n}(\Sigma_0))$ converges (in law) to $(\mathcal{N}^{b,\kappa}, \mathcal{N}^{b,\kappa}(\Sigma_0))$ (this is done in Lemma 6.19 in [SSS14] for the case $\kappa_n, \kappa = 0$. The same type of argument carries over for the killed Brownian net). Proposition 4.3 then follows by a direct application of Lemma 4.3 (taking $X, Y = \mathcal{N}^{b,\kappa}$). The rest of the section is dedicated to the proofs of Lemmas 4.5 and 4.6.

Proof of Lemma 4.5. In Lemma 4.2, we showed that

(1) $(\xi_n^0(T), S_{\epsilon_n}(\mathcal{U}^{b_n}))$ converges to $(\xi^0(T), \mathcal{N}^b)$ in law.

In [NRS15], we showed that the following statements hold in distribution:

(2) $\{(x, \theta_n^1) \text{ killing point} : x \in \xi_n^1(\theta_n^1) \cap (-L, L)\}$ converges to the singleton $(x, \theta^1) \in \mathcal{M}^k(0)$,

(3) and finally

$$\theta_n^1 \rightarrow \theta^1, \text{ and } \xi_n^0(\theta_n^1) \implies \xi^0(\theta^1). \tag{4.7}$$

The second item can be found in Proposition 4.4(1) and Lemma 4.4 in [NRS15]. The third item is the content of Corollary 4.2 in [NRS15]. Further, analogously to Theorem 2.6, a closer look at the proofs shows that the latter results were shown by constructing a coupling between $\{(S_{\epsilon_n}(\mathcal{U}^{b_n}), S_{\epsilon_n}(\mathcal{U}^{b_n,\kappa_n}))\}_{n>0}$ and $(\mathcal{N}^b, \mathcal{N}^{b,\kappa})$ such that under this coupling, the convergence of the random variables above and the convergence of $S_{\epsilon_n}(\mathcal{U}^{b_n,\kappa_n})$ to $\mathcal{N}^{b,\kappa}$ hold jointly in probability.

Using Lemma 4.3, this shows the convergence (in law) of $\{(S_{\epsilon_n}(\mathcal{U}^{b_n,\kappa_n}), \theta_n^1, \xi_n^0(\theta_n^1))\}$ to its continuum counterpart, and from there it is easy to see that $(S_{\epsilon_n}(\mathcal{U}^{b_n,\kappa_n}), \bar{\mathcal{L}}_n^1) \rightarrow (\mathcal{N}^{b,\kappa}, \bar{\mathcal{L}}^1)$. □

Proof of Lemma 4.6. Combing Proposition 4.5, Lemma 4.5, together with the convergence of $\{(S_{\epsilon_n}(\mathcal{U}^{b_n,\kappa_n}), \theta_n^1, \xi_n^0(\theta_n^1))\}$ to $(\mathcal{N}^{b,\kappa}, \theta^1, \xi^0(\theta^1))$ — see previous lemma — we have

$$\forall t \geq 0, \{(S_{\epsilon_n}(\mathcal{U}^{b_n}), \bar{R}_n(0,t))\} \rightarrow (\mathcal{N}^b, \bar{R}(0,t)), \quad (S_{\epsilon_n}(\mathcal{U}^{b_n,\kappa_n}), \bar{\mathcal{L}}_n^1) \rightarrow (\mathcal{N}^{b,\kappa}, \bar{\mathcal{L}}^1),$$
$$\text{and} \quad \{(S_{\epsilon_n}(\mathcal{U}^{b_n,\kappa_n}), \theta_n^1, \xi_n^0(\theta_n^1))\} \rightarrow (\mathcal{N}^{b,\kappa}, \theta^1, \xi^0(\theta^1)) \text{ in law.}$$

Furthermore, by Theorem 2.6, $S_{\epsilon_n}(\mathcal{U}^{b_n}, \mathcal{U}^{b_n, \kappa_n})$ converges to $(\mathcal{N}^b, \mathcal{N}^{b,\kappa})$. By the Skorohod representation theorem and Lemma 4.3 with $(X, Y) = (\mathcal{N}^b, \mathcal{N}^{b,\kappa})$, this implies that there exists a coupling between the discrete and continuum levels such that (1) $S_{\epsilon_n}(\mathcal{U}^{b_n})$ converges to $\mathcal{N}^b$ a.s., (2) $\bar{\mathcal{L}}_n^1$ converges to $\bar{\mathcal{L}}^1$ a.s., (3) the convergence statement $\{(S_{\epsilon_n}(\mathcal{U}^{b_n, \kappa_n}), \theta_n^1, \xi_n^0(\theta_n^1))\}$ hold a.s., and (4) such that $\bar{R}_n(0, t) \implies \bar{R}(0, t)$ a.s. for any positive rational value of t and for $t = T$. We will work under this coupling for the rest of the proof.

Next, we first note that

$$\forall t \in \mathbb{Q}^+, t = T, \quad R(0, t) \cap \{(x, s) \ : \ s \in [0, t], \ x = \pm L\} = \emptyset \text{ a.s.} \tag{4.8}$$

(This is a direct consequence of the fact that $R(0, t)$ is locally finite a.s. and translation invariance of the Brownian net along the x-axis). Since $\forall t \in \mathbb{Q}^+, t = T, \ \bar{R}_n(0, t) \implies \bar{R}(0, t)$ a.s., this implies that

$$\forall t \in \mathbb{Q}^+, t = T, \quad \sum_{z \in R_n(0,t) \cap \square_{0,t}^L} \delta_{z, d_n^0(z)} \implies \sum_{z \in R(0,t) \cap \square_{0,t}^L} \delta_{z, d^0(z)} \text{ a.s..} \tag{4.9}$$

(4.9) when $t = T$ entails that Lemma 4.6 holds for almost every realization on $\{\theta^1 > T\}$. In order to complete the proof of Lemma 4.6, it remains to show that

$$\sum_{z \in R_n(0,\theta_n^1) \cap \square_{0,\theta_n^1}^L} \delta_{z, d_n^0(z)} \implies \sum_{z \in R(0,\theta^1) \cap \square_{0,\theta^1}^L} \delta_{z, d^0(z)} \text{ a.s..} \tag{4.10}$$

We will proceed in three steps.

Step 1. Since $\mathcal{N}^b$ is a set of equicontinuous paths a.s., for almost every realization of our coupling, there exists $L' \in \mathbb{Q}^+$ with $L' > L$ (and L' random) such that any path in $\mathcal{N}^b(\Sigma_0)$ hitting a point in $R(0, \theta^1) \cap \square_{0,\theta^1}^L$ remains in the box $\square_{0,\theta^1}^{L'}$ between time 0 and θ^1, and further, under our coupling, the same holds at the discrete level for n large enough. In the following, we show that for almost every realization of our coupling, there exists a (random) rational $q < \theta^1$ such that

$$R(0, \theta^1) \cap \square_{q,\theta^1}^{L'} = \quad \emptyset, \tag{4.11}$$
$$\text{and for } n \text{ large enough}, \quad R_n(0, \theta_n^1) \cap \square_{q,\theta_n^1}^{L'} = \quad \emptyset.$$

Since under our coupling, $\theta_n^1 \to \theta^1$ a.s., it is enough to show that

$$\inf\{x < \theta^1 \ : \ R(0, \theta^1) \cap \square_{x,\theta^1}^{L'} = \emptyset\} \quad < \quad \theta^1,$$
$$\limsup_{n \to \infty} \ \inf\{x < \theta_n^1 \ : \ R_n(0, \theta_n^1) \cap \square_{x,\theta_n^1}^{L'} = \emptyset\} \quad < \quad \theta^1 \text{ a.s.} \tag{4.12}$$

The first property simply follows from the fact that $R(0, \theta^1)$ is locally finite (which was established in the proof of Proposition 3.1). Let us consider a realization such that the $\limsup$ above is equal to θ^1 and let us show that it must belong to a set of measure zero. Going to a subsequence if necessary, there is a sequence $q_n \uparrow \theta^1$ such that

$$R_n(0, \theta_n^1) \cap \square_{q_n,\theta_n^1}^{L'} \neq \emptyset.$$

In particular, there must exist a sequence of discrete separation points $\{z_n\}$ with $z_n \in R(0, \theta_n^1) \cap \square_{q_n,\theta_n^1}^{L'}$, and $\pi_n^1, \pi_n^2 \in S_{\epsilon_n}(\mathcal{U}^{b_n}(\Sigma_0))$ such that π_n^1 and π_n^2 pass through the point z_n but attain two distinct points (x_n^1, θ_n^1) and (x_n^2, θ_n^1) at time θ_n^1. Recall that $\xi^0(\theta^1)$ is locally finite a.s. and that under our coupling

$$\xi_n^0(\theta_n^1) \implies \xi^0(\theta^1) \text{ a.s.}$$

In particular, the sets $\xi_n^0(\theta_n^1)$ are sparse, in the sense that distinct points must remain at a macroscopic distance from each other. Let us assume w.l.o.g. that the realization under consideration belongs to the set under which the previous limiting statement holds. Since the two distinct points x_n^1, x_n^2 alluded to earlier belong to $\xi_n^0(\theta^1)$, this entails that the family of paths $\{\pi_n^1, \pi_n^2\}_n$ is not equi-continuous (the two paths π_n^1 and π_n^2 separate at time arbitrarily close to θ_n^1 and take two macroscopically distinct values at time θ_n^1). We already argued that $\cup_n S_{\epsilon_n}(\mathcal{U}^{b_n})$ is a set of equicontinuous paths a.s. As a consequence, the event $\{\limsup_{n\to\infty} \ \inf\{x < \theta_n^1 \ : \ R_n(0,\theta_n^1) \cap \square_{x,\theta_n^1}^{L'} = \emptyset\} = \theta^1\}$ has measure 0. It follows that (4.12) (and thus (4.11)) holds a.s., as claimed earlier.

Step 2. For every $q, L' \in \mathbb{Q}^+$, define $E_{q,L'}$ to be the set of realizations such that (1) for n large enough, any path in $S_{\epsilon_n}(\mathcal{U}^{b_n})(\Sigma_0)$ hitting a point in $R_n(0,\theta_n^1) \cap \square_{0,\theta^1}^L$ remains in the box $\square_{0,\theta^1}^{L'}$ and (2) $\theta^1 > q$, and (3) (4.11) holds. Note that for any realization in $E_{q,L'}$, any point in $R(0,\theta^1) \cap \square_{0,\theta^1}^L$ must belong to the rectangle $\square_{0,q}^L$.

In Step 2, our goal is to show that for almost every realization in $E_{q,L'}$,

$$\sum_{z \in R_n(0,\theta_n^1) \ \cap \square_{0,q}^L} \delta_{z,d_n^0(z)} \implies \sum_{z \in R(0,\theta^1) \ \cap \square_{0,q}^L} \delta_{z,d^0(z)} \quad \text{a.s..} \tag{4.13}$$

First, under our coupling,

$$\sum_{z \in R_n(0,q) \cap \square_{0,q}^L} \delta_{z,d_n^0(z)} \implies \sum_{z \in R(0,q) \ \cap \square_{0,q}^L} \delta_{z,d^0(z)} \ \text{a.s..} \tag{4.14}$$

Next, on $E_{q,L'}$, since $q < \theta^1$, $R(0,\theta^1) \cap \square_{0,q}^L$ is a subset of $R(0,q) \ \cap \square_{0,q}^L$: if $t < q$, any two paths separating at t up to time θ^1 must also separate up to time q. Let us now show that for almost every realization in $E_{q,L'}$:

$$\forall z_n = (x_n, t_n) \ \in \ R_n(0,q) \text{ s.t. } z_n \ \to \ z = (x,t) \in R(0,q), \text{ then}$$
$$z \ \in \ R(0,\theta^1) \ \text{iff} \ z_n \in R_n(0,\theta_n^1) \text{ for infinitely many } n. \tag{4.15}$$

Combining (4.14) with (4.15) yields (4.13). The rest of Step 2 is dedicated to the proof of (4.15) (thus showing (4.13)).

First, let us assume that $z \in R(0,q)$ and that z also belongs to $R(0,\theta^1)$. We now show that z_n (as defined in (4.15)) belongs to $R(0,\theta_n^1)$ for n large enough. Let $\pi^1, \pi^2 \in \mathcal{N}^b(\Sigma_0)$ passing through z and separating up to θ^1. Since $S_{\epsilon_n}(\mathcal{U}^{b_n}(\Sigma_0))$ converges to $\mathcal{N}^b(\Sigma_0)$ a.s., there are two sequences π_n^1 and π_n^2 in $S_{\epsilon_n}(\mathcal{U}^{b_n}(\Sigma_0))$ such that π_n^i converges to π^i, for $i = 1, 2$. By Lemma 4.1, $d^0(z) > 0$ a.s. and further, under our coupling $d_n^0(z_n)$ converges to $d^0(z)$ a.s.: in other words, z_n must remain macroscopically isolated from the left and from the right. Since $\pi_n^1(t_n) - \pi_n^2(t_n) \to 0$, this implies that

$$\pi_n^1(t_n) = \pi_n^2(t_n) \ \text{ for } n \text{ large enough.}$$

Next, recall that π^1 and π^2 separate at z up to time θ^1. By the latter property, π_n^1 and π_n^2 separate up to θ_n^1 for large enough n at a point z_n' with $\rho(z_n', z_n) \to 0$ as $n \to \infty$. However, because $R(0,q)$ is locally finite and $R_n(0,q) \implies R(0,q)$ a.s., z_n' and z_n must coincide after a finite rank. This proves that if $z \in R(0,q)$, but z also belongs to $R(0,\theta^1)$, then z_n also belongs to $R(0,\theta_n^1)$ for n large enough.

Next, let us assume that $z_n \in R_n(0,q)$ and also belongs to $R_n(0,\theta_n^1)$ for infinitely many n. Going to a subsequence if necessary, we can assume that z_n belongs to $R_n(0,\theta_n^1)$ for every n. Let π_n^1 and π_n^2 in $S_{\epsilon_n}(\mathcal{U}^{b_n}(\Sigma_0))$ separating at z_n up to time θ_n^1. In order to prove (4.15), we need to show that z belongs to $R(0,\theta^1)$. Going to a subsequence if necessary, π_n^i converges to $\pi^i \in \mathcal{N}^b(\Sigma_0)$ for $i = 1, 2$. Furthermore, since $d^0(z) > 0$ a.s., arguing as in

the previous paragraph, the π^i's must go through the point z. It remains to prove that π^1 and π^2 separate up to time θ^1 after passing through z. First, since $\xi^0(\theta^1)$ is locally finite a.s., and since $\xi_n^0(\theta_n^1)$ converges to $\xi^0(\theta^1)$ a.s., the set of $\xi_n^0(\theta_n^1)$ remains "sparse" as n goes to ∞. Since $\pi_n^1(\theta_n^1) \neq \pi_n^2(\theta_n^1)$, this entails

$$\lim_n \pi_n^1(\theta_n^1) - \pi_n^2(\theta_n^1) > 0,$$

which is equivalent to $\pi^1(\theta^1) \neq \pi^2(\theta^1)$.

We now need to prove that π^1 and π^2 never coincide on the time interval (t, θ^1) (where t is the time coordinate of z). Let us assume by contradiction that π^1 and π^2 meet again on this time interval. Since $\pi^1(\theta^1) \neq \pi^2(\theta^1)$, the two paths must separate at another point $z' = (x', t')$, with $t < t' < \theta^1$. Since we are only considering realizations in $E_{q,L'}$, we must have $t' < q$ (as already mentioned at the beginning of Step 2). Let $z_n' \in R_n(0, q)$ converging to z'. Since $d^0(z') > 0$ and $d_n^0(z_n')$ converges to $d^0(z')$, this implies that π_1^n and π_2^n must also go through z_n' after a finite rank. But this would contradict the fact that π_n^1 and π_n^1 separate at z_n. This ends the proof of (4.13).

Step 3. We are now ready to prove (4.10). For every $q, L' \in \mathbb{Q}^+$, and for almost every realization in $E_{q,L'}$:

$$\sum_{z \in R(0,\theta^1)\,\cap\,\square_{0,\theta^1}^L} \delta_{z,d^0(z)} \;=\; \sum_{z \in R(0,\theta^1)\,\cap\,\square_{0,q}^L} \delta_{z,d^0(z)} \;+\; \sum_{z \in R(0,\theta^1)\,\cap\,\square_{q,\theta^1}^L} \delta_{z,d^0(z)}$$
$$=\; \sum_{z \in R(0,\theta^1)\,\cap\,\square_{0,q}^L} \delta_{z,d^0(z)}\,.$$

Since the analogous identity holds at the discrete level for n large enough, (4.13) implies that for almost every realization in $E_{q,L'}$

$$\sum_{z \in R(0,\theta_n^1)\,\cap\,\square_{0,\theta_n^1}^L} \delta_{z,d_n^0(z)} \implies \sum_{z \in R(0,\theta^1)\,\cap\,\square_{0,\theta^1}^L} \delta_{z,d^0(z)} \quad \text{a.s.}$$

Finally, since $\mathbb{P}(\cup_{q,L' \in \mathbb{Q}^+} E_{q,L'}) = 1$ by Step 1, (4.10) follows. This completes the proof of Lemma 4.6. $\qquad\square$

4.4 Proof of Proposition 4.4

Going to a subsequence, we can assume w.l.o.g. that the sign of t_n remains constant as n varies. In the following, we will treat the case $\forall n \in \mathbb{N}$, $t_n \leq 0$. The other case can be handled along the same lines. We decompose the proof of Proposition 4.4 into three lemmas.

Lemma 4.7. Let $\gamma > 0$ and define the event

$$D_{\gamma,n} = \{\forall k \in \mathbb{Z} \text{ s.t. } |(2k+1)\epsilon_n| \leq \gamma, \; G_n^{((2k+1)\epsilon_n,0)}(T)$$
$$\text{and } G_n^{(0,0)}(T) \text{ only differ by their root}\}.$$

Then $\liminf_{\gamma \to 0} \liminf_{n \to \infty} \mathbb{P}(D_{\gamma,n}) = 1$.

Proof. Define

$$\gamma_n := \sup\{(2k+1)\epsilon_n : k \in \mathbb{N}, \; (2k+1)\epsilon_n \leq \gamma\}.$$

and let $l_{\gamma,n}$ (resp., $r_{\gamma,n}$) be the leftmost (resp., rightmost) path in $S_{\epsilon_n}(\mathcal{U}^{b_n})$ starting from $-\gamma_n$ (resp., γ_n). Finally, we set

$$\tau_{\gamma,n} := \inf\{t > 0 \; : \; r_{\gamma,n}(t) = l_{\gamma,n}(t)\}.$$

Let $L = 1$ in the definition of θ_n^1. Any path in $S_{\epsilon_n}(\mathcal{U}^{b_n,\kappa_n})$ starting on $[-\gamma_n, \gamma_n]$ at time 0 remains squeezed between $l_{\gamma,n}$ and $r_{\gamma,n}$, which easily implies that

$$D'_{\gamma,n} = \{\tau_{\gamma,n} \le \theta_n^1, \ |r_{\gamma,n}(u)|, |l_{\gamma,n}(u)| < 1, \forall u \in [0, \tau_{\gamma,n}]\}$$

is a subset of $D_{\gamma,n}$. (Indeed, the properties characterizing $D'_{\gamma,n}$ ensure that any path in the killed Brownian net starting at time 0 at $\bar{x}_n \in [-\gamma_n, \gamma_n]$ connects the root $(\bar{x}_n, 0)$ to the point $(r(\tau_{\gamma,n}), \tau_{\gamma,n}) = (l(\tau_{\gamma,n}), \tau_{\gamma,n}))$. On the other hand, as n go to $+\infty$, $l_{\gamma,n}$ and $r_{\gamma,n}$ converge to drifted Brownian motions starting from $-\gamma$ and γ respectively, those two Brownian motions being independent until they meet. This easily implies:

$$\forall \delta > 0, \ \lim_{\gamma \to 0} \lim_{n \to \infty} \mathbb{P}(\tau_{\gamma,n} \le \delta, \ |r_{\gamma,n}(u)|, |l_{\gamma,n}(u)| < 1, \forall u \in [0, \tau_{\gamma,n}]) \ = \ 1,$$

whereas θ_n^1 converges in distribution to θ^1, with $\theta_1 > 0$ a.s.. This implies that

$$\lim_{\gamma \to 0} \lim_{n \to \infty} \ \mathbb{P}(D'_{\gamma,n}) \ = \ 1,$$

and since $D'_{\gamma,n} \subseteq D_{\gamma,n}$, this completes the proof of the lemma. $\qquad\square$

Lemma 4.8. Let $\gamma > 0$ and define $F_{\gamma,n} \ = \ \{\forall \pi \in S_{\epsilon_n}(\mathcal{U}^{b_n})(z_n), \ |\pi(0)| < \gamma/2\}$. Then $\liminf_{n \to \infty} \mathbb{P}(F_{\gamma,n}) \ = \ 1$.

Proof. This is a direct consequence of the equicontinuity of the paths in $\cup_n S_{\epsilon_n}(\mathcal{U}^{b_n})$. $\quad\square$

Lemma 4.9. Define $H_n \ = \ \{\forall \pi \in S_{\epsilon_n}(\mathcal{U}^{b_n,\kappa_n})(z_n), \ e_\pi > 0\}$. Then $\liminf_{n \to \infty} \mathbb{P}(H_n) = 1$.

Proof. Again, we let $L = 1$ in the definition of θ_n^1. Using translation invariance (both in time and space), we have

$$
\begin{aligned}
\mathbb{P}(H_n) \ &= \ \mathbb{P}(\forall \pi \in S_{\epsilon_n}(\mathcal{U}^{b_n,\kappa_n})(0,0), \ e_\pi > -t_n) \\
&\ge \ \mathbb{P}(\forall \pi \in S_{\epsilon_n}(\mathcal{U}^{b_n,\kappa_n})(0,0), \ |\pi(u)| \le 1 \ \forall u \in [0, -t_n], \ \theta_n^1 > -t_n).
\end{aligned}
$$

Finally, the conclusion follows from the equicontinuity of $\cup_n S_{\epsilon_n}(\mathcal{U}^{b_n,\kappa_n})$ and the fact that θ_1^n goes to θ_1, with $\theta^1 > 0$ a.s.. $\qquad\square$

We are now ready to complete the proof of Proposition 4.4. It is not hard to check that for every $\gamma > 0$,

$$D_{\gamma,n} \cap F_{\gamma,n} \cap H_n \subset \{G_n^{z_n}(T) \text{ and } G_n^{(0,0)}(T) \text{ are isomorphic}\}$$

and thus,

$$\mathbb{P}(\{G_n^{z_n}(T) \text{ and } G_n^{(0,0)}(T) \text{ are isomorphic}\}^c) \ \le \ \mathbb{P}(D_{\gamma,n}^c) \ + \ \mathbb{P}(F_{\gamma,n}^c) \ + \ \mathbb{P}(H_n^c).$$

Letting successively n and γ go to ∞ and 0 respectively completes the proof of Proposition 4.4.

5 Appendix

In this section we present three models that can be written as a voter model perturbation (VMP). In the following, we use the same notation as in Section 1.2.

5.1 Spatial stochastic Lotka-Volterra model

As in [CDP13], we consider a discrete version of the stochastic Lotka-Volterra model as introduced by Pacala and Neuhauser [NP00]. It is a spin system with each site taking value in $\{0,1\}$. At each time step, the dynamics can easily be described by a two steps procedure (between t and $t+1$). First, the particle at x dies with probability $\alpha(1 - m_\epsilon(1 - \eta(x)) \, f_{x,\eta}(1 - \eta(x)))$, where m_ϵ is a function from $\{0,1\}$ to $(0,1)$ and $\alpha \in [0,1]$. Secondly, if it dies, it is replaced by the type of one its neighbor chosen according to the transition kernel K.

For the sake of illustration, we show that when $m_\epsilon(1) = m_\epsilon(0) = \epsilon > 0$ (symbiotic symmetric case) the Lotka-Volterra model can be written as a perturbation of the voter model. Other cases (i.e. when m_ϵ is not constant and might be negative) can be treated in a similar fashion.

Proposition 5.1. The symmetric symbiotic Lotka—Volterra model can be written as a perturbation of the voter model.

Proof. We will show that the symmetric symbiotic Lotka—Volterra model can be written as a perturbation of the voter model

$$P^\epsilon_{x,\eta} \;=\; (1 - \epsilon)\bar{P}^v_{x,\eta} \;+ \epsilon B_{x,\eta} \tag{5.1}$$

with

$$\bar{P}^v_{x,\eta} \;=\; (1 - \alpha)\delta_{\eta(x)} + \alpha \sum_i f_{x,\eta}(i)\delta_i$$

and $B_{x,\eta}$ is a boundary noise with

$$
\begin{aligned}
Q &= \delta_0 \times K(0,dy) \times K(0,dy) \\
g_{0,1,0} = g_{0,0,1} &= \frac{1}{2}\alpha\delta_1 + (1 - \frac{1}{2}\alpha)\delta_0 \quad \text{and } g_{0,i,j} = \delta_0 \text{ otherwise.} \\
g_{1,1,0} = g_{1,0,1} &= \frac{1}{2}\alpha\delta_0 + (1 - \frac{1}{2}\alpha)\delta_1 \quad \text{and } g_{1,i,j} = \delta_1 \text{ otherwise}
\end{aligned}
$$

(Note that there is no bulk noise in this model.) For $\eta(x) = 0$, we have

$$
\begin{aligned}
P^\epsilon_{x,\eta}(1) &= \alpha(1 - \epsilon f_{x,\eta}(1))f_{x,\eta}(1) \\
P^\epsilon_{x,\eta}(0) &= (1 - \alpha(1 - \epsilon f_{x,\eta}(1))) + \alpha(1 - \epsilon f_{x,\eta}(1))f_{x,\eta}(0)
\end{aligned}
$$

which can be rewritten as

$$
\begin{aligned}
P^\epsilon_{x,\eta}(1) &= (1 - \epsilon)[\alpha f_{x,\eta}(1)] \;+\; \epsilon[\alpha f_{x,\eta}(1) - \alpha f_{x,\eta}(1)^2] \\
&= (1 - \epsilon)[\alpha f_{x,\eta}(1)] \;+\; \epsilon[\alpha f_{x,\eta}(0)f_{x,\eta}(1)] \\
P^\epsilon_{x,\eta}(0) &= (1 - \epsilon)[\alpha f_{x,\eta}(0) + (1 - \alpha)] \;+\; \epsilon[1 - \alpha f_{x,\eta}(0)f(1)].
\end{aligned}
$$

One can get a symmetric expression for $\eta(x) = 1$. From there, one can directly check that (5.1) holds. □

5.2 The q colors Potts model

In this example, we show how our particular scaling for the voter model perturbation emerges naturally when considering a stochastic Potts model at low temperature. Stochastic one-dimensional Potts model are Markov processes $\{\eta_t\}$ in discrete or continuous time taking values in the state space $\{1, ..., q\}^{\mathbb{Z}}$ with an invariant distribution equal (or closely related) to the Gibbs measure (at inverse temperature β) with formal Hamiltonian H given by

$$H(\eta) = \sum_{x\in\mathbb{Z}} \delta_{\eta(x)\neq\eta(x+1)}, \tag{5.2}$$

where $\eta(x)$ is the x coordinate of the configuration η. Interpreting $\{1, ..., q\}$ as a set of colors, H simply counts the number of boundaries between the color-clusters of the system. In dimension 1, it is well known that for $\beta < \infty$ the Gibbs measure is unique, and that there is no phase transition for the (static) Potts model.

We will primarily be concerned with the following discrete time model in which at each integer time, the values of $\eta_t(x)$ for $x \in \mathbb{Z}$ all update simultaneously with the following probabilities

$$w_\beta = \frac{2(e^\beta - 1)}{q + 2(e^\beta - 1)}, b_\beta = \frac{(e^\beta - 1)^2 q}{((e^{2\beta} - 1) + q)(q + 2(e^\beta - 1))}, \kappa_\beta = \frac{q}{e^{2\beta} + (q - 1)}$$

This particular dynamics qualifies as a perturbation of the voter model, with the parameter ϵ corresponding to $e^{-\beta}$ and further

$$k_\beta e^{2\beta} \to q \quad \text{and} \quad b_\beta e^\beta \to q/2.$$

Remark 5.2. The scaling of the branching and killing parameters coincides with the scaling of Theorem 1.1(3)(i).

Later in this section, we will discuss the relation of the Potts model Gibbs measure to this discrete time process. But first, we motivate the choice of transition probabilities by considering a continuous time voter model perturbation with transition rates given by w_β, b_β and k_β

Proposition 5.3. Let us consider the continuous time Markov process with transition rates $(w_\beta, b_\beta, \kappa_\beta)$ – the transitions corresponding to a walk, branch and kill move respectively (see end of Section 1.2) – and let $\{B_{x,\eta}\}$ for $\eta(x-1) \neq \eta(x+1)$ and p be the uniform distribution on $\{1, \cdots, q\}$ and $B_{x,\eta} = \delta_{\eta(x-1)}$ when $\eta(x-1) = \eta(x+1)$. This model defines a contiuous time q-states Potts model at inverse temperature which is reversible with respect to Gibbs measure.

Proof. When a Poisson clock rings at site x at time t that site is updated based on the values of spins at the two nearest neighbor sites. Therefore in the detailed balance equations needed for reversibility $\eta(x-1)$ and $\eta(x+1)$ are fixed while $\eta_{t-}(x)$ makes a transition to $\eta_t(x)$. For $\bar{c}_i = (c_\ell, c_i, c_r)$ the color configuration in $\{1, \ldots, q\}^3$ of $(\eta_{t-}(x-1), \eta_{t-}(x), \eta_{t-}(x+1))$, c_f the color of $\eta_t(x)$, with $\bar{c}_f = (c_\ell, c_f, c_r)$, the detailed balance equations for the rates q of the transitions $\bar{c}_i \to \bar{c}_f$ and $\bar{c}_f \to \bar{c}_i$ are

$$q(\bar{c}_i \to \bar{c}_f)/q(\bar{c}_f \to \bar{c}_i) = \exp(-\beta\Delta H(\bar{c}_i \to \bar{c}_f)), \tag{5.3}$$

where

$$\Delta H(\bar{c}_i \to \bar{c}_f) = H(\bar{c}_f) - H(\bar{c}_i) = (\delta_{c_f \neq c_\ell} - \delta_{c_i \neq c_\ell}) + (\delta_{c_f \neq c_r} - \delta_{c_i \neq c_r}). \tag{5.4}$$

One can readily check that (5.3) is satisfied for our particular choice of the triplet $(w_\epsilon, b_\epsilon, \kappa_\epsilon)$. $\square$

Before returning to our discrete time model where all vertices update simultaneously, we consider a model in which, say, even (resp., odd) vertices update at even (resp., odd) integer times. We leave it as an exercise for the reader to convince herself that the detailed balance calculation in the proposition above for continuous time model implies the same for each time step in this (alternating) discrete time model.

For the (simultaneously updating) discrete model, we note two elementary properties: (A) if one restricts attention to the even (resp., odd) space-time sublattice of $\mathbb{Z} \times \mathbb{N}$, these two restricted stochastic processes are independent of each other. (B) The odd sublattice restriction of this dynamics is identical to the odd sublattice restriction of the

alternating discrete model. These two properties imply that an invariant measure for simultaneously updating process is $P \times P'$ where P (resp., P') is the Gibbs measure on $\mathbb{Z}$ restricted to $\mathbb{Z}_{even}$ ($\mathbb{Z}_{odd}$).

5.3 Noisy biased voter model

We start with a general description of this model. Let $(\kappa_\epsilon, b_\epsilon, \alpha)$ be three non-negative numbers with $\alpha \in [0,1]$. In a continuous time setting, the noisy biased voter model is a spin system with transition rates at x:

$$\text{transition to } 1 \qquad (1 + b_\epsilon) f_{x,1}(\eta) + \kappa_\epsilon (1 - \alpha),$$
$$\text{transition to } 2 \qquad f_{2,\eta}(x) + \kappa_\epsilon \alpha.$$

A discrete time analog of such system consists in updating each site x at each discrete time step according to the following rule

$$\text{take on color 1 with probabilty } P^\epsilon_{x,\eta}(1) = \frac{(1 + b_\epsilon) f_{1,\eta}(x) + (1 - \alpha)\kappa_\epsilon}{1 + b_\epsilon f_{1,\eta}(x) + \kappa_\epsilon},$$

$$\text{take on color 2 with probability } P^\epsilon_{x,\eta}(2) = \frac{f_{2,\eta}(x) + \alpha\kappa_\epsilon}{1 + b_\epsilon f_{1,\eta}(x) + \kappa_\epsilon}, \qquad (5.5)$$

Proposition 5.4. Let $b, k \geq 0$, and let $\kappa_\epsilon = \epsilon^2 \kappa$ and $b_\epsilon = \epsilon b$. In the case $d = 1$, the previous model is a voter model perturbation.

Proof. We show that the biased voter model is a voter model perturbation with a boundary nucleation part given by

$$Q = K(0, dy) \times K(0, dy) \times K(0, dy)$$
$$g_{2,2,2} = \delta_2, \ g_{1,1,1} = \delta_1 \qquad (5.6)$$
$$g_{1,2,2} = g_{2,1,2} = g_{2,2,1} = \tfrac{1}{3}\delta_2 + \tfrac{2}{3}\delta_1$$
$$g_{2,1,1} = g_{1,2,1} = g_{1,1,2} = \tfrac{b_\epsilon}{3}\delta_2 + (1 - \tfrac{b_\epsilon}{3})\delta_1 \qquad (5.7)$$

and a bulk nucleation part given by

$$p^\epsilon(2) = \alpha, \ \ , p^\epsilon(1) = 1 - \alpha,$$

while the rate of branching is b_ϵ and the killing rate is κ_ϵ. Expanding (5.5) in the ϵ parameter, we get that

$$P^\epsilon_{x,\eta}(1) = (f_{x,\eta}(1) + r_\epsilon(f_{x,\eta}(1))) + b_\epsilon f_{x,\eta}(1) \left(1 - (1 + b_\epsilon) f_{x,\eta}(1) + b_\epsilon f_{1,\epsilon}\right)$$
$$\qquad + \kappa_\epsilon((1 - \alpha) - f_{x,\eta}(1))$$
$$P^\epsilon_{x,\eta}(2) = (f_{x,\eta}(2) - r_\epsilon(f_{x,\eta}(1))) + b_\epsilon f_{2,\epsilon}(b_\epsilon f_{x,\eta}(1)^2 - f_{x,\eta}(1)) + \kappa_\epsilon(\alpha - f_{x,\eta}(2)). \quad (5.8)$$

where r_ϵ is a function such that $r_\epsilon(f_{x,\eta}(1)) = O(\epsilon^3)$ and for ϵ small enough, for every value of $f_{x,\eta}(1) \in [0,1]$:

$$f_{x,\eta}(1) + r_\epsilon(f_{x,\eta}(1))/w_\epsilon, \ \ f_{x,\eta}(2) - r_\epsilon(f_{x,\eta}(1))/w_\epsilon \geq 0.$$

After some straightforward manipulations, this can be rewritten under the following form

$$P^\epsilon_{x,\eta}(1) = w_\epsilon \left(f_{x,\eta}(1) + r_\epsilon(f_{x,\eta}(1))/w_\epsilon\right)$$
$$\qquad + b_\epsilon \left(f_{x,\eta}(1)^3 + 3f_{x,\eta}(2)^2 f_{x,\eta}(1) \times \frac{2}{3} + 3f_{x,\eta}(1)^2 f_{x,\eta}(2) \times \left(1 - \frac{b_\epsilon}{3}\right)\right) + \kappa_\epsilon(1 - \alpha)$$
$$P^\epsilon_{x,\eta}(2) = w_\epsilon \left(f_{x,\eta}(2) - r_\epsilon(f_{x,\eta}(1))/w_\epsilon\right)$$
$$\qquad + b_\epsilon \left(f_{x,\eta}(2)^3 + 3f_{x,\eta}(2)^2 f_{x,\eta}(1) \times \frac{1}{3} + 3f_{x,\eta}(1)^2 f_{x,\eta}(2) \times \frac{b_\epsilon}{3}\right) + \kappa_\epsilon \alpha,$$

where $w_\epsilon = 1 - b_\epsilon - \kappa_\epsilon$.

Under this condition, one can readily check that the noisy voter model can be written as a voter model perturbation with a boundary and bulk nucleations chosen as described above. $\qquad \square$

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