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Functional central limit theorem for subgraph counting processes*

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Abstract

The objective of this study is to investigate the limiting behavior of a subgraph counting process built over random points from an inhomogeneous Poisson point process on $\mathbb{R}^d$. The subgraph counting process we consider counts the number of subgraphs having a specific shape that exist outside an expanding ball as the sample size increases. As underlying laws, we consider distributions with either a regularly varying tail or an exponentially decaying tail. In both cases, the nature of the resulting functional central limit theorem differs according to the speed at which the ball expands. More specifically, the normalizations in the central limit theorems and the properties of the limiting Gaussian processes are all determined by whether or not an expanding ball covers a region - called a weak core - in which the random points are highly densely scattered and form a giant geometric graph.

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1 Introduction

The history of random geometric graphs started with Gilbert's 1961 study ([16]) and, since then, it has received much attention both in theory and applications. More formally, given a finite set $\mathcal{X} \subset \mathbb{R}^d$ and a real number $r > 0$, the geometric graph $G(\mathcal{X}, r)$ is defined as an undirected graph with vertex set $\mathcal{X}$ and edges $[x, y]$ for all pairs $x, y \in \mathcal{X}$ for which $\|x - y\| \leq r$. The theory of geometric graphs has been applied mainly in large communication network analysis, in which the connectivity of network agents strongly depends on the distance between them; see [12], [30], and Chapter 3 of [18]. On the purely theoretical side of random geometric graphs, the monograph [23] is probably the best known resource. It covers a wide range of topics, such as the asymptotics of the number of subgraphs with a specific shape, the vertex degree, the clique number,

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the formation of a giant component, etc. From among these interesting subjects, the present study focuses on constructing the functional central limit theorem (FCLT) for the number of subgraphs isomorphic to a predefined connected graph Γ of finite vertices.

A typical setup in [23] is as follows. Let $\mathcal{X}_n$ be a set of random points on $\mathbb{R}^d$. Typically, this will be either an i.i.d. random sample of n points from f, or an inhomogeneous Poisson point process with intensity nf, where f is a probability density. We assume that the threshold radius r_n depends on n and decreases to 0 as $n \to \infty$, but we do not impose any restrictive assumptions on f except for boundedness. Then, the asymptotic behavior of the subgraph counts given by

$$G_n := \sum_{\mathcal{Y} \subset \mathcal{X}_n} \mathbf{1}\big\{G(\mathcal{Y}, r_n) \cong \Gamma\big\}, \tag{1.1}$$

($\cong$ denotes graph isomorphism, and Γ is a fixed connected graph) splits into three different regimes. First, if $nr_n^d \to 0$, called the *subcritical* or *sparse* regime, the distribution of subgraphs isomorphic to Γ is sparse, and these subgraphs are mostly observed as isolated components. If $nr_n^d \to \xi \in (0, \infty)$, called the *critical* or *thermodynamic* regime, for which r_n decreases to 0 at a slower rate than the subcritical regime, many of the isolated subgraphs in $G(\mathcal{X}_n, r_n)$ become connected to one another. Finally, if $nr_n^d \to \infty$ (the *supercritical* regime), the subgraphs are very highly connected and create a large component.

Historically, the research on the limiting behavior of subgraph counts of the type (1.1) dates back to the studies of [17], [29], and [31], in all of which mainly the subcritical regime was treated. Furthermore, [7] adopted an approach based on the martingale CLT for U-statistics and proved a CLT under various conditions on f and r_n. Relying on the so-called Stein-Chen method, a set of extensive results for all three regimes was nicely summarized in Chapter 3 of [23].

Recently, as a higher-dimensional analogue of a random geometric graph, there has been growing interest in the asymptotics of the so-called random Cěch complex. See, for example, [19], [20], and [32], while [11] provides an elegant review of that direction. Similarly to subgraph counts in (1.1), the behavior of random Čech complexes splits, once again, into three different regimes as above. In particular, [20] mainly investigated sparse regime (i.e., $nr_n^d \to 0$), while the main focus of [32] was the thermodynamic regime (i.e., $nr_n^d \to \xi \in (0, \infty)$) in which complexes are large and highly connected.

Somewhat parallel to (1.1), but more important for the study on the geometric features of extreme sample clouds, is an alternative that we explore in this paper. To set this up, we introduce a growing sequence $R_n \to \infty$ and a threshold radius $t > 0$. The following quantity, $G_n(t)$ counts the number of subgraphs in $G(\mathcal{X}_n, t)$ isomorphic to Γ that exist outside a centered ball in $\mathbb{R}^d$ with radius R_n:

$$G_n(t) := \sum_{\mathcal{Y} \subset \mathcal{X}_n} \mathbf{1}\big\{G(\mathcal{Y}, t) \cong \Gamma\big\} \times \mathbf{1}\big\{m(\mathcal{Y}) \geq R_n\big\}, \tag{1.2}$$

where $m(x_1, \dots, x_k) = \min_{1 \le i \le k} ||x_i||$, $x_i \in \mathbb{R}^d$, and $\| \cdot \|$ is the usual Euclidean norm.

From the viewpoint of extreme value theory (EVT), it is important to investigate limit theorems for $G_n(t)$. Indeed, over the last decade or so there have been numerous papers treating geometric descriptions of multivariate extremes, among them [4], [5], and [6]. In particular, Poisson limits of point processes possessing a U-statistic structure were investigated by [13] and [28], the latter also treating a number of examples in stochastic geometry. The main references for EVT are [15], [26], and [14].

The asymptotic behavior of (1.2) has been partially explored in [22], where a growing sequence R_n is taken in such a way that (1.2) has Poisson limits as $n \to \infty$. The main contribution in [22] is the discovery of a certain layered structure consisting of a collection of "rings" around the origin with each ring containing extreme random points which exhibit different geometric and topological behavior. The objective of the current study is to develop a fuller description of this ring-like structure, at least in a geometric graph model, by establishing a variety of FCLTs which describe geometric graph formation between the rings.

By construction, the subgraph counts (1.2) can be viewed as generating a stochastic process in the parameter $t \geq 0$, while a process-level extension in (1.1) is much less obvious. Then, while (1.2) captures the dynamic evolution of geometric graphs as t varies, (1.1) only describes the static geometry. Thus, the limits in the FCLT for (1.2) are intrinsically Gaussian processes, rather than one-dimensional Gaussian distributions.

One of the main results of this paper is that the limiting Gaussian processes can be classified into three distinct categories, according to how rapidly R_n grows. The most important condition for this classification is whether or not a ball centered at the origin with radius R_n, denoted by $B(0, R_n)$, asymptotically covers a *weak core*. Weak cores are balls, centered at the origin with growing radii as n increases, in which the random points are densely scattered and form a highly connected geometric graph. This notion, along with the related notion of a *core*, play a crucial role for the classification of the limiting Gaussian processes. Indeed, if $B(0, R_n)$ grows so that it asymptotically covers a weak core, then the geometric graph outside $B(0, R_n)$ is "sparse" with many small disconnected components. In this case, the limit is denoted as the difference between two time-changed Brownian motions. In contrast, if $B(0, R_n)$ is asymptotically covered by a weak core, the geometric graph in the area between the outside of $B(0, R_n)$ and inside of a weak core becomes "dense", and, accordingly, the limit becomes a degenerate Gaussian process with deterministic sample paths. Finally if $B(0, R_n)$ coincides with a weak core, then the limiting Gaussian process possesses more complicated structure and are even non-self-similar. The probabilistic properties of the limiting Gaussian processes are summarized in Tables 1 and 2 at the end of Section 3.

We want to emphasize that the nature of the FCLT depends not only on the growth rate of R_n but also the tail property of f. This is in complete contrast to (1.1), because, as seen in Chapter 3 of [23], the proper normalization, limiting Gaussian distribution, etc. of the CLT are all robust to whether f has a heavy or a light tail. In this paper, we particularly deal with the distributions of regularly varying tails and (sub)exponential tails. However, we are not basically concerned with any distribution with a superexponential tail, e.g., a multivariate normal distribution. The details of the FCLT in that case remain for a future study.

The remainder of the paper is organized as follows. First, in Section 2 we provide a formal definition of the subgraph counting process. Section 3 gives an overview of what was shown in the previous work [22] and what will be shown in this paper. Subsequently, in Section 4 we focus on the case in which the underlying density has a regularly varying tail, including power-law tails, and prove the required FCLT. We also investigate the properties of the limiting Gaussian processes, in particular, in terms of self-similarity and sample path continuity. In Section 5, we do the same when the underlying density has an exponentially decaying tail. To distinguish densities via their tail properties, we need basic tools in EVT. In essence, the properties of the limiting Gaussian processes are determined by how rapidly R_n grows to infinity, as well as how rapidly the tail of f decays. Finally, Section 6 carefully examines both cores and weak cores for a large class of densities.

Before commencing the main body of the paper, we remark that all the random points in this paper are assumed to be generated by an inhomogeneous Poisson point process on $\mathbb{R}^d$ with intensity nf. In our opinion, the FCLT in the main theorem can be carried over to a usual i.i.d. random sample setup by a standard "de-Poissonization" argument;

see Section 2.5 in [23]. This is, however, a little more technical and challenging, and therefore, we decided to concentrate on the simpler setup of an inhomogeneous Poisson point process. Furthermore we consider only spherically symmetric distributions. Although the spherical symmetry assumption is far from being crucial, we adopt it to avoid unnecessary technicalities.

2 Subgraph counting process

Let $(X_i,\, i \geq 1)$ be i.i.d. $\mathbb{R}^d$-valued random variables with spherically symmetric probability density f. Given a Poisson random variable N_n with mean n, independent of $(X_i,\, i \geq 1)$, denote by $\mathcal{P}_n = \{X_1, X_2, \dots, X_{N_n}\}$ a Poisson point process with $|\mathcal{P}_n| := N_n$. We choose a positive integer k, which remains fixed hereafter. We take $k \geq 2$, unless otherwise stated, because many of the functions and objects to follow are degenerate in the case of $k = 1$.

Let Γ be a fixed connected graph of k vertices and G represent a geometric graph; $\cong$ denotes graph isomorphism. To avoid an unnecessary triviality, we assume, in the following, that Γ is *feasible*, that is, $\mathbb{P}\big(G(\{X_1, \dots, X_k\}, r) \cong \Gamma\big) > 0$ for some $r > 0$. We define

$$h(x_1, \dots, x_k) := \mathbf{1}\big\{G\big(\{x_1, \dots, x_k\},\, 1\big) \cong \Gamma\big\}\,, \quad x_1, \dots, x_k \in \mathbb{R}^d\,.$$

Next, we define a collection of indicators $(h_t,\, t \geq 0)$ by

$$h_t(x_1, \dots, x_k) := h(x_1/t, \dots, x_k/t) = \mathbf{1}\big\{G\big(\{x_1, \dots, x_k\},\, t\big) \cong \Gamma\big\}\,, \tag{2.1}$$

from which one can capture the manner in which a geometric graph dynamically evolves as the threshold radius t varies. Note, in particular, that $h_1(x_1, \dots, x_k) = h(x_1, \dots, x_k)$.

Clearly h_t is shift invariant:

$$h_t(x_1, \dots, x_k) = h_t(x_1 + y, \dots, x_k + y)\,, \quad x_1, \dots, x_k, y \in \mathbb{R}^d\,, \tag{2.2}$$

and, further,

$$h_t(0, x_1, \dots, x_{k-1}) = 0 \ \text{ if } ||x_i|| > kt \text{ for some } i = 1, \dots, k-1\,. \tag{2.3}$$

The latter condition implies that $h_t(x_1, \dots, x_k) = 1$ only when all the points $x_1, \dots, x_k$ are close enough to each other.

Moreover h_t can be decomposed as follows. Suppose that Γ has k vertices and j edges for some $j \in \big\{k-1, \dots, k(k-1)/2\big\}$. Letting A_ℓ be the set of all connected graphs on k vertices and ℓ edges (up to graph isomorphism), define for $x_1, \dots, x_k \in \mathbb{R}^d$,

$$h_t^+(x_1, \dots, x_k) := h_t(x_1, \dots, x_k) + \sum_{\ell=j+1}^{k(k-1)/2} \sum_{\Gamma' \in A_\ell} \mathbf{1}\big\{G\big(\{x_1, \dots, x_k\}, t\big) \cong \Gamma'\big\}\,,$$

$$h_t^-(x_1, \dots, x_k) := \sum_{\ell=j+1}^{k(k-1)/2} \sum_{\Gamma' \in A_\ell} \mathbf{1}\big\{G\big(\{x_1, \dots, x_k\}, t\big) \cong \Gamma'\big\}\,.$$

Note that $h_t^+(x_1, \dots, x_k) = 1$ if and only if a geometric graph $G\big(\{x_1, \dots, x_k\}, t\big)$ either coincides with Γ (up to graph isomorphism) or has more than j edges, while $h_t^-(x_1, \dots, x_k) = 1$ only when $G\big(\{x_1, \dots, x_k\}, t\big)$ has more than j edges. It is then elementary to check that $h_t^{\pm}$ are both indicators, taking values 0 or 1, and satisfying, for all $x_1, \dots, x_k \in \mathbb{R}^d$ and $0 \leq s \leq t$,

$$h_t(x_1, \dots, x_k) = h_t^+(x_1, \dots, x_k) - h_t^-(x_1, \dots, x_k)\,, \tag{2.4}$$

$$h_s^+(x_1, \dots, x_k) \leq h_t^+(x_1, \dots, x_k)\,, \tag{2.5}$$

$$h_s^-(x_1, \dots, x_k) \leq h_t^-(x_1, \dots, x_k)\,.$$

$$h_t^{\pm}(0, x_1, \dots, x_{k-1}) = 0 \ \text{ if } ||x_i|| > kt \text{ for some } i = 1, \dots, k-1\,. \tag{2.6}$$

In addition, since h_t is an indicator, it is always the case that

$$h_t^-(x_1,\dots,x_k) \le h_t^+(x_1,\dots,x_k)\,.$$

The objective of this study is to establish a functional central limit theorem (FCLT) of the *subgraph counting process* defined by

$$G_n(t) := \sum_{\mathcal{Y}\subset\mathcal{P}_n} h_t(\mathcal{Y})\,\mathbf{1}\big\{m(\mathcal{Y}) \ge R_n\big\}\,, \quad t\ge 0\,, \tag{2.7}$$

where h_t is given in (2.1), $m(x_1,\dots,x_k) = \min_{1\le i\le k} ||x_i||$, $x_i \in \mathbb{R}^d$, and $(R_n,\, n\ge 1)$ is a properly chosen normalizing sequence tending to infinity. Note that (2.7) counts the number of subgraphs in $G(\mathcal{P}_n, t)$ isomorphic to Γ that lie completely outside of $B(0,R_n)$. More concrete definitions of (R_n) are given in the subsequent sections, where the sequence is shown to be dependent on the tail decay rate of f.

The current work is motivated by extreme value theory (EVT). EVT studies, as its name suggests, the extremal behavior of stochastic processes; in our context, we are interested in the spatial distribution of subgraphs lying far away from the origin. For this reason, we do not treat the case in which the density f has a bounded support with R_n tending to a positive and finite constant.

Another important concept related to (2.7) is the number of "components" of $G(\mathcal{P}_n,t)$ isomorphic to Γ. In this case, the resulting FCLT may partially share the same normalizing constants and limiting Gaussian processes as the FCLT for (2.7), at least when R_n grows sufficiently fast. The present paper, however, does not treat this quantity, because the proof of the FCLT, especially that of tightness, involves much more technicalities due to an extra condition that all Γ-subgraphs must be disconnected from the rest of $\mathcal{P}_n$.

Subgraph counts are one of the most basic quantities in random geometric graph theory. Nevertheless, the study on subgraph counts is a good starting point for more advanced and general objects. One of the most natural and interesting objects is a random Čech complex, which can be regarded as a higher-dimensional version of a random geometric graph. Then, some variants of (2.7), which counts the number of homological holes, are known as a *Betti number* in algebraic topology. The relevant article in this direction is [21] which discusses the asymptotic behavior of Betti numbers relating to the tail of probability distributions. In addition, (2.7) can also be seen as a special case of "local" U-statistics, and therefore, for a future work, it would be interesting to construct FCLTs for a general class of U-statistics. The related publications include, e.g., [24], [25], and [9], though these articles do not necessarily focus on the tail of probability distributions.

3 Annuli structure

The objective of this short section is to clarify what is already known and what is new in this paper. Without any real loss of generality, we will do this via two simple examples, one of which treats a power-law density and the other a density with a (sub)exponential tail. Before this, however, we introduce two important notions.

Definition 3.1. *([1]) Given an inhomogeneous Poisson point process $\mathcal{P}_n$ in $\mathbb{R}^d$ with a spherically symmetric density f, a centered ball $B(0,R_n)$, with $R_n\to\infty$, is called a core if*

$$B(0,R_n) \subset \bigcup_{X\in\mathcal{P}_n\cap B(0,R_n)} B(X,1)\,. \tag{3.1}$$

In other words, a core is a centered ball in which random points are densely scattered, so that placing unit balls around them covers the ball itself. We usually wish to seek the largest possible value of R_n such that (3.1) occurs asymptotically with probability 1. A related notion, the *weak core*, plays a more decisive role in characterizing the FCLT proven in this paper. It is shown later that a weak core is generally larger but close in size to a core of maximum size.

Definition 3.2. *Let f be a spherically symmetric density on $\mathbb{R}^d$ and $e_1 = (1, 0, \dots, 0) \in \mathbb{R}^d$. A weak core is a centered ball $B(0, R_n^{(w)})$ such that $nf(R_n^{(w)} e_1) \to 1$ as $n \to \infty$.*

Example 3.3. Consider the power-law density

$$f(x) = C/\big(1 + ||x||^{\alpha}\big)\,, \quad x \in \mathbb{R}^d, \tag{3.2}$$

for some $\alpha > d$ and normalizing constant C. Using this density, we see how random geometric graphs are formed in all of $\mathbb{R}^d$. First, according to [1], there exists a sequence $R_n^{(c)} \sim \text{constant} \times (n/\log n)^{1/\alpha}$, $n \to \infty$ such that, if $R_n \le R_n^{(c)}$, (3.1) occurs asymptotically with probability 1. In addition, as for the radius of a weak core, it suffices to take $R_n^{(w)} = (Cn)^{1/\alpha}$. Although $R_n^{(w)}$ grows faster than $R_n^{(c)}$, they are seen to be "close" to each other in the sense that they have the same regular variation exponent, $1/\alpha$.

Beyond a weak core, however, the formation of random geometric graphs drastically varies. In fact, the exterior of a weak core can be divided into annuli of different radii, at which many isolated subgraphs of finite vertices are asymptotically placed in a specific fashion. To be more precise, let us fix connected graphs Γ_k with k vertices for $k = 2, 3, \dots$ and let

$$R_{k,n}^{(p)} := \big(Cn\big)^{1/(\alpha - d/k)},$$

which in turn implies that $R_n^{(w)} \ll \cdots \ll R_{k,n}^{(p)} \ll R_{k-1,n}^{(p)} \ll \cdots \ll R_{2,n}^{(p)}$, and

$$n^k \big(R_{k,n}^{(p)}\big)^d f\big(R_{k,n}^{(p)} e_1\big)^k \to 1\,, \quad n \to \infty\,.$$

Under this circumstance, [22] considered the subgraph counts given by

$$\sum_{\mathcal{Y} \subset \mathcal{P}_n} \mathbf{1}\big\{G(\mathcal{Y}, t) \cong \Gamma_k\big\} \times \mathbf{1}\big\{m(\mathcal{Y}) \ge R_{k,n}^{(p)}\big\}\,, \tag{3.3}$$

and showed that (3.3) weakly converges to a Poisson distribution for each fixed t. To be more specific on the geometric side, let $\text{Ann}(K, L)$ be an annulus with inner radius K and outer radius L. Then, we have, in an asymptotic sense,

- Outside $B\big(0, R_{2,n}^{(p)}\big)$, there are finitely many graphs isomorphic to Γ_2, but none isomorphic to $\Gamma_3, \Gamma_4, \dots$.
- Outside $B\big(0, R_{3,n}^{(p)}\big)$, equivalently inside $\text{Ann}\big(R_{3,n}^{(p)}, R_{2,n}^{(p)}\big)$, there are infinitely many graphs isomorphic to Γ_2 and finitely many graphs isomorphic to Γ_3, but none isomorphic to $\Gamma_4, \Gamma_5, \dots$.

In general,

- Outside $B\big(0, R_{k,n}^{(p)}\big)$, equivalently inside $\text{Ann}\big(R_{k,n}^{(p)}, R_{k-1,n}^{(p)}\big)$, there are infinitely many graphs isomorphic to $\Gamma_2, \dots, \Gamma_{k-1}$ and finitely many graphs isomorphic to Γ_k, but none isomorphic to $\Gamma_{k+1}, \Gamma_{k+2}, \dots$ etc.

Section 4 of the current paper considers the subgraph counts of the form

$$\sum_{\mathcal{Y} \subset \mathcal{P}_n} \mathbf{1}\big\{G(\mathcal{Y}, t) \cong \Gamma_k\big\} \times \mathbf{1}\big\{m(\mathcal{Y}) \ge R_n\big\}\,, \tag{3.4}$$

where (R_n) satisfies

$$n^k R_n^d f(R_n e_1)^k \to \infty\,, \quad n \to \infty\,, \tag{3.5}$$

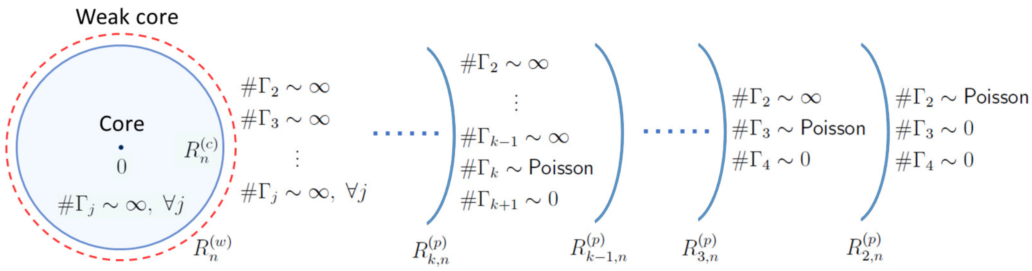

Figure 1: Layered structure of random geometric graphs. For the density (3.2), $R_n^{(c)}$ and $R_n^{(w)}$ are regularly varying sequences with exponent α^{-1}. $R_{k,n}^{(p)}$ is also a regularly varying sequence with exponent $(\alpha - d/k)^{-1}$. We study the FCLT for (3.4) in three different regimes, i.e., (i) $nf(R_n e_1) \to 0$, (ii) $nf(R_n e_1) \to \xi \in (0,\infty)$, and (iii) $nf(R_n e_1) \to \infty$. In relation to other radii, they are respectively equivalent to (i) $R_n^{(w)} \ll R_n \ll R_{k,n}^{(p)}$, (ii) $R_n \sim R_n^{(w)}$, and (iii) $R_n \ll R_n^{(w)}$.

in which case, $R_n \ll R_{k,n}^{(p)}$. Equivalently, $B(0,R_n)$ is covered by $B\big(0,R_{k,n}^{(p)}\big)$ as $n \to \infty$. We may, therefore, expect that infinitely many subgraphs isomorphic to Γ_k appear asymptotically outside $B(0,R_n)$. This in turn implies that, instead of a Poisson limit theorem, the FCLT governs the limiting behavior of the subgraph counting process. More directly, slightly modifying the argument in Theorem 2 of [1], we see that the mean of (3.4) asymptotically behaves as a constant multiple of $n^k R_n^d f(R_n e_1)^k$, which itself diverges as $n \to \infty$. This also indicates that the subgraph counting process (3.4) obeys a FCLT.

As the analog of the setup for (1.1), when deriving an FCLT, the behavior of (3.4) splits into three different regimes:

$$(i)\ nf(R_n e_1) \to 0\,,\quad (ii)\ nf(R_n e_1) \to \xi \in (0,\infty)\,,\quad (iii)\ nf(R_n e_1) \to \infty\,.$$

Specifically, if $nf(R_n e_1) \to 0$ (i.e., $B(0,R_n)$ contains a weak core), many isolated components of subgraphs isomorphic to Γ_k are distributed outside $B(0,R_n)$. If $nf(R_n e_1) \to \xi \in (0,\infty)$ (i.e., $B(0,R_n)$ agrees with a weak core), the subgraphs isomorphic to Γ_k outside $B(0,R_n)$ begin to be connected to one another. In particular, observing that $\lim_{k\to\infty} R_{k,n}^{(p)} = R_n^{(w)}$ for all n, we see that

- Outside of $B(0,R_n^{(w)})$, there are infinitely many graphs isomorphic to Γ_j for every $j = 2,3,\dots$.

If $nf(R_n e_1) \to \infty$ (i.e., $B(0,R_n)$ is contained in a weak core), the subgraphs isomorphic to Γ_k outside $B(0,R_n)$ are further increasingly connected and form a large component.

In Section 4, we will see that the nature of the FCLT, including the normalizing constants and the properties of the limiting Gaussian processes, differs according to which regime one considers. Combing the results on the FCLT and the Poissonian results in [22] produces a complete picture of the annuli structure formed by heavy tailed random variables.

Example 3.4. Next, we turn to a density with a (sub)exponential tail

$$f(x) = Ce^{-||x||^\tau/\tau}, \quad x \in \mathbb{R}^d\,,\ 0 < \tau \le 1\,.$$

for which the radius of a maximum core is given by

$$R_n^{(c)} = \big(\tau \log n - \tau \log\log(\tau\log n)^{1/\tau} + \text{constant}\big)^{1/\tau};$$

	$nf(R_ne_1) \to 0$	$nf(R_ne_1) \to \xi$	$nf(R_ne_1) \to \infty$
Regularly varying tail	$d(k-1)/2$	Non-SS	$d(k-1)$
Subexponential tail	$d(k-1)/2$	Non-SS	$d(k-1)$
Exponential tail	Non-SS	Non-SS	Non-SS

Table 1: Self-similarity exponents of the limiting Gaussian processes. A regularly varying tail is a notion generalizing a power-law tail. Non-SS means that the process is non-self-similar. A zero limit of $nf(R_ne_1)$ is equivalent to the case in which a ball $B(0, R_n)$ contains a weak core, and $nf(R_ne_1) \to \infty$ if and only if $B(0, R_n)$ is contained in a weak core. If $nf(R_ne_1) \to \xi \in (0, \infty)$, then $B(0, R_n)$ agrees with a weak core (up to multiplicative constants).

see [1] and [22]. In addition, one can take $R_n^{(w)} = \big(\tau \log n + \tau \log C\big)^{1/\tau}$. We have that $R_n^{(c)} < R_n^{(w)}$ for sufficiently large n, but these radii are asymptotically equal. As in the previous example, the exterior of a weak core is characterized by the same kind of layer structure, for which the description in Figure 1 applies, except for the change in the values of $R_{k,n}^{(p)}$. Letting

$$R_{k,n}^{(p)} = \big(\tau \log n + k^{-1}(d-\tau) \log(\tau \log n) + \tau \log C\big)^{1/\tau},$$

we have, in an asymptotic sense, $R_n^{(w)} \ll \cdots \ll R_{k,n}^{(p)} \ll R_{k-1,n}^{(p)} \ll \cdots \ll R_{2,n}^{(p)}$, and

$$n^k \big(R_{k,n}^{(p)}\big)^{d-\tau} f\big(R_{k,n}^{(p)} e_1\big)^k \to 1 \,, \quad n \to \infty \,.$$

Then, it was shown in [22] that (3.3) converges weakly to a Poisson distribution for each fixed t.

In Section 5 of this paper, taking (R_n) such that $n^k R_n^{d-\tau} f(R_n e_1)^k \to \infty$, we establish a FCLT for the subgraph counting process (2.7). To this end, our argument has to be split, once again, into the three different regimes:

$$(i)\ nf(R_ne_1) \to 0 \,, \quad (ii)\ nf(R_ne_1) \to \xi \in (0,\infty) \,, \quad (iii)\ nf(R_ne_1) \to \infty \,.$$

As in the last example, three different Gaussian limits may appear depending on the regime. This completes the full description of the annuli structure formed by random variables with an exponentially decaying tail, when combined with the Poisson limit theorems in [22].

Before proceeding to the next section, we would like to quickly overview the properties of limiting Gaussian processes in the FCLT in terms of self-similarity and some representation results. The probabilistic features of the limiting Gaussian processes crucially differ whether $B(0, R_n)$ contains a weak core or not. In addition, the tail decay rate of an underlying density plays a decisive role as well. More detailed arguments are presented in the subsequent sections.

4 Heavy tail case

4.1 The setup

In this section, we explore the case in which the underlying density f on $\mathbb{R}^d$ has a heavy tail under a more general setup than that in Example 3.3. Let S_{d-1} be a $(d-1)$-dimensional unit sphere in $\mathbb{R}^d$. We assume that the density has a regularly varying tail

	$nf(R_ne_1) \to 0$	$nf(R_ne_1) \to \xi$	$nf(R_ne_1) \to \infty$
Regularly varying tail	Difference of time-changed Brownian motions	New	Degenerate Gaussian process
Subexponential tail	Difference of time-changed Brownian motions	New	Degenerate Gaussian process
Exponential tail	Difference of time-changed Brownian motions	New	New

Table 2: Representation results on the limiting Gaussian processes. "New" implies that the limit constitutes a new class of Gaussian processes.

(at infinity) in the sense that for any $\theta \in S_{d-1}$ (equivalently, for some $\theta \in S_{d-1}$ because of the spherical symmetry of f), and for some $\alpha > d$,

$$\lim_{r\to\infty} \frac{f(rt\theta)}{f(r\theta)} = t^{-\alpha} \text{ for every } t > 0\,.$$

Denoting by $RV_{-\alpha}$ a collection of regularly varying functions (at infinity) of exponent $-\alpha$, the above is written as

$$f \in RV_{-\alpha}\,. \tag{4.1}$$

Clearly, a power-law density in Example 3.3 satisfies (4.1). Let $k \geq 2$ be an integer that remains fixed throughout this section. We remark that many of the functions and objects are dependent on k, but the dependence may not be stipulated by subscripts (or superscripts). Choosing the sequence $R_n \to \infty$ so that

$$n^k R_n^d f(R_ne_1)^k \to \infty \text{ as } n \to \infty\,, \tag{4.2}$$

we consider the subgraph counting process given in (2.7), whose behavior is, as argued in Example 3.3, expected to be governed by a FCLT.

The scaling constants for the FCLT, denoted by τ_n, are shown to depend on the limit value of $nf(R_ne_1)$ as $n \to \infty$. More precisely, we take

$$\tau_n := \begin{cases} n^k R_n^d f(R_ne_1)^k & \text{if } nf(R_ne_1) \to 0\,, \\ R_n^d & \text{if } nf(R_ne_1) \to \xi \in (0,\infty)\,, \\ n^{2k-1} R_n^d f(R_ne_1)^{2k-1} & \text{if } nf(R_ne_1) \to \infty\,. \end{cases} \tag{4.3}$$

The reason for which we need three different normalizations is deeply related to the connectivity of a random geometric graph. To explain this, we need the notion of a *weak core*; see Definition 3.2 for the formal definition. The main point is that the density of random points between the outside and inside of a weak core is completely different. In essence, random points inside a weak core are highly densely scattered, and the corresponding random geometric graph forms a single giant component. Beyond a weak core, however, random points are distributed sparsely, and as a result, we observe many isolated geometric graphs of smaller size. This disparity between the outside and inside of a weak core requires different normalizations in (τ_n). In Section 6, a more detailed study in this direction is presented.

4.2 Limiting Gaussian processes and the FCLT

We introduce a family of Gaussian processes which function as the building blocks for the limiting Gaussian processes in the FCLT. For $\ell = 1, \dots, k$, let

$$B_\ell = \frac{s_{d-1}}{\ell!\big((k-\ell)!\big)^2\big(\alpha(2k-\ell)-d\big)}\,,$$

where s_{d-1} is a surface area of the $(d-1)$-dimensional unit sphere in $\mathbb{R}^d$.

For $\ell = 2, \dots, k$, write λ_ℓ for the Lebesgue measure on $(\mathbb{R}^d)^{\ell-1}$, and denote by G_ℓ a *Gaussian $B_\ell\lambda_\ell$-noise*, such that

$$G_\ell(A) \sim \mathcal{N}\big(0, B_\ell\lambda_\ell(A)\big)$$

for measurable sets $A \subset (\mathbb{R}^d)^{\ell-1}$ with $\lambda_\ell(A) < \infty$, and if $A \cap B = \emptyset$, then $G_\ell(A)$ and $G_\ell(B)$ are independent. For $\ell = 1$, we define G_1 as a Gaussian random variable with zero mean and variance B_1. We assume that $G_1, \dots, G_k$ are independent.

For $\ell = 2, \dots, k-1$, we define Gaussian processes $\mathbf{V}_\ell = \big(V_\ell(t),\, t \ge 0\big)$ by

$$V_\ell(t) := \int_{(\mathbb{R}^d)^{\ell-1}} \int_{(\mathbb{R}^d)^{k-\ell}} h_t(0,\mathbf{y},\mathbf{z})\, d\mathbf{z}\, G_\ell(d\mathbf{y}), \quad t \ge 0.$$

In addition, if $\ell = k$, define

$$V_k(t) := \int_{(\mathbb{R}^d)^{k-1}} h_t(0,\mathbf{y}) G_k(d\mathbf{y}),$$

and if $\ell = 1$, set

$$V_1(t) := \int_{(\mathbb{R}^d)^{k-1}} h_t(0,\mathbf{z})\, d\mathbf{z}\, G_1 = t^{d(k-1)} \int_{(\mathbb{R}^d)^{k-1}} h_1(0,\mathbf{z})\, d\mathbf{z}\, G_1.$$

Note that $\mathbf{V}_1$ is a degenerate Gaussian process with deterministic sample paths. These processes later turn out to be the building blocks of the weak limits in the main theorem.

The covariance function of the process $\mathbf{V}_\ell$ is given by

$$\begin{aligned} L_\ell(t,s) &:= \mathbb{E}\big\{V_\ell(t)V_\ell(s)\big\} \qquad (4.4)\\ &= B_\ell \int_{(\mathbb{R}^d)^{\ell-1}} d\mathbf{y} \int_{(\mathbb{R}^d)^{k-\ell}} d\mathbf{z}_2 \int_{(\mathbb{R}^d)^{k-\ell}} d\mathbf{z}_1\, h_t(0,\mathbf{y},\mathbf{z}_1)\, h_s(0,\mathbf{y},\mathbf{z}_2)\,, \quad t,s \ge 0 \end{aligned}$$

(if $\ell = k$, we take $\mathbf{z}_i = \emptyset$, $i = 1,2$, and if $\ell = 1$, we set $\mathbf{y} = \emptyset$).

Using the decomposition (2.4), we can express $\mathbf{V}_\ell$ as the difference between two Gaussian processes; that is, for $\ell = 2, \dots, k-1$,

$$\begin{aligned} V_\ell(t) &= \int_{(\mathbb{R}^d)^{\ell-1}} \int_{(\mathbb{R}^d)^{k-\ell}} h_t^+(0,\mathbf{y},\mathbf{z})\, d\mathbf{z}\, G_\ell(d\mathbf{y}) - \int_{(\mathbb{R}^d)^{\ell-1}} \int_{(\mathbb{R}^d)^{k-\ell}} h_t^-(0,\mathbf{y},\mathbf{z})\, d\mathbf{z}\, G_\ell(d\mathbf{y})\\ &:= V_\ell^+(t) - V_\ell^-(t). \end{aligned}$$

The same decomposition is feasible in an analogous manner for $\mathbf{V}_1$ and $\mathbf{V}_k$.

The following proposition shows that the processes $\mathbf{V}_k^+$ and $\mathbf{V}_k^-$ can be represented as a time-changed Brownian motion.

Proposition 4.1. *The process $\mathbf{V}_k^+$ can be expressed as*

$$\big(V_k^+(t),\, t \ge 0\big) \stackrel{d}{=} \Big(B\big(K_k^+\, t^{d(k-1)}\big),\, t \ge 0\Big),$$

where B is the standard Brownian motion, and $K_k^+ := B_k \int_{(\mathbb{R}^d)^{k-1}} h_1^+(0,\mathbf{y})d\mathbf{y}$.

Replacing K_k^+ with $K_k^- := B_k \int_{(\mathbb{R}^d)^{k-1}} h_1^-(0,\mathbf{y})d\mathbf{y}$, we obtain the same statement for $\mathbf{V}_k^-$.

Proof. It is enough to verify that the covariance functions on both sides coincide. It follows from (2.5) that for $0 \le s \le t$,

$$\begin{aligned}\mathbb{E}\big\{V_k^+(t)V_k^+(s)\big\} &= B_k \int_{(\mathbb{R}^d)^{k-1}} h_t^+(0,\mathbf{y})\, h_s^+(0,\mathbf{y})d\mathbf{y} \\ &= s^{d(k-1)}K_k^+ \\ &= \mathbb{E}\big\{B(K_k^+\, t^{d(k-1)})B(K_k^+\, s^{d(k-1)})\big\}.\end{aligned}$$

□

We also claim that the process $\mathbf{V}_\ell$ is self-similar and has a.s. Hölder continuous sample paths. Recall that a stochastic process $\big(X(t),\, t\ge 0\big)$ is said to be self-similar with exponent H if

$$\big(X(ct_i),\, i=1,\dots,k\big) \stackrel{d}{=} \big(c^H X(t_i),\, i=1,\dots,k\big)$$

for any $c>0$, $t_1,\dots,t_k\ge 0$, and $k\ge 1$.

Proposition 4.2. *(i) For $\ell=1,\dots,k$, the process $\mathbf{V}_\ell$ is self similar with exponent $H=d(2k-\ell-1)/2$.*

(ii) For $\ell=1,\dots,k$ and every $T>0$, $\big(V_\ell(t),\, 0\le t\le T\big)$ has a modification, the sample paths of which are Hölder continuous of any order in $[0,1/2)$.

Proof. We can immediately prove (i) by the scaling property

$$L_\ell(ct,cs) = c^{d(2k-\ell-1)}L_\ell(t,s)\,, \quad t,s\ge 0\,,\ c>0\,.$$

As for (ii), the statement is obvious for $\ell=1$ or $\ell=k$; therefore, we take $\ell\in\{2,\dots,k-1\}$. By Gaussianity,

$$\mathbb{E}\Big\{\big(V_\ell(t)-V_\ell(s)\big)^{2m}\Big\} = \prod_{i=1}^m (2i-1)\left(\mathbb{E}\Big\{\big(V_\ell(t)-V_\ell(s)\big)^2\Big\}\right)^m,\ \ m=1,2,\dots \tag{4.5}$$

We now show that there exists a constant $C>0$, which depends on T, such that

$$\mathbb{E}\Big\{\big(V_\ell(t)-V_\ell(s)\big)^2\Big\} \le C(t-s)\ \text{ for all } 0\le s\le t\le T\,. \tag{4.6}$$

By virtue of the decomposition $\mathbf{V}_\ell = \mathbf{V}_\ell^+ - \mathbf{V}_\ell^-$, showing (4.6) for each of $\mathbf{V}_\ell^+$ and $\mathbf{V}_\ell^-$ suffices. We handle $\mathbf{V}_\ell^+$ only, since $\mathbf{V}_\ell^-$ can be treated in the same manner. We have

$$\begin{aligned}\mathbb{E}\Big\{\big(V_\ell^+(t)-V_\ell^+(s)\big)^2\Big\} = B_\ell \int_{(\mathbb{R}^d)^{\ell-1}} d\mathbf{y} \int_{(\mathbb{R}^d)^{k-\ell}} d\mathbf{z}_2 \int_{(\mathbb{R}^d)^{k-\ell}} d\mathbf{z}_1 \big\{h_t^+(0,\mathbf{y},\mathbf{z}_1) - h_s^+(0,\mathbf{y},\mathbf{z}_1)\big\} \\ \times \big\{h_t^+(0,\mathbf{y},\mathbf{z}_2) - h_s^+(0,\mathbf{y},\mathbf{z}_2)\big\}\,.\end{aligned}$$

Because of (2.6), the above integral is not altered if the integral domain is restricted to $(\mathbb{R}^d)^{\ell-1}\times(\mathbb{R}^d)^{k-\ell}\times\big(B(0,kT)\big)^{k-\ell}$. In addition, by (2.5), there exist constants $C_1, C_2>0$, both depending on T, such that

$$\begin{aligned}\mathbb{E}\Big\{\big(V_\ell^+(t)-V_\ell^+(s)\big)^2\Big\} &\le C_1 \int_{(\mathbb{R}^d)^{\ell-1}} \int_{(\mathbb{R}^d)^{k-\ell}} \big\{h_t^+(0,\mathbf{y},\mathbf{z}) - h_s^+(0,\mathbf{y},\mathbf{z})\big\} d\mathbf{z}d\mathbf{y} \\ &= C_1 \int_{(\mathbb{R}^d)^{\ell-1}} \int_{(\mathbb{R}^d)^{k-\ell}} h_1^+(0,\mathbf{y},\mathbf{z})d\mathbf{z}d\mathbf{y}\, \big(t^{d(k-1)} - s^{d(k-1)}\big) \\ &\le C_2 \int_{(\mathbb{R}^d)^{\ell-1}} \int_{(\mathbb{R}^d)^{k-\ell}} h_1^+(0,\mathbf{y},\mathbf{z})d\mathbf{z}d\mathbf{y}\, (t-s)\ \text{ for all } 0\le s\le t\le T\,,\end{aligned}$$

which verifies (4.6).

Combining (4.5) and (4.6), we have that for some $C_3 > 0$,

$$\mathbb{E}\Big\{\big(V_\ell(t) - V_\ell(s)\big)^{2m}\Big\} \le C_3(t-s)^m \ \text{ for all } 0 \le s \le t \le T\,.$$

It now follows from the Kolmogorov continuity theorem that there exists a modification of $\big(V_\ell(t),\, 0 \le t \le T\big)$, the sample paths of which are Hölder continuous of any order in $\big[0, (m-1)/(2m)\big)$. Since m is arbitrary, we are done by letting $m \to \infty$. □

We are now ready to state the FCLT for the subgraph counting process, suitably scaled and centered in such a way that

$$X_n(t) = \tau_n^{-1/2}\big(G_n(t) - \mathbb{E}\{G_n(t)\}\big)\,, \quad t \ge 0\,.$$

In the following, $\Rightarrow$ denotes weak convergence. All weak convergence hereafter are in the space $\mathcal{D}[0,\infty)$ of right-continuous functions with left limits. The proof of the theorem is deferred to Section 7.1. Since Γ is assumed to be a feasible subgraph, we have that $G_n(t) > 0$ with positive probability for any $n \ge 1$ and $t > 0$, and thus, the resulting FCLTs are always non-trivial.

Theorem 4.3. *Assume that the probability density f has a regularly varying tail as in* (4.1).

(i) *If $nf(R_n e_1) \to 0$ as $n \to \infty$, then*

$$\big(X_n(t),\, t \ge 0\big) \Rightarrow \big(V_k(t),\, t \ge 0\big) \ \text{ in } \mathcal{D}[0,\infty)\,.$$

(ii) *If $nf(R_n e_1) \to \xi \in (0,\infty)$ as $n \to \infty$, then*

$$\big(X_n(t),\, t \ge 0\big) \Rightarrow \left(\sum_{\ell=1}^{k} \xi^{2k-\ell} V_\ell(t),\, t \ge 0\right) \ \text{ in } \mathcal{D}[0,\infty)\,.$$

(iii) *If $nf(R_n e_1) \to \infty$ as $n \to \infty$, then*

$$\big(X_n(t),\, t \ge 0\big) \Rightarrow \big(V_1(t),\, t \ge 0\big) \ \text{ in } \mathcal{D}[0,\infty)\,.$$

The processes $\mathbf{V}_1, \dots, \mathbf{V}_k$ can be viewed as the building blocks of the limiting Gaussian processes; however, how many and which ones contribute to the limit depends on whether the ball $B(0, R_n)$ covers a weak core or not. If $B(0, R_n)$ covers a weak core, equivalently, $nf(R_n e_1) \to 0$, then $\mathbf{V}_k$ is the only process remaining in the limit. Although, as seen in Proposition 4.1, $\mathbf{V}_k$ is generally represented as the difference in two time-changed Brownian motions, it can be denoted as a *single* time-changed Brownian motion when h_t is increasing in t, i.e., $h_s(\mathcal{Y}) \le h_t(\mathcal{Y})$ for all $0 \le s \le t$, $\mathcal{Y} \in (\mathbb{R}^d)^k$. This is the case when Γ is a complete graph, in which case the negative part h_t^- is identically zero. In contrast, the process $\mathbf{V}_1$, a degenerate Gaussian process with deterministic sample paths, only appears in the limit when $B(0, R_n)$ is contained in a weak core, i.e., $nf(R_n e_1) \to \infty$. Finally, if $B(0, R_n)$ agrees with a weak core (up to multiplicative constants), all of the processes $\mathbf{V}_1, \dots, \mathbf{V}_k$ contribute to the limit. Interestingly, only in this case, do the weak limits become non-self-similar.

5 Exponentially decaying tail case

5.1 The setup

This section develops the FCLT of the subgraph counting process suitably scaled and centered, when the underlying density on $\mathbb{R}^d$ possesses an exponentially decaying tail. Typically, in the spirit of extreme value theory, a class of multivariate densities with exponentially decaying tails can be formulated by the so-called *von Mises functions*. See for example, [3] and [4]. In particular, in the one-dimensional case ($d = 1$), the von Mises function plays a decisive role in the characterization of the max-domain of attraction of the Gumbel law. See Proposition 1.4 in [26]. We assume that the density f on $\mathbb{R}^d$ is given by

$$f(x) = L\big(||x||\big) \exp\big\{-\psi\big(||x||\big)\big\}\,, \quad x \in \mathbb{R}^d. \tag{5.1}$$

Here, $\psi : \mathbb{R}_+ \to \mathbb{R}$ is a function of C^2-class and is referred to as a von Mises function, so that

$$\psi'(z) > 0, \quad \psi(z) \to \infty, \quad \big(1/\psi'\big)'(z) \to 0 \tag{5.2}$$

as $z \to z_\infty \in (0, \infty]$. In this paper, we restrict ourselves to an unbounded support of the density, i.e., $z_\infty \equiv \infty$. For notational ease, we introduce the function $a(z) = 1/\psi'(z)$, $z > 0$. Since $a'(z) \to 0$ as $z \to \infty$, the Cesàro mean of a' converges as well:

$$\frac{a(z)}{z} = \frac{1}{z}\int_0^z a'(r)dr \to 0\,, \quad \text{as } z \to \infty\,. \tag{5.3}$$

Suppose that a measurable function $L : \mathbb{R}_+ \to \mathbb{R}_+$ is *flat* for a, that is,

$$\frac{L\big(t + a(t)v\big)}{L(t)} \to 1 \ \text{ as } t \to \infty \text{ uniformly on bounded } v\text{-sets.} \tag{5.4}$$

This condition implies that L behaves as a constant locally in the tail of f, and thus, only ψ plays a dominant role in the characterization of the tail of f. Here, we need to put an extra technical condition on L. Namely, there exist $\gamma \geq 0$, $z_0 > 0$, and $C \geq 1$ such that

$$\frac{L(zt)}{L(z)} \leq C\,t^\gamma \ \text{ for all } t > 1,\ z \geq z_0\,. \tag{5.5}$$

Since L is negligible in the tail of f, it seems reasonable to classify the density (5.1) in terms of the limit of a. If $a(z) \to \infty$ as $z \to \infty$, we say that f belongs to a class of densities with *subexponential* tail, because the tail of f decays more slowly than that of an exponential distribution. Conversely, if $a(z) \to 0$ as $z \to \infty$, f is said to have a *superexponential* tail, and if $a(z) \to c \in (0, \infty)$, we say that f has an exponential tail. To be more specific about the difference in tail behaviors, let us consider a slightly more general example than that in Example 3.4, for which $f(x) = L\big(||x||\big) \exp\big\{-||x||^\tau/\tau\big\}$, $\tau > 0$, $x \in \mathbb{R}^d$. Clearly, the parameter τ is associated with the speed at which f vanishes in the tail. Observe that $a(z) = z^{1-\tau} \to \infty$ as $z \to \infty$ if $0 < \tau < 1$, and therefore in this case, f has a subexponential tail. If $\tau > 1$, $a(z)$ decreases to 0, in which case f has a superexponential tail.

An important assumption throughout most of this study is that there exists $c \in (0, \infty]$ such that

$$a(z) \to c \ \text{ as } z \to \infty\,. \tag{5.6}$$

In view of the classification described above, (5.6) eliminates the possibility of densities with superexponential tail. As discovered in [22] and [1], random points drawn from a superexponential law hardly form isolated geometric graphs outside a core, whereas random points coming from a subexponential law do constitute a layer of isolated geometric graphs outside a core. Accordingly, it is highly likely that the nature of the FCLT differs according to whether the underlying density has a superexponential or a subexponential tail. The present work focuses on the (sub)exponential tail case, and more detailed studies on a superexponential tail case remain for future work.

To realize a more formal set up, let $k \geq 2$ be an integer, which remains fixed for the remainder of this section; however, once again, note that many of the functions and objects are implicitly dependent on k. Define the sequence $R_n \to \infty$, so that

$$n^k a(R_n) R_n^{d-1} f(R_n e_1)^k \to \infty\,, \quad n \to \infty\,. \tag{5.7}$$

Defining an alternative sequence $R_{k,n}^{(p)} \to \infty$ for which

$$n^k a\big(R_{k,n}^{(p)}\big)\big(R_{k,n}^{(p)}\big)^{d-1} f\big(R_{k,n}^{(p)} e_1\big)^k \to 1\,, \quad n \to \infty\,,$$

the subgraph counting process using $R_{k,n}^{(p)}$ is known to weakly converge to a Poisson distribution; see [22]. Since R_n in (5.7) grows more slowly than $R_{k,n}^{(p)}$, i.e., $R_n / R_{k,n}^{(p)} \to 0$, we may expect that an FCLT plays a decisive role in the asymptotic behavior of a subgraph counting process.

As in the last section, we now want to recall the notion of a *weak core*. Let $R_n^{(w)} \to \infty$ be a sequence such that $nf(R_n^{(w)} e_1) \to 1$ as $n \to \infty$. Then, we say that a ball $B\big(0, R_n^{(w)}\big)$ is a weak core. We have to change, once again, the scaling constants τ_n of the FCLT, depending on whether $B(0, R_n)$ covers a weak core or not. More specifically, we define

$$\tau_n := \begin{cases} n^k a(R_n) R_n^{d-1} f(R_n e_1)^k & \text{if } nf(R_n e_1) \to 0\,, \\ a(R_n) R_n^{d-1} & \text{if } nf(R_n e_1) \to \xi \in (0,\infty)\,, \\ n^{2k-1} a(R_n) R_n^{d-1} f(R_n e_1)^{2k-1} & \text{if } nf(R_n e_1) \to \infty\,. \end{cases} \tag{5.8}$$

5.2 Limiting Gaussian processes and the FCLT

The objective of this subsection is to formulate the limiting Gaussian processes and the FCLT. Let

$$D_\ell = \frac{s_{d-1}}{\ell!\,\big((k-\ell)!\big)^2}\,, \quad \ell = 1, \dots, k, \tag{5.9}$$

and let H_ℓ be a Gaussian μ_ℓ-noise, where the μ_ℓ for $\ell = 2, \dots, k$, satisfy

$$\mu_\ell(d\rho\, d\mathbf{y}) = D_\ell\, e^{-\ell\rho - c^{-1}\sum_{i=1}^{\ell-1}\langle e_1, y_i\rangle} \times \mathbf{1}\big\{\, \rho + c^{-1}\langle e_1, y_i\rangle \geq 0,\ i = 1, \dots, \ell-1 \,\big\}\, d\rho\, d\mathbf{y}, \quad \rho \geq 0, \mathbf{y} \in (\mathbb{R}^d)^{\ell-1},$$

and

$$\mu_1(d\rho) = D_1\, e^{-\rho} d\rho, \quad \rho \geq 0,$$

where $c \in [0,\infty)$ is determined in (5.6). Moreover, we assume that $H_1, \dots, H_k$ are independent.

We now define a collection of Gaussian processes needed for the construction of the limits in the FCLT. For $\ell = 2, \dots, k-1$, we define

$$W_\ell(t) := \int_{[0,\infty)\times(\mathbb{R}^d)^{\ell-1}} \int_{(\mathbb{R}^d)^{k-\ell}} e^{-\sum_{i=1}^{k-\ell}\left(\rho + c^{-1}\langle e_1, z_i\rangle\right)} \times \mathbf{1}\big\{\, \rho + c^{-1}\langle e_1, z_i\rangle \geq 0,\ i = 1, \dots, k-\ell \,\big\}\, h_t(0, \mathbf{y}, \mathbf{z})\, d\mathbf{z}\, H_\ell(d\rho\, d\mathbf{y}),$$

and, further,

$$W_1(t) := \int_0^\infty \int_{(\mathbb{R}^d)^{k-1}} e^{-\sum_{i=1}^{k-1}\left(\rho + c^{-1}\langle e_1, z_i\rangle\right)} \times \mathbf{1}\big\{\, \rho + c^{-1}\langle e_1, z_i\rangle \geq 0,\ i = 1, \dots, k-1 \,\big\}\, h_t(0, \mathbf{z})\, d\mathbf{z}\, H_1(d\rho),$$

$$W_k(t) := \int_{[0,\infty)\times(\mathbb{R}^d)^{k-1}} h_t(0, \mathbf{y})\, H_k(d\rho\, d\mathbf{y}).$$

As we did in Section 4.2, by the decomposition $h_t = h_t^+ - h_t^-$, one can write the process $\mathbf{W}_\ell$ as the corresponding difference $\mathbf{W}_\ell = \mathbf{W}_\ell^+ - \mathbf{W}_\ell^-$ for $\ell = 1, \dots, k$.

It is easy to compute the covariance function of $\mathbf{W}_\ell$. We have, for $\ell = 1, \dots, k$ and $t, s \ge 0$,

$$\begin{aligned} M_\ell(t,s) &:= \mathbb{E}\big\{ W_\ell(t) W_\ell(s) \big\} \qquad (5.10) \\ &= D_\ell \int_0^\infty \int_{(\mathbb{R}^d)^{2k-\ell-1}} e^{-(2k-\ell)\rho - c^{-1}\sum_{i=1}^{2k-\ell-1} \langle e_1, y_i \rangle} \\ &\quad \times \mathbf{1}\big\{ \rho + c^{-1}\langle e_1, y_i \rangle \ge 0\,,\ i = 1, \dots, 2k-\ell-1 \big\} h_{t,s}^{(\ell)}(0, \mathbf{y})\, d\mathbf{y}\, d\rho\,, \end{aligned}$$

where

$$h_{t,s}^{(\ell)}(0, y_1, \dots, y_{2k-\ell-1}) := h_t(0, y_1, \dots, y_{k-1})\, h_s(0, y_1, \dots, y_{\ell-1}, y_k, \dots, y_{2k-\ell-1})\,, \qquad (5.11)$$

and, in particular, we set

$$h_s(0, y_1, \dots, y_{\ell-1}, y_k, \dots, y_{2k-\ell-1}) := \begin{cases} h_s(0, y_k, \dots, y_{2k-2}) & \text{if } \ell = 1\,, \\ h_s(0, y_1, \dots, y_{k-1}) & \text{if } \ell = k\,. \end{cases}$$

It is important to note that if $a(z) \to \infty$ as $z \to \infty$, then M_ℓ coincides with L_ℓ given in (4.4) up to multiplicative factors, i.e.,

$$M_\ell(t,s) = \big(\alpha - d(2k-\ell)^{-1}\big) L_\ell(t,s), \ \ t, s \ge 0.$$

This in turn implies that

$$\mathbf{W}_\ell \stackrel{d}{=} \big(\alpha - d(2k-\ell)^{-1}\big)^{1/2} \mathbf{V}_\ell,$$

in which case, there is nothing to explore here, because the properties of $\mathbf{V}_\ell$ have already been studied in Section 4.2.

In contrast, if $a(z) \to c \in (0, \infty)$ as $z \to \infty$, then M_ℓ does not directly relate to L_ℓ as above, and, consequently, the process $\mathbf{W}_\ell$ exhibits properties different to those of $\mathbf{V}_\ell$. For example, although one may anticipate, as the analog of the process $\mathbf{V}_1$, that $\mathbf{W}_1$ is a degenerate Gaussian process, this is no longer the case.

Proposition 5.1. *Suppose that $a(z) \to c \in (0, \infty)$ as $z \to \infty$.*

(i) $\mathbf{W}_1$ *is a non-degenerate Gaussian process.*

(ii) *For $\ell = 1, \dots, k$, $\mathbf{W}_\ell$ is non-self-similar.*

Proof. If $a(z) \to c \in (0, \infty)$ as $z \to \infty$, then $M_1(t,s)$ cannot be decomposed into a function of t and a function of s, and therefore, $\mathbf{W}_1$ is non-degenerate.

As for (ii), M_ℓ does not match L_ℓ at all and it loses the scale invariance, meaning that $\mathbf{W}_\ell$ is non-self-similar. □

Similarly to Proposition 4.1, however, the process $\mathbf{W}_k (= \mathbf{W}_k^+ - \mathbf{W}_k^-)$ can be denoted in law as the difference between two time-changed Brownian motions, regardless of whether $a(z) \to \infty$ or $a(z) \to c \in (0, \infty)$ as $z \to \infty$. Furthermore, the sample paths of $\mathbf{W}_\ell$ are Hölder continuous.

Proposition 5.2. *Irrespective of the limit of a, the following two results hold.*

(i) *The process $\mathbf{W}_k^+$ can be represented in law as*

$$\big(W_k^+(t),\, t \ge 0\big) \stackrel{d}{=} \bigg(B\Big(\int_{[0,\infty) \times (\mathbb{R}^d)^{k-1}} h_t^+(0, \mathbf{y})\, \mu_k(d\rho\, d\mathbf{y}) \Big),\, t \ge 0 \bigg),$$

where B is the standard Brownian motion.

The same statement holds for $\mathbf{W}_k^-$, by replacing h_t^+ with h_t^-.

(ii) *For $\ell = 1, \dots, k$, and every $T > 0$, $\big(W_\ell(t),\, 0 \le t \le T\big)$ has a modification, the sample paths of which are Hölder continuous of any order in $[0, 1/2)$.*

Proof. The proof of (i) is very similar to that in Proposition 4.1, so we omit it. The proof of (ii) is analogous to that in Proposition 4.2 (ii); we have only to show that for some $C > 0$,

$$\mathbb{E}\Big\{\big(W_\ell(t) - W_\ell(s)\big)^2\Big\} \le C(t-s) \;\text{ for all } 0 \le s \le t \le T\,.$$

Because of the decomposition $\mathbf{W}_\ell = \mathbf{W}_\ell^+ - \mathbf{W}_\ell^-$, it suffices to prove the above for each $\mathbf{W}_\ell^+$ and $\mathbf{W}_\ell^-$. We check only the case of $\mathbf{W}_\ell^+$. We see that

$$\begin{aligned}
&\mathbb{E}\Big\{\big(W_\ell^+(t) - W_\ell^+(s)\big)^2\Big\} \\
&= \int_{[0,\infty)\times(\mathbb{R}^d)^{\ell-1}} \bigg(\int_{(\mathbb{R}^d)^{k-\ell}} e^{-\sum_{i=1}^{k-\ell}\left(\rho + c^{-1}\langle e_1, z_i\rangle\right)} \mathbf{1}\big\{\, \rho + c^{-1}\langle e_1, z_i\rangle \ge 0,\; i = 1,\dots,k-\ell\,\big\} \\
&\qquad\qquad \times \big(h_t^+(0,\mathbf{y},\mathbf{z}) - h_s^+(0,\mathbf{y},\mathbf{z})\big)\, d\mathbf{z}\bigg)^2 \mu_\ell(d\rho\, d\mathbf{y}) \\
&\le D_\ell\, B_\ell^{-1}\, \mathbb{E}\Big\{\big(V_\ell^+(t) - V_\ell^+(s)\big)^2\Big\}.
\end{aligned}$$

The rest of the argument is completely the same as Proposition 4.2 (ii). □

Now, we can state the FCLT of the centered and scaled subgraph counting process

$$X_n(t) = \tau_n^{-1/2}\big(G_n(t) - \mathbb{E}\{G_n(t)\}\big)\,, \quad t \ge 0\,,$$

where the normalizing sequence (R_n) satisfies (5.7) and (τ_n) is defined in (5.8). The proof of the theorem is presented in Section 7.2.

We have now officially presented all the limiting Gaussian processes of this paper, so it would be beneficial for the readers to return to Tables 1 and 2 in Section 3 to overview their properties once again. These tables indicate that the limiting Gaussian processes are somewhat special when f has an exponential tail. For example, in this case, the limits always lose self-similarity, regardless of the asymptotics of $nf(R_ne_1)$, whereas, in the regularly varying or the subexponential tail case, the self-similarity is lost only when $nf(R_ne_1)$ converges to a positive and finite constant. Furthermore, when $nf(R_ne_1) \to \infty$, a non-degenerate limit appears only in the exponential tail case. Finally, we remark that as in the last section, the resulting FCLTs are necessarily non-trivial, because Γ is a feasible subgraph.

Theorem 5.3. *Assume that the density* (5.1) *satisfies* (5.2), (5.4), (5.5)*, and* (5.6).

(i) *If* $nf(R_ne_1) \to 0$ *as* $n \to \infty$*, then*

$$\big(X_n(t),\, t \ge 0\big) \Rightarrow \big(W_k(t),\, t \ge 0\big) \;\text{ in } \mathcal{D}[0,\infty)\,.$$

(ii) *If* $nf(R_ne_1) \to \xi \in (0,\infty)$ *as* $n \to \infty$*, then*

$$\big(X_n(t),\, t \ge 0\big) \Rightarrow \left(\sum_{\ell=1}^{k} \xi^{2k-\ell} W_\ell(t),\, t \ge 0\right) \;\text{ in } \mathcal{D}[0,\infty)\,.$$

(iii) *If* $nf(R_ne_1) \to \infty$ *as* $n \to \infty$*, then*

$$\big(X_n(t),\, t \ge 0\big) \Rightarrow \big(W_1(t),\, t \ge 0\big) \;\text{ in } \mathcal{D}[0,\infty)\,.$$

6 Graph connectivity in weak core

We start this section by recalling the *weak core*, which was defined as a centered ball $B\big(0, R_n^{(w)}\big)$ such that $nf\big(R_n^{(w)}e_1\big) \to 1$ as $n \to \infty$. In addition, we need the relevant notion, the *core*, which was defined in Definition 3.1. Recall that, given a Poisson point process $\mathcal{P}_n$ on $\mathbb{R}^d$, a core is a centered ball $B(0, R_n)$ such that

$$B(0,R_n) \subset \bigcup_{X \in \mathcal{P}_n \cap B(0,R_n)} B(X,1)\,. \tag{6.1}$$

In the following, we seek the largest possible sequence $R_n \to \infty$ such that the event (6.1) occurs asymptotically with probability 1, and subsequently, it is shown that the largest possible core and a weak core are "close" in size. However, the degree of this closeness depends on the tail of an underlying density f, and therefore, we divide the argument into two cases.

We first assume that the density f on $\mathbb{R}^d$ is spherically symmetric and has a regularly varying tail, as in (4.1). For increased clarity, we place an extra condition that $p(r) := f(re_1)$ is eventually non-increasing in r, that is, p is non-increasing on (r_0, ∞) for some large $r_0 > 0$. In this case, the radius of a weak core is, clearly, given by

$$R_n^{(w)} = \left(\frac{1}{p}\right)^{\leftarrow}(n) := \inf\left\{ s : \left(\frac{1}{p}\right)(s) \geq n \right\}. \tag{6.2}$$

Proposition 6.1. *Suppose that $p \in RV_{-\alpha}$ for some $\alpha > d$ and p is eventually non-increasing. Define*

$$R_n^{(c)} = \left(\frac{1}{p}\right)^{\leftarrow}\left(\frac{\delta_1 n}{\log n - \delta_2 \log\log n}\right) \tag{6.3}$$

with $\delta_1 \in \big(0, \alpha/(2^d d^{d/2+1})\big)$ and $\delta_2 \in (0,1)$. If $R_n \leq R_n^{(c)}$, then

$$\mathbb{P}\left(B(0,R_n) \subset \bigcup_{X \in \mathcal{P}_n \cap B(0,R_n)} B(X,1) \right) \to 1\,, \quad n \to \infty\,. \tag{6.4}$$

Furthermore, the sequences $(R_n^{(c)})$ in (6.3) *and $(R_n^{(w)})$ in* (6.2) *are both regularly varying sequences with exponent $1/\alpha$, and*

$$\frac{R_n^{(c)}}{R_n^{(w)}} - \left(\frac{\delta_1}{\log n - \delta_2 \log\log n}\right)^{1/\alpha} \to 0\,, \quad n \to \infty\,. \tag{6.5}$$

One can obtain a parallel result when the underlying density has an exponentially decaying tail, as in (5.1). We simplify the situation a bit by assuming

$$f(x) = C \exp\big\{-\psi\big(||x||\big)\big\}, \quad x \in \mathbb{R}^d\,, \tag{6.6}$$

where C is a normalizing constant and $\psi : \mathbb{R}_+ \to \mathbb{R}$ is of C^2-class and satisfies $\psi \in RV_v$ (at infinity) for some $v > 0$ and $\psi' > 0$. It should be noted that we are permitting the case $v > 1$, implying that, unlike in the previous section, we do not rule out densities with superexponential tail. Evidently, the radius of a weak core is given by

$$R_n^{(w)} = \psi^{\leftarrow}(\log n + \log C)\,. \tag{6.7}$$

Proposition 6.2. *Assume that a probability density f on $\mathbb{R}^d$ is given by* (6.6)*. Define*

$$R_n^{(c)} = \psi^{\leftarrow}(\log n - \log\log\log n - \delta_1 - \delta_2)\,, \tag{6.8}$$

where $\delta_1 = d\log 2 - \log v + (1 + d/2)\log d - \log C$ and $\delta_2 > 0$. If $R_n \leq R_n^{(c)}$, then

$$\mathbb{P}\left(B(0,R_n) \subset \bigcup_{X \in \mathcal{P}_n \cap B(0,R_n)} B(X,1) \right) \to 1\,, \quad n \to \infty\,. \tag{6.9}$$

Furthermore, the sequences $(R_n^{(c)})$ *in* (6.8) *and* $(R_n^{(w)})$ *in* (6.7) *are close in size in the sense of*

$$\frac{R_n^{(c)}}{R_n^{(w)}} - \left(1 - \frac{\log\log\log n + \delta_1 + \delta_2 + \log C}{\log Cn}\right)^{1/v} \to 0\,, \quad n \to \infty\,. \tag{6.10}$$

The following result is needed as preparation for the proof of these propositions. The proof may be obtained by slightly modifying the proof of Theorem 2.1 in [1], but we repeat the argument in order for this paper to be self-contained.

Lemma 6.3. *Given a spherically symmetric density* f *on* $\mathbb{R}^d$*, suppose that* $p(r) = f(re_1)$ *is eventually non-increasing. Let* $g = 1/(2d^{1/2})$*. Suppose, in addition, that there exists a sequence* $R_n \nearrow \infty$ *such that* $d \log R_n - g^d n f(R_n e_1) \to -\infty$ *as* $n \to \infty$*. Then,*

$$\mathbb{P}\left(B(0, R_n) \subset \bigcup_{X \in \mathcal{P}_n \cap B(0,R_n)} B(X, 1)\right) \to 1\,, \quad n \to \infty\,. \tag{6.11}$$

Proof. For $\rho > 0$, let $\mathcal{Q}(\rho)$ be a collection of cubes with grid g that are contained in $B(0, \rho)$. Then,

$$\big\{\, Q \cap \mathcal{P}_n \neq \emptyset \text{ for all } Q \in \mathcal{Q}(\rho) \,\big\} \subset \Big\{\, B(0,\rho) \subset \bigcup_{X \in \mathcal{P}_n \cap B(0,\rho)} B(X,1) \,\Big\}$$

for all $\rho > 0$ and $n \geq 1$. It now suffices to show that

$$\mathbb{P}\big(\, Q \cap \mathcal{P}_n = \emptyset \text{ for some } Q \in \mathcal{Q}(R_n) \,\big) \to 0\,, \quad n \to \infty\,.$$

This probability is estimated from above by

$$\sum_{Q \in \mathcal{Q}(R_n)} \mathbb{P}(Q \cap \mathcal{P}_n = \emptyset) = \sum_{Q \in \mathcal{Q}(R_n)} \exp\Big\{-n \int_Q f(x)dx\Big\}$$
$$\leq \sum_{Q \in \mathcal{Q}(R_n)} \exp\big\{-ng^d f(R_n e_1)\big\} \leq g^{-d} R_n^d \exp\big\{-g^d n f(R_n e_1)\big\}\,.$$

At the first inequality, we used the fact that p is eventually non-increasing. Clearly, the rightmost term vanishes as $n \to \infty$. □

Proof. (*proof of Proposition 6.1*) Observe that the assumption $p \in RV_{-\alpha}$ implies $(1/p)^{\leftarrow} \in RV_{1/\alpha}$, e.g., Proposition 2.6 (v) in [27]. Thus, (6.5) readily follows from the uniform convergence of regularly varying functions; see Proposition 2.4 in [27]. By Lemma 6.3, it suffices to verify that $d \log R_n^{(c)} - g^d n f\big(R_n^{(c)} e_1\big) \to -\infty$ as $n \to \infty$. Since $0 < \delta_2 < 1$, we have

$$d \log R_n^{(c)} \leq d \left[\log \left(\frac{1}{p}\right)^{\leftarrow} \left(\frac{\delta_1 n}{\log n - \delta_2 \log\log n}\right)\right] \left[\log \left(\frac{\delta_1 n}{\log n - \delta_2 \log n \log n}\right)\right]^{-1}$$
$$\times \left(\log \delta_1 + \log n - \delta_2 \log\log n\right),$$

and $g^d n f\big(R_n^{(c)} e_1\big) = g^d \delta_1^{-1} (\log n - \delta_2 \log\log n)$. Using Proposition 2.6 (i) in [27],

$$d \left[\log \left(\frac{1}{p}\right)^{\leftarrow} \left(\frac{\delta_1 n}{\log n - \delta_2 \log\log n}\right)\right] \left[\log \left(\frac{\delta_1 n}{\log n - \delta_2 \log n \log n}\right)\right]^{-1} - g^d \delta_1^{-1}$$
$$\to d\alpha^{-1} - g^d \delta_1^{-1} < 0\,, \quad n \to \infty\,.$$

At the last inequality, we applied the constraint in δ_1. Therefore, we have $d \log R_n^{(c)} - g^d n f\big(R_n^{(c)} e_1\big) \to -\infty$, $n \to \infty$, as requested. □

Proof. (*proof of Proposition 6.2*) Since $\psi^{\leftarrow} \in RV_{1/v}$, it is easy to show (6.10), and therefore, we prove only that $d \log R_n^{(c)} - g^d n f\big(R_n^{(c)} e_1\big) \to -\infty$ as $n \to \infty$. We see that

$$d \log R_n^{(c)} \le d \log \psi^{\leftarrow}(\log n) \sim d v^{-1} \log \log n \,, \quad n \to \infty \,,$$

and that $g^d n f\big(R_n^{(c)} e_1\big) = g^d C e^{\delta_1 + \delta_2} \log \log n$. By virtue of the constraints in δ_1 and δ_2, we have $d v^{-1} - g^d C e^{\delta_1 + \delta_2} < 0$; thus, the claim is proved. ☐

Remark 6.4. The proof of Lemma 6.3 merely estimated the probability in (6.11) from below. Therefore, it seems to be possible that in the propositions above, (6.4) and (6.9) may hold for the sequence $R_n \nearrow \infty$ growing more quickly than $R_n^{(c)}$ but more slowly than $R_n^{(w)}$, i.e., $R_n^{(c)} \le R_n \le R_n^{(w)}$; it is unknown, however, to what extent we can make R_n closer to $R_n^{(w)}$.

7 Proof of main results

This section presents the proof of the main results of this paper. The proof is, however, rather long, and therefore, it is divided into several parts. All the supplemental ingredients necessary are collected in the Appendix, most of which are cited from [23].

Let $\text{Ann}(K, L)$ be an annulus of inner radius K and outer radius L. For $x_1, \dots, x_k \in \mathbb{R}^d$, define $\text{Max}(x_1, \dots, x_k)$ as the function selecting an element with the largest distance from the origin. That is, $\text{Max}(x_1, \dots, x_k) = x_i$ if $||x_i|| = \max_{1 \le j \le k} ||x_j||$. If multiple x_j's achieve the maximum, we choose an element with the smallest subscript.

In the following, $\mathcal{Y}, \mathcal{Y}', \mathcal{Y}_i$, etc. always represent a finite collection of d-dimensional real vectors. We use the following shorthand notations. That is, for $\mathbf{x} = (x_1, \dots, x_m) \in (\mathbb{R}^d)^m$, $x \in \mathbb{R}^d$, and $\mathbf{y} = (y_1, \dots, y_{m-1}) \in (\mathbb{R}^d)^{m-1}$,

$$\begin{aligned} f(\mathbf{x}) &:= f(x_1) \cdots f(x_m) \,, \\ f(x + \mathbf{y}) &:= f(x + y_1) \cdots f(x + y_{m-1}) \,, \\ h_t(0, \mathbf{y}) &:= h_t(0, y_1, \dots, y_{m-1}) \;\; \text{etc.} \end{aligned}$$

Regarding the indicator $h_t : (\mathbb{R}^d)^k \to \{0, 1\}$ given in (2.1), the following notations are used to save space.

$$h_{t,s}(\mathbf{x}) := h_t(\mathbf{x}) - h_s(\mathbf{x}) \,, \;\; 0 \le s \le t \,, \; \mathbf{x} \in (\mathbb{R}^d)^k, \tag{7.1}$$

$$h^{\pm}_{t,s}(\mathbf{x}) := h^{\pm}_t(\mathbf{x}) - h^{\pm}_s(\mathbf{x}) \,, \;\; 0 \le s \le t \,, \; \mathbf{x} \in (\mathbb{R}^d)^k,$$

$$h_{n,t,s}(\mathbf{x}) := h_{t,s}(\mathbf{x}) \, \mathbf{1}\big\{ m(\mathbf{x}) \ge R_n \big\} \,, \;\; 0 \le s \le t \,, \; \mathbf{x} \in (\mathbb{R}^d)^k, \tag{7.2}$$

and for $\ell \in \{0, \dots, k\}$,

$$h^{(\ell)}_{t,s}(\mathbf{x}) := h_t(x_1, \dots, x_k) \, h_s(x_1, \dots, x_\ell, x_{k+1}, \dots, x_{2k-\ell}) \,, \;\; t, s \ge 0 \,, \; \mathbf{x} \in (\mathbb{R}^d)^{2k-\ell}. \tag{7.3}$$

In particular, we set

$$h_s(x_1, \dots, x_\ell, x_{k+1}, \dots, x_{2k-\ell}) := \begin{cases} h_s(x_{k+1}, \dots, x_{2k}) & \text{if } \ell = 0 \,, \\ h_s(x_1, \dots, x_k) & \text{if } \ell = k \,. \end{cases}$$

In Section 7.1, we use, for $1 \le K < L \le \infty$, $n \in \mathbb{N}_+$ and $t \ge 0$,

$$\begin{aligned} h_{n,t,K,L}(\mathbf{x}) &:= h_t(\mathbf{x}) \, \mathbf{1}\big\{ m(\mathbf{x}) \ge R_n \,, \; \text{Max}(\mathbf{x}) \in \text{Ann}(K R_n, L R_n) \big\} \,, \\ h^{\pm}_{n,t,K,L}(\mathbf{x}) &:= h^{\pm}_t(\mathbf{x}) \, \mathbf{1}\big\{ m(\mathbf{x}) \ge R_n \,, \; \text{Max}(\mathbf{x}) \in \text{Ann}(K R_n, L R_n) \big\} \,. \end{aligned}$$

The same notations are retained for Section 7.2 to represent, for $0 \le K < L \le \infty$, $n \in \mathbb{N}_+$ and $t \ge 0$,

$$h_{n,t,K,L}(\mathbf{x}) := h_t(\mathbf{x})\,\mathbf{1}\big\{m(\mathbf{x}) \ge R_n\,,\ a(R_n)^{-1}\big(\mathrm{Max}(\mathbf{x}) - R_n\big) \in [K,L)\big\}\,,$$
$$h^{\pm}_{n,t,K,L}(\mathbf{x}) := h^{\pm}_t(\mathbf{x})\,\mathbf{1}\big\{m(\mathbf{x}) \ge R_n\,,\ a(R_n)^{-1}\big(\mathrm{Max}(\mathbf{x}) - R_n\big) \in [K,L)\big\}\,.$$

Finally, C^* denotes a generic positive constant, which may change between lines and does not depend on n.

In the following, we divide the argument into two subsections. Section 7.1 treats the case in which the underlying density has a regularly varying tail; our goal is to prove Theorem 4.3. Subsequently Section 7.2 provides the proof of Theorem 5.3, where the density is assumed to have an exponentially decaying tail. Before the specific subsections, however, we show some preliminary results, which are commonly used for the tightness proof in both subsections.

Lemma 7.1. *Let $h_t : (\mathbb{R}^d)^k \to \{0,1\}$ be an indicator given in* (2.1). *Fix $T > 0$. Then, we have for $\ell \in \{1,\dots,k\}$,*

$$\int_{(\mathbb{R}^d)^{\ell-1}} d\mathbf{y} \int_{(\mathbb{R}^d)^{k-\ell}} d\mathbf{z}_2 \int_{(\mathbb{R}^d)^{k-\ell}} d\mathbf{z}_1 h^{+}_{t,s}(0,\mathbf{y},\mathbf{z}_1)\, h^{+}_{s,r}(0,\mathbf{y},\mathbf{z}_2) \le C^*(t-s)(s-r)\,, \tag{7.4}$$
$$\int_{(\mathbb{R}^d)^{\ell-1}} d\mathbf{y} \int_{(\mathbb{R}^d)^{k-\ell}} d\mathbf{z}_2 \int_{(\mathbb{R}^d)^{k-\ell}} d\mathbf{z}_1 h^{-}_{t,s}(0,\mathbf{y},\mathbf{z}_1)\, h^{-}_{s,r}(0,\mathbf{y},\mathbf{z}_2) \le C^*(t-s)(s-r)$$

for all $0 \le r \le s \le t \le T$.

Proof. We only prove the first inequality. If $\ell = 1$ or $\ell = k$, the claim is trivial, and therefore, we can take $2 \le \ell \le k-1$. It follows from (2.6) that the integral in (7.4) is not altered if the integral domain is restricted to $\big(B(0,kT)\big)^{\ell-1} \times \big(B(0,kT)\big)^{k-\ell} \times \big(B(0,kT)\big)^{k-\ell}$. With λ being the Lebesgue measure on $(\mathbb{R}^d)^{k-\ell}$, we see that for every $\mathbf{y} \in (\mathbb{R}^d)^{\ell-1}$,

$$\begin{aligned}
\int_{(B(0,kT))^{k-\ell}} h^{+}_{t,s}(0,\mathbf{y},\mathbf{z})d\mathbf{z} &= \lambda\big\{\mathbf{z} \in \big(B(0,kT)\big)^{k-\ell} : h^{+}_t(0,\mathbf{y},\mathbf{z}) = 1\,,\ h^{+}_s(0,\mathbf{y},\mathbf{z}) = 0\big\} \\
&\le \lambda\big\{\mathbf{z} \in \big(B(0,kT)\big)^{k-\ell} : s < ||z_i - z_j|| \le t \text{ for some } i \ne j\big\} \\
&\quad + \lambda\big\{\mathbf{z} \in \big(B(0,kT)\big)^{k-\ell} : s < ||z_i - y_j|| \le t \text{ for some } i,j\big\} \\
&\quad + \lambda\big\{\mathbf{z} \in \big(B(0,kT)\big)^{k-\ell} : s < ||z_i|| \le t \text{ for some } i\big\} \\
&\quad + \lambda\big\{\mathbf{z} \in \big(B(0,kT)\big)^{k-\ell} : s < ||y_i - y_j|| \le t \text{ for some } i \ne j\big\} \\
&\quad + \lambda\big\{\mathbf{z} \in \big(B(0,kT)\big)^{k-\ell} : s < ||y_i|| \le t \text{ for some } i\big\}.
\end{aligned} \tag{7.5}$$

Observe that for $i \ne j$,

$$\lambda\big\{\mathbf{z} \in \big(B(0,kT)\big)^{k-\ell} : s < ||z_i - z_j|| \le t\big\} \le (kT)^{d(k-\ell-1)}(\omega_d)^{k-\ell}(t^d - s^d)\,,$$

where ω_d is the volume of the d-dimensional unit ball. Since the second and the third terms on the rightmost term in (7.5) have the same upper bound, we ultimately obtain

$$\begin{aligned}
&\int_{(B(0,kT))^{k-\ell}} h^{+}_{t,s}(0,\mathbf{y},\mathbf{z})d\mathbf{z} \\
&\qquad \le C^*\Big(t^d - s^d + \sum_{i,j=1,\ i\ne j}^{\ell-1} \mathbf{1}\big\{s < ||y_i - y_j|| \le t\big\} + \sum_{i=1}^{\ell-1} \mathbf{1}\big\{s < ||y_i|| \le t\big\}\Big).
\end{aligned}$$

Therefore, the integral in (7.4) is bounded above by

$$C^* \int_{\left(B(0,kT)\right)^{\ell-1}} \Bigg(t^d - s^d + \sum_{i,j=1,\, i\neq j}^{\ell-1} \mathbf{1}\big\{ s < ||y_i - y_j|| \le t \big\} + \sum_{i=1}^{\ell-1} \mathbf{1}\big\{ s < ||y_i|| \le t \big\} \Bigg) \times \Bigg(s^d - r^d + \sum_{i,j=1,\, i\neq j}^{\ell-1} \mathbf{1}\big\{ r < ||y_i - y_j|| \le s \big\} + \sum_{i=1}^{\ell-1} \mathbf{1}\big\{ r < ||y_i|| \le s \big\} \Bigg) d\mathbf{y}\,.$$

An elementary calculation shows that for all $i, j, i', j' \in \{1, \dots, \ell-1\}$ with $i > j$ and $i' > j'$,

$$\int_{\left(B(0,kT)\right)^{\ell-1}} \mathbf{1}\big\{ s < ||y_i - y_j|| \le t \big\}\, \mathbf{1}\big\{ r < ||y_{i'} - y_{j'}|| \le s \big\} d\mathbf{y} \le C^*(t^d - s^d)(s^d - r^d) \le C^*(t-s)(s-r)$$

In particular, if $i = i'$ and $j = j'$, the integral is identically zero. Applying the same manipulation to the integral of other cross-terms, we can conclude the claim of the lemma. □

7.1 Regularly varying tail case

Under the setup of Theorem 4.3, we first define the subgraph counting process with restricted domain. For $1 \le K < L \le \infty$, $n \in \mathbb{N}_+$, and $t \ge 0$, let

$$\begin{aligned} G_{n,K,L}(t) &= \sum_{\mathcal{Y} \subset \mathcal{P}_n} h_t(\mathcal{Y})\, \mathbf{1}\big\{ m(\mathcal{Y}) \ge R_n\,,\ \mathrm{Max}(\mathcal{Y}) \in \mathrm{Ann}(KR_n, LR_n) \big\} \\ &:= \sum_{\mathcal{Y} \subset \mathcal{P}_n} h_{n,t,K,L}(\mathcal{Y})\,, \end{aligned}$$

and

$$\begin{aligned} G^{\pm}_{n,K,L}(t) &= \sum_{\mathcal{Y} \subset \mathcal{P}_n} h^{\pm}_t(\mathcal{Y})\, \mathbf{1}\big\{ m(\mathcal{Y}) \ge R_n\,,\ \mathrm{Max}(\mathcal{Y}) \in \mathrm{Ann}(KR_n, LR_n) \big\} \\ &:= \sum_{\mathcal{Y} \subset \mathcal{P}_n} h^{\pm}_{n,t,K,L}(\mathcal{Y})\,, \end{aligned}$$

where (R_n) satisfies (4.2). For the special case $K = 1$ and $L = \infty$, we simply denote $G_n(t) = G_{n,1,\infty}(t)$ and $G^{\pm}_n(t) = G^{\pm}_{n,1,\infty}(t)$. The subgraph counting processes, centered and scaled, for which we prove the FCLT, are given by

$$\begin{aligned} X_n(t) &= \tau_n^{-1/2}\Big(G_n(t) - \mathbb{E}\big\{ G_n(t) \big\} \Big)\,, \\ X^{\pm}_n(t) &= \tau_n^{-1/2}\Big(G^{\pm}_n(t) - \mathbb{E}\big\{ G^{\pm}_n(t) \big\} \Big)\,, \end{aligned} \tag{7.6}$$

where (τ_n) is determined by (4.3) according to which regime is considered. The first proposition below computes the covariances of $\big(G_{n,K,L}(t)\big)$.

Proposition 7.2. *Assume the conditions of Theorem 4.3. Let $1 \le K < L \le \infty$.*

(i) *If $nf(R_n e_1) \to 0$ as $n \to \infty$, then*

$$\tau_n^{-1} \mathit{Cov}\big(G_{n,K,L}(t), G_{n,K,L}(s) \big) \to (K^{d-\alpha k} - L^{d-\alpha k}) L_k(t,s)\,, \quad n \to \infty\,.$$

(ii) *If $nf(R_n e_1) \to \xi \in (0,\infty)$ as $n \to \infty$, then*

$$\tau_n^{-1} \mathit{Cov}\big(G_{n,K,L}(t), G_{n,K,L}(s) \big) \to \sum_{\ell=1}^{k} (K^{d-\alpha(2k-\ell)} - L^{d-\alpha(2k-\ell)}) \xi^{2k-\ell} L_\ell(t,s)\,, \quad n \to \infty\,.$$

(iii) *If $nf(R_n e_1) \to \infty$ as $n \to \infty$, then*

$$\tau_n^{-1} \mathit{Cov}\big(G_{n,K,L}(t), G_{n,K,L}(s) \big) \to (K^{d-\alpha(2k-1)} - L^{d-\alpha(2k-1)}) L_1(t,s)\,, \quad n \to \infty\,.$$

Proof. We start by writing

$$\begin{aligned}
&\mathbb{E}\big\{G_{n,K,L}(t)\,G_{n,K,L}(s)\big\}\\
&= \sum_{\ell=0}^{k} \mathbb{E}\Big\{ \sum_{\mathcal{Y}_1\subset\mathcal{P}_n} \sum_{\mathcal{Y}_2\subset\mathcal{P}_n} h_{n,t,K,L}(\mathcal{Y}_1)\, h_{n,s,K,L}(\mathcal{Y}_2)\, \mathbf{1}\big\{ |\mathcal{Y}_1\cap\mathcal{Y}_2| = \ell \big\}\Big\}\\
&:= \sum_{\ell=0}^{k} \mathbb{E}\{I_\ell\}\,.
\end{aligned}$$

For $\ell = 0$, applying Palm theory (see the Appendix) twice,

$$\begin{aligned}
\mathbb{E}\{I_0\} &= \frac{n^{2k}}{(k!)^2}\, \mathbb{E}\big\{h_{n,t,K,L}(X_1,\dots,X_k)\, h_{n,s,K,L}(X_{k+1},\dots,X_{2k})\big\}\\
&= \mathbb{E}\big\{G_{n,K,L}(t)\big\}\mathbb{E}\big\{G_{n,K,L}(s)\big\}\,.
\end{aligned}$$

Therefore, the multiple applications of Palm theory yield

$$\begin{aligned}
&\mathrm{Cov}\big(G_{n,K,L}(t)\,, G_{n,K,L}(s)\big) = \sum_{\ell=1}^{k} \mathbb{E}\{I_\ell\}\\
&= \sum_{\ell=1}^{k} \frac{n^{2k-\ell}}{\ell!\big((k-\ell)!\big)^2}\, \mathbb{E}\Big\{h_{n,t,K,L}(\mathcal{Y}_1)\, h_{n,s,K,L}(\mathcal{Y}_2)\, \mathbf{1}\big\{ |\mathcal{Y}_1\cap\mathcal{Y}_2| = \ell \big\}\Big\}.
\end{aligned}$$

Define for $\ell \in \{1,\dots,k\}$,

$$\begin{aligned}
C_n^{(\ell)}(K,L) := \big\{\mathbf{x}\in(\mathbb{R}^d)^{2k-\ell} : &\ \mathrm{Max}(x_1,\dots,x_k)\in \mathrm{Ann}(KR_n, LR_n)\,,\\
&\mathrm{Max}(x_1,\dots,x_\ell,x_{k+1},\dots,x_{2k-\ell})\in\mathrm{Ann}(KR_n,LR_n)\big\}\,.
\end{aligned}$$

By the change of variables $\mathbf{x}\to(x, x+\mathbf{y})$ with $\mathbf{x}\in(\mathbb{R}^d)^{2k-\ell}$, $x\in\mathbb{R}^d$, $\mathbf{y}\in(\mathbb{R}^d)^{2k-\ell-1}$, together with invariance (2.2), while recalling notation (7.3),

$$\begin{aligned}
&\mathbb{E}\Big\{h_{n,t,K,L}(\mathcal{Y}_1)\, h_{n,s,K,L}(\mathcal{Y}_2)\, \mathbf{1}\big\{ |\mathcal{Y}_1\cap\mathcal{Y}_2| = \ell \big\}\Big\}\\
&= \int_{(\mathbb{R}^d)^{2k-\ell}} f(\mathbf{x})\, \mathbf{1}\big\{m(\mathbf{x})\ge R_n\big\}\, h_{t,s}^{(\ell)}(\mathbf{x})\, \mathbf{1}\big\{\mathbf{x}\in C_n^{(\ell)}(K,L)\big\} d\mathbf{x}\\
&= \int_{\mathbb{R}^d}\int_{(\mathbb{R}^d)^{2k-\ell-1}} f(x)\, f(x+\mathbf{y})\, \mathbf{1}\big\{m(x,x+\mathbf{y})\ge R_n\big\}\, h_{t,s}^{(\ell)}(0,\mathbf{y})\\
&\quad\times \mathbf{1}\big\{(x,x+\mathbf{y})\in C_n^{(\ell)}(K,L)\big\} d\mathbf{y} dx\,.
\end{aligned}$$

The polar coordinate transform $x\to(r,\theta)$ and an additional change of variable $\rho\to r/R_n$ yield

$$\begin{aligned}
&\mathbb{E}\Big\{h_{n,t,K,L}(\mathcal{Y}_1)\, h_{n,s,K,L}(\mathcal{Y}_2)\, \mathbf{1}\big\{ |\mathcal{Y}_1\cap\mathcal{Y}_2| = \ell \big\}\Big\} \qquad (7.7)\\
&= R_n^d f(R_n e_1)^{2k-\ell} \int_{S_{d-1}} J(\theta) d\theta \int_1^\infty d\rho \int_{(\mathbb{R}^d)^{2k-\ell-1}} d\mathbf{y}\, \rho^{d-1}\, \frac{f(R_n\rho e_1)}{f(R_n e_1)}\\
&\quad\times \prod_{i=1}^{2k-\ell-1} \frac{f\big(||R_n\rho\theta + y_i|| e_1\big)}{f(R_n e_1)}\, \mathbf{1}\big\{||\rho\theta + y_i/R_n||\ge 1\big\}\, h_{t,s}^{(\ell)}(0,\mathbf{y})\\
&\quad\times \mathbf{1}\big\{(R_n\rho\theta, R_n\rho\theta+\mathbf{y})\in C_n^{(\ell)}(K,L)\big\}\,,
\end{aligned}$$

where S_{d-1} denotes the $(d-1)$-dimensional unit sphere in $\mathbb{R}^d$ and $J(\theta)$ is the usual Jacobian

$$J(\theta) = \sin^{k-2}(\theta_1)\sin^{k-3}(\theta_2)\cdots\sin(\theta_{k-2})\,.$$

Note that by the regular variation of f (with exponent $-\alpha$), for every $\rho > 1$, $\theta \in S_{d-1}$, and y_i's,

$$\frac{f(R_n\rho e_1)}{f(R_n e_1)} \to \rho^{-\alpha}, \qquad \prod_{i=1}^{2k-\ell-1} \frac{f\big(||R_n\rho\theta + y_i||e_1\big)}{f(R_n e_1)} \to \rho^{-\alpha(2k-\ell-1)}, \quad n \to \infty \tag{7.8}$$

and, furthermore,

$$\mathbf{1}\big\{(R_n\rho\theta, R_n\rho\theta + \mathbf{y}) \in C_n^{(\ell)}(K,L)\big\} \to \mathbf{1}\{K \le \rho \le L\}, \quad n \to \infty. \tag{7.9}$$

Substituting (7.8) and (7.9) back into (7.7), while supposing temporarily that the dominated convergence theorem is applicable, we may conclude that

$$\begin{aligned} &\mathrm{Cov}\big(G_{n,K,L}(t), G_{n,K,L}(s)\big) \\ &\sim \sum_{\ell=1}^{k} n^{2k-\ell} R_n^d f(R_n e_1)^{2k-\ell} \big(K^{d-\alpha(2k-\ell)} - L^{d-\alpha(2k-\ell)}\big) L_\ell(t,s), \quad n \to \infty. \end{aligned} \tag{7.10}$$

Observe that the limit value of $nf(R_n e_1)$ completely determines which term on the right hand side of (7.10) is dominant. If $nf(R_n e_1) \to 0$, then the kth term,i.e., $\ell = k$, in the sum grows fastest, while the first term, i.e., $\ell = 1$, grows fastest when $nf(R_n e_1) \to \infty$. Moreover, if $nf(R_n e_1) \to \xi \in (0,\infty)$, then all the terms in the sum grow at the same rate. This concludes the claim of the proposition.

It now remains to establish an integrable upper bound for the application of the dominated convergence theorem. First, condition (2.3) provides

$$h_{t,s}^{(\ell)}(0,\mathbf{y}) \le \mathbf{1}\big\{||y_i|| \le k(t+s),\ i = 1,\dots,2k-\ell-1\big\}.$$

Next, appealing to Potter's bound ,e.g., Proposition 2.6 (ii) in [27], for every $\xi \in (0, \alpha - d)$ and sufficiently large n,

$$\frac{f(R_n\rho e_1)}{f(R_n e_1)} \mathbf{1}\{\rho \ge 1\} \le (1+\xi)\,\rho^{-\alpha+\xi}\,\mathbf{1}\{\rho \ge 1\}$$

and

$$\prod_{i=1}^{2k-\ell-1} \frac{f(||R_n\rho\theta + y_i||e_1)}{f(R_n e_1)} \mathbf{1}\big\{||\rho\theta + y_i/R_n|| \ge 1\big\} \le (1+\xi)^{2k-\ell-1}.$$

Since $\int_1^\infty \rho^{d-1-\alpha+\xi} d\rho < \infty$, we are allowed to apply the dominated convergence theorem. □

The next proposition proves the weak convergence of Theorem 4.3 in a finite-dimensional sense.

Proposition 7.3. *Assume the conditions of Theorem 4.3. Then, weak convergences $(i) - (iii)$ in the theorem hold in a finite-dimensional sense. Furthermore, let $\mathbf{X}_n^\pm$ be the processes defined in (7.6). Then, the following results also hold in a finite-dimensional sense.*

(i) If $nf(R_n e_1) \to 0$ as $n \to \infty$, then

$$(\mathbf{X}_n^+, \mathbf{X}_n^-) \Rightarrow (\mathbf{V}_k^+, \mathbf{V}_k^-). \tag{7.11}$$

(ii) If $nf(R_n e_1) \to \xi \in (0,\infty)$ as $n \to \infty$, then

$$(\mathbf{X}_n^+, \mathbf{X}_n^-) \Rightarrow \left(\sum_{\ell=1}^{k} \xi^{2k-\ell}\mathbf{V}_\ell^+, \sum_{\ell=1}^{k} \xi^{2k-\ell}\mathbf{V}_\ell^-\right). \tag{7.12}$$

(iii) If $nf(R_n e_1) \to \infty$ *as* $n \to \infty$*, then*

$$(\mathbf{X}_n^+, \mathbf{X}_n^-) \Rightarrow (\mathbf{V}_1^+, \mathbf{V}_1^-)\,. \tag{7.13}$$

The limiting Gaussian processes $(\mathbf{V}_\ell^+, \mathbf{V}_\ell^-)$*,* $\ell = 1, \dots, k$ *are all defined in Section 4.2.*

Proof. The proofs of (7.11), (7.12), and (7.13) are a bit more technical, but are very similar to the corresponding results in Theorem 4.3; therefore, we check only finite-dimensional weak convergences in Theorem 4.3. The argument here is closely related to that in Theorem 3.9 of [23], for which we rely on the so-called Cramér-Wold device. For $0 \le t_1 < \cdots < t_m < \infty$, $a_1, \dots, a_m \in \mathbb{R}$ and $m \ge 1$, define $S_n := \sum_{j=1}^m a_j G_n(t_j)$. For $K > 1$, S_n can be further decomposed into two parts:

$$\begin{aligned} S_n &= \sum_{j=1}^m a_j G_{n,1,K}(t_j) + \sum_{j=1}^m a_j G_{n,K,\infty}(t_j) \\ &:= T_n^{(K)} + U_n^{(K)}\,. \end{aligned}$$

We define a constant γ_K as follows in accordance with the limit of $nf(R_n e_1)$.

$$\gamma_K := \begin{cases} \sum_{i=1}^m \sum_{j=1}^m a_i a_j (1 - K^{d-\alpha k}) L_k(t_i, t_j) & \text{if } nf(R_n e_1) \to 0\,, \\ \sum_{i=1}^m \sum_{j=1}^m a_i a_j \sum_{\ell=1}^k (1 - K^{d-\alpha(2k-\ell)}) \xi^{2k-\ell} L_\ell(t_i, t_j) & \text{if } nf(R_n e_1) \to \xi \in (0,\infty)\,, \\ \sum_{i=1}^m \sum_{j=1}^m a_i a_j (1 - K^{d-\alpha(2k-1)}) L_1(t_i, t_j) & \text{if } nf(R_n e_1) \to \infty\,. \end{cases}$$

Moreover, $\gamma := \lim_{K\to\infty} \gamma_K$. It directly follows from Proposition 7.2 that, regardless of the regime we consider,

$$\tau_n^{-1} \text{Var}\{T_n^{(K)}\} \to \gamma_K\,, \quad \tau_n^{-1} \text{Var}\{U_n^{(K)}\} \to \gamma - \gamma_K \quad \text{as } n \to \infty\,.$$

For the completion of the proof, we ultimately need to show that

$$\tau_n^{-1/2} \big(S_n - \mathbb{E}\{S_n\} \big) \Rightarrow N(0, \gamma)\,.$$

By the standard approximation argument given on p. 64 of [23], it suffices to show that

$$\tau_n^{-1/2} \Big(T_n^{(K)} - \mathbb{E}\big\{T_n^{(K)}\big\} \Big) \Rightarrow \mathrm{N}(0, \gamma_K) \ \text{ for every } K > 1\,; \tag{7.14}$$

equivalently,

$$\frac{T_n^{(K)} - \mathbb{E}\big\{T_n^{(K)}\big\}}{\sqrt{\text{Var}\big\{T_n^{(K)}\big\}}} \Rightarrow N(0,1) \ \text{ for every } K > 1\,. \tag{7.15}$$

Let $(Q_\ell : \ell \in \mathbb{N})$ be a collection of unit cubes covering $\mathbb{R}^d$. Define

$$V_n := \big\{ \ell \in \mathbb{N} : Q_\ell \cap \text{Ann}(R_n, KR_n) \neq \emptyset \big\}\,,$$

where we have that $|V_n| \le C^* R_n^d$.

Then, $T_n^{(K)}$ can be partitioned as follows.

$$\begin{aligned} T_n^{(K)} &= \sum_{\ell \in V_n} \sum_{j=1}^m a_j \sum_{\mathcal{Y} \subset \mathcal{P}_n} h_{t_j}(\mathcal{Y})\, \mathbf{1}\big\{ m(\mathcal{Y}) \ge R_n,\ \text{Max}(\mathcal{Y}) \in \text{Ann}(R_n, KR_n) \cap Q_\ell \big\} \\ &:= \sum_{\ell \in V_n} \eta_{\ell,n}\,. \end{aligned}$$

For $i, j \in V_n$, we put an edge between i and j (write $i \sim j$) if $i \neq j$ and the distance between Q_i and Q_j are less than $2kt_m$. Then, $(V_n, \sim)$ gives a *dependency graph* with respect to $(\eta_{\ell,n},\, \ell \in V_n)$; that is, for any two disjoint subsets I_1, I_2 of V_n with no edges connecting I_1 and I_2, $(\eta_{\ell,n},\, \ell \in I_1)$ is independent of $(\eta_{\ell,n},\, \ell \in I_2)$. Notice that the maximum degree of $(V_n, \sim)$ is at most finite.

Writing Φ for a distribution function of the standard normal distribution, it follows from Stein's method for normal approximation (see Theorem 2.4 in [23]) that for all $\lambda \in \mathbb{R}$,

$$\left| \mathbb{P}\left(\frac{T_n^{(K)} - \mathbb{E}\{T_n^{(K)}\}}{\sqrt{\mathrm{Var}\{T_n^{(K)}\}}} \le \lambda \right) - \Phi(\lambda) \right|$$
$$\le C^* \sqrt{R_n^d \max_{\ell \in V_n} \frac{\mathbb{E}\big|\eta_{\ell,n} - \mathbb{E}\{\eta_{\ell,n}\}\big|^3}{\Big(\mathrm{Var}\{T_n^{(K)}\}\Big)^{3/2}}} + C^* \sqrt{R_n^d \max_{\ell \in V_n} \frac{\mathbb{E}\big|\eta_{\ell,n} - \mathbb{E}\{\eta_{\ell,n}\}\big|^4}{\Big(\mathrm{Var}\{T_n^{(K)}\}\Big)^{2}}}$$

Thus, (7.15) immediately follows if we can show that for $p = 3, 4$,

$$R_n^d \max_{\ell \in V_n} \frac{\mathbb{E}\big|\eta_{\ell,n} - \mathbb{E}\{\eta_{\ell,n}\}\big|^p}{\Big(\mathrm{Var}\{T_n^{(K)}\}\Big)^{p/2}} \to 0 \ \text{ as } n \to \infty\,. \tag{7.16}$$

Since the proof for showing this varies depending on the limit of $nf(R_n e_1)$, we divide the argument into three different cases. Suppose first that $nf(R_n e_1) \to 0$ as $n \to \infty$. Let $Z_{\ell,n}$ denote the number of points in $\mathcal{P}_n$ lying in

$$\mathrm{Tube}(Q_\ell; kt_m) := \left\{ x \in \mathbb{R}^d : \inf_{y \in Q_\ell} ||x - y|| \le kt_m \right\}.$$

Then, $Z_{\ell,n}$ has a Poisson distribution with mean $n \int_{\mathrm{Tube}(Q_\ell; kt_m)} f(z)dz$. Using Potter's bound, we see that $Z_{\ell,n}$ is stochastically dominated by another Poisson random variable Z_n with mean $C^* nf(R_n e_1)$. Observing that

$$|\eta_{\ell,n}| \le C^* \binom{Z_{\ell,n}}{k},$$

we have, for $q = 1, 2, 3, 4$,

$$\mathbb{E}|\eta_{\ell,n}|^q \le C^* \mathbb{E} \binom{Z_{\ell,n}}{k}^q \le C^* \mathbb{E} \binom{Z_n}{k}^q \le C^* \big(nf(R_n e_1)\big)^k,$$

where in the last step we used the assumption $nf(R_n e_1) \to 0$.

It now follows that for $p = 3, 4$,

$$\max_{\ell \in V_n} E\big|\eta_{\ell,n} - \mathbb{E}\{\eta_{\ell,n}\}\big|^p \le C^* \big(nf(R_n e_1)\big)^k.$$

Therefore,

$$R_n^d \max_{\ell \in V_n} \frac{\mathbb{E}\big|\eta_{\ell,n} - \mathbb{E}\{\eta_{\ell,n}\}\big|^p}{\Big(\mathrm{Var}\{T_n^{(K)}\}\Big)^{p/2}} \le C^* R_n^d \frac{\big(nf(R_n e_1)\big)^k}{\big(n^k R_n^d f(R_n e_1)^k \gamma_K\big)^{p/2}}$$
$$= \frac{C^*}{\gamma_K^{p/2}} \big(n^k R_n^d f(R_n e_1)^k\big)^{1-p/2} \to 0\,, \quad n \to \infty\,,$$

where the last convergence follows from (4.2).

In the case of $nf(R_n e_1) \to \xi \in (0, \infty)$, the argument for proving (7.16) is very similar to, or even easier than, the previous case, so we omit it.

Finally, suppose that $nf(R_n e_1) \to \infty$ as $n \to \infty$. We begin by establishing an appropriate upper bound for the fourth moment expectation

$$\mathbb{E}\big|\eta_{\ell,n} - \mathbb{E}\{\eta_{\ell,n}\}\big|^4 = \sum_{j=0}^{4} \binom{4}{j} (-1)^j \mathbb{E}\{\eta_{\ell,n}^j\}\big(\mathbb{E}\{\eta_{\ell,n}\}\big)^{4-j}. \tag{7.17}$$

Letting

$$g_{\ell,n}(\mathcal{Y}) := \sum_{j=1}^{m} a_j h_{t_j}(\mathcal{Y})\,\mathbf{1}\big\{m(\mathcal{Y}) \ge R_n,\ \mathrm{Max}(\mathcal{Y}) \in \mathrm{Ann}(R_n, KR_n) \cap Q_\ell\big\}\,,$$

we see that for every $j \in \{0, \dots, 4\}$,

$$F_n(j) := \mathbb{E}\{\eta_{\ell,n}^j\}\big(\mathbb{E}\{\eta_{\ell,n}\}\big)^{4-j}$$

can be denoted as the expectation of a quadruple sum

$$\mathbb{E}\left\{\sum_{\mathcal{Y}_1 \subset \mathcal{P}_n^{(1)}} \sum_{\mathcal{Y}_2 \subset \mathcal{P}_n^{(2)}} \sum_{\mathcal{Y}_3 \subset \mathcal{P}_n^{(3)}} \sum_{\mathcal{Y}_4 \subset \mathcal{P}_n^{(4)}} g_{\ell,n}(\mathcal{Y}_1)\, g_{\ell,n}(\mathcal{Y}_2)\, g_{\ell,n}(\mathcal{Y}_3)\, g_{\ell,n}(\mathcal{Y}_4)\right\}, \tag{7.18}$$

where, for every $i \neq j$, either $\mathcal{P}_n^{(i)} = \mathcal{P}_n^{(j)}$ or $\mathcal{P}_n^{(i)}$ is an independent copy of $\mathcal{P}_n^{(j)}$. By definition, each $\mathcal{Y}_i$ is a finite collection of d-dimensional vectors. If, in particular, $|\mathcal{Y}_1 \cup \mathcal{Y}_2 \cup \mathcal{Y}_3 \cup \mathcal{Y}_4| = 4k$, i.e., any two of $\mathcal{Y}_i$, $i = 1, \dots, 4$ have no common elements, then the Palm theory given in the Appendix reveals that (7.18) is equal to $\big(\mathbb{E}\{\eta_{\ell,n}\}\big)^4$. Then, in this case, their overall contribution to (7.17) is identically zero, because

$$\sum_{j=0}^{4} \binom{4}{j} (-1)^j \big(\mathbb{E}\{\eta_{\ell,n}\}\big)^4 = 0\,.$$

Next, suppose that $|\mathcal{Y}_1 \cup \mathcal{Y}_2 \cup \mathcal{Y}_3 \cup \mathcal{Y}_4| = 4k - 1$, i.e., there is a pair $(\mathcal{Y}_i, \mathcal{Y}_j)$, $i \neq j$ having exactly one element in common and no other common elements between $\mathcal{Y}_i$'s are present. In this case, (7.18) can be written as

$$\frac{n^{2k-1}}{\big((k-1)!\big)^2}\, \mathbb{E}\Big\{g_{\ell,n}(\mathcal{Y}_1)\, g_{\ell,n}(\mathcal{Y}_2)\, \mathbf{1}\big\{|\mathcal{Y}_1 \cap \mathcal{Y}_2| = 1\big\}\Big\} \left(\frac{n^k}{k!}\mathbb{E}\big\{g_{\ell,n}(\mathcal{Y})\big\}\right)^2. \tag{7.19}$$

In particular, (7.19) appears once in $F_n(2)$, $\binom{3}{2}$ times in $F_n(3)$, and $\binom{4}{2}$ times in $F_n(4)$. Thus, the total contribution to (7.17) sums up to

$$\left\{\binom{4}{2}(-1)^2 + \binom{4}{3}(-1)^3\binom{3}{2} + \binom{4}{4}(-1)^4\binom{4}{2}\right\} \times (7.19) = 0\,.$$

We may assume, therefore, that $|\mathcal{Y}_1 \cup \mathcal{Y}_2 \cup \mathcal{Y}_3 \cup \mathcal{Y}_4| \le 4k - 2$. Let us start with $|\mathcal{Y}_1 \cup \mathcal{Y}_2 \cup \mathcal{Y}_3 \cup \mathcal{Y}_4| = 4k - 2$, where we shall examine in particular the case in which $\mathcal{P}_n^{(1)} = \mathcal{P}_n^{(2)} = \mathcal{P}_n^{(3)} = \mathcal{P}_n^{(4)}$, $|\mathcal{Y}_1 \cap \mathcal{Y}_2| = 2$ and no other common elements between $\mathcal{Y}_i$'s exist. The argument for the other cases will be omitted because they can be handled in the same manner. Then, by Palm theory, (7.18) is equal to

$$\frac{n^{2k-2}}{2\big((k-2)!\big)^2}\, \mathbb{E}\Big\{g_{\ell,n}(\mathcal{Y}_1)\, g_{\ell,n}(\mathcal{Y}_2)\, \mathbf{1}\big\{|\mathcal{Y}_1 \cap \mathcal{Y}_2| = 2\big\}\Big\} \left(\frac{n^k}{k!}\mathbb{E}\big\{g_{\ell,n}(\mathcal{Y})\big\}\right)^2. \tag{7.20}$$

Because of Potter's bound, together with the fact that Q_ℓ intersects with $\mathrm{Ann}(R_n, KR_n)$,

$$\begin{aligned}&\Big|\mathbb{E}\Big\{g_{\ell,n}(Y_1)\, g_{\ell,n}(\mathcal{Y}_2)\, \mathbf{1}\big\{|\mathcal{Y}_1 \cap \mathcal{Y}_2| = 2\big\}\Big\}\Big| \\ &\quad \le C^* \Big(\mathbb{P}\big\{X_1 \in \mathrm{Tube}(Q_\ell; kt_m)\big\}\Big)^{2k-2} \le C^* f(R_n e_1)^{2k-2}.\end{aligned}$$

Similarly, we can obtain

$$\Big|\mathbb{E}\big\{g_{\ell,n}(\mathcal{Y})\big\}\Big| \le C^* f(R_n e_1)^k,$$

and therefore, the absolute value of (7.20), equivalently that of (7.18), is bounded above by $C^*\big(nf(R_n e_1)\big)^{4k-2}$.

A similar argument proves that if $|\mathcal{Y}_1 \cup \mathcal{Y}_2 \cup \mathcal{Y}_3 \cup \mathcal{Y}_4| = 4k - q$ for some $q \ge 3$, the absolute value of (7.18) is bounded above by $C^*\big(nf(R_n e_1)\big)^{4k-q}$. Putting these facts altogether, while recalling $nf(R_n e_1) \to \infty$ as $n \to \infty$, we may conclude that

$$\mathbb{E}\big|\eta_{\ell,n} - \mathbb{E}\{\eta_{\ell,n}\}\big|^4 \le C^*\big(nf(R_n e_1)\big)^{4k-2}.$$

Now, it is easy to check (7.16).

In terms of the third moment expectation $\mathbb{E}\big|\eta_{\ell,n} - \mathbb{E}\{\eta_{\ell,n}\}\big|^3$, we apply Hölder's inequality to obtain

$$\mathbb{E}\big|\eta_{\ell,n} - \mathbb{E}\{\eta_{\ell,n}\}\big|^3 \le \Big(\mathbb{E}\big|\eta_{\ell,n} - \mathbb{E}\{\eta_{\ell,n}\}\big|^4\Big)^{3/4} \le C^*\big(nf(R_n e_1)\big)^{3k-3/2}.$$

Again, it is easy to prove (7.16).

Now, we have obtained a CLT in (7.14) as required, regardless of the limit of $nf(R_n e_1)$. □

Proof of Theorem 4.3. The last proposition has justified finite-dimensional weak convergence of the processes $(\mathbf{X}_n)$ and $(\mathbf{X}_n^+, \mathbf{X}_n^-)$. The proof of Theorem 4.3 will be complete, provided that the tightness of $(\mathbf{X}_n)$ is verified in the space $\mathcal{D}[0,\infty)$ equipped with the Skorohod J_1-topology; see [8]. To this aim, it suffices to show that $(\mathbf{X}_n^+)$ and $(\mathbf{X}_n^-)$ are both tight in $\mathcal{D}[0,\infty)$. To see this, suppose that $\mathbf{X}_n^+$ and $\mathbf{X}_n^-$ were tight in $\mathcal{D}[0,\infty)$. Then, a joint process $(\mathbf{X}_n^+, \mathbf{X}_n^-)$ is tight as well in $\mathcal{D}[0,\infty) \times \mathcal{D}[0,\infty)$, which is endowed with the product topology. Because of the already established finite-dimensional weak convergence of $(\mathbf{X}_n^+, \mathbf{X}_n^-)$, every subsequential limit of $(\mathbf{X}_n^+, \mathbf{X}_n^-)$ coincides with the limiting process in Proposition 7.3. This in turn implies the weak convergence of $(\mathbf{X}_n^+, \mathbf{X}_n^-)$ in $\mathcal{D}[0,\infty) \times \mathcal{D}[0,\infty)$. Using the basic fact that the map $(x, y) \to x - y$ from $\mathcal{D}[0,\infty) \times \mathcal{D}[0,\infty)$ to $\mathcal{D}[0,\infty)$ is continuous at $(x, y) \in \mathcal{C}[0,\infty) \times \mathcal{C}[0,\infty)$, while recalling that the limits in Proposition 7.3 all have continuous sample paths, the continuous mapping theorem gives weak convergence of $\mathbf{X}_n = \mathbf{X}_n^+ - \mathbf{X}_n^-$ in $\mathcal{D}[0,\infty)$.

In the following, we prove the tightness of $(\mathbf{X}_n^+)$ only, because the argument for $(\mathbf{X}_n^-)$ is the same as that for $(\mathbf{X}_n^+)$. By a standard argument for the $\mathcal{D}$-space (see, e.g., Chapter 16 in [8]), it is enough to show the tightness in the space $\mathcal{D}[0, L]$ for every $L > 0$. For notational ease, we omit the superscript "+" from all the functions and objects during the proof. By Theorem 13.5 of [8], it is sufficient to show that there exists $B > 0$ such that

$$\mathbb{E}\Big\{\big(X_n(t) - X_n(s)\big)^2\big(X_n(s) - X_n(r)\big)^2\Big\} \le B(t-r)^2$$

for all $0 \le r \le s \le t \le L$ and $n \ge 1$.

For typographical convenience, we use shorthand notations (7.1), (7.2), and further,

$$\xi_{n,t,s} := \sum_{\mathcal{Y} \subset \mathcal{P}_n} h_{n,t,s}(\mathcal{Y})\,.$$

Then,

$$\begin{aligned}&\mathbb{E}\Big\{\big(X_n(t)-X_n(s)\big)^2\big(X_n(s)-X_n(r)\big)^2\Big\}\\&\quad=\tau_n^{-2}\mathbb{E}\Big\{\big(\xi_{n,t,s}-\mathbb{E}\{\xi_{n,t,s}\}\big)^2\big(\xi_{n,s,r}-\mathbb{E}\{\xi_{n,s,r}\}\big)^2\Big\}\\&\quad=\tau_n^{-2}\sum_{p=0}^{2}\sum_{q=0}^{2}\binom{2}{p}\binom{2}{q}(-1)^{p+q}F_n(p,q)\,,\end{aligned}$$

where

$$F_n(p,q)=\mathbb{E}\{\xi_{n,t,s}^p\xi_{n,s,r}^q\}\big(\mathbb{E}\{\xi_{n,t,s}\}\big)^{2-p}\big(\mathbb{E}\{\xi_{n,s,r}\}\big)^{2-q}.$$

Note that for every $p,q\in\{0,1,2\}$, $F_n(p,q)$ can be represented by

$$\mathbb{E}\left\{\sum_{\mathcal{Y}_1\subset\mathcal{P}_n^{(1)}}\sum_{\mathcal{Y}_2\subset\mathcal{P}_n^{(2)}}\sum_{\mathcal{Y}_3\subset\mathcal{P}_n^{(3)}}\sum_{\mathcal{Y}_4\subset\mathcal{P}_n^{(4)}}h_{n,t,s}(\mathcal{Y}_1)\,h_{n,t,s}(\mathcal{Y}_2)\,h_{n,s,r}(\mathcal{Y}_3)\,h_{n,s,r}(\mathcal{Y}_4)\right\},\tag{7.21}$$

where, for every $i\neq j$, either $\mathcal{P}_n^{(i)}=\mathcal{P}_n^{(j)}$ or $\mathcal{P}_n^{(i)}$ is an independent copy of $\mathcal{P}_n^{(j)}$.

According to the Palm theory given in the Appendix, if $|\mathcal{Y}_1\cup\mathcal{Y}_2\cup\mathcal{Y}_3\cup\mathcal{Y}_4|=4k$, i.e., any two of $\mathcal{Y}_i$ have no common elements, then (7.21) reduces to $\big(\mathbb{E}\{\xi_{n,t,s}\}\big)^2\big(\mathbb{E}\{\xi_{n,s,r}\}\big)^2$. Then, an overall contribution in this case identically vanishes, since

$$\sum_{p=0}^{2}\sum_{q=0}^{2}\binom{2}{p}\binom{2}{q}(-1)^{p+q}\big(\mathbb{E}\{\xi_{n,t,s}\}\big)^2\big(\mathbb{E}\{\xi_{n,s,r}\}\big)^2=0\,.$$

In the following, we examine the case in which at least one common element exists between $\mathcal{Y}_i$'s. First, for $\ell=1,\dots,k$, we count the number of times

$$\mathbb{E}\Big\{\sum_{\mathcal{Y}_1\subset\mathcal{P}_n}\sum_{\mathcal{Y}_2\subset\mathcal{P}_n}h_{n,t,s}(\mathcal{Y}_1)\,h_{n,t,s}(\mathcal{Y}_2)\,\mathbf{1}\big\{|\mathcal{Y}_1\cap\mathcal{Y}_2|=\ell\big\}\Big\}\big(\mathbb{E}\{\xi_{n,s,r}\}\big)^2\tag{7.22}$$

appears in each $F_n(p,q)$. Indeed, (7.22) appears only once in $F_n(2,0)$, $F_n(2,1)$, and $F_n(2,2)$. Therefore, the total contribution amounts to

$$\left[\binom{2}{2}\binom{2}{0}(-1)^{2+0}+\binom{2}{2}\binom{2}{1}(-1)^{2+1}+\binom{2}{2}\binom{2}{2}(-1)^{2+2}\right]\times(7.22)=0\,.$$

Similarly, for every $\ell=1,\dots,k$, no contribution is made by

$$\mathbb{E}\Big\{\sum_{\mathcal{Y}_1\subset\mathcal{P}_n}\sum_{\mathcal{Y}_2\subset\mathcal{P}_n}h_{n,s,r}(\mathcal{Y}_1)\,h_{n,s,r}(\mathcal{Y}_2)\,\mathbf{1}\big\{|\mathcal{Y}_1\cap\mathcal{Y}_2|=\ell\big\}\Big\}\big(\mathbb{E}\{\xi_{n,t,s}\}\big)^2.$$

Subsequently, for $\ell=1,\dots,k$, we explore the presence of

$$\mathbb{E}\Big\{\sum_{\mathcal{Y}_1\subset\mathcal{P}_n}\sum_{\mathcal{Y}_2\subset\mathcal{P}_n}h_{n,t,s}(\mathcal{Y}_1)\,h_{n,s,r}(\mathcal{Y}_2)\,\mathbf{1}\big\{|\mathcal{Y}_1\cap\mathcal{Y}_2|=\ell\big\}\Big\}\mathbb{E}\{\xi_{n,t,s}\}\mathbb{E}\{\xi_{n,s,r}\}\,.\tag{7.23}$$

One can immediately check that (7.23) appears once in $F_n(1,1)$, twice in $F_n(2,1)$, twice in $F_n(1,2)$, and four times in $F_n(2,2)$. However, their total contribution disappears again, because

$$\begin{aligned}\Bigg[&\binom{2}{1}\binom{2}{1}(-1)^{1+1}+\binom{2}{2}\binom{2}{1}(-1)^{2+1}\cdot 2\\&\qquad+\binom{2}{1}\binom{2}{2}(-1)^{1+2}\cdot 2+\binom{2}{2}\binom{2}{2}(-1)^{2+2}\cdot 4\Bigg]\times(7.23)=0\,.\end{aligned}$$

Next, let $\ell_i \in \{0, \dots, k\}$, $i = 1, 2, 3$, $\ell \in \{2, \dots, 2k\}$ such that at least two of ℓ_i's are non-zero, so that we should examine the appearance of

$$\mathbb{E}\Big\{ \sum_{\mathcal{Y}_1 \subset \mathcal{P}_n} \sum_{\mathcal{Y}_2 \subset \mathcal{P}_n} \sum_{\mathcal{Y}_3 \subset \mathcal{P}_n} h_{n,t,s}(\mathcal{Y}_1)\, h_{n,t,s}(\mathcal{Y}_2)\, h_{n,s,r}(\mathcal{Y}_3) \\ \times \mathbf{1}\big\{ |\mathcal{Y}_1 \cap \mathcal{Y}_2| = \ell_1,\, |\mathcal{Y}_1 \cap \mathcal{Y}_3| = \ell_2,\, |\mathcal{Y}_2 \cap \mathcal{Y}_3| = \ell_3,\, |\mathcal{Y}_1 \cup \mathcal{Y}_2 \cup \mathcal{Y}_3| = 3k - \ell \big\} \Big\} \, \mathbb{E}\{\xi_{n,s,r}\}\,. \tag{7.24}$$

This actually appears once in $F_n(2,1)$ and twice in $F_n(2,2)$; therefore, their overall contribution is

$$\left[\binom{2}{2}\binom{2}{1}(-1)^{2+1} + \binom{2}{2}\binom{2}{2}(-1)^{2+2} \cdot 2 \right] \times (7.24) = 0\,.$$

For the same reason, we can ignore the presence of

$$\mathbb{E}\Big\{ \sum_{\mathcal{Y}_1 \subset \mathcal{P}_n} \sum_{\mathcal{Y}_2 \subset \mathcal{P}_n} \sum_{\mathcal{Y}_3 \subset \mathcal{P}_n} h_{n,t,s}(\mathcal{Y}_1)\, h_{n,s,r}(\mathcal{Y}_2)\, h_{n,s,r}(\mathcal{Y}_3) \\ \times \mathbf{1}\big\{ |\mathcal{Y}_1 \cap \mathcal{Y}_2| = \ell_1,\, |\mathcal{Y}_1 \cap \mathcal{Y}_3| = \ell_2,\, |\mathcal{Y}_2 \cap \mathcal{Y}_3| = \ell_3,\, |\mathcal{Y}_1 \cup \mathcal{Y}_2 \cup \mathcal{Y}_3| = 3k - \ell \big\} \Big\} \, \mathbb{E}\{\xi_{n,t,s}\}\,.$$

where $\ell_i \in \{0, \dots, k\}$, $i = 1, 2, 3$, $\ell \in \{2, \dots, 2k\}$ such that at least two of ℓ_i's are non-zero.

Putting these calculations altogether, we find that the tightness follows, once we can show that there exists $B > 0$ such that

$$\tau_n^{-2}\mathbb{E}\Big\{ \sum_{\mathcal{Y}_1 \subset \mathcal{P}_n} \sum_{\mathcal{Y}_2 \subset \mathcal{P}_n} \sum_{\mathcal{Y}_3 \subset \mathcal{P}_n} \sum_{\mathcal{Y}_4 \subset \mathcal{P}_n} h_{n,t,s}(\mathcal{Y}_1)\, h_{n,t,s}(\mathcal{Y}_2)\, h_{n,s,r}(\mathcal{Y}_3)\, h_{n,s,r}(\mathcal{Y}_4) \\ \times \mathbf{1}\big\{ \text{each } \mathcal{Y}_i \text{ has at least one common elements with} \\ \text{at least one of the other three} \big\} \Big\} \le B(t-r)^2 \tag{7.25}$$

for all $0 \le r \le s \le t \le L$ and $n \ge 1$. We need to check only the following possibilities.
[I] $\ell := |\mathcal{Y}_1 \cap \mathcal{Y}_2| \in \{1, \dots, k\}$, $\ell' := |\mathcal{Y}_3 \cap \mathcal{Y}_4| \in \{1, \dots, k\}$, and $(\mathcal{Y}_1 \cup \mathcal{Y}_2) \cap (\mathcal{Y}_3 \cup \mathcal{Y}_4) = \emptyset$.
[II] $\ell := |\mathcal{Y}_2 \cap \mathcal{Y}_3| \in \{1, \dots, k\}$, $\ell' := |\mathcal{Y}_1 \cap \mathcal{Y}_4| \in \{1, \dots, k\}$, and $(\mathcal{Y}_2 \cup \mathcal{Y}_3) \cap (\mathcal{Y}_1 \cup \mathcal{Y}_4) = \emptyset$.
[III]. Each $\mathcal{Y}_i$ has at least one common element with at least one of the other three, but neither [I] or [II] is true.

For example, if $|\mathcal{Y}_1 \cap \mathcal{Y}_2| = 2$, $|\mathcal{Y}_1 \cap \mathcal{Y}_3| = 3$, $|\mathcal{Y}_2 \cap \mathcal{Y}_4| = 1$, and there are no other common elements between $\mathcal{Y}_i$'s, then it falls into category [III], where, unlike [I] or [II], the expectation in (7.25) can no longer be separated by the Palm theory.

Denoting by A the left-hand side of (7.25), let us start with case [I]. As a result of Palm theory,

$$\begin{aligned} A &= \tau_n^{-1} \frac{n^{2k-\ell}}{\ell!\big((k-\ell)!\big)^2}\, \mathbb{E}\Big\{ h_{n,t,s}(\mathcal{Y}_1)\, h_{n,t,s}(\mathcal{Y}_2)\, \mathbf{1}\big\{|\mathcal{Y}_1 \cap \mathcal{Y}_2| = \ell\big\} \Big\} \\ &\quad \times \tau_n^{-1} \frac{n^{2k-\ell'}}{\ell'!\big((k-\ell')!\big)^2}\, \mathbb{E}\Big\{ h_{n,s,r}(\mathcal{Y}_3)\, h_{n,s,r}(\mathcal{Y}_4)\, \mathbf{1}\big\{|\mathcal{Y}_3 \cap \mathcal{Y}_4| = \ell'\big\} \Big\} \\ &:= A_1 \times A_2\,. \end{aligned}$$

Proceeding as in the calculation of Proposition 7.2, we obtain

$$A_1 \le C^* \tau_n^{-1} n^{2k-\ell} R_n^d f(R_n e_1)^{2k-\ell} \int_{(\mathbb{R}^d)^{\ell-1}} d\mathbf{y} \int_{(\mathbb{R}^d)^{k-\ell}} d\mathbf{z}_2 \int_{(\mathbb{R}^d)^{k-\ell}} d\mathbf{z}_1 \, h_{t,s}(0,\mathbf{y},\mathbf{z}_1)\, h_{t,s}(0,\mathbf{y},\mathbf{z}_2)\,, \tag{7.26}$$

$$A_2 \le C^* \tau_n^{-1} n^{2k-\ell'} R_n^d f(R_n e_1)^{2k-\ell'} \times \int_{(\mathbb{R}^d)^{\ell'-1}} d\mathbf{y} \int_{(\mathbb{R}^d)^{k-\ell'}} d\mathbf{z}_2 \int_{(\mathbb{R}^d)^{k-\ell'}} d\mathbf{z}_1 \, h_{s,r}(0,\mathbf{y},\mathbf{z}_1)\, h_{s,r}(0,\mathbf{y},\mathbf{z}_2)\,. \tag{7.27}$$

Notice that h_t is increasing in t in the sense of (2.5) (recall that the superscript "+" is suppressed during the proof). It also follows from (2.6) that the triple integral in (7.26) is unchanged if the integral domain is restricted to $\big(B(0,kL)\big)^{\ell-1} \times \big(B(0,kL)\big)^{k-\ell} \times \big(B(0,kL)\big)^{k-\ell}$. Therefore, with λ being the Lebesgue measure on $(\mathbb{R}^d)^{k-\ell}$,

$$\begin{aligned}
&\int_{(\mathbb{R}^d)^{\ell-1}} d\mathbf{y} \int_{(\mathbb{R}^d)^{k-\ell}} d\mathbf{z}_2 \int_{(\mathbb{R}^d)^{k-\ell}} d\mathbf{z}_1 \, h_{t,s}(0,\mathbf{y},\mathbf{z}_1)\, h_{t,s}(0,\mathbf{y},\mathbf{z}_2) \\
&\quad \le \lambda\Big\{\big(B(0,kL)\big)^{k-\ell}\Big\} \int_{(\mathbb{R}^d)^{\ell-1}} \int_{(\mathbb{R}^d)^{k-\ell}} h_{t,s}(0,\mathbf{y},\mathbf{z}) d\mathbf{y} d\mathbf{z} \\
&\quad = \lambda\Big\{\big(B(0,kL)\big)^{k-\ell}\Big\} \big(t^{d(k-1)} - s^{d(k-1)}\big) \int_{(\mathbb{R}^d)^{k-1}} h_1(0,\mathbf{y}) d\mathbf{y} \\
&\quad \le C^*(t-r)\,.
\end{aligned}$$

Applying the same manipulation to the triple integral in (7.27), we obtain

$$A \le C^* \tau_n^{-2} n^{4k-\ell-\ell'} R_n^{2d} f(R_n e_1)^{4k-\ell-\ell'} (t-r)^2.$$

It remains to check that $\sup_n \tau_n^{-2} n^{4k-\ell-\ell'} R_n^{2d} f(R_n e_1)^{4k-\ell-\ell'} < \infty$, which is, however, easy to prove, irrespective of the definition of τ_n. Now case [I] is done.

Next, we turn to case [II]. As a consequence of the same operation as in [I], we obtain the same upper bound for A up to multiplicative constants.

Finally, we proceed to case [III]. Let $\ell := 4k - |\mathcal{Y}_1 \cup \mathcal{Y}_2 \cup \mathcal{Y}_3 \cup \mathcal{Y}_4|$; then, it must be that $3 \le \ell \le 3k$. It follows from Palm theory that

$$A = C^* \tau_n^{-2} n^{4k-\ell} \mathbb{E}\big\{h_{n,t,s}(\mathcal{Y}_1)\, h_{n,t,s}(\mathcal{Y}_2)\, h_{n,s,r}(\mathcal{Y}_3)\, h_{n,s,r}(\mathcal{Y}_4)\big\}$$

with $(\mathcal{Y}_1, \dots, \mathcal{Y}_4)$ satisfying requirements in case [III]. In particular, $(\mathcal{Y}_1 \cup \mathcal{Y}_2) \cap (\mathcal{Y}_3 \cup \mathcal{Y}_4)$ must be non-empty; hence, we may assume without loss of generality that $\mathcal{Y}_1 \cap \mathcal{Y}_3 \ne \emptyset$. Set $\ell' := |\mathcal{Y}_1 \cap \mathcal{Y}_3| \in \{1, \dots, k\}$. By (2.5) and (2.6), we have

$$A \le C^* \tau_n^{-2} n^{4k-\ell} R_n^d f(R_n e_1)^{4k-\ell} \times \int_{(\mathbb{R}^d)^{\ell'-1}} d\mathbf{y} \int_{(\mathbb{R}^d)^{k-\ell'}} d\mathbf{z}_2 \int_{(\mathbb{R}^d)^{k-\ell'}} d\mathbf{z}_1 \, h_{t,s}(0,\mathbf{y},\mathbf{z}_1)\, h_{t,s}(0,\mathbf{y},\mathbf{z}_2)\,,$$

Because of Lemma 7.1,

$$A \le C^* \tau_n^{-2} n^{4k-\ell} R_n^d f(R_n e_1)^{4k-\ell} (t-r)^2.$$

Once again, verifying

$$\sup_n \tau_n^{-2} n^{4k-\ell} R_n^d f(R_n e_1)^{4k-\ell} < \infty$$

is elementary, and hence, we have completed the proof of (7.25) as required. □

7.2 Exponentially decaying tail case

We start by defining a subgraph counting process with restricted domain. For $0 \le K < L \le \infty$, we define

$$\begin{aligned} G_{n,K,L}(t) &= \sum_{\mathcal{Y} \subset \mathcal{P}_n} h_t(\mathcal{Y})\, \mathbf{1}\bigl\{ m(\mathcal{Y}) \ge R_n\,,\ a(R_n)^{-1}\bigl(\mathrm{Max}(\mathcal{Y}) - R_n\bigr) \in [K, L) \bigr\} \\ &:= \sum_{\mathcal{Y} \subset \mathcal{P}_n} h_{n,t,K,L}(\mathcal{Y})\,, \end{aligned}$$

and

$$\begin{aligned} G^{\pm}_{n,K,L}(t) &= \sum_{\mathcal{Y} \subset \mathcal{P}_n} h^{\pm}_t(\mathcal{Y})\, \mathbf{1}\bigl\{ m(\mathcal{Y}) \ge R_n\,,\ a(R_n)^{-1}\bigl(\mathrm{Max}(\mathcal{Y}) - R_n\bigr) \in [K, L) \bigr\} \\ &:= \sum_{\mathcal{Y} \subset \mathcal{P}_n} h^{\pm}_{n,t,K,L}(\mathcal{Y})\,, \end{aligned}$$

where (R_n) satisfies (5.7). For the special case $K = 0$ and $L = \infty$, we denote $G_n(t) = G_{n,0,\infty}(t)$ and $G^{\pm}_n(t) = G^{\pm}_{n,0,\infty}(t)$. The centered and scaled versions of the subgraph counting process are

$$X_n(t) = \tau_n^{-1/2}\Bigl(G_n(t) - \mathbb{E}\bigl\{G_n(t)\bigr\} \Bigr)\,, \tag{7.28}$$

$$X^{\pm}_n(t) = \tau_n^{-1/2}\Bigl(G^{\pm}_n(t) - \mathbb{E}\bigl\{G^{\pm}_n(t)\bigr\} \Bigr)\,, \tag{7.29}$$

where (τ_n) is given in (5.8). As seen in the regularly varying tail case, we first need to know the growing rate of the covariances of $G_{n,K,L}(t)$. Before presenting the results, we introduce for $\ell = 1, \dots, k$,

$$\begin{aligned} M_{\ell,K,L}(t,s) := D_\ell \int_0^\infty \int_{(\mathbb{R}^d)^{2k-\ell-1}} & e^{-(2k-\ell)\rho - c^{-1}\sum_{i=1}^{2k-\ell-1}\langle e_1, y_i\rangle} \\ & \times \mathbf{1}\bigl\{ \mathbf{y} \in E^{(\ell)}_{K,L}(\rho, e_1) \bigr\}\, h^{(\ell)}_{t,s}(0, \mathbf{y})\, d\mathbf{y} d\rho\,,\ \ t, s \ge 0\,, \end{aligned}$$

where D_ℓ is given in (5.9), $h^{(\ell)}_{t,s}(0, \mathbf{y})$ is defined in (5.11), and for $\rho > 0$ and $\theta \in S_{d-1}$,

$$\begin{aligned} E^{(\ell)}_{K,L}(\rho, \theta) = \Bigl\{ \mathbf{y} \in (\mathbb{R}^d)^{2k-\ell-1} : & \,\rho + c^{-1}\langle \theta, y_i\rangle \ge 0\,,\ i = 1, \dots, 2k-\ell-1\,, \\ & K \le \max\bigl\{ \rho\,, \rho + c^{-1} \max_{i=1,\dots,k-1} \langle \theta, y_i\rangle \bigr\} < L\,, \\ & K \le \max\bigl\{ \rho\,, \rho + c^{-1} \max_{i=1,\dots,\ell-1,k,\dots,2k-\ell-1} \langle \theta, y_i\rangle \bigr\} < L \Bigr\}\,. \end{aligned}$$

Note that $M_{\ell,0,\infty}(t,s)$ completely matches (5.10).

Proposition 7.4. *Assume the conditions of Theorem 5.3. Let $0 \le K < L \le \infty$.*

(i) *If $nf(R_n e_1) \to 0$ as $n \to \infty$, then*

$$\tau_n^{-1}\mathit{Cov}\bigl(G_{n,K,L}(t), G_{n,K,L}(s)\bigr) \to M_{k,K,L}(t,s)\,, \quad n \to \infty\,.$$

(ii) *If $nf(R_n e_1) \to \xi \in (0,\infty)$ as $n \to \infty$, then*

$$\tau_n^{-1}\mathit{Cov}\bigl(G_{n,K,L}(t), G_{n,K,L}(s)\bigr) \to \sum_{\ell=1}^k \xi^{2k-\ell} M_{\ell,K,L}(t,s)\,, \quad n \to \infty\,.$$

(iii) *If $nf(R_n e_1) \to \infty$ as $n \to \infty$, then*

$$\tau_n^{-1}\mathit{Cov}\bigl(G_{n,K,L}(t), G_{n,K,L}(s)\bigr) \to M_{1,K,L}(t,s)\,, \quad n \to \infty\,.$$

Proof. As argued in Proposition 7.2, with the multiple applications of Palm theory, one can write

$$\begin{aligned}&\mathrm{Cov}\big(G_{n,K,L}(t)\,,G_{n,K,L}(s)\big)\\&=\sum_{\ell=1}^{k}\frac{n^{2k-\ell}}{\ell!\big((k-\ell)!\big)^2}\,\mathbb{E}\Big\{h_{n,t,K,L}(\mathcal{Y}_1)\,h_{n,s,K,L}(\mathcal{Y}_2)\,\mathbf{1}\big\{\,|\mathcal{Y}_1\cap\mathcal{Y}_2|=\ell\,\big\}\Big\}.\end{aligned}$$

Define for $\ell\in\{1,\dots,k\}$,

$$\begin{aligned}F_n^{(\ell)}(K,L):=\big\{\mathbf{x}\in(\mathbb{R}^d)^{2k-\ell}:a(R_n)^{-1}\big(\mathrm{Max}(x_1,\dots,x_k)-R_n\big)\in[K,L)\,,\\a(R_n)^{-1}\big(\mathrm{Max}(x_1,\dots,x_\ell,x_{k+1},\dots,x_{2k-\ell})-R_n\big)\in[K,L)\big\}\,.\end{aligned}$$

By the change of variables $\mathbf{x}\to(x,x+\mathbf{y})$ with $\mathbf{x}\in(\mathbb{R}^d)^{2k-\ell}$, $x\in\mathbb{R}^d$, $\mathbf{y}\in(\mathbb{R}^d)^{2k-\ell-1}$, together with invariance (2.2),

$$\begin{aligned}&\mathbb{E}\Big\{h_{n,t,K,L}(\mathcal{Y}_1)\,h_{n,s,K,L}(\mathcal{Y}_2)\,\mathbf{1}\big\{\,|\mathcal{Y}_1\cap\mathcal{Y}_2|=\ell\,\big\}\Big\}\\&=\int_{(\mathbb{R}^d)^{2k-\ell}}f(\mathbf{x})\,\mathbf{1}\big\{m(\mathbf{x})\ge R_n\big\}\,h_{t,s}^{(\ell)}(\mathbf{x})\,\mathbf{1}\big\{\mathbf{x}\in F_n^{(\ell)}(K,L)\big\}d\mathbf{x}\\&=\int_{\mathbb{R}^d}\int_{(\mathbb{R}^d)^{2k-\ell-1}}f(x)\,f(x+\mathbf{y})\,\mathbf{1}\big\{m(x,x+\mathbf{y})\ge R_n\big\}\,h_{t,s}^{(\ell)}(0,\mathbf{y})\\&\quad\times\mathbf{1}\big\{(x,x+\mathbf{y})\in F_n^{(\ell)}(K,L)\big\}d\mathbf{y}dx\,.\end{aligned}$$

Let J_k denote the last integral. Further calculation by the polar coordinate transform $x\to(r,\theta)$ with $J(\theta)=|\partial x/\partial\theta|$ and the change of variable $\rho=a(R_n)^{-1}(r-R_n)$ yields

$$\begin{aligned}J_k=&a(R_n)R_n^{d-1}f(R_ne_1)^{2k-\ell}\int_{S^{d-1}}J(\theta)d\theta\int_0^\infty d\rho\int_{(\mathbb{R}^d)^{2k-\ell-1}}d\mathbf{y}\\&\times\left(1+\frac{a(R_n)}{R_n}\rho\right)^{d-1}\frac{f\Big(\big(R_n+a(R_n)\rho\big)e_1\Big)}{f(R_ne_1)}\\&\times\prod_{i=1}^{2k-\ell-1}f(R_ne_1)^{-1}f\Big(\|(R_n+a(R_n)\rho)\theta+y_i\|e_1\Big)\,\mathbf{1}\Big\{\|(R_n+a(R_n)\rho)\theta+y_i\|\ge R_n\Big\}\\&\times\mathbf{1}\Big\{\big((R_n+a(R_n)\rho)\theta,\,(R_n+a(R_n)\rho)\theta+\mathbf{y}\big)\in F_n^{(\ell)}(K,L)\Big\}\,h_{t,s}^{(\ell)}(0,\mathbf{y})\,,\end{aligned}\tag{7.30}$$

where S^{d-1} is the $(d-1)$-dimensional unit sphere in $\mathbb{R}^d$.

The following expansion is applied frequently in the following. For each $i=1,\dots,2k-\ell-1$,

$$\Big\|\big(R_n+a(R_n)\rho\big)\theta+y_i\Big\|=R_n+a(R_n)\rho+\langle\theta,y_i\rangle+\gamma_n(\rho,\theta,y_i)\,,$$

so that $\gamma_n(\rho,\theta,y_i)\to0$ uniformly in $\rho>0$, $\theta\in S^{d-1}$, and $\|y_i\|\le k(t+s)$.

For the application of the dominated convergence theorem, we need to compute the limit of the expression under the integral sign, while establishing an integrable upper bound. We first calculate the limit of the indicator functions. For every $\rho>0$, $\theta\in S^{d-1}$, and $\|y_i\|\le k(t+s)$, $i=1,\dots,2k-\ell-1$,

$$\begin{aligned}&\prod_{i=1}^{2k-\ell-1}\mathbf{1}\Big\{\|(R_n+a(R_n)\rho)\theta+y_i\|\ge R_n\Big\}\\&\qquad\times\mathbf{1}\Big\{\big((R_n+a(R_n)\rho)\theta,\,(R_n+a(R_n)\rho)\theta+\mathbf{y}\big)\in F_n^{(\ell)}(K,L)\Big\}\\&\qquad\to\mathbf{1}\big\{\mathbf{y}\in E_{K,L}^{(\ell)}(\rho,\theta)\,\big\}\,,\quad n\to\infty\,.\end{aligned}$$

Next, it is clear that for every $\rho > 0$, $\big(1 + a(R_n)\rho/R_n\big)^{d-1}$ tends to 1 as $n \to \infty$ (see (5.3)) and is bounded above by $2\big(\max\{1,\rho\}\big)^{d-1}$.

As for the ratio of the densities in the second line of (7.30), we use the basic fact that $1/a$ is flat for a, that is, as $n \to \infty$,

$$\frac{a(R_n)}{a\big(R_n + a(R_n)v\big)} \to 1\,, \quad \text{uniformly on bounded } v\text{-sets}; \tag{7.31}$$

see p142 in [15] for details. Noting that L is also flat for a, we have for every $\rho > 0$,

$$\begin{aligned}\frac{f\Big(\big(R_n + a(R_n)\rho\big)e_1\Big)}{f(R_n e_1)} &= \frac{L\big(R_n + a(R_n)\rho\big)}{L(R_n)}\, \exp\Big\{-\psi\big(R_n + a(R_n)\rho\big) + \psi(R_n)\Big\} \\ &= \frac{L\big(R_n + a(R_n)\rho\big)}{L(R_n)}\, \exp\Big\{-\int_0^\rho \frac{a(R_n)}{a\big(R_n + a(R_n)r\big)} dr\Big\} \\ &\to e^{-\rho}, \quad \text{as } n \to \infty\,.\end{aligned}$$

To provide an upper bound for the ratio of the densities, let $\big(q_m(n),\, m \ge 0, n \ge 1\big)$ be a sequence defined by

$$q_m(n) = a(R_n)^{-1}\Big(\psi^{\leftarrow}\big(\psi(R_n) + m\big) - R_n\Big)\,,$$

equivalently,

$$\psi\big(R_n + a(R_n)q_m(n)\big) = \psi(R_n) + m\,.$$

Then, for $\epsilon \in \big(0, (d + \gamma(2k-\ell))^{-1}\big)$, there exists an integer $N_\epsilon \ge 1$ such that

$$q_m(n) \le e^{m\epsilon}/\epsilon \;\; \text{for all } n \ge N_\epsilon, m \ge 0\,.$$

For the proof of this assertion, the reader may refer to Lemma 5.2 in [3]; see also Lemma 4.7 of [22]. Because of the fact that ψ is non-decreasing, we have, for sufficiently large n,

$$\begin{aligned}&\exp\Big\{-\psi\big(R_n + a(R_n)\rho\big) + \psi(R_n)\Big\}\,\mathbf{1}\{\rho > 0\} \\ &= \sum_{m=0}^\infty \mathbf{1}\big\{q_m(n) < \rho \le q_{m+1}(n)\big\}\, \exp\Big\{-\psi\big(R_n + a(R_n)\rho\big) + \psi(R_n)\Big\} \\ &\le \sum_{m=0}^\infty \mathbf{1}\big\{0 < \rho \le \epsilon^{-1}e^{(m+1)\epsilon}\big\}\, e^{-m}.\end{aligned}$$

Using the bound in (5.5),

$$L(R_n)^{-1}L\big(R_n + a(R_n)\rho\big)\mathbf{1}\{\rho > 0\} \le C\left(1 + \frac{a(R_n)}{R_n}\rho\right)^\gamma \le 2C\big(\max\{\rho,1\}\big)^\gamma.$$

Combining these bounds,

$$\frac{f\Big(\big(R_n + a(R_n)\rho\big)e_1\Big)}{f(R_n e_1)}\,\mathbf{1}\{\rho > 0\} \le 2C\big(\max\{\rho,1\}\big)^\gamma \sum_{m=0}^\infty \mathbf{1}\big\{0 < \rho \le \epsilon^{-1}e^{(m+1)\epsilon}\big\}\, e^{-m}.$$

Finally, we turn to

$$\begin{aligned}\prod_{i=1}^{2k-\ell-1} \frac{f\Big(\|(R_n + a(R_n)\rho)\theta + y_i\|e_1\Big)}{f(R_n e_1)} = \prod_{i=1}^{2k-\ell-1} &\frac{L\Big(R_n + a(R_n)\big(\rho + \xi_n(\rho,\theta,y_i)\big)\Big)}{L(R_n)} \\ &\times \exp\left\{-\int_0^{\rho + \xi_n(\rho,\theta,y_i)} \frac{a(R_n)}{a\big(R_n + a(R_n)r\big)}\, dr\right\},\end{aligned}$$

where

$$\xi_n(\rho,\theta,y)=\frac{\langle\theta,y\rangle+\gamma_n(\rho,\theta,y)}{a(R_n)}\,.$$

Since $c=\lim_{n\to\infty}a(R_n)>0$,

$$A:=\sup_{\substack{n\ge1,\ \rho>0,\\ \theta\in S^{d-1},\ \|y\|\le k(t+s)}}\big|\xi_n(\rho,\theta,y)\big|<\infty\,.$$

Therefore, because of the uniform convergence in (7.31), for every $\rho>0$, $\theta\in S^{d-1}$, and $\|y_i\|\le k(t+s)$,

$$\prod_{i=1}^{2k-\ell-1}\frac{f\Big(\|(R_n+a(R_n)\rho)\theta+y_i\|e_1\Big)}{f(R_ne_1)}\to\exp\big\{-(2k-\ell-1)\rho-c^{-1}\sum_{i=1}^{2k-\ell-1}\langle\theta,y_i\rangle\big\}.$$

Subsequently, on the set

$$\begin{aligned}&\Big\{\|(R_n+a(R_n)\rho)\theta+y_i\|\ge R_n\,,\ i=1,\dots,2k-\ell-1\Big\}\\&=\big\{\rho+\xi_n(\rho,\theta,y_i)\ge0,\ i=1,\dots,2k-\ell-1\big\}\,,\end{aligned}$$

we have an obvious upper bound

$$\prod_{i=1}^{2k-\ell-1}\exp\left\{-\int_0^{\rho+\xi_n(\rho,\theta,y_i)}\frac{a(R_n)}{a\big(R_n+a(R_n)r\big)}\,dr\right\}\le1$$

from which, together with (5.5), we see that

$$\begin{aligned}\prod_{i=1}^{2k-\ell-1}\frac{f\Big(\|(R_n+a(R_n)\rho)\theta+y_i\|e_1\Big)}{f(R_ne_1)}&\le\prod_{i=1}^{2k-\ell-1}C\left(1+\frac{a(R_n)}{R_n}\big(\rho+\xi_n(\rho,\theta,y_i)\big)\right)^{\gamma}\\&\le C^*\big(\max\{\rho,1\}\big)^{\gamma(2k-\ell-1)}.\end{aligned}$$

From the argument thus far, for every $\rho>0$, $\theta\in S^{d-1}$, and $\|y_i\|\le k(t+s)$, $i=1,\dots,2k-\ell-1$, the expression under the integral sign in (7.30) eventually converges to

$$e^{-(2k-\ell)\rho-c^{-1}\sum_{i=1}^{2k-\ell-1}\langle\theta,y_i\rangle}\,\mathbf{1}\big\{\mathbf{y}\in E^{(\ell)}_{K,L}(\rho,\theta)\big\}\,h^{(\ell)}_{t,s}(0,\mathbf{y})\,,$$

while it possesses an upper bound of the form

$$C^*\big(\max\{\rho,1\}\big)^{d-1+\gamma(2k-\ell)}\sum_{m=0}^{\infty}\mathbf{1}\big\{0<\rho\le\epsilon^{-1}e^{(m+1)\epsilon}\big\}\,e^{-m}h^{(\ell)}_{t,s}(0,\mathbf{y})$$

for sufficiently large n. Because of the restriction in ϵ, it is elementary to check that

$$\int_0^\infty\big(\max\{\rho,1\}\big)^{d-1+\gamma(2k-\ell)}\sum_{m=0}^{\infty}\mathbf{1}\big\{0<\rho\le\epsilon^{-1}e^{(m+1)\epsilon}\big\}e^{-m}d\rho<\infty\,.$$

As a result of the dominated convergence theorem, we have obtained, as $n\to\infty$,

$$\begin{aligned}J_k&\sim a(R_n)R_n^{d-1}f(R_ne_1)^{2k-\ell}\int_{S_{d-1}}J(\theta)d\theta\int_0^\infty d\rho\int_{(\mathbb{R}^d)^{2k-\ell-1}}d\mathbf{y}\\&\quad\times e^{-(2k-\ell)\rho-c^{-1}\sum_{i=1}^{2k-\ell-1}\langle\theta,y_i\rangle}\,\mathbf{1}\big\{\mathbf{y}\in E^{(\ell)}_{K,L}(\rho,\theta)\big\}\,h^{(\ell)}_{t,s}(0,\mathbf{y})\\&=a(R_n)R_n^{d-1}f(R_ne_1)^{2k-\ell}\ell!\big((k-\ell)!\big)^2M_{\ell,K,L}(t,s)\,,\end{aligned}$$

where the last step follows from the rotation invariance of $h_\cdot$. Hence, we have

$$\mathrm{Cov}\big(G_{n,K,L}(t)\,, G_{n,K,L}(s)\big) \sim \sum_{\ell=1}^{k} n^{2k-\ell} a(R_n) R_n^{d-1} f(R_n e_1)^{2k-\ell} M_{\ell,K,L}(t,s)\,, \quad n \to \infty\,.$$

If $nf(R_n e_1) \to 0$, then the kth term in the sum is asymptotically dominant, and therefore, statement (i) of the theorem is complete. However, the first term becomes dominant when $nf(R_n e_1) \to \infty$, in which case, statement (iii) is established. In addition, if $nf(R_n e_1) \to \xi \in (0,\infty)$, all the terms in the sum grow at the same rate, and this completes statement (ii). □

Subsequently, we show the results on finite-dimensional weak convergence of $\mathbf{X}_n$ and $(\mathbf{X}_n^+, \mathbf{X}_n^-)$ defined in (7.28) and (7.29), which somewhat parallel those of Proposition 7.3. The reader may return to Section 5.2 to recall the definition and properties of the limit $(\mathbf{W}_\ell^+, \mathbf{W}_\ell^-)$. We omit their proofs, since the argument in Proposition 7.3 does apply again with minor modifications.

Proposition 7.5. *Assume the conditions of Theorem 5.3. Then, weak convergences $(i) - (iii)$ in the theorem hold in a finite-dimensional sense. Furthermore, the following results also hold in a finite-dimensional sense.*

(i) *If $nf(R_n e_1) \to 0$ as $n \to \infty$, then*

$$(\mathbf{X}_n^+, \mathbf{X}_n^-) \Rightarrow (\mathbf{W}_k^+, \mathbf{W}_k^-)\,.$$

(ii) *If $nf(R_n e_1) \to \xi \in (0,\infty)$ as $n \to \infty$, then*

$$(\mathbf{X}_n^+, \mathbf{X}_n^-) \Rightarrow \left(\sum_{\ell=1}^{k} \xi^{2k-\ell} \mathbf{W}_\ell^+,\, \sum_{\ell=1}^{k} \xi^{2k-\ell} \mathbf{W}_\ell^- \right)\,.$$

(iii) *If $nf(R_n e_1) \to \infty$ as $n \to \infty$, then*

$$(\mathbf{X}_n^+, \mathbf{X}_n^-) \Rightarrow (\mathbf{W}_1^+, \mathbf{W}_1^-)\,.$$

Proof of Theorem 5.3. For the same reason as discussed in the proof of Theorem 4.3, it suffices to prove that $(\mathbf{X}_n^+)$ and $(\mathbf{X}_n^-)$ are both tight in $\mathcal{D}[0,\infty)$. We only prove the tightness of $(\mathbf{X}_n^+)$, while suppressing the superscript "+" from the functions and objects involved during the proof. Proceeding completely in the same manner as the proof of Theorem 4.3, we have only to show that there exists $B > 0$ such that

$$\begin{aligned} \tau_n^{-2} \mathbb{E}\Big\{ \sum_{\mathcal{Y}_1 \subset \mathcal{P}_n} \sum_{\mathcal{Y}_2 \subset \mathcal{P}_n} \sum_{\mathcal{Y}_3 \subset \mathcal{P}_n} \sum_{\mathcal{Y}_4 \subset \mathcal{P}_n} & h_{n,t,s}(\mathcal{Y}_1)\, h_{n,t,s}(\mathcal{Y}_2)\, h_{n,s,r}(\mathcal{Y}_3)\, h_{n,s,r}(\mathcal{Y}_4) \\ & \times \mathbf{1}\big\{\text{each } \mathcal{Y}_i \text{ has at least one common elements} \\ & \qquad \text{with at least one of the other three}\big\}\Big\} \le B(t-r)^2 \end{aligned} \tag{7.32}$$

for all $0 \le r \le s \le t \le L$ and $n \ge 1$. There are three possibilities to be discussed.
[I] $\ell := |\mathcal{Y}_1 \cap \mathcal{Y}_2| \in \{1,\dots,k\}$, $\ell' := |\mathcal{Y}_3 \cap \mathcal{Y}_4| \in \{1,\dots,k\}$, and $(\mathcal{Y}_1 \cup \mathcal{Y}_2) \cap (\mathcal{Y}_3 \cup \mathcal{Y}_4) = \emptyset$.
[II] $\ell := |\mathcal{Y}_2 \cap \mathcal{Y}_3| \in \{1,\dots,k\}$, $\ell' := |\mathcal{Y}_1 \cap \mathcal{Y}_4| \in \{1,\dots,k\}$, and $(\mathcal{Y}_2 \cup \mathcal{Y}_3) \cap (\mathcal{Y}_1 \cup \mathcal{Y}_4) = \emptyset$.
[III]. Each $\mathcal{Y}_i$ has at least one common element with at least one of the other three, but neither [I] or [II] is true.

Let B be the left hand side of (7.32). As for case [I], by mimicking the argument in the proof of Theorem 4.3, we obtain

$$B \le C^* \tau_n^{-2} n^{4k-\ell-\ell'} a(R_n)^2 R_n^{2(d-1)} f(R_n e_1)^{4k-\ell-\ell'} (t-r)^2 \le C^*(t-r)^2,$$

which proves (7.32). Since we can deal with [Ⅱ] in an analogous way, we can turn to case [Ⅲ]. Letting $\ell := 4k - |\mathcal{Y}_1 \cup \mathcal{Y}_2 \cup \mathcal{Y}_3 \cup \mathcal{Y}_4| \in \{3, \dots, 3k\}$, the same argument as in the proof of Theorem 4.3 yields

$$B \le C^* \tau_n^{-2} n^{4k-\ell} a(R_n) R_n^{d-1} f(R_n e_1)^{4k-\ell} (t-r)^2 \le C^* (t-r)^2$$

which verifies (7.32). □

8 Appendix

We collect supplemental but important results for the completion of the main theorems. This result is known as the Palm theory of Poisson point processes, which is applied a number of times throughout the proof.

Lemma 8.1. *(Palm theory for Poisson point processes, [2], Corollary B.2 in [10], see also Theorem 1.6 in [23]) Let (X_i) be i.i.d. $\mathbb{R}^d$-valued random variables with common density f. Let $\mathcal{P}_n$ be a Poisson point process on $\mathbb{R}^d$ with intensity nf. Let $h(\mathcal{Y})$, $h_i(\mathcal{Y})$, $i = 1, 2, 3, 4$ be measurable bounded functions defined for $\mathcal{Y} \in (\mathbb{R}^d)^k$. Then,*

$$\mathbb{E}\Big\{ \sum_{\mathcal{Y} \subset \mathcal{P}_n} h(\mathcal{Y}) \Big\} = \frac{n^k}{k!} \mathbb{E}\big\{ h(\mathcal{Y}) \big\} ,$$

and for every $\ell \in \{0, \dots, k\}$,

$$\begin{aligned} &\mathbb{E}\Big\{ \sum_{\mathcal{Y}_1 \subset \mathcal{P}_n} \sum_{\mathcal{Y}_2 \subset \mathcal{P}_n} h_1(\mathcal{Y}_1)\, h_2(\mathcal{Y}_2)\, \mathbf{1}\big\{ |\mathcal{Y}_1 \cap \mathcal{Y}_2| = \ell \big\} \Big\} \\ &\quad = \frac{n^{2k-\ell}}{\ell! \big((k-\ell)! \big)^2} \, \mathbb{E}\Big\{ h_1(\mathcal{Y}_1)\, h_2(\mathcal{Y}_2)\, \mathbf{1}\big\{ |\mathcal{Y}_1 \cap \mathcal{Y}_2| = \ell \big\} \Big\} . \end{aligned}$$

Moreover, for every $\ell_1, \ell_2, \ell_3 \in \{0, \dots, k\}$ and $\ell \in \{0, \dots, 2k\}$, there exists a constant $C > 0$, which depends only on ℓ_i, ℓ, and k such that

$$\begin{aligned} &\mathbb{E}\Big\{ \sum_{\mathcal{Y}_1 \subset \mathcal{P}_n} \sum_{\mathcal{Y}_2 \subset \mathcal{P}_n} \sum_{\mathcal{Y}_3 \subset \mathcal{P}_n} h_1(\mathcal{Y}_1)\, h_2(\mathcal{Y}_2)\, h_3(\mathcal{Y}_3) \\ &\qquad \times \mathbf{1}\big\{ |\mathcal{Y}_1 \cap \mathcal{Y}_2| = \ell_1,\, |\mathcal{Y}_1 \cap \mathcal{Y}_3| = \ell_2,\, |\mathcal{Y}_2 \cap \mathcal{Y}_3| = \ell_3,\, |\mathcal{Y}_1 \cup \mathcal{Y}_2 \cup \mathcal{Y}_3| = 3k - \ell \big\} \Big\} \\ &\quad = C n^{3k-\ell} \mathbb{E}\Big\{ h_1(\mathcal{Y}_1)\, h_2(\mathcal{Y}_2)\, h_3(\mathcal{Y}_3) \\ &\qquad \times \mathbf{1}\big\{ |\mathcal{Y}_1 \cap \mathcal{Y}_2| = \ell_1,\, |\mathcal{Y}_1 \cap \mathcal{Y}_3| = \ell_2,\, |\mathcal{Y}_2 \cap \mathcal{Y}_3| = \ell_3,\, |\mathcal{Y}_1 \cup \mathcal{Y}_2 \cup \mathcal{Y}_3| = 3k - \ell \big\} \Big\} . \end{aligned}$$

Similarly, for $\ell_{i,j} \in \{0, \dots, k\}$, $m_{p,q,r} \in \{0, \dots, k\}$, $i, j, p, q, r \in \{1, 2, 3, 4\}$ with $i \ne j, p \ne q, p \ne r, q \ne r$, and $\ell \in \{0, \dots, 3k\}$, there exists a constant $C > 0$, which depends only on $\ell_{i,j}$, $m_{p,q,r}$, ℓ, and k such that

$$\begin{aligned} &\mathbb{E}\Big\{ \sum_{\mathcal{Y}_1 \subset \mathcal{P}_n} \sum_{\mathcal{Y}_2 \subset \mathcal{P}_n} \sum_{\mathcal{Y}_3 \subset \mathcal{P}_n} \sum_{\mathcal{Y}_4 \subset \mathcal{P}_n} h_1(\mathcal{Y}_1)\, h_2(\mathcal{Y}_2)\, h_3(\mathcal{Y}_3)\, h_4(\mathcal{Y}_4) \\ &\qquad \times \mathbf{1}\big\{ |\mathcal{Y}_i \cap \mathcal{Y}_j| = \ell_{i,j},\, i, j \in \{1, 2, 3, 4\}, i \ne j\,, \\ &\qquad\qquad |\mathcal{Y}_p \cap \mathcal{Y}_q \cap \mathcal{Y}_r| = m_{p,q,r},\, p, q, r \in \{1, 2, 3, 4\},\, p \ne q, p \ne q, q \ne r, \\ &\qquad\qquad |\mathcal{Y}_1 \cup \mathcal{Y}_2 \cup \mathcal{Y}_3 \cup \mathcal{Y}_4| = 4k - \ell \big\} \Big\} \\ &\quad = C n^{4k-\ell} \mathbb{E}\Big\{ h_1(\mathcal{Y}_1)\, h_2(\mathcal{Y}_2)\, h_3(\mathcal{Y}_3)\, h_4(\mathcal{Y}_4) \end{aligned}$$

$$\times \mathbf{1}\Big\{|\mathcal{Y}_i \cap \mathcal{Y}_j| = \ell_{i,j},\, i,j \in \{1,2,3,4\}, i \neq j\,,\\ |\mathcal{Y}_p \cap \mathcal{Y}_q \cap \mathcal{Y}_r| = m_{p,q,r},\, p,q,r \in \{1,2,3,4\},\, p \neq q, p \neq q, q \neq r,\\ |\mathcal{Y}_1 \cup \mathcal{Y}_2 \cup \mathcal{Y}_3 \cup \mathcal{Y}_4| = 4k - \ell\Big\}\,.$$

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