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Inversion, duality and Doob h-transforms for self-similar Markov processes *

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Abstract

We show that any $\mathbb{R}^d \setminus \{0\}$-valued self-similar Markov process X, with index $\alpha > 0$ can be represented as a path transformation of some Markov additive process (MAP) (θ, ξ) in $S_{d-1} \times \mathbb{R}$. This result extends the well known Lamperti transformation. Let us denote by $\widehat{X}$ the self-similar Markov process which is obtained from the MAP $(\theta, -\xi)$ through this extended Lamperti transformation. Then we prove that $\widehat{X}$ is in weak duality with X, with respect to the measure $\pi(x/\|x\|)\|x\|^{\alpha-d}dx$, if and only if (θ, ξ) is reversible with respect to the measure $\pi(ds)dx$, where $\pi(ds)$ is some σ-finite measure on S_{d-1} and dx is the Lebesgue measure on $\mathbb{R}$. Moreover, the dual process $\widehat{X}$ has the same law as the inversion $(X_{\gamma_t}/\|X_{\gamma_t}\|^2, t \geq 0)$ of X, where γ_t is the inverse of $t \mapsto \int_0^t \|X\|_s^{-2\alpha}\, ds$. These results allow us to obtain excessive functions for some classes of self-similar Markov processes such as stable Lévy processes.

Keywords: self-similar Markov processes; Markov additive processes; time change; inversion; duality; Doob h-transform.
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1 Introduction

There exist many ways to construct the three dimensional Bessel process from Brownian motion. It is generally defined as the strong solution of a stochastic differential equation driven by Brownian motion or as the norm of the three dimensional Brownian motion. It can also be obtained by conditioning Brownian motion to stay positive. Then there are several path transformations. Let us focus on the following example.

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Theorem A (M. Yor, [29])**.** Let $\{(B_t^0)_{t\ge 0}, \mathbb{P}_x\}$ and $\{(R_t)_{t\ge 0}, \mathrm{P}_x\}$, $x > 0$, be respectively the standard Brownian motion absorbed at 0 and the three dimensional Bessel process. Then $\{(R_t)_{t\ge 0}, \mathrm{P}_x\}$ can be constructed from $\{(B_t^0)_{t\ge 0}, \mathbb{P}_x\}$ through the following path transformation:

$$\{(R_t)_{t\ge 0}, \mathrm{P}_x\} = \{(1/B^0_{\gamma_t})_{t\ge 0}, \mathbb{P}_{1/x}\},$$

where $\gamma_t = \inf\{s : \int_0^s \frac{du}{(B_u^0)^4} > t\}$.

This result was actually obtained in higher dimension in [29] where the law of the time changed inversion of d-dimensional Brownian motion is fully described. Recalling that three dimensional Bessel process is a Doob h-transform of Brownian motion absorbed at 0, the following result can be considered as a counterpart of Theorem A for isotropic stable Lévy processes.

Theorem B (K. Bogdan, T. Żak, [6])**.** Let $\{(X_t)_{t\ge 0}, \mathbb{P}_x\}$, $x \in \mathbb{R}^d \setminus \{0\}$ be a d-dimensional, isotropic stable Lévy process with index $\alpha \in (0, 2]$, which is absorbed at its first hitting time of 0. Then the process

$$\{(X_{\gamma_t}/\|X\|^2_{\gamma_t})_{t\ge 0}, \mathbb{P}_{x/\|x\|^2}\}\,, \tag{1.1}$$

where $\gamma_t = \inf\{s : \int_0^s \frac{du}{\|X\|_u^{2\alpha}} > t\}$, is the Doob h-transform of X with respect to the positive excessive function $x \mapsto \|x\|^{\alpha-d}$.

When $d = 1$ and $\alpha > 1$, Yano [28] showed that the h-process which is involved in Theorem B can be interpreted as the Lévy process $\{(X_t)_{t\ge 0}, \mathbb{P}_x\}$, conditioned to avoid 0, see also Pantí [24]. Then more recently Kyprianou [20] and Kyprianou, Rivero, Satitkanitkul [21] proved that Theorem B is actually valid for any real valued stable Lévy process.

Comparing Theorems A and B, we notice that they are concerned with the same path transformation of some Markov process, and that the resulting Markov process can be obtained as a Doob h-transform of the initial process. Then one is naturally tempted to look for a general principle which would allow us to prove an overall result in an appropriate framework. It clearly appears that the self-similarity property is essential in these path transformations. Therefore a first step in our approach was an in-depth study of the structure of self-similar Markov processes. This led us to an extension of the famous Lamperti representation. The latter is the object of the next section, see Theorem 2.3, and represents one of our main results. It asserts that any self-similar Markov process absorbed at 0 can be represented as a time changed Markov additive process and actually provides a one-to-one relationship between these two classes of processes.

Then Section 3 is devoted to the study of the time changed inversion (1.1) when $\{(X_t)_{t\ge 0}, \mathbb{P}_x\}$ is any self-similar Markov process absorbed at 0. Another important step in our reasoning is the characterization of self-similar Markov processes $\{(X_t)_{t\ge 0}, \mathbb{P}_x\}$, which are in duality with the time changed inversion $\{(X_{\gamma_t}/\|X\|^2_{\gamma_t})_{t\ge 0}, \mathbb{P}_{x/\|x\|^2}\}$, see Theorem 3.7. We show that a necessary and sufficient condition for this to hold is that the underlying Markov additive process in the Lamperti representation satisfies a condition of reversibility. Some important classes of processes satisfying this condition are also described. The results of this section extend those obtained by Graversen and Vuolle-Apiala in [15].

It remains to appeal to some link between duality and Doob h-transform. More specifically in Section 4, we recover Theorems A and B, and Theorem 4 in [20] as consequences of Theorem 3.7 in Section 3 and the simple observation that if two Markov processes are in duality between themselves then it is also the case for their Doob h-transforms. This general principle actually applies to large classes of self-similar Markov processes and allows us to obtain excessive functions attached to them. We end the paper by reviewing the examples of conditioned stable Lévy processes, free Bessel processes and Dunkl processes. Let us finally emphasize that another incentive for our work was the recent paper from Alili, Graczyk and Żak [2], where some relationships between inversions and h-processes are provided in the framework of diffusions.

2 Lamperti representation of $\mathbb{R}^d \setminus \{0\}$-valued ssMp's

All Markov processes considered in this work are standard processes for which there is a reference measure. Let us first briefly recall these definitions from Section I.9 and Chapter V of [5]. A standard Markov process $Z = (Z_t)_{t\geq 0}$ is a strong Markov process, with values in some state space $E_\delta = E \cup \{\delta\}$, where E is a locally compact space with a countable base, δ is some isolated extra state and E_δ is endowed with its topological Borel σ-field. The process Z is defined on some completed, filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, (P_x)_{x\in E_\delta})$, where $P_x(Z_0 = x) = 1$, for all $x \in E_\delta$.The state δ is absorbing, that is $Z_t = \delta$, for all $t \geq \zeta(Z) := \inf\{t : Z_t = \delta\}$ and $\zeta(Z)$ will be called the lifetime of Z. The paths of Z are assumed to be right continuous on $[0, \infty)$. Moreover, they have left limits and are quasi-left continuous on $[0, \zeta)$. Finally, we assume that there is a reference measure, that is a σ-finite measure $\mu(dy)$ on E such that for each $x \in E$, the potential measure $E_x(\int_0^\zeta \mathbb{1}_{\{Z_t\in dy\}}\, dt)$ of Z is equivalent to $\mu(dy)$. We will generally omit to mention $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0})$ and in what follows, a process satisfying the above properties will be denoted by $\{Z, P_x\}$ and will simply be referred to as an E-valued Markov process absorbed at δ. (Note that absorption may or may not hold with positive probability.)

In all this work, we fix an integer $d \geq 1$ and we denote by $\|x\|$ the Euclidean norm of $x \in \mathbb{R}^d$. We also denote by S_{d-1} the sphere of $\mathbb{R}^d$, where $S_{d-1} = \{-1, +1\}$ if $d = 1$. Let H be a locally compact subspace of $\mathbb{R}^d \setminus \{0\}$. An H-valued Markov process $\{X, \mathbb{P}_x\}$ absorbed at 0, which satisfies the following scaling property: there exists an index $\alpha \geq 0$ such that for all $a > 0$ and $x \in H$,

$$\{X, \mathbb{P}_x\} = \{(aX_{a^{-\alpha}t},\, t \geq 0), \mathbb{P}_{a^{-1}x}\}\,, \tag{2.1}$$

is called an H-valued self-similar Markov process (ssMp for short). The scaling property implies in particular that H should satisfy $H = aH$, for any $a > 0$. Therefore the space H is necessarily a cone of $\mathbb{R}^d \setminus \{0\}$, that is a set of the form

$$H := \phi(S \times \mathbb{R})\,, \tag{2.2}$$

where S is some locally compact subspace of S_{d-1} and ϕ is the homeomorphism, $\phi : S_{d-1} \times \mathbb{R} \to \mathbb{R}^d \setminus \{0\}$ defined by $\phi(y, z) = ye^z$. Henceforth, H and S will be any locally compact subspaces of $\mathbb{R}^d \setminus \{0\}$ and S_{d-1} respectively, which are related to each other by (2.2). Note that when $\{X, \mathbb{P}_x\}$ is an H-valued ssMp with index α, the process $(Y, P_y) := \left(\frac{X}{\|X\|^2}, \mathbb{P}_{\frac{x}{\|x\|^2}}\right)$ is an H-valued ssMp with index $-\alpha$. This remark entails that all the general results of this paper are valid for ssMp's with nonpositive indices. However, since most of our applications are concerned with ssMp's having nonnegative indices, we will always assume that $\alpha \geq 0$. The main result of this section asserts that ssMp's can be obtained as time changed Markov additive processes (MAP) which we now define.

A MAP $\{(\theta, \xi), P_{y,z}\}$ is an $S \times \mathbb{R}$-valued Markov process absorbed at some extra state δ, such that for any $y \in S$, $z \in \mathbb{R}$, $s, t \geq 0$, and for any positive measurable function f, defined on $S \times \mathbb{R}$,

$$E_{y,z}(f(\theta_{t+s}, \xi_{t+s} - \xi_t), t + s < \zeta_p \,|\, \mathcal{F}_t) = E_{\theta_t,0}(f(\theta_s, \xi_s), s < \zeta_p)\mathbb{1}_{\{t<\zeta_p\}}\,, \tag{2.3}$$

where we set $\zeta_p := \zeta(\theta, \xi)$ for the lifetime of $\{(\theta, \xi), P_{y,z}\}$, in order to avoid heavy notation. Let us now stress the following important remarks.

Remark 2.1. Let $\{(\theta,\xi),P_{y,z}\}$ be any MAP and set $\theta_t=\delta'$, $t\ge\zeta_p$, for some extra state δ'. Then according to the definition of MAP's, for any fixed $z\in\mathbb{R}$, the process $\{\theta,P_{y,z}\}$ is an S-valued Markov process absorbed at δ', such that $P_{y,z}(\theta_0=y)=1$, for all $y\in S$ and whose transition semigroup does not depend on z.

Remark 2.2. If for any $z\in\mathbb{R}$, the law of the process $(\xi_t,0\le t<\zeta_p)$ under $P_{y,z}$ does not depend on $y\in S$, then (2.3) entails that the latter is a possibly killed Lévy process such that $P_{y,z}(\xi_0=z)=1$, for all $z\in\mathbb{R}$. In particular, ζ_p is exponentially distributed and its mean does not depend on y,z. Moreover, when ζ_p is finite, the process $(\xi_t,0\le t<\zeta_p)$ admits almost surely a left limit at its lifetime. Examples of such MAP's can be constructed by coupling any S-valued Markov process θ with any independent real valued Lévy process ξ and by killing the couple (θ,ξ) at an independent exponential time. Isotropic MAP's also satisfy this property. These cases are described in Section 3, see parts 2. and 3. of Proposition 3.2, Definition 3.3 and the remark which follows.

MAP's taking values in general state spaces were introduced in [14] and [11]. We also refer to Chapter XI 2.a in [3] for an account on MAP's in the case where θ is valued in a finite set. In this particular setting, they are also accurately described in the articles [19] and [12], see Sections A.1 and A.2 in [12], which inspired the following extension of Lamperti representation.

Theorem 2.3. *Let $\alpha\ge 0$ and $\{(\theta,\xi),P_{y,z}\}$ be a MAP in $S\times\mathbb{R}$, with lifetime ζ_p and absorbing state δ. Define the process X by*

$$X_t=\begin{cases}\theta_{\tau_t}e^{\xi_{\tau_t}}\,, & \text{if}\quad t<\int_0^{\zeta_p}\exp(\alpha\xi_s)\,ds\,,\\ 0\,, & \text{if}\quad t\ge\int_0^{\zeta_p}\exp(\alpha\xi_s)\,ds\,,\end{cases}$$

where τ_t is the time change $\tau_t=\inf\{s:\int_0^s e^{\alpha\xi_u}\,du>t\}$, for $t<\int_0^{\zeta_p}e^{\alpha\xi_s}\,ds$. Define the probability measures $\mathbb{P}_x=P_{x/\|x\|,\log\|x\|}$, for $x\in H$ and $\mathbb{P}_0=P_\delta$. Then the process $\{X,\mathbb{P}_x\}$ is an H-valued ssMp, with index α and lifetime $\int_0^{\zeta_p}\exp(\alpha\xi_s)\,ds$.

Conversely, let $\{X,\mathbb{P}_x\}$ be an H-valued ssMp, with index $\alpha\ge 0$ and denote by ζ_c its lifetime. Define the process (θ,ξ) by

$$\begin{cases}\xi_t=\log\|X\|_{A_t}\quad\text{and}\quad\theta_t=\frac{X_{A_t}}{\|X\|_{A_t}}\,, & \text{if}\quad t<\int_0^{\zeta_c}\frac{ds}{\|X_s\|^\alpha}\,,\\ (\xi_t,\theta_t)=\delta\,,\quad\text{if}\quad t\ge\int_0^{\zeta_c}\frac{ds}{\|X_s\|^\alpha}\,,\end{cases}$$

where δ is some extra state, and A_t is the time change $A_t=\inf\{s:\int_0^s\frac{du}{\|X_u\|^\alpha}>t\}$, for $t<\int_0^{\zeta_c}\frac{ds}{\|X_s\|^\alpha}$. Define the probability measures, $P_{y,z}:=\mathbb{P}_{ye^z}$, for $y\in S$, $z\in\mathbb{R}$ and $P_\delta=\mathbb{P}_0$. Then the process $\{(\theta,\xi),P_{y,z}\}$ is a MAP in $S\times\mathbb{R}$, with lifetime $\int_0^{\zeta_c}\frac{ds}{\|X_s\|^\alpha}$.

Proof. Let $(\mathcal{G}_t)_{t\ge0}$ be the filtration corresponding to the probability space on which the MAP $\{(\theta,\xi),P_{y,z}\}$ is defined. Then the process $\{Y,\mathbb{P}_x\}$, where $\mathbb{P}_x$ is defined in the statement and $Y_t=\theta_t e^{\xi_t}$, for $t<\zeta_p$ and $Y_t=0$, for $t\ge\zeta_p$, is the image of (θ,ξ) through an obvious one to one measurable mapping, say $\phi_\delta:(S_{d-1}\times\mathbb{R})\cup\{\delta\}\to\mathbb{R}^d$. Hence it is clearly a standard process, as defined in the beginning of this section, in the filtration $(\mathcal{G}_t)_{t\ge0}$. Moreover, if ν is the reference measure of $\{(\theta,\xi),P_{y,z}\}$, then $\nu\circ\phi_\delta^{-1}$ is a reference measure for $\{Y,\mathbb{P}_x\}$. Now define τ_t as in the statement if $t<\int_0^{\zeta_p}e^{\alpha\xi_s}\,ds$, and set $\tau_t=\infty$ and $X_{\tau_t}=0$, if $t\ge\int_0^{\zeta_p}e^{\alpha\xi_s}\,ds$. Since $e^{\xi_s}=\|Y_s\|$, $(\tau_t)_{t\ge0}$ is the right continuous inverse of the continuous, additive functional $t\mapsto\int_0^{t\wedge\zeta_p}\|Y_s\|^\alpha\,ds$ of $\{Y,\mathbb{P}_x\}$, which is strictly increasing on $(0,\zeta_p)$. It follows from part (v) of Exercise (2.11), in Chapter V of [5], that $\{X,\mathbb{P}_x\}$ is a standard process in the filtration $(\mathcal{G}_{\tau_t})_{t\ge0}$. Finally, note that $\zeta_c:=\int_0^{\zeta_p}e^{\alpha\xi_s}\,ds$ is the lifetime of X. Then we derive from the identity

$\mathbb{E}_x(\int_0^{\zeta_c} \mathbb{I}_{\{X_t\in dy\}}\, dt) = \|y\|^\alpha \mathbb{E}_x(\int_0^{\zeta_p} \mathbb{I}_{\{Y_t\in dy\}}\, dt)$ and the fact that for all $x \in H$, $\|Y_t\| > 0$, $\mathbb{P}_x$-a.s., for $t < \zeta_p$, that $\nu \circ \phi_\delta^{-1}$ is also reference measure for $\{X, \mathbb{P}_x\}$.

Now we check the scaling property as follows. Let $a > 0$, then for $t < a^\alpha \int_0^{\zeta_p} e^{\alpha\xi_s}\, ds$,

$$\tau_{a^{-\alpha}t} = \inf\{s : \int_0^s e^{\alpha(\ln a+\xi_v)}\, dv > t\}\,.$$

Let us set $\xi_t^{(a)} = \ln a + \xi_t$, then with obvious notation, $\tau_{a^{-\alpha}t} = \tau_t^{(a)}$ and

$$X_{a^{-\alpha}t} = a^{-1}\theta_{\tau_t^{(a)}} \exp(\xi^{(a)}_{\tau_t^{(a)}})\,.$$

But the equality $\{(\theta, \xi^{(a)}), P_{y,-\ln a+z}\} = \{(\theta, \xi), P_{y,z}\}$ follows from the definition (2.3) of MAP's, so that with $x = ye^z$, $a^{-1}x = ye^{-\ln a+z}$, $\mathbb{P}_x = P_{y,z}$ and $\mathbb{P}_{a^{-1}x} = P_{y,-\ln a+z}$, we have

$$\{(aX_{a^{-\alpha}t},\, t \ge 0), \mathbb{P}_{a^{-1}x}\} = \{(X_t,\, t \ge 0), \mathbb{P}_x\}\,.$$

Conversely, let $\{X, \mathbb{P}_x\}$ be a ssMp with index α. Then we prove that the process $\{(\theta, \xi), P_{y,z}\}$ of the statement is a standard process which admits a reference measure through the same arguments as in the direct part of the proof. We only have to check that this process is a MAP. Let $(\mathcal{F}_t)_{t\ge0}$ be the filtration of the probability space on which $\{X, \mathbb{P}_x\}$ is defined. Define A_t as in the statement if $t < \int_0^{\zeta_c} \frac{ds}{\|X_s\|^\alpha}$, set $A_t = \infty$, if $t \ge \int_0^{\zeta_c} \frac{ds}{\|X_s\|^\alpha}$ and note that for each t, A_t is a stopping time of $(\mathcal{F}_t)_{t\ge0}$.Then let us prove that $\{(\theta, \xi), P_{y,z}\}$ is a MAP in the filtration $\mathcal{G}_t := \mathcal{F}_{A_t}$. We denote the usual shift operator by S_t and note that for all $s, t \ge 0$,

$$A_{t+s} = A_t + S_{A_t}(A_s)\,.$$

Set $\zeta_p = \int_0^{\zeta_c} \frac{ds}{\|X_s\|^\alpha}$. Then from the strong Markov property of $\{X, \mathbb{P}_x\}$ applied at the stopping time A_t, we obtain from the definition of $\{(\theta, \xi), P_{y,z}\}$ in the statement, that for any positive, Borel function f,

$$\begin{aligned}
& E_{\frac{x}{\|x\|},\log\|x\|}(f(\theta_{t+s}, \xi_{t+s} - \xi_t), t + s < \zeta_p \,|\, \mathcal{G}_t) \\
= \;& \mathbb{E}_x\left(f\left(S_{A_t}\left(\frac{X_{A_s}}{\|X\|_{A_s}}\right), \log\frac{S_{A_t}(\|X\|_{A_s})}{\|X\|_{A_t}}\right), A_t + S_{A_t}(A_s) < \zeta_c \,|\, \mathcal{G}_t\right) \\
= \;& \mathbb{E}_{X_{A_t}}\left(f\left(\frac{X_{A_s}}{\|X\|_{A_s}}, \log\frac{\|X\|_{A_s}}{z}\right), A_s < \zeta_c\right)_{z=\|X\|_{A_t}} \mathbb{I}_{\{A_t<\zeta_c\}} \\
= \;& \mathbb{E}_{\frac{X_{A_t}}{\|X\|_{A_t}}}\left(f\left(\frac{X_{A_s}}{\|X\|_{A_s}}, \log\|X\|_{A_s}\right), A_s < \zeta_c\right)\mathbb{I}_{\{A_t<\zeta_c\}} \\
= \;& E_{\theta_t,0}(f(\theta_{t+s}, \xi_{t+s} - \xi_t), t + s < \zeta_p)\mathbb{I}_{\{t<\zeta_p\}}\,,
\end{aligned}$$

where the third equality follows from the self-similarity property of $\{X, \mathbb{P}_x\}$. We have obtained (2.3) and the theorem is proved. □

This theorem provides a one-to-one correspondence between ssMp's with index $\alpha \ge 0$ and MAP's, in the general setting of standard processes which have a reference measure. We emphasize that $\{X, \mathbb{P}_x\}$ and $\{(\theta, \xi), P_{y,z}\}$ can have very broad behaviours at their lifetimes. For instance, $\{X, \mathbb{P}_x\}$ can have a finite lifetime ζ_c, but may or may not have a left limit at ζ_c. Moreover, whether or not ζ_c is finite, either $\int_0^{\zeta_c} \frac{ds}{\|X_s\|^\alpha} = \infty$ and $\{(\theta, \xi), P_{y,z}\}$ has infinite lifetime or $\int_0^{\zeta_c} \frac{ds}{\|X_s\|^\alpha} < \infty$ and $\{(\theta, \xi), P_{y,z}\}$ has finite lifetime $\zeta_p = \int_0^{\zeta_c} \frac{ds}{\|X_s\|^\alpha}$ and may or may not have a left limit at ζ_p. However, in all commonly studied cases, the processes $\{X,\mathbb{P}_x\}$ and $\{(\theta,\xi),P_{y,z}\}$ admit almost surely a left limit at their lifetime.

For instance, if $d=1$ and $S=\{1\}$, then it follows from (2.3) that ξ is a possibly killed real Lévy process. Hence our result implies Theorem 4.1 of Lamperti [22], who proved that all positive self-similar Markov processes can be obtained as exponentials of time changed Lévy process. In this case, in order to describe the behaviour of $\{X,\mathbb{P}_x\}$ at its lifetime, it suffices to note from general properties of Lévy processes that $\int_0^{\zeta_p}\exp(\alpha\xi_s)\,ds=\infty$ if and only if ξ is an unkilled Lévy process such that $\limsup\xi_t=\infty$, almost surely.

More generally, whenever S is a finite set, for all z, $\{\theta,P_{y,z}\}$ is a possibly absorbed continuous time Markov chain. As we have already observed, the law of this Markov chain does not depend on z. Then it is plain from the definition that between two successive jump times of θ, the process ξ behaves like a Lévy process. Therefore, if $n=\text{card}(S)$, then the law of $\{(\theta,\xi),P_{y,z}\}$ is characterized by the intensity matrix $Q=(q_{ij})_{i,j\in S}$ of θ, n non killed Lévy processes $\xi^{(1)},\dots,\xi^{(n)}$, and the real valued random variables Δ_{ij}, such that $\Delta_{ii}=0$ and where, for $i\neq j$, Δ_{ij} represents the size of the jump of ξ when θ jumps from i to j. More specifically, the law of $\{(\theta,\xi),P_{y,z}\}$ is given by

$$E_{i,0}(e^{u\xi_t},\theta_t=j)=(e^{A(u)t})_{i,j}\,,\quad i,j\in S\,,\quad u\in\mathbb{C}\,, \tag{2.4}$$

where $A(u)$ is the matrix,

$$A(u)=\text{diag}(\psi_1(u),\dots,\psi_n(u))+(q_{ij}G_{i,j}(u))_{i,j\in S}\,,$$

$\psi_1,\dots,\psi_n$ are the characteristic exponents of the Lévy processes $\xi^{(i)}$, $i=1,\dots,n$, that is $E(e^{u\xi_1^{(i)}})=e^{\psi_i(u)}$, and $G_{i,j}(u)=E(\exp(u\Delta_{i,j})$. We refer to Section XI.2 of [3] and Sections A.1 and A.2 of [12] for more details. Note that when $d=1$, any $\mathbb{R}\setminus\{0\}$-valued ssMp absorbed at 0 is represented by such a MAP. The case where the intensity matrix Q is irreducible has been intensively studied in [8], [19] and [12]. We emphasize that in the present paper, irreducibility is not required.

We end this section with an application of Theorem 2.3 to a construction of ssMp's which are not killed when they hit 0. Let $\{X,\mathbb{P}_x\}$ be an $\mathbb{R}^d$-valued Markov process satisfying the scaling property (2.1) with $\alpha>0$, and assume that it has an infinite lifetime. This means in particular that $\{X,\mathbb{P}_x\}$ can possibly hit 0 without being absorbed, like real Brownian motion for instance. In Corollary 2.4 we give a path construction of $\{X,\mathbb{P}_x\}$ by adding an extra coordinate to X in order to obtain a new ssMp, with values in $\mathbb{R}^{d+1}$, which never hits 0 so that Theorem 2.3 can be applied.

More specifically, let us consider the trivial real valued ssMp $\{Y,\mathbf{P}_y\}$, whose law is defined by $\mathbf{E}_y(f(Y_t))=f(\text{sgn}(y)(|y|^\alpha+t)^{1/\alpha})$, for $y\in\mathbb{R}\setminus\{0\}$. Then clearly the process $\{(X,Y),\mathbb{P}_x\otimes\mathbf{P}_y\}$ is an $\mathbb{R}^{d+1}$-valued ssMp which never hits 0. Hence, from Theorem 2.3, it admits a representation from a MAP $\{(\theta,\xi),P_{y,z}\}$ in $S_d\times\mathbb{R}$, such that $\int_0^{\zeta_p}e^{\alpha\xi_s}\,ds=\infty$, $P_{y,z}$-a.s. for all y,z. Therefore, the process $\{X,\mathbb{P}_x\}$ can be represented as a functional of this MAP for all $t\in[0,\infty)$ and since Y is deterministic, this MAP is itself a functional of $\{X,\mathbb{P}_x\}$.

Corollary 2.4. *Let $\{X,\mathbb{P}_x\}$ be an $\mathbb{R}^d$-valued Markov process satisfying the scaling property (2.1) with $\alpha>0$ and assume that its lifetime is infinite $\mathbb{P}_x$-a.s. for all $x\in\mathbb{R}^d\setminus\{0\}$. Then the process*

$$\begin{cases}\xi_t=\log(\|X\|_{A_t}^2+A_t^{2/\alpha})^{1/2}\ \text{ and }\ \theta_t=\frac{(X_{A_t},A_t^{1/\alpha})}{(\|X\|_{A_t}^2+A_t^{2/\alpha})^{1/2}}\,,\ \text{ if }\ t<\int_0^\infty\frac{ds}{(\|X\|_s^2+s^{2/\alpha})^{1/2}}\,,\\ (\xi_t,\theta_t)=\delta\,,\ \text{ if }\ t\geq\int_0^\infty\frac{ds}{(\|X\|_s^2+s^{2/\alpha})^{1/2}}\,,\end{cases}$$

where δ is an extra state and $A_t=\inf\{s:\int_0^s\frac{du}{(\|X\|_u^2+u^{2/\alpha})^{\alpha/2}}>t\}$, is a MAP in $S_d\times\mathbb{R}$, with lifetime ζ_p, such that $\int_0^{\zeta_p}e^{\alpha\xi_s}\,ds=\infty$, $P_{y,z}$-a.s. for all y,z.

Moreover, the process $\{X, \mathbb{P}_x\}$ can be represented as follows:

$$X_t = \bar{\theta}_{\tau_t} e^{\xi_{\tau_t}}\,, \quad t \geq 0\,,$$

where $\tau_t = \inf\{s : \int_0^s e^{\alpha\xi_u}\, du > t\}$, $\bar{\theta} = (\theta^{(1)}, \dots, \theta^{(d)})$ and $\theta^{(i)}$ is the i-th coordinate of θ.

3 Inversion and duality of ssMp's

Recall that two E-valued Markov processes absorbed at δ with respective semigroups $(P_t)_{t\geq 0}$ and $(\widehat{P}_t)_{t\geq 0}$ are in weak duality with respect to some σ-finite measure $m(dx)$ if for all positive measurable functions f and g,

$$\int_E g(x) P_t f(x)\, m(dx) = \int_E f(x) \widehat{P}_t g(x)\, m(dx)\,. \tag{3.1}$$

Duality holds when moreover, P_t and $\widehat{P}_t$ are absolutely continuous with respect to $m(dx)$. However, we will make an abuse of language by simply saying that they are in duality, whenever they are in weak duality. With the convention that all measurable functions on E vanish at the isolated point δ, duality is sometimes defined by $\int_{E\cup\{\delta\}} g(x)P_t f(x)\, m(dx) = \int_{E\cup\{\delta\}} f(x)\widehat{P}_t g(x)\, m(dx)$, which is equivalent to (3.1). We refer to Chapter 13 in [10] where duality of standard processes is fully described.

For a MAP $\{(\theta,\xi), P_{y,z}\}$, we denote by $\{(\theta,-\xi), P_{y,-z}\}$ the process with lifetime ζ_p, obtained from $\{(\theta,\xi), P_{y,z}\}$ simply by replacing ξ by its negative. Then $\{(\theta,-\xi), P_{y,-z}\}$ is clearly a standard process, with an obvious reference measure, which satisfies (2.3). Hence, $\{(\theta,-\xi), P_{y,-z}\}$ is a MAP. In this section, we will focus on MAP's $\{(\theta,\xi), P_{y,z}\}$ such that $\{(\theta,\xi), P_{y,z}\}$ and $\{(\theta,-\xi), P_{y,-z}\}$ are in weak duality with respect to the measure $\pi(ds)dx$ on $S\times\mathbb{R}$, where $\pi(ds)$ is some σ-finite measure on S and dx is the Lebesgue measure on $\mathbb{R}$. We will need on the following characterisation of this duality.

Lemma 3.1. *The MAP's $\{(\theta,\xi), P_{y,z}\}$ and $\{(\theta,-\xi), P_{y,-z}\}$ are in duality with respect to the measure $\pi(ds)dx$ if and only if the following identity between measures*

$$P_{y,0}(\theta_t \in dy_1, \xi_t \in dz)\, \pi(dy) = P_{y_1,0}(\theta_t \in dy, \xi_t \in dz)\, \pi(dy_1)\,, \tag{3.2}$$

holds on $S\times\mathbb{R}\times S$.

We call (3.2) the reversibility property of the MAP $\{(\theta,\xi), P_{y,z}\}$, or equivalently we will say that $\{(\theta,\xi), P_{y,z}\}$ is reversible (with respect to the measure $\pi(ds)dx$).

Proof. We can write for all nonnegative Borel functions f and g on $(S\times\mathbb{R})\cup\{\delta\}$ which vanish at δ,

$$\begin{aligned}
&\int_{S\times\mathbb{R}} f(y,z) E_{y,z}(g(\theta_t,\xi_t))\, \pi(dy) dz \\
&= \int_S E_{y,0}\left(\int_{\mathbb{R}} f(y,z) g(\theta_t, \xi_t + z)\, dz, t < \zeta_p\right) \pi(dy) \\
&= \int_S \int_{\mathbb{R}} E_{y,0}\left(f(y, z_1 - \xi_t) g(\theta_t, z_1), t < \zeta_p\right)\, \pi(dy)\, dz_1 \\
&= \int_{S^2} \int_{\mathbb{R}^2} f(y, z_1 - u) g(y_1, z_1) P_{y,0}(\theta_t \in dy_1, \xi_t \in du)\, \pi(dy)\, dz_1 \\
&= \int_{S^2} \int_{\mathbb{R}^2} f(y, z_1 - u) g(y_1, z_1) P_{y_1,0}(\theta_t \in dy, \xi_t \in du)\, \pi(dy_1)\, dz_1 \\
&= \int_{S\times\mathbb{R}} g(y_1, z_1) E_{y_1,-z_1}(f(\theta_t, -\xi_t))\, \pi(dy_1) dz_1\,,
\end{aligned}$$

where the first identity follows from the definition of MAP's, the second one is obtained from a change of variables and the fourth identity is due to (3.2). Then comparing the first and the last term of the above identities, we obtain duality between $\{(\theta,\xi),P_{y,z}\}$ and $\{(\theta,-\xi),P_{y,-z}\}$ with respect to the measure $\pi(dy)dz$. Conversely, we prove that the latter duality implies (3.2) from the same computation. □

By integrating (3.2) over the variable z, it appears from Lemma 3.1 that if a MAP $\{(\theta,\xi),P_{y,z}\}$ is reversible with respect to the measure $\pi(ds)dx$, then the Markov process $\{\theta,P_{y,z}\}$ is in duality with itself with respect to $\pi(ds)$, which is also sometimes called the reversibility property of $\{\theta,P_{y,z}\}$ and justifies our terminology for $\{(\theta,\xi),P_{y,z}\}$.

In the next proposition, we give sufficient conditions for a MAP to be reversible. As will be seen later on, each case corresponds to a well known class of ssMp's via the representation of Theorem 2.3.

Proposition 3.2. *In each of the following three cases, the reversibility condition* (3.2) *is satisfied.*

1. *Assume that S is finite. Then the MAP $\{(\theta,\xi),P_{y,z}\}$ is reversible if and only if $\{\theta,P_{y,z}\}$ is in duality with itself with respect to some measure $(\pi_i,\, i\in S)$ and the random variables Δ_{ij} introduced in* (2.4) *are such that $\Delta_{ij}\stackrel{(d)}{=}\Delta_{ji}$, for all $i,j\in S$.*
2. *The transition probabilities of the MAP $\{(\theta,\xi),P_{y,z}\}$ have the following particular form:*
$$\begin{cases} P_{y,z}(\theta_t\in dy_1,\xi_t\in dz_1)=e^{-\lambda t}P_y^{\theta'}(\theta'_t\in dy_1)P_z^{\xi'}(\xi'_t\in dz_1)\,,\\ P_{y,z}((\theta_t,\xi_t)=\delta)=1-e^{-\lambda t}\,,\end{cases}\tag{3.3}$$
for all $t\ge 0$, $(y,z),(y_1,z_1)\in S\times H$, where $\lambda>0$ is some constant, $\{\xi',P_z^{\xi'}\}$ is any non killed real Lévy process and $\{\theta',P_y^{\theta'}\}$ is any Markov process on S with infinite lifetime. Moreover, $\{\theta',P_y^{\theta'}\}$ is in duality with itself with respect to some σ-finite measure $\pi(ds)$.
3. *$S=S_{d-1}$ and for any orthogonal transformation T of S_{d-1} and for all $(y,z)\in S_{d-1}\times\mathbb{R}$, $\{(\theta,\xi),P_{y,z}\}=\{(T(\theta),\xi),P_{T^{-1}(y),z}\}$. In this case, $\pi(ds)$ is the Lebesgue measure on S_{d-1}.* (*When $d=1$, the Lebesgue measure on S_{d-1} is to be understood as the discrete symmetric measure on* $\{-1,+1\}$.)

Proof. 1. Recall the notation of Section 2. Then duality with itself of $\{\theta,P_{y,z}\}$ with respect to some measure $(\pi_i,\, i\in S)$ (i.e. reversibility) is equivalent to $\pi_iq_{ij}=\pi_jq_{ji}$, for all $i,j\in S$. Since $\Delta_{ij}\stackrel{(d)}{=}\Delta_{ji}$, we have $\pi_iG(u)_{ij}=\pi_jG(u)_{ji}$, which implies
$$\pi_iE_{i,0}(e^{u\xi_t},\theta_t=j)=\pi_jE_{j,0}(e^{u\xi_t},\theta_t=i)\,,\tag{3.4}$$
and proves that $\{(\theta,\xi),P_{y,z}\}$ is reversible with respect to $(\pi_i,\, i\in S)$. Conversely, (3.4) with $u=0$ implies that $\{\theta,P_{y,z}\}$ is reversible with respect to $(\pi_i,\, i\in S)$ and furthermore that $\Delta_{ij}\stackrel{(d)}{=}\Delta_{ji}$, for all $i,j\in S$.

2. We easily check that the law defined in (3.3) is that of a MAP. Then (3.2) follows directly from the particular form of this law.

3. Then we prove the result in the isotropic case. Let us denote by $O(d)$ the orthogonal group and by $H(dt)$ the Haar measure on this group. It is known that since $O(d)$ acts transitively on the sphere S_{d-1}, then for any $y_1\in S_{d-1}$ and any positive Borel function f, defined on S_{d-1},
$$\int_{O(d)}f(hy_1)H(dh)=\int_{S_{d-1}}f(y)\,dy\,.\tag{3.5}$$

Let g be another positive Borel function defined on S_{d-1} and $\lambda \in \mathbb{R}$. Assume moreover that $f(\delta) = g(\delta) = 0$, then from the assumption and (3.5), for any $y_1 \in S_{d-1}$,

$$\begin{aligned}\int_{S_{d-1}} g(y)E_{y,0}(e^{i\lambda\xi_t}f(\theta_t))dy &= \int_{O(d)} g(hy_1)E_{hy_1,0}(e^{i\lambda\xi_t}f(\theta_t))H(dh)\\ &= \int_{O(d)} g(hy_1)E_{y_1,0}(e^{i\lambda\xi_t}f(h^{-1}\theta_t), t<\zeta_p)H(dh)\\ &= E_{y_1,0}\left(e^{i\lambda\xi_t}\int_{O(d)} g(hy_1)f(h^{-1}\theta_t)H(dh), t<\zeta_p\right).\end{aligned}$$

Let $h' \in O(d)$ such that $y_1 = h'\theta_t$ and make the change of variables $h'' = hh'$, then since $H(dh)$ is the Haar measure, we have

$$\begin{aligned}&E_{y_1,0}\left(e^{i\lambda\xi_t}\int_{O(d)} g(hy_1)f(h^{-1}\theta_t)H(dh), t<\zeta_p\right)\\ =\ &E_{y_1,0}\left(e^{i\lambda\xi_t}\int_{O(d)} g(hh'\theta_t)f(h^{-1}h'^{-1}y_1)H(dh), t<\zeta_p\right)\\ =\ &E_{y_1,0}\left(e^{i\lambda\xi_t}\int_{O(d)} g(h''\theta_t)f(h''^{-1}y_1)H(dh''), t<\zeta_p\right)\\ =\ &\int_{S_{d-1}} f(y)E_{y,0}(e^{i\lambda\xi_t}g(\theta_t))dy\,,\end{aligned}$$

where we have used the assumption and (3.5) again in the last equality. Then comparing the first and the last term in the above equalities gives (3.2). ☐

Examples given in this proposition lead to the definition of two important classes of ssMp's: those which satisfy the skew product property and those which are isotropic.

Definition 3.3. *Let $\{(\theta,\xi), P_{y,z}\}$ and $\{X, \mathbb{P}_x\}$ be respectively a MAP and a ssMp which are related to each other through the representation of Theorem 2.3.*

Then we say that $\{(\theta,\xi), P_{y,z}\}$ and $\{X, \mathbb{P}_x\}$ satisfy the skew product property if the transition probabilities of the MAP $\{(\theta,\xi), P_{y,z}\}$ have the form (3.3). (Note that we do not require the process $\{\theta', P^{\theta'}_y\}$ involved in Proposition 3.2 to be in duality with itself, and hence the process $\{(\theta,\xi), P_{y,z}\}$ will have the skew product and reversibility properties if and only if (3.3) is satisfied and $\{\theta', P^{\theta'}_y\}$ is in duality with itself.)

We say that $\{(\theta,\xi), P_{y,z}\}$ and $\{X, \mathbb{P}_x\}$ are isotropic if the MAP $\{(\theta,\xi), P_{y,z}\}$ satisfies conditions of part 3 of Proposition 3.2.

Note that this common definition of isotropy for $\{(\theta,\xi), P_{y,z}\}$ and $\{X, \mathbb{P}_x\}$ relies on the fact that for any orthogonal transformation T of S_{d-1}, $(\theta_t,\xi_t)_{t\ge0} \overset{(d)}{=} (T(\theta_t),\xi_t)_{t\ge0}$, under $P_{y,z}$, for all $(y,z) \in S_{d-1}\times\mathbb{R}$ if and only if $(X_t)_{t\ge0} \overset{(d)}{=} (T(X)_t)_{t\ge0}$ under $\mathbb{P}_x$, for all $x \in \mathbb{R}^d \setminus \{0\}$. Let us also stress some facts in the following remarks.

Remark 3.4. If $\{X, \mathbb{P}_x\}$ is isotropic, then its norm is a positive ssMp. Hence from the Lamperti representation of $\|X\|$, the process ξ appearing in the representation of $\{X, \mathbb{P}_x\}$, given in Theorem 2.3 is a possibly killed Lévy process. This fact can also be derived directly from the identity $\{(\theta,\xi), P_{y,z}\} = \{(T(\theta),\xi), P_{T^{-1}(y),z}\}$, where T is any transformation of $O(d)$, and Remark 2.2.

Remark 3.5. From an obvious extension of part 3 of Proposition 3.2, if there exists a one-to-one measurable transformation $\varphi : S \to S$ such that the MAP $\{(\varphi(\theta),\xi), P_{y,z}\}$ is isotropic, then the MAP $\{(\theta,\xi), P_{y,z}\}$ is reversible with respect to the measure $d\varphi^{-1}(s)dx$, where $d\varphi^{-1}(s)$ is the image by φ of the Lebesgue measure on S_{d-1}.

Remark 3.6. Lemma 3.1 provides a very simple means to construct a non reversible MAP. Indeed, it suffices to consider a non reversible continuous time Markov chain with values in a finite set of S_{d-1} and to couple it with any independent Lévy process, in the same way as in (3.3).

Before stating the main result of this section, we need the following definitions. Let $\pi(dy)$ be any σ-finite measure on S, then we define the measure $\Lambda_\pi(dx)$ on H as the image of the measure $\pi(dy)dz$ by the function ϕ defined in (2.2). Now let $\{\theta,\xi\}$ be a MAP which is reversible with respect to the measure $\pi(dy)dz$. As already seen in the proof of Theorem 2.3, with the probability measures $\mathbb{P}_x = P_{x/\|x\|,\log\|x\|}$, for $x\in H$ and $\mathbb{P}_0 = P_\delta$, the process $\{Y,\mathbb{P}_x\}$, where $Y_t=\theta_t e^{\xi_t}$, is an H-valued Markov process absorbed at 0. Then following [25], see also p.240 in [26], for $\alpha>0$, we define the measure ν_π^α associated to the additive functional $t\mapsto A_t := \int_0^t \|Y_s\|^\alpha\,ds$ of $\{Y,\mathbb{P}_x\}$ by

$$\int_H f(x)\nu_\pi^\alpha(dx) = \lim_{t\downarrow 0}\int_H \mathbb{E}_x\left(t^{-1}\int_0^t f(Y_s)dA_s\right)\Lambda_\pi(dx)\,,$$

where f is any positive Borel function, that is

$$\nu_\pi^\alpha(dx) = \|x\|^\alpha \Lambda_\pi(dx)\,. \tag{3.6}$$

In the special case where $\pi(dy)$ is absolutely continuous with density $\pi(y)$, the measure ν_π^α can be explicited as

$$\nu_\pi^\alpha(dx) = \pi(x/\|x\|)\|x\|^{\alpha-d}\,dx\,. \tag{3.7}$$

The following result is called *Riesz-Bogdan-Żak transform* in [20], in the context of 1-dimensional stable processes.

Theorem 3.7. *Let $\{X,\mathbb{P}_x\}$ be a ssMp with values in H, with index $\alpha\ge 0$ and lifetime ζ_c, and let $\{(\theta,\xi),P_{y,z}\}$ be the MAP which is associated to $\{X,\mathbb{P}_x\}$ through the transformation of Theorem 2.3. Define the process*

$$\widehat{X}_t = \begin{cases} \frac{X_{\gamma_t}}{\|X\|^2_{\gamma_t}}\,, & \text{if } \ t<\int_0^{\zeta_c}\frac{ds}{\|X_s\|^{2\alpha}}\,,\\ 0\,, & \text{if } \ t\ge\int_0^{\zeta_c}\frac{ds}{\|X_s\|^{2\alpha}}\,, \end{cases} \tag{3.8}$$

where γ_t is the time change $\gamma_t=\inf\left\{s:\int_0^s\|X\|_u^{-2\alpha}du>t\right\}$, for $t<\int_0^{\zeta_c}\frac{ds}{\|X_s\|^{2\alpha}}$. Define also the probability measures $\widehat{\mathbb{P}}_x := \mathbb{P}_{x/\|x\|^2}$, for $x\in H$ and $\widehat{\mathbb{P}}_0 := \mathbb{P}_0$. Then the process $\{\widehat{X},\widehat{\mathbb{P}}_x\}$ is a ssMp with values in H, with index α and lifetime $\int_0^{\zeta_c}\frac{ds}{\|X_s\|^{2\alpha}}$. The MAP which is associated to $\{\widehat{X},\widehat{\mathbb{P}}_x\}$ through the transformation of Theorem 2.3 is $\{(\theta,-\xi),P_{y,-z}\}$. Moreover, $\{X,\mathbb{P}_x\}$ and $\{\widehat{X},\widehat{\mathbb{P}}_x\}$ are in duality with respect to the measure $\nu_\pi^\alpha(dx)$ defined in (3.6) if and only if the MAP $\{(\theta,\xi),P_{y,z}\}$ is reversible with respect to the measure $\pi(ds)dx$.

Proof. Recall the definition of the MAP $\{(\theta,-\xi),P_{y,-z}\}$ given before Lemma 3.1. Then let us define $\{\widehat{Y},\widehat{\mathbb{P}}_x\}$, where $\widehat{\mathbb{P}}_x$ is as in the statement and $\widehat{Y}_t := \theta_t e^{-\xi_t}$, if $t<\zeta_p$ and $\widehat{Y}_t=0$, if $t\ge\zeta_p$. From the same arguments as those developed at the beginning of the proof of Theorem 2.3, we obtain that $\{\widehat{Y},\widehat{\mathbb{P}}_x\}$ is a standard Markov process, which possesses a reference measure. Now set $\hat\tau_t=\inf\{s:\int_0^s e^{-\alpha\xi_u}\,du>t\}$ and define

$$\begin{cases} Z_t=\theta_{\hat\tau_t}e^{-\xi_{\hat\tau_t}}\,, & \text{if } \ t<\int_0^{\zeta_p}e^{-\alpha\xi_u}\,du\,,\\ Z_t=0\,, & \text{if } \ t\ge\int_0^{\zeta_p}e^{-\alpha\xi_u}\,du\,. \end{cases}$$

Since the process $\{Z,\widehat{\mathbb{P}}_x\}$ is constructed from the MAP $\{(\theta,-\xi),P_{y,-z}\}$ through the transformation of Theorem 2.3, we derive from this theorem that it is a ssMp with index α, absorbed at 0. Moreover, it is clearly H-valued. Now let us check that for $t < \int_0^{\zeta_p} e^{-\alpha\xi_u}\,du$, $Z_t = \frac{X_{\gamma_t}}{\|X\|^2_{\gamma_t}}$. First note that from a change of variables, $\int_0^s \exp(-2\alpha\xi_{\tau_u})\,du = \int_0^{\tau_s} \exp(-\alpha\xi_u)\,du$, so that with $\zeta_c = \int_0^{\zeta_p} \exp(\alpha\xi_u)\,du$, we obtain $\int_0^{\zeta_c} \frac{ds}{\|X_s\|^{2\alpha}} = \int_0^{\zeta_p} e^{-\alpha\xi_u}\,du$. Moreover it follows from the definitions that for $t < \int_0^{\zeta_p} e^{-\alpha\xi_u}\,du$,

$$\begin{aligned}\gamma_t = \inf\{s, \int_0^s \exp(-2\alpha\xi_{\tau_u})\,du > t\} &= \inf\{s, \int_0^{\tau_s} \exp(-\alpha\xi_u)\,du > t\}\\ &= \tau^{-1}(\hat{\tau}_t)\,,\end{aligned}$$

where τ^{-1} is the right continuous inverse of τ. Therefore,

$$\begin{aligned}\frac{X_{\gamma_t}}{\|X\|^2_{\gamma_t}} &= \theta_{\tau(\gamma_t)} e^{-\xi_{\tau(\gamma_t)}}\\ &= \theta_{\hat{\tau}_t} e^{-\xi_{\hat{\tau}_t}}\,.\end{aligned}$$

Then assume that the MAP $\{(\theta,\xi), P_{y,z}\}$ is reversible with respect to the measure $\pi(ds)dx$. Recall from the proof of Theorem 2.3, the definition of the standard process $\{Y, \mathbb{P}_x\}$ which is constructed from the MAP $\{(\theta,\xi), P_{y,z}\}$. From the assumption, $\{Y, \mathbb{P}_x\}$ and $\{\widehat{Y}, \widehat{\mathbb{P}}_x\}$ are in duality with respect to the measure $\Lambda_\pi(dx)$. Indeed, we derive from obvious changes of variables and the definition of $\Lambda_\pi(dx)$ that for any positive Borel functions f and g,

$$\int_{S\times\mathbb{R}} f(ye^z) E_{y,z}(g(\theta_t e^{\xi_t}))\,\pi(dy)\,dz = \int_H f(x)\mathbb{E}_x(g(Y_t))\,\Lambda_\pi(dx)\,,$$

and

$$\int_{S\times\mathbb{R}} g(ye^z) E_{y,-z}(f(\theta_t e^{-\xi_t}))\,\pi(dy)\,dz = \int_H g(x)\widehat{\mathbb{E}}_x(f(\widehat{Y}_t))\Lambda_\pi(dx)\,.$$

Since $\{(\theta,\xi), P_{y,z}\}$ is reversible, from Lemma 3.1 the MAP's $\{(\theta,\xi), P_{y,z}\}$ and $\{(\theta,-\xi), P_{y,-z}\}$ are in duality with respect to the measure $\pi(dy)\,dz$ and we derive from the above identities that

$$\int_H f(x)\mathbb{E}_x(g(Y_t))\,\Lambda_\pi(dx) = \int_H g(x)\widehat{\mathbb{E}}_x(f(\widehat{Y}_t))\Lambda_\pi(dx)\,,$$

which proves the duality between $\{Y, \mathbb{P}_x\}$ and $\{\widehat{Y}, \widehat{\mathbb{P}}_x\}$, with respect to the measure $\Lambda_\pi(dx)$. Then from this duality and Theorem 4.5, p. 241 in [26], we deduce that $\{X, \mathbb{P}_x\}$ and $\{\widehat{X}, \widehat{\mathbb{P}}_x\}$ are Markov processes which are in duality with respect to the measure $\nu_\pi^\alpha(dx) = \|x\|^\alpha \Lambda_\pi(dx)$.

Conversely if $\{X, \mathbb{P}_x\}$ and $\{\widehat{X}, \widehat{\mathbb{P}}_x\}$ are in duality with respect to the measure $\nu_\pi^\alpha(dx)$, then Theorem 4.5, p. 241 in [26] can be applied to $\{X, \mathbb{P}_x\}$ and $\{\widehat{X}, \widehat{\mathbb{P}}_x\}$ and to the additive functionals $t \mapsto \int_0^t \frac{ds}{\|X_s\|^\alpha}$ and $t \mapsto \int_0^t \frac{ds}{\|\widehat{X}_s\|^\alpha}$ which are strictly increasing, on $[0,\zeta_c)$ and $[0,\hat{\zeta}_c)$, respectively, where $\hat{\zeta}_c = \int_0^{\zeta_c} \frac{ds}{\|X_s\|^{2\alpha}}$ is the lifetime of $\widehat{X}$. Then by reading the above arguments in reverse order, we prove that the MAP $\{(\theta,\xi), P_{y,z}\}$ is reversible with respect to the measure $\pi(ds)dx$. □

Remark 3.8. It readily follows from the previous proof that the transformation of Theorem 3.7 is invertible, namely, with obvious notation:

$$X_t = \begin{cases} \frac{\widehat{X}_{\hat{\gamma}_t}}{\|\widehat{X}\|^2_{\hat{\gamma}_t}}\,, & \text{if} \quad t < \int_0^{\hat{\zeta}_c} \frac{ds}{\|\widehat{X}_s\|^{2\alpha}}\,,\\ 0\,, & \text{if} \quad t \geq \int_0^{\hat{\zeta}_c} \frac{ds}{\|\widehat{X}_s\|^{2\alpha}}\,.\end{cases}$$

Remark 3.9. In the case where $\{X, \mathbb{P}_x\}$ is a positive ssMp (i.e. $d = 1$ and $S = \{1\}$), the fact that $\{X, \mathbb{P}_x\}$ is in duality with respect to the positive ssMp associated with the Lévy process $-\xi$, in the Lamperti representation, was proved in [4]. It is clear that any positive ssMp satisfies the skew product property according to Definition 3.3, so in this case the result follows from Proposition 3.2 and Theorem 3.7.

Remark 3.10. From Proposition 3.2, Theorem 3.7 also applies when $\{X, \mathbb{P}_x\}$ is an isotropic ssMp. The existence of a dual process with respect to the measure $\|x\|^{\alpha-d}\,dx$ in the isotropic case was already obtained in [15], where the proof relies on the wrong observation that isotropic ssMp's also satisfies the skew product property. In [23] the authors proved that actually if $\{(\theta,\xi), P_{y,z}\}$ is isotropic then θ and ξ are independent (i.e. $\{(\theta,\xi), P_{y,z}\}$ satisfies the skew product property) if and only if the processes (ξ_{τ_t}) and (θ_t) do not jump together, $P_{y,z}$-a.s. for all y, z. As noticed in [23], it is not the case of isotropic stable Lévy processes which will be studied in Section 4.

Duality of ssMp's with themselves is also an interesting property and jointly with the duality of Theorem 3.7 it allows us to obtain excessive functions, as shown in the next section. The next proposition which will be useful later on, asserts that this self duality holds if and only if the same property holds for the underlying MAP.

Proposition 3.11. *Let $\{X, \mathbb{P}_x\}$ and $\{(\theta,\xi), P_{y,z}\}$ be as in Theorem 2.3. Then the ssMp $\{X, \mathbb{P}_x\}$ is in duality with itself with respect to some measure $M(dx)$ on H if and only if the MAP $\{(\theta,\xi), P_{y,z}\}$ is in duality with itself with respect to the image $\eta(dy, dz)$ on $S \times \mathbb{R}$ of the measure $\|x\|^{-\alpha}M(dx)$ through the function ϕ^{-1}, where ϕ is defined in (2.2).*

In particular, if $M(dx)$ has a density which can be split as the product of an angular and a radial part, that is

$$M(dx) = \pi(x/\|x\|)r(\|x\|)\,dx\,,$$

where π and r are nonnegative Borel functions which are respectively defined on S and $\mathbb{R}_+$, then the measure η on $S \times \mathbb{R}$ has the following form,

$$\eta(dy, dz) = \pi(y)e^{(d-\alpha)z}r(e^z)\,dydz\,.$$

If moreover the MAP $\{(\theta,\xi), P_{y,z}\}$ satisfies the skew product property, then the Markov process $\{\theta, P_{y,z}\}$ on S is in duality with itself with respect to the measure $\pi(y)\,dy$ on S.

Proof. Recall the notation of the proof of Theorem 3.7. Then for any positive Borel functions f and g, we have

$$\int_H f(x)\mathbb{E}_x(g(X_t))\,M(dx) = \int_H g(x)\mathbb{E}_x(f(X_t))M(dx)\,.$$

Applying Theorem 4.5, p. 241 in [26], we obtain that

$$\int_H f(x)\mathbb{E}_x(g(Y_t))\,\|x\|^{-\alpha}M(dx) = \int_H g(x)\mathbb{E}_x(f(Y_t))\|x\|^{-\alpha}M(dx)\,,$$

which gives from a change of variables,

$$\int_{S\times\mathbb{R}} g(ye^z)E_{y,z}(f(\theta_t e^{\xi_t}))\,\eta(dy,dz) = \int_{S\times\mathbb{R}} f(ye^z)E_{y,z}(g(\theta_t e^{\xi_t}))\,\eta(dy,dz).$$

The other assertions are straightforward. □

4 Inversion and Doob h-transforms for ssMp's

We begin this section with a simple observation on the relationship between duality and Doob h-transform of Markov processes.

Lemma 4.1. *Let $(P_t^{(1)})$, $(P_t^{(2)})$ and $(P_t^{(3)})$ be the semigroups of three H-valued Markov processes absorbed at 0. Assume that $(P_t^{(1)})$ and $(P_t^{(2)})$ are in weak duality with respect to $h_1(x)\,dx$ and that $(P_t^{(1)})$ and $(P_t^{(3)})$ are in weak duality with respect to $h_2(x)\,dx$, where h_1 and h_2 are positive and continuous functions. Assume moreover that $P_t^{(2)}$ and $P_t^{(3)}$ are Feller semigroups on H.*

Then $h := h_1/h_2$ is excessive for $(P_t^{(3)})$ and the Markov process with semigroup $(P_t^{(2)})$ is an h-process of the Markov process with semigroup $(P_t^{(3)})$, with respect to the function h, that is

$$P_t^{(2)}g(x) = \frac{1}{h(x)}P_t^{(3)}(hg)(x)\,, \tag{4.1}$$

for all $t \geq 0$, $x \in H$ and all positive Borel functions g.

Conversely, if (4.1) holds and if $(P_t^{(1)})$ and $(P_t^{(2)})$ are in weak duality with respect to $h_1(x)\,dx$, then $(P_t^{(1)})$ and $(P_t^{(3)})$ are in weak duality with respect to $h_2(x)\,dx$.

Proof. From the assumptions, for all positive Borel functions f and g,

$$\begin{aligned}\int_H P_t^{(1)}f(x)g(x)\,h_1(x)dx &= \int_H f(x)P_t^{(2)}g(x)\,h_1(x)dx\,, \\ \int_H P_t^{(1)}f(x)g(x)\,h_2(x)dx &= \int_H f(x)P_t^{(3)}g(x)\,h_2(x)dx\,.\end{aligned} \tag{4.2}$$

Replacing g by $\frac{h_1}{h_2}g$ in both members of the second equality gives for all f,

$$\int_H P_t^{(1)}f(x)g(x)\,h_1(x)dx = \int_H f(x)\frac{h_2}{h_1}(x)P_t^{(3)}\frac{h_1}{h_2}g(x)\,h_1(x)dx\,. \tag{4.3}$$

Identifying the second members of (4.2) and (4.3), yields (4.1) in the case where g and $\frac{h_1}{h_2}g$ are bounded and continuous, thanks to the Feller property. Then we extend (4.1) to all positive Borel functions from classical arguments.

The converse is proved in the same way. □

Then as an application of this lemma and Theorem 3.7 we can recover the results recalled in the introduction, that is Theorem 1 in [6] and Theorem 4 in [20].

Corollary 4.2. *Let $\{X, \mathbb{P}_x\}$ be a d-dimensional stable Lévy process, with index $\alpha \in (0, 2]$, which is absorbed at its first hitting time of 0 (absorption actually holds with probability 1 if $d = 1$ and $\alpha > 1$, and with probability 0 in the other cases). Recall the definition of the process $\{\widehat{X}, \widehat{\mathbb{P}}_x\}$, from Theorem 3.7.*

1. *If $d = 1$ and if $\{X, \mathbb{P}_x\}$ is not spectrally one sided, then the process $\{\widehat{X}, \widehat{\mathbb{P}}_x\}$ is an h-process of $\{-X, \mathbb{P}_{-x}\}$ with respect to the function $x \mapsto \pi(x/|x|)|x|^{\alpha-1}$, where $(\pi(-1), \pi(+1))$ is the invariant measure of the first coordinate of the MAP associated to $\{X, \mathbb{P}_x\}$.*
2. *If $d > 1$ and $\{X, \mathbb{P}_x\}$ is isotropic, then the process $\{\widehat{X}, \widehat{\mathbb{P}}_x\}$ is an h-process of $(X, \mathbb{P}_x)$ with respect to the function $x \mapsto \|x\|^{\alpha-d}$.*

Proof. When $d = 1$ and $\{X, \mathbb{P}_x\}$ is not spectrally one sided, we derive from Corollary 11, Section 4.1 in [8], that the MAP associated to the stable Lévy process $\{X, \mathbb{P}_x\}$ satisfies $\Delta_{ij} \overset{(d)}{=} \Delta_{ji}$, $i, j \in \{-1, +1\}$, with the notation introduced in (2.4). Moreover, since the continuous time Markov chain $\{\theta, P_{y,z}\}$ is irreducible and takes only two values, it is reversible with respect to some measure $(\pi(-1), \pi(+1))$. Then we derive from part 1. of Proposition 3.2 that the MAP $\{(\theta, \xi), P_{y,z}\}$ is reversible. Hence we can apply Theorem 3.7 which ensures that $\{X, \mathbb{P}_x\}$ and $\{\widehat{X}, \widehat{\mathbb{P}}_x\}$ are in duality with respect to the function $x \mapsto \pi(x/|x|)|x|^{\alpha-1}$.

If $d \geq 1$ and the process is isotropic, then again we derive from Proposition 3.2 and Theorem 3.7 that $\{X, \mathbb{P}_x\}$ and $\{\widehat{X}, \widehat{\mathbb{P}}_x\}$ are in duality with respect to the function $x \mapsto \|x\|^{\alpha-d}$.

Finally it is well known that stable Lévy processes satisfy the Feller property. Then from the path construction in Theorem 3.7 and homogeneity of the increments of $\{X, \mathbb{P}_x\}$, we derive that the process $\{\widehat{X}, \widehat{\mathbb{P}}_x\}$ is itself a Feller process on $\mathbb{R}^d \setminus \{0\}$. Moreover it is well known that $\{X, \mathbb{P}_x\}$ and $\{-X, \mathbb{P}_{-x}\}$ are in duality with respect to the Lebesgue measure (when $d = 1$ and $\alpha > 1$, this duality is inherited from the duality between the non absorbed Lévy processes with respect to the Lebesgue measure on $\mathbb{R}$). It remains to apply Lemma 4.1 in order to conclude in both cases $d = 1$ and $d > 1$. $\square$

Note that in the case $d = 1$, the probability measure $(\pi(-1), \pi(+1))$ has been found explicitly in Corollary 11 of [8] and in Theorem 4 of [20]. Moreover, the h-process involved in part 1 of Corollary 4.2 has been extensively investigated in [28] and [24], where for $\alpha > 1$, it is identified as the real Lévy process $\{-X, \mathbb{P}_{-x}\}$ conditioned to avoid 0.

Theorem 3.7 and Lemma 4.1 can be applied in the same way as in Corollary 4.2, to any reversible ssMp $\{X, \mathbb{P}_x\}$ provided $\{\widehat{X}, \widehat{\mathbb{P}}_x\}$ has the Feller property on H and $\{X, \mathbb{P}_x\}$ is in duality with some other Feller process on H. Let us give three examples.

A. Conditioned Lévy processes: Let $\{X, \mathbb{P}_x\}$ be a one dimensional stable Lévy process, with index $\alpha \in (0, 2)$ and let us denote by X^0 the process X which is absorbed at 0 when it first hits the negative halfline, that is

$$X_t^0 = X_t \mathbb{1}_{\{t<\tau^-\}}\,, \quad t \geq 0\,,$$

where $\tau^- = \inf\{t : X_t < 0\}$. It is well known, see [7] and the references therein, that the functions $h_1(x) = x^{\alpha(1-\rho)}$ and $h_2(x) = x^{\alpha(1-\rho)-1}$, where $\rho = \mathbb{P}_0(X_1 > 0)$, are respectively invariant and excessive for the process $(X^0, \mathbb{P}_x)$. We denote the Doob h-transforms of $(X^0, \mathbb{P}_x)$ associated to h_1 and h_2, respectively by

$$\{X^\uparrow, \mathbb{P}_x^\uparrow\} \quad \text{and} \quad \{X^\searrow, \mathbb{P}_x^\searrow\}\,.$$

The process $\{X^\uparrow, \mathbb{P}_x^\uparrow\}$ is known as the Lévy process $\{X, \mathbb{P}_x\}$ conditioned to stay positive. It is a positive ssMp with index α, which satisfies $\lim_{t\to\infty} X_t = +\infty$, $\mathbb{P}_x$-a.s., for all $x > 0$. In particular this process never hits 0. In the case of Brownian motion, that is for $\alpha = 2$, it corresponds to the three dimensional Bessel process. The process $\{X^\searrow, \mathbb{P}_x^\searrow\}$ is called the process $\{X, \mathbb{P}_x\}$ conditioned to hit 0 continuously. It is also a positive ssMp with index α. Its absorption time ζ_c is finite and satisfies $X_{\zeta_c-} = 0$, $\mathbb{P}_x^\searrow$-a.s. for all $x > 0$. Note that in the present case, $H = (0, \infty)$.

Then let us set $Y := -X$ and denote by P_x, $x \in \mathbb{R}$, the family of probability measures associated to Y. We denote by $\{Y^\uparrow, \mathrm{P}_x^\uparrow\}$ and $\{Y^\searrow, \mathrm{P}_x^\searrow\}$ the process $\{Y, \mathrm{P}_x\}$ conditioned to stay positive and conditioned to hit 0 continuously, respectively. Since $\{Y, \mathrm{P}_x\}$ is a stable Lévy process, these processes enjoy the same properties as $\{X^\uparrow, \mathbb{P}_x^\uparrow\}$ and $\{X^\searrow, \mathbb{P}_x^\searrow\}$.

The process $\{X, \mathbb{P}_x\}$ conditioned to stay positive is related to the process $\{Y, \mathrm{P}_x\}$ conditioned to hit 0 continuously as follows.

Corollary 4.3. *The process $\{X^\uparrow, \mathbb{P}_x^\uparrow\}$ can by obtained from the paths of the process $\{Y^\searrow, \mathrm{P}_x^\searrow\}$, through the following transformation*:

$$\{X^\uparrow, \mathbb{P}_x^\uparrow\} = \{(1/Y_{\gamma_t}^\searrow)_{t\geq 0}, \mathrm{P}_{1/x}^\searrow\}\,,$$

where the time change γ_t is defined by

$$\gamma_t = \inf\left\{s : \int_0^s \frac{du}{(Y_u^\searrow)^{2\alpha}} > t\right\}, \quad t \geq 0\,.$$

Proof. As a positive ssMp, the process $\{Y^{\searrow}, \mathrm{P}_x^{\searrow}\}$ satisfies the skew product property. Therefore it is reversible and from Theorem 3.7, $\{Y^{\searrow}, \mathrm{P}_x^{\searrow}\}$ and $\{(1/Y_{\gamma_t}^{\searrow})_{t\geq 0}, \mathrm{P}_{1/x}^{\searrow}\}$ are in duality with respect to the measure $x^{\alpha-1}\,dx$. Moreover, $\{Y^{\searrow}, \mathrm{P}_x^{\searrow}\}$ and $\{X^{\uparrow}, \mathbb{P}_x^{\uparrow}\}$ are also in duality with respect to the measure $x^{\alpha-1}\,dx$. Indeed, let us denote by Y^0 the process Y absorbed at 0 when it first hits the negative half-line. Then $\{X^0, \mathbb{P}_x\}$ and $\{Y^0, \mathrm{P}_x\}$ are in duality with respect to the Lebesgue measure on $(0,\infty)$ (this duality is inherited from the duality between the Lévy processes $\{X, \mathbb{P}_x\}$ and $\{Y, \mathrm{P}_x\}$ with respect to the Lebesgue measure on $\mathbb{R}$). Therefore, for any positive Borel functions f and g,

$$\begin{aligned}\int_0^\infty g(x)\mathrm{E}_x^{\searrow}(f(Y_t^{\searrow}))\,x^{\alpha-1}\,dx &= \int_0^\infty g(x)\mathrm{E}_x(Y_t^{\alpha\rho-1}f(Y_t), t<\tau_Y^-)\,x^{\alpha(1-\rho)}\,dx\\ &= \int_0^\infty f(x)\mathbb{E}_x(X_t^{\alpha(1-\rho)}g(X_t), t<\tau^-)\,x^{\alpha\rho-1}\,dx\\ &= \int_0^\infty f(x)\mathbb{E}_x^{\uparrow}(g(X_t^{\uparrow}))\,x^{\alpha-1}\,dx\,,\end{aligned}$$

where $\tau_Y^- = \inf\{t : Y_t < 0\}$ in right hand side of the above equality.

Then as positive ssMp's, $\{X^{\uparrow}, \mathbb{P}_x^{\uparrow}\}$ and $\{(1/Y_{\gamma_t}^{\searrow})_{t\geq 0}, \mathrm{P}_{1/x}^{\searrow}\}$ satisfy the Feller property on $(0,\infty)$, see Theorem 2.1 in [22]. Therefore applying Lemma 4.1, we conclude that $\{X^{\uparrow}, \mathbb{P}_x^{\uparrow}\}$ is an h process of the process $\{(1/Y_{\gamma_t}^{\searrow})_{t\geq 0}, \mathrm{P}_{1/x}^{\searrow}\}$ with respect to the function which is identically equal to 1, so that both processes are equal. □

It is straightforward that the same relationship exists between the processes $\{Y^{\uparrow}, \mathrm{P}_x^{\uparrow}\}$ and $\{X^{\searrow}, \mathbb{P}_x^{\searrow}\}$. Let us also note that when $\alpha > 1$ and $\{X, \mathbb{P}_x\}$ has no positive jumps, then $\{Y^{\searrow}, \mathrm{P}_x^{\searrow}\} = \{Y^0, \mathrm{P}_x\}$. Therefore Corollary 4.3 and its proof allow us to complete part 1 of Corollary 4.2, where completely asymmetric Lévy processes are excluded, by stating that $\{\widehat{X}, \widehat{\mathbb{P}}_x\}$ is an h-process of $\{-X, \mathbb{P}_{-x}\}$ with respect to the function $x \mapsto x^{\alpha-1}$. We emphasize that here we consider $\{\widehat{X}, \widehat{\mathbb{P}}_x\}$ and $\{-X, \mathbb{P}_{-x}\}$ as positive ssMp's, that is $H = (0,\infty)$. Then in this case, $\{\widehat{X}, \widehat{\mathbb{P}}_x\}$ corresponds to the process $\{-X, \mathbb{P}_{-x}\}$ conditioned to stay positive. This remark also allows us to recover Theorem A. Note also that Corollaries 4.2 and 4.3 provide constructions of Lévy processes conditioned to avoid 0 as times changes and h-processes of the original process. Then recently, in Theorems 2.1 and 2.2 of [21], further representations for general real ssMp's conditioned to avoid 0 are given.

B. Free d-dimensional Bessel processes: Let $X = \{(X_1(t), X_2(t), .., X_d(t)), t \geq 0\}$, where $X_i(t)$ are independent BES(δ) processes of dimension $\delta > 0$ and let us consider the H-valued ssMp $\{X, \mathbb{P}_x\}$ absorbed at 0, where $H = (0,\infty)^d$. It is well known that for each $i = 1, \dots, d$, X_i is in duality with itself with respect to its speed measure $m_i(dx_i) = x_i^{1+2\nu}/|\nu|dx_i$ when $\nu \neq 0$ and $m_i(dx_i) = x_i dx_i$ when $\nu = 0$, where $\nu = \frac{\delta}{2} - 1$. This entails that $\{X, \mathbb{P}_x\}$ is in duality with itself, with respect to the measure $D(x)\,dx$, where

$$D(x) = \prod_{i=1}^d x_i^{\delta-1} = ||x||^{d\delta-d}\prod_{i=1}^d\left(\frac{x_i}{||x||}\right)^{\delta-1}\,, \quad x \in H\,. \tag{4.4}$$

Let us observe that the process $\{X, \mathbb{P}_x\}$ satisfies the skew product property. Indeed we can argue similarly as for the skew product decomposition of Brownian motion, see e.g. Chapter 7.15 in [17]. The generator of $\{X, \mathbb{P}_x\}$ is equal to $L = \sum_{i=1}^d L_{x_i}$, where $L_{x_i} = \frac{1}{2}\frac{\partial^2}{\partial x_i^2} + \frac{1}{2}\frac{\delta-1}{x_i}\frac{\partial}{\partial x_i}$. Then we compute the spherical decomposition of L. Set $u = u(r,\sigma)$, where $r = \|x\|$ and $\sigma = \frac{x}{\|x\|}$ are the spherical coordinates in $\mathbb{R}^d$ and let Δ_σ be the Laplacian on the unit sphere S_{d-1}. By the well-known formula $\Delta = \frac{\partial^2}{\partial r^2} + \frac{d-1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\Delta_\sigma$

and by the chain rule, we obtain

$$Lu = \frac{1}{2}\frac{\partial^2 u}{\partial r^2} + \frac{1}{2}\frac{\delta d - 1}{r}\frac{\partial u}{\partial r} + \frac{1}{2r^2}\left[\Delta_\sigma u + \frac{\delta - 1}{2}(\sigma^{-1} - d\sigma)\cdot\nabla_\sigma u\right],$$

where $(\sigma^{-1})_i = \sigma_i^{-1}$. Note that $L_r := \frac{1}{2}\frac{\partial^2 u}{\partial r^2} + \frac{\delta d-1}{2r}\frac{\partial u}{\partial r}$ is the generator of $\|X\| \sim$BES(δd).

We deduce from Propositions 3.2 and 3.11 that the free Bessel process $\{X, \mathbb{P}_x\}$ is reversible, with respect to the measure with density $\pi(y) = \prod_{i=1}^d y_i^{\delta-1}, \;\; y \in S_{d-1}\cap H$. Indeed, it satisfies the skew product property and it is self-dual with respect to the measure $D(x)dx$ which splits as the product of an angular part and a radial part, see (4.4). Therefore Lemma 4.1 and Theorem 3.7 can be applied to the free Bessel process $\{X, \mathbb{P}_x\}$ and we obtain the following result.

Corollary 4.4. *Let $\{X, \mathbb{P}_x\}$ be a free Bessel process with values in $(0,\infty)^d$ and absorbed at 0. Recall the definition of the process $\{\widehat{X}, \widehat{\mathbb{P}}_x\}$ from Theorem 3.7. Then the processes $\{X, \mathbb{P}_x\}$ and $\{\widehat{X}, \widehat{\mathbb{P}}\}$ are in duality with respect to the measure with density*

$$\prod_{i=1}^d \left(\frac{x_i}{\|x\|}\right)^{\delta-1}\|x\|^{2-d}.$$

Moreover, the process $\{\widehat{X}, \widehat{\mathbb{P}}_x\}$ is a Doob h-transform of $\{X, \mathbb{P}_x\}$ with respect to the excessive function $h(x) = \|x\|^{2-d\delta}$.

Remark 4.5. It is easy to check that for $x \in H$ one has $Lh = 0$, i.e. h is L-harmonic on its domain. However, h is not X-invariant when $d\delta > 2$, i.e. $(||X_t||^{2-d\delta}, t \geq 0)$ is a strict local martingale. See for instance the discussion on p. 330 of [13].

Remark 4.6. For $\delta > 2$, we can give the following realization of the MAP corresponding to the free Bessel process $\{X, \mathbb{P}_x\}$. There exists a d-dimensional Brownian motion $(W^{(1)}, W^{(2)}, \cdots, W^{(d)})$ such that

$$\xi_t = \sum_{j=1}^d \int_0^t \theta_s^{(j)} dW_s^{(j)} + \left(\frac{d\delta}{2} - 1\right)t$$

and (θ_t) satisfies the SDE system

$$d\theta_t^{(i)} = dW_t^{(i)} - \theta_t^{(i)}\sum_{j=1}^d \theta_t^{(j)} dW_t^{(j)} + \left(\frac{\delta-1}{2}\frac{1}{\theta_t^{(i)}} - \frac{d\delta - 1}{2}\theta_t^{(i)}\right)dt, \quad i = 1, 2, \ldots, d.$$

The processes (ξ_t) and (θ_t) are independent. The proof of this MAP representation relies on the SDE representations of the processes $X_i(t)$.

C. Dunkl processes: In what follows, we recall and use some properties of the Dunkl processes which may be found in Chapters 2 and 3 of [9].

Let R be a finite root system on $\mathbb{R}^d$, i.e. a finite subset $R \subset \mathbb{R}^d\backslash\{0\}$, satisfying $\sigma_\alpha(R) = R$ for all $\alpha \in R$, where σ_α are the symmetries with respect to the hyperplanes $\{x|\ \alpha\cdot x = 0\}$. Let also R^+ be a positive subsystem of R and k be a non-negative function on R, called *multiplicity function*. The generator of the Dunkl process $\{X, \mathbb{P}_x\}$ is $L_k := \frac{1}{2}\Delta_k$ where $\Delta_k = \sum_{i=1}^d T_i^2$ is the Dunkl Laplacian and $T_i f(x) := \partial_i f(x) + \sum_{\alpha\in R^+} k(\alpha)\alpha_i \frac{f(x) - f(\sigma_\alpha x)}{\alpha\cdot x}$, $i = 1, 2, \cdots, d$, are the Dunkl derivatives.

Dunkl processes are ssMp's with index 2. In Dunkl analysis, an important role is played by the so-called Dunkl weight function $\omega_k(x) = \prod_{\alpha\in R}|\alpha\cdot x|^{k(\alpha)}$ and the constant $\gamma = \gamma(k) = \sum_{\alpha\in R^+} k(\alpha)$. We see that $\omega_k(x)$ is homogeneous of order 2γ.

Let us also mention that Dunkl processes have the skew product property: this fact was proved by Chybiryakov in [9], see Theorem 8, p.156 therein. Moreover, the radial part $R_t = \|X_t\|$ is a $\mathrm{BES}(d+2\gamma)$ process.

We also observe that the Dunkl process $\{X, \mathbb{P}_x\}$ is self-dual with respect to the measure $M(dx) = \omega_k(x)\,dx$. This follows from the formula for the Dunkl transition function, see (23) p.120 in [9],

$$p_t^{(k)}(x,y) = \frac{1}{c_k t^{\gamma+d/2}} \exp\left(-\frac{\|x\|^2+\|y\|^2}{2t}\right) D_k\left(\frac{x}{\sqrt{t}}, \frac{y}{\sqrt{t}}\right) \omega_k(y)\,, \tag{4.5}$$

where D_k is the Dunkl kernel. The only non-symmetric factor in (4.5) is $\omega_k(y)$; hence the kernel $p_t^{(k)}(x,y)\omega_k(x)$ is symmetric in x and y.

The density of the self-duality measure M factorizes as

$$\omega_k(x) = \omega_k(x/\|x\|)\|x\|^{2\gamma}.$$

By Proposition 3.11, the Dunkl process $\{X, \mathbb{P}_x\}$ is reversible with respect to the measure $\pi(y) = \omega_k(y),\ y \in S_{d-1}$.

Note that contrary to the case of Brownian motion, the Dunkl process $\{X, \mathbb{P}_x\}$ with $k \neq 0$ is non-isotropic. Indeed, the process $\{X, \mathbb{P}_x\}$ always jumps from a state y to a symmetric state $\sigma_\alpha(y)$. Thus, like free Bessel processes, Dunkl processes are a class of reversible non-isotropic self-similar processes. Then we derive the next corollary as a consequence of Theorem 3.7.

Corollary 4.7. *Let $\{X, \mathbb{P}_x\}$ be a Dunkl process in $\mathbb{R}^d \setminus \{0\}$ and absorbed at 0. Recall the definition of the process $\{\widehat{X}, \widehat{\mathbb{P}}_x\}$ from Theorem 3.7. The processes $\{X, \mathbb{P}_x\}$ and $\{\widehat{X}, \widehat{\mathbb{P}}_x\}$ are in duality with respect to the measure with density*

$$\omega_k(x/\|x\|)\|x\|^{2-d}.$$

Moreover, the process $\{\widehat{X}, \widehat{\mathbb{P}}_x\}$ is a Doob h-transform of $\{X, \mathbb{P}_x\}$ with respect to the excessive function $h(x) = \|x\|^{2-d-2\gamma}$.

Remark 4.8. The function h of Corollary 4.7 is always Dunkl-harmonic in the sense that $\Delta_k h = 0$. This follows from the form of the Dunkl Laplacian in polar coordinates, see [27]. This is confirmed by the well-known fact that $(h(X_t), t \le \zeta_c)$ is a local martingale which is a true martingale only when $d + 2\gamma \le 2$.

Finally, let us mention that new classes of stochastic processes satisfying a property analogous to those of Corollaries 4.2, 4.3, 4.4 and 4.7 may be found in the recent work [1].

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