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Double roots of random polynomials with integer coefficients

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Abstract

We consider random polynomials whose coefficients are independent and identically distributed on the integers. We prove that if the coefficient distribution has bounded support and its probability to take any particular value is at most $\frac{1}{2}$, then the probability of the polynomial to have a double root is dominated by the probability that either 0, 1, or -1 is a double root up to an error of $o(n^{-2})$. We also show that if the support of the coefficients' distribution excludes 0, then the double root probability is $O(n^{-2})$. Our result generalizes a similar result of Peled, Sen and Zeitouni [13] for Littlewood polynomials.

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1 Introduction

Let $n \in \mathbb{N}$ and let $(\xi_j)_{0\leq j}$ be a sequence of independent and identically distributed random variables taking values in $\mathbb{Z}$. Define the random polynomial $P = P_n$ by

$$P(z) := \sum_{j=0}^{n} \xi_j z^j. \tag{1.1}$$

In a previous paper, Peled, Sen and Zeitouni [13] showed that if the random variables are supported on $\{-1, 0, +1\}$ with $\max_{x\in\{-1,0,1\}} \mathbb{P}(\xi_0 = x) < \frac{1}{\sqrt{3}}$, then the probability of P to have a double root in the complex plane is same as having a double root at $0, \pm 1$ up to an error of $o(n^{-2})$. In this paper, we extend the result for more general integer-valued random variables. Our main result is the following.

Theorem 1.1. *Suppose the coefficient distribution satisfies the following conditions.*

$$\textit{There exists constant } M \geq 1 \textit{ such that } \mathbb{P}(|\xi_0| \leq M) = 1. \tag{1.2}$$

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$$\max_{x\in\mathbb{Z}} \mathbb{P}\Big(\xi_0 = x\Big) \le \frac{1}{2}. \tag{1.3}$$

Then we have

$$\mathbb{P}\Big(P \text{ has a double root}\Big) = \mathbb{P}\Big(P \text{ has a double root at either } 0, -1 \text{ or } 1\Big) + o(n^{-2}), \tag{1.4}$$

as $n \to \infty$. *Moreover, if* $\mathbb{P}(\xi_0 = 0) = 0$, *then* $\mathbb{P}\Big(P \text{ has a double root}\Big) = O(n^{-2})$.

We make a few remarks about the above theorem.

(a) When $\mathbb{P}(\xi_0 = 0) = 0$, the upper bound in Theorem 1.1 is sharp. When ξ_i's are i.i.d. ± 1 symmetric Bernoulli and $(n+1)$ is divisible by 4, then it was shown in [13] that the probability of having a double root is $\Theta(n^{-2})$.

(b) We can have a better error bound if we allow the possibility of having double roots at other low-degree roots of unity. More precisely, our proof can be modified to show that for any fixed $d \ge 1$,

$$\begin{aligned}&\mathbb{P}\Big(P \text{ has a double root}\Big)\\ &= \mathbb{P}\Big(P \text{ has a double root at } 0 \text{ or some roots of unity of degree at most } d\Big) + o(n^{-2d}).\end{aligned}$$

(c) The bounded support Condition (1.2) can be weaken with minor modifications of our arguments. We did not pursue that here for the sake of simplicity. On the other hand, we do not know how to relax Condition (1.3) on the size of the maximum of atom and it seems that the current bound $\frac{1}{2}$ is a limitation of our proof. In fact, we believe that both conditions are unnecessary and that the result (1.4) should hold for any non-degenerate integer-valued coefficient distribution.

(d) Even though some parts of our proof closely follow the lines of arguments from the paper of Peled, Sen and Zeitouni [13], extending the result to general integer-valued coefficients, however, poses a few significant challenges. For example, to handle high-degree double roots, we need a key anti-concentration estimate for $P(\pm 2)$ given in the form of Theorem 1.2. When the coefficients are ± 1-valued, the map $(a_0, \dots, a_n) \mapsto \sum_{i=1} a_i 2^i : \{-1, 1\}^{n+1} \to \mathbb{Z}$ is one-to-one, which immediately implies the bound that

$$\mathbb{P}(P(\pm 2) = m) \le \big(\max_{x=\pm 1} \mathbb{P}(\xi_0 = x)\big)^{n+1}.$$

The paper [13] made use of the above simple observation. But for more general integer-valued coefficients, we lose such one-to-one property, which makes proving Theorem 1.2 nontrivial. One consequence of this difficulty is that the result here requires the maximal atom of the coefficient distribution to be at most $\frac{1}{2}$ while in [13] atoms up to $\frac{1}{\sqrt{3}}$ could be handled.

Moreover, the argument used in [13] to deal with low degree roots does not carry over either. In [13], P was always a monic polynomial and hence its roots were algebraic integers. For algebraic integers, one can use some partial result (see, e.g., Dobrowolski [5]) on Lehmer's conjecture to show that any non-zero algebraic integer is either a root of unity or has a conjugate which is a bit far (depending on its degree) away from unit circle. In low degree case, [13] made use of this dichotomy of algebraic integers. In contrast, in our case we also have to deal with non-monic P, so its roots are algebraic numbers in general. Such dichotomy is not available for algebraic numbers. For example, there are algebraic numbers which are not a root of unity and all of its conjugates lie on the unit circle. This requires new methods for handling such roots which are given in sections 5 and 6.

A key instrument in the proof of the theorem, which may be of independent interest, is the following anti-concentration bound.

Theorem 1.2. *Under Condition* (1.3) *there exists* $\varepsilon > 0$ *such that for all* $n \in \mathbb{N}$ *large enough we have*

$$\max_{m\in\mathbb{Z}} \mathbb{P}\Big(P(\pm 2) = m\Big) \le 2^{-n(\frac{1}{2}+\varepsilon)}.$$

We proceed as follows. In Section 1.1 we provide some notation and reduce Theorem 1.1 to several key lemmata. In Section 2 we prove Theorem 1.2. Each subsequent section is then dedicated to the proof of one of the key lemmata stated in Section 1.1.

1.1 Proof overview

Preliminaries. Recall that a real number α is called *algebraic* if it is a root of a polynomial with rational coefficients. Let $\mathbb{A}$ denote the set of algebraic numbers. The *minimal polynomial* of $\alpha \in \mathbb{A}$ is the unique least degree monic polynomial in $\mathbb{Q}[X]$ with a root at α. The *algebraic degree* of α is the degree of the minimal polynomial of α, which we denote by $\deg(\alpha)$. A real number α is said to be an *algebraic integer* if all the coefficients of its minimal polynomial are integers.

We define $\Lambda(\alpha)$, the *house* of α, by

$$\Lambda(\alpha) = \max_{j\in\{1,\dots,\deg(\alpha)\}} |\alpha_j|,$$

where $\alpha_1 = \alpha, \dots, \alpha_{\deg(\alpha)}$ are the conjugates of α, i.e., the roots of the minimal polynomial of α.

We further define the *associated minimal polynomial* of α in $\mathbb{Z}[x]$ to be the unique polynomial in $\mathbb{Z}[x]$ of degree $\deg(\alpha)$ with a root at α, whose leading coefficient is positive and whose coefficients are coprime.

Main lemmata. The proof of Theorem 1.1 breaks into several cases. In what follows in this subsection, we let P be as in Theorem 1.1 with coefficient distribution satisfying (1.2) and (1.3).

We first consider the probability of having a double root of algebraic degree and prove the following result.

Lemma 1.3 (high degree)**.** *Given any* $B > 0$*, there exists a constant* $C_0 > 0$ *such that*

$$\mathbb{P}(P \text{ has a double root } \alpha \text{ with } \deg(\alpha) \ge C_0 \log n) = O(n^{-B}).$$

The proof of Lemma 1.3 follows the line of arguments given in [13], which, in turn, was based on idea that appeared in a work of Filaseta and Konyagin [7]. However, several modifications are needed when dealing with general integer-valued coefficients. Most crucially, we need a new anti-concentration bound (Theorem 1.2) that consumes the bulk of our effort. Let us point out here that Theorem 1.2 is the only place where Assumption (1.3) is crucially used.

By virtue of Lemma 1.3, we now have to deal with potential double roots with low algebraic degree, more precisely, with degree at most $C_0 \log n$. In the next lemma we show that the probability that P has a root at an algebraic numbers of low degree such that one of its conjugates lying at a distance of at least $\Omega((\log n)^{-1})$ from the unit circle is negligible.

Lemma 1.4 (low degree roots far away from the unit circle)**.** *For every* $B > 0$ *and* $C_0 > 0$*, there exists* $C_1 > 0$ *such that*

$$\mathbb{P}\Big(P \text{ has a root } \alpha \ : \ \deg(\alpha) \le C_0 \log n \text{ and } \Lambda(\alpha) > 1 + \tfrac{C_1}{\log n}\Big) = O(e^{-\frac{Bn}{\log n}}).$$

For the proof, we use a simple sparsification of P to bound the root probability for each fixed low-degree algebraic number lying far away from the unit circle and then employ a rather crude union bound. After Lemma 1.4, we next deal with the low degree double roots with small house (i.e. all of their conjugates lying close to the unit circle). We break this into two cases. First we consider the case when the degree of the root is at least 5 and we show that

Lemma 1.5 (low degree roots close to the unit circle)**.** *For every $C_0 > 0$ and $C_1 > 0$, we have*

$$\mathbb{P}\left(P \text{ has a root } \alpha \ : \ 4 < \deg(\alpha) \le C_0 \log n \text{ and } \Lambda(\alpha) \le 1 + \tfrac{C_1}{\log n}\right) = o(n^{-2}).$$

From a standard application of inverse Littlewood-Offord type results, it follows that for any fixed algebraic number α of degree at least 5, $\mathbb{P}(P(\alpha) = 0) = O_\varepsilon(n^{-5/2+\varepsilon})$, for any $\varepsilon > 0$. This is shown in Lemma 5.1. More importantly, to prove Lemma 1.5, we need to count the number of algebraic numbers α such that $deg(\alpha) \le C_0 \log n$ and $\Lambda(\alpha) \le 1 + \frac{C_1}{\log n}$. Towards this direction, we show in Lemma 5.2 that they are at most $o(n^\varepsilon)$ in number for any $\varepsilon > 0$. The counting estimate makes heavy use of a result of Dubickas [6].

Finally, the next lemma takes care of the potential double roots that have degree at most 4 (excluding $0, \pm 1$) and have small house.

Lemma 1.6 (roots with degree at most 4)**.** *For every $C_0 > 0$ and $C_1 > 0$, we have*

$$\mathbb{P}\left(P \text{ has a double root } \alpha \neq 0, \pm 1 : \ \deg(\alpha) \le 4 \text{ and } \Lambda(\alpha) \le 1 + \tfrac{C_1}{\log n}\right) = o(n^{-2})$$

It is not hard to see that if α is a root of P for large enough n satisfying the conditions that $\deg(\alpha) = O(1)$ and $\Lambda(\alpha) = o(1)$, then it must be a unimodular root, i.e., all of the conjugates of α must lie on the unit circle. Now if α is a root of unity, we closely follow [13] to bound the probability of having a double root α which involves an application of an anti-concentration bound due to Sárközi and Szemerédi [14]. However, when α is unimodular but not a root of unity, we need a new argument to bound the probability of having a double root at α. In fact, in Lemma 6.1 we show that $\mathbb{P}(P(\alpha) = 0) = O(n^{-5/2})$. The argument relies on a powerful anti-concentration bound by Halász [8].

Clearly, the first assertion of Theorem 1.1 is an immediate consequence of lemmata 1.3, 1.4, 1.5, 1.6. To prove the second assertion of Theorem 1.1, note that since $P(\xi_0 = 0) = 0$, with probability one, P can not have a root at 0. So, we need to show that

$$\mathbb{P}(P \text{ has a double root at } \pm 1) = O(n^{-2}).$$

The above bound follows from an application of an inverse Littlewood-Offord result from [18, Theorem 2.5]. For details, see Lemma A.5 in [4] where the same has been proved under the assumption that ξ_0 has bounded $(2+\varepsilon)$ moment. This completes the proof of Theorem 1.1.

2 Anti-concentration of $P(\pm 2)$

In this section we prove Theorem 1.2. As an important first step, we will find a very useful a characterization of integer-valued measures with max-atom bounded by $\frac{1}{2}$ in terms of mixture of two-point distributions.

2.1 Bernoulli mixture

A probability measure μ is said to be a *(unbiased) Bernoulli measure* if $\mu = \frac{1}{2}\delta_a + \frac{1}{2}\delta_b$, where $a \neq b \in \mathbb{Z}$ and δ_x is the Dirac measure at x. A countable mixture of unbiased

Bernoulli measures is simply said to be a *Bernoulli mixture*. In other words, a probability measure μ is a Bernoulli mixture if it can be written as

$$\mu = \frac{1}{2}\sum_{i=1}^{\infty} t_i(\delta_{a_i} + \delta_{b_i})$$

where $t_i \geq 0$ satisfy $\sum_i t_i = 1$ and $a_i \neq b_i \in \mathbb{Z}$ for each i.

Note that if the distribution of a random variable ξ is a Bernoulli mixture, then there exists a random vector (I, Δ) on $\mathbb{Z} \times \mathbb{N}$, such that

$$\xi \stackrel{d}{=} I + B\Delta, \tag{2.1}$$

where B is a $\mathrm{Ber}(\frac{1}{2})$ random variable, independent from both I and Δ. With a slight abuse of notation, we will also call such a random variable ξ a Bernoulli mixture.

The following proposition gives a useful characterization for Bernoulli mixtures.

Proposition 2.1 (Bernoulli mixture)**.** *An integer-valued random variable ξ is a Bernoulli mixture if and only if it satisfies $\max_{x\in\mathbb{Z}} \mathbb{P}\big(\xi = x\big) \leq 1/2$.*

Clearly, the necessary part of Proposition 2.1 is trivial. Most of the reminder of Section 2 is dedicated to proving the sufficient part.

Let μ be a non-negative positive finite measure on $\mathbb{Z}$. It induces a unique total order $(\pi_i^\mu)_{i\in\mathbb{N}}$ on $\mathbb{Z}$ such that $w_i^\mu := \mu(\pi_i^\mu)$ are monotone non-increasing (i.e., $w_i^\mu \geq w_j^\mu$ if $i < j$) and $\pi_i^\mu < \pi_j^\mu$ if $w_i^\mu = w_j^\mu$. Then μ can be expressed as

$$\mu = \sum_{i\in\mathbb{Z}} w_i^\mu \delta_{\pi_i^\mu}.$$

We write $\mathcal{M}$ for the collection of non-negative finite measures μ on the integers (including the null measure), which satisfy $w_1^\mu \leq \mu(\mathbb{Z})/2$. Also, for any non-null finite measure μ on $\mathbb{Z}$, we denote by $\bar{\mu}$ the normalized probability measure $\bar{\mu}(\cdot) := \mu(\cdot)/\mu(\mathbb{Z})$.

To prove Proposition 2.1 we use the following couple of lemmata.

Lemma 2.2. *If $\mu \in \mathcal{M}$ is non-null and μ is supported on at most 3 integers, then $\bar{\mu}$ is a mixture of at most 3 Bernoulli measures.*

Proof. We write

$$\bar{\mu} = w_1\delta_{\pi_1} + w_2\delta_{\pi_2} + w_3\delta_{\pi_3}$$

where $w_1 \geq w_2 \geq w_3$ and $\sum_{i=1}^3 w_i = 1$. We then give the explicit decomposition:

$$\begin{aligned}\mu = (w_1 + w_2 - w_3)\Big(\tfrac{1}{2}\delta_{\pi_1} + \tfrac{1}{2}\delta_{\pi_2}\Big) + (w_1 + w_3 - w_2)\Big(\tfrac{1}{2}\delta_{\pi_1} + \tfrac{1}{2}\delta_{\pi_3}\Big)\\ + (w_2 + w_3 - w_1)\Big(\tfrac{1}{2}\delta_{\pi_2} + \tfrac{1}{2}\delta_{\pi_3}\Big).\end{aligned}$$

It is now straightforward to check that each of the weights is non-negative and that equality indeed holds. □

Lemma 2.3. *Let $k \geq 4$ be an integer. Every $\mu \in \mathcal{M}$ can be written as*

$$\mu = \nu + \beta,$$

where either β is the null measure or $\bar{\beta}$ is a Bernoulli measure, and $\nu \in \mathcal{M}$ and satisfies $\nu(\pi_k^\mu) = 0$.

Proof. Set $\beta := w_k^\mu \delta_{\pi_1^\mu} + w_k^\mu \delta_{\pi_k^\mu}$ and $\nu := \mu - \beta$. It only remains to check that $\nu \in \mathcal{M}$, that is, the fact that $w_1^\nu \le \nu(\mathbb{Z})/2$. To see this, observe that $\pi_1^\nu \in \{\pi_1^\mu, \pi_2^\mu\}$. If $\pi_1^\nu = \pi_1^\mu$, then, since $\mu \in \mathcal{M}$, we have

$$w_1^\nu = w_1^\mu - w_k^\mu \le \frac{\mu(\mathbb{Z})}{2} - w_k^\mu = \frac{\nu(\mathbb{Z})}{2}.$$

On the other hand, if $\pi_1^\nu = \pi_2^\mu$, we get that

$$w_1^\nu = w_2^\mu \le \frac{w_1^\mu + w_2^\mu + w_3^\mu - w_k^\mu}{2} = \frac{\nu(\pi_1^\mu) + \nu(\pi_2^\mu) + \nu(\pi_3^\mu)}{2} \le \frac{\nu(\mathbb{Z})}{2}.$$

The lemma follows. □

Proof of Proposition 2.1. Write μ_1 for the distribution of ξ. Define a decreasing sequence of finite measures $(\mu_i)_{i\in\mathbb{N}}$ on $\mathbb{Z}$ inductively as follows. Suppose μ_i has already been defined and $\mu_i \in \mathcal{M}$. An application of Lemma 2.3 to μ_i with $k = 4$ yields the decomposition $\mu_i = \beta_i + \mu_{i+1}$ with $\mu_{i+1}(\pi_4^{\mu_i}) = 0$ where β_i is either the null measure or $\bar{\beta}_i$ is a Bernoulli measure and $\mu_{i+1} \in \mathcal{M}$. This defines the measure μ_{i+1}. Since $(\mu_i)_{i\in\mathbb{N}}$ is a decreasing sequence of finite measures, it has a limiting measure (possibly null) which we denote by μ_∞. Thus we write

$$\mu = \sum_{i\in\mathbb{N}} \beta_i + \mu_\infty.$$

All that remains in order to prove the proposition is to show that μ_∞ is supported on at most 3 integers, and then apply Lemma 2.2.

To that end, assume, if possible that, there exists four distinct integers a_1, a_2, a_3, a_4 such that $\mu_\infty(a_i) > 0$ for all i. Set $c := \min\{\mu_\infty(a_i) : 1 \le i \le 4\} > 0$. For each $i \in \mathbb{N}$, define the set

$$L_i := \{x \in \mathbb{Z} : \mu_i(x) \ge c\}.$$

Since $\mu_i \downarrow \mu_\infty$, $L_i \supseteq L_{i+1}$ and $a_1, \dots, a_4 \in L_i$ for each i. Thus $\pi_4^{\mu_i} \in L_i$ and hence, by the definition of the measure μ_{i+1}, we have $L_i \subseteq L_{i+1} \setminus \{\pi_4^{\mu_i}\}$. This implies that $|L_{i+1}| < |L_i|$ for each i. Since $|L_1| < \infty$, this contradicts the fact that $|L_i| \ge 4$ for each i. Hence, μ_∞ is supported on at most 3 integers. □

Using Proposition 2.1 we may reduce Theorem 1.2 to the following proposition.

Proposition 2.4. *Let $(X_i)_{1\le i\le n}$ be i.i.d. random variables whose distribution is a Bernoulli mixture. Then there exists $\varepsilon > 0$ such that for $n \in \mathbb{N}$ large enough and every sign sequence $(\sigma_i)_{1\le i\le n}$ with $\sigma_i = \pm 1$, the following holds.*

$$\max_{m\in\mathbb{Z}} \mathbb{P}\big(\sum_{i=1}^n 2^i \sigma_i X_i = m\big) \le 2^{-n(\frac{1}{2}+\varepsilon)}.$$

2.2 Proof of Proposition 2.4

In this section we prove Proposition 2.4. Throughout the section we fix a sign sequence $(\sigma_i)_{1\le i\le n}$ with $\sigma_i = \pm 1$. In the course of the proof we shall make several claims whose proofs are given in sections 2.2 and 2.3.

For $k \in \mathbb{Z}$ we write $L(k)$ for the leading power of 2 in the factorization of k, i.e., $L(k) = \max\{l \in \mathbb{Z}_+ : 2^l \text{ divides } k\}$. Since $(X_i)_{1\le i\le n}$ are i.i.d. Bernoulli mixtures, following the representation (2.1), we can express X_i as

$$X_i = I_i + B_i \Delta_i,$$

where $(I_i, \Delta_i)_{1\le i\le n}$ are i.i.d. random vectors in $\mathbb{Z}\times\mathbb{N}$ and $(B_i)_{1\le i\le n}$ are i.i.d. $\mathrm{Ber}(\frac{1}{2})$ random variables, which are independent from $(I_i,\Delta_i)_{1\le i\le n}$. Define

$$p_{\max} := \max_{m\in\mathbb{Z}} \mathbb{P}\big(\sum_{i=1}^{n} 2^i\sigma_i X_i = m\big).$$

The following claim yields an useful upper bound on $p_{\max}$.

Claim 2.5. *Let $(B_i)_{1\le i\le n}$ be i.i.d. $\mathrm{Ber}(\frac{1}{2})$ random variables and let $(b_i)_{1\le i\le n}$ and $(d_i)_{1\le i\le n}$ be any two sequences of integers. Then*

$$\max_{m\in\mathbb{Z}} \mathbb{P}\Big(\sum_{i=1}^{n} b_i + d_iB_i = m\Big) \le 2^{-|\{L(d_i)\,:\,1\le i\le n\}|}.$$

Proof of Claim 2.5. Let $k := |\{L(d_i) \;:\; 1\le i\le n\}|$. Let us assume, without loss of generality, that the values of $L(d_1), L(d_2), \dots, L(d_k)$ are distinct and moreover, $L(d_1) < L(d_2) < \cdots < L(d_k)$. Now, by conditioning on the random variables $B_{k+1}, B_{k+2}, \dots, B_n$, we have

$$\max_{m\in\mathbb{Z}} \mathbb{P}\Big(\sum_{i=1}^{n} b_i + d_iB_i = m\Big) = \max_{m\in\mathbb{Z}} \mathbb{P}\Big(\sum_{i=1}^{n} d_iB_i = m\Big) \le \max_{m\in\mathbb{Z}} \mathbb{P}\Big(\sum_{i=1}^{k} d_iB_i = m\Big).$$

It would now suffice to show that for all $m\in\mathbb{Z}$ we have

$$\mathbb{P}\big(\sum_{i=1}^{k} d_iB_i = m\big) \le 2^{-k}.$$

To see this, it would be enough to show that $\sum_{i=1}^{k} d_iB_i$ takes distinct values for every choice of values of $(B_i)_{1\le i\le k}$ in $\{0,1\}^k$. Indeed, let $(a_i)_{1\le i\le k}$ and $(a'_i)_{1\le i\le k}$ be two distinct vectors of $\{0,1\}^k$, and let $r = \min\{i\in\mathbb{N} : a_i \ne a'_i\}$. By definition,

$$\sum_{i=1}^{k} d_ia_i \not\equiv \sum_{i=1}^{k} d_ia'_i \pmod{2^{L(d_r)+1}},$$

and therefore the corresponding sums are distinct. □

Applying Claim 2.5 we have,

$$p_{\max} \le \sum_{s=1}^{n} \mathbb{P}\Big(\big|\{L(2^i\sigma_i\Delta_i) \;:\; 1\le i\le n\}\big| = s\Big)2^{-s}.$$

Observing that $L(2^i\sigma_i\Delta_i) = L(2^i\Delta_i)$ for all i, it would suffice to show that for large enough n we have,

$$\sum_{s=1}^{n} \mathbb{P}\Big(\big|\{L(2^i\Delta_i) \;:\; 1\le i\le n\}\big| = s\Big)2^{-s} \le 2^{-n(1/2+\varepsilon)}.$$

Here and in the rest of the proof we let ε be a small positive constant, chosen to satisfy various constraints which are specified along the proof.

Taking $w_i := L(\Delta_i)$ for $1\le i\le n$, and $W := |\{i+w_i \;:\; 1\le i\le n\}|$, we may rewrite the above inequality as

$$\sum_{s=1}^{n} \mathbb{P}\big(W = s\big)2^{-s} \le 2^{-n(1/2+\varepsilon)}. \tag{2.2}$$

By rewriting the LHS of (2.2) as

$$\sum_{1\le s<n(1/2+\varepsilon)} \mathbb{P}\big(W=s\big)2^{-s} + \sum_{n(1/2+\varepsilon)\le s\le n} \mathbb{P}\big(W=s\big)2^{-s},$$

we observe that

$$\sum_{s=1}^{n} \mathbb{P}\big(W=s\big)2^{-s} < n\left(\max_{\alpha\in(0,1/2+\varepsilon)} \mathbb{P}\big(W=\alpha n\big)2^{-\alpha n} + 2^{-n(1/2+\varepsilon)}\right). \tag{2.3}$$

Plugging (2.3) into (2.2) we get that it would be enough to show the existence of $\varepsilon>0$ such that for large enough n,

$$\max_{\alpha\in(0,1/2+\varepsilon)} \mathbb{P}\big(W=\alpha n\big)2^{-\alpha n} \le 2^{-n(1/2+\varepsilon)}.$$

Multiplying both sides by $2^{n/2}$ it reduces to showing that for n sufficiently large,

$$\max_{\alpha\in(0,1/2+\varepsilon)} \mathbb{P}\big(W=\alpha n\big)2^{n(1/2-\alpha)} \le 2^{-\varepsilon n}. \tag{2.4}$$

In order to show (2.4), we use the following lemma.

Lemma 2.6. *Let $(w_i)_{1\le i\le n}$ be i.i.d. non-negative integer-valued random variables, and define $W=|\{i+w_i \;:\; 1\le i\le n\}|$. Then the following holds.*

(a) For any $\alpha\in(0,1)$ with $\alpha n\in\mathbb{N}$, we have

$$\mathbb{P}\Big(W=\alpha n\Big) \le \binom{n}{\alpha n}\alpha^n \le \left(\frac{\alpha}{1-\alpha}\right)^{n(1-\alpha)}.$$

(b) Furthermore, there exists $\delta',\varepsilon'>0$ depending on α and the law of w_1 such that if $\alpha\in(1/2-\delta',1/2+\delta')$, then

$$\mathbb{P}\Big(W=\alpha n\Big) \le e^{-\varepsilon' n}\left(\frac{\alpha}{1-\alpha}\right)^{n(1-\alpha)}.$$

Proving Lemma 2.6 is the main technical step in the proof of Proposition 2.4, and we devote Section 2.3 to its proof.

The following claim captures two technical properties of the bound obtained in Lemma 2.6.

Claim 2.7. *For $n\in\mathbb{N}$, we define a function $f_n:(0,1)\to\mathbb{R}_+$ as*

$$f_n(\alpha) := \left(\frac{\alpha}{1-\alpha}\right)^{n(1-\alpha)} 2^{n(1/2-\alpha)}.$$

Then the following hold.

(a) There exists $c_0>1/2$ such that $f_n(\alpha)$ is strictly monotone increasing in $(0,c_0)$.

(b) Let c_0 be as in part (a). Then for any $c>0$ there exists $0<\delta<c_0-\frac{1}{2}$ such that $f_n(\frac{1}{2}+\delta)<2^{cn}$.

Proof of Claim 2.7. Notice that $f_n(\alpha)=f_1(\alpha)^n$, and $f_1(\alpha)>0$ for all $\alpha\in(0,1)$. For Part (a) it is therefore enough to show that

$$f_1(\alpha) := \left(\frac{\alpha}{1-\alpha}\right)^{(1-\alpha)} 2^{(1/2-\alpha)}$$

is strictly monotone increasing. Taking logarithm it is enough to show that

$$g(\alpha) := \log f_1(\alpha) = (1-\alpha)(\log\alpha - \log(1-\alpha)) + (\frac{1}{2} - \alpha)\log 2,$$

is strictly monotone increasing. Differentiate g to get

$$g'(\alpha) = -\log\alpha + \log(1-\alpha) + \frac{1}{\alpha} - \log 2.$$

For $\alpha \le \frac{1}{2}$, $\log(1-\alpha) > \log\alpha$ and $\frac{1}{\alpha} > \log 2$, and thus $g'(\alpha) > 0$. By continuity of g' at $\alpha = \frac{1}{2}$, there exists $c_0 > \frac{1}{2}$ such that $f'(\alpha) > 0$ also for $\alpha \in [\frac{1}{2}, c_0)$, as required.

For part (b), notice that $f_1(\frac{1}{2}) = 1$. Let $c > 0$ be given. By continuity of f_1, there exists $\delta \in (0, c_0 - 1/2)$ such that

$$f_1\left(\tfrac{1}{2} + \delta\right) < 2^c.$$

Thus, for all $n \in \mathbb{N}$ we have $f_n(\frac{1}{2}+\delta) = f_1(\frac{1}{2}+\delta)^n < 2^{cn}$, as required. □

Finally we are fully equipped to demonstrate the existence of $\varepsilon > 0$ such that (2.4) holds. Let δ', ε' be as in part (b) of Lemma 2.6 and let c_0 be as in part (a) of Claim 2.7. By part (b) of Claim 2.7, applied to $c = \varepsilon'/2$, there exists $\delta \in (0, c_0 - \frac{1}{2})$ such that $f_n\big(\frac{1}{2}+\delta\big) < 2^{\frac{\varepsilon' n}{2}}$. We take $\varepsilon := \min(c_0 - \frac{1}{2}, \delta', \delta, \frac{\varepsilon'}{2})$. We are thus left with verifying (2.4).

Applying Part (a) of Lemma 2.6 and part (a) of Claim 2.7, we obtain

$$\begin{aligned} I_1 &:= \max_{\alpha\in(0,1/2-\varepsilon]} \mathbb{P}\big(W = \alpha n\big) 2^{n(1/2-\alpha)} \le \max_{\alpha\in(0,1/2-\varepsilon]} f_n(\alpha) = f_n\big(\tfrac{1}{2}-\varepsilon\big) \\ &= 2^{n\left((\frac{1}{2}+\varepsilon)\log_2\left(\frac{\frac{1}{2}-\varepsilon}{\frac{1}{2}+\varepsilon}\right)+\varepsilon\right)} < 2^{n\left((\frac{1}{2}+\varepsilon)\log_2\left(1-\frac{2\varepsilon}{\frac{1}{2}+\varepsilon}\right)+\varepsilon\right)} < 2^{-\left(\frac{2}{\log 2}-1\right)\varepsilon n} < 2^{-\varepsilon n}, \end{aligned} \tag{2.5}$$

using the inequality $\log_2(1-x) < -\frac{x}{\log 2}$ for $x > 0$. From part (b) of Lemma 2.6 we obtain

$$\begin{aligned} I_2 &:= \max_{\alpha\in(1/2-\varepsilon,1/2+\varepsilon)} \mathbb{P}\big(W_n = \alpha n\big) 2^{n(1/2-\alpha)} \le 2^{-\varepsilon' n} \max_{\alpha\in(1/2-\varepsilon,1/2+\varepsilon)} f_n(\alpha) = 2^{-\varepsilon' n} f_n\big(\tfrac{1}{2}+\varepsilon\big) \\ &\le 2^{-\varepsilon' n} f_n\big(\tfrac{1}{2}+\delta\big) \le 2^{-\varepsilon' n} 2^{\frac{\varepsilon' n}{2}} = 2^{-\frac{\varepsilon' n}{2}} \le 2^{-\varepsilon n}. \end{aligned}$$

Therefore $\max(I_1, I_2) < 2^{-\varepsilon n}$ and we obtain (2.4), as required. □

We remark that if our interest was limited to obtaining the theorem for the case $\varepsilon = 0$, it would have been possible to use only the first part of Lemma 2.6, which is, as will become evident, easier to obtain.

2.3 Proof of Lemma 2.6

In this section we prove Lemma 2.6. In the proof we keep using the notation introduced in the previous section. We assume $\alpha n \in \mathbb{N}$. Let $\mathbb{Z}_n := \{0, 1, 2, \ldots, n-1\}$.

2.3.1 Proof of item (a)

In order to bound $\mathbb{P}\Big(W = \alpha n\Big)$, we use

$$\mathbb{P}\Big(W = \alpha n\Big) \le \mathbb{P}\Big(\big|\{i + w_i \pmod n \ : \ 1 \le i \le n\}\big| \le \alpha n\Big) \tag{2.6}$$

For a set $A \subset \mathbb{Z}_n$, we write

$$U(A) := \mathbb{P}\Big(\{i + w_i \pmod n \ : \ 1 \le i \le n\} \subset A\Big). \tag{2.7}$$

We then intend to show the following.

$$\text{For every } A \subset \mathbb{Z}_n \text{ of size } |A| = \alpha n, \text{ we have } U(A) \leq \alpha^n. \tag{2.8}$$

Part (a) of Lemma 2.6 would follow from (2.8) since

$$\mathbb{P}\Big(W = \alpha n\Big) \leq \sum_{A:|A|=\alpha n} U(A) \leq \binom{n}{\alpha n}\alpha^n \leq \left(\frac{\alpha}{1-\alpha}\right)^{n(1-\alpha)}, \tag{2.9}$$

where the leftmost inequality uses (2.6), the middle one uses (2.8) and a union bound, and the rightmost one follows from the well-known inequality of the binomial coefficient $\binom{n}{k} \leq \frac{n^n}{k^k(n-k)^{n-k}}$.

Towards showing (2.8), let $A \subset \mathbb{Z}_n$ be a set of size $|A| = \alpha n$. Observe that, by the fact that w_i's are i.i.d., we have

$$\mathbb{P}\big(\{i + w_i \pmod n \ : \ 1 \leq i \leq n\} \subset A\big) = \prod_{i=1}^{n} \mathbb{P}\big(i + w_i \pmod n \in A\big).$$

We further observe that for every $a \in A$, we have

$$\begin{aligned}\sum_{i=1}^{n} \mathbb{P}\Big(i + w_i \equiv a \pmod n\Big) &= \sum_{i=1}^{n} \mathbb{P}\Big(w_i \equiv a - i \pmod n\Big) \\ &= \sum_{i=1}^{n} \mathbb{P}\Big(w_1 \equiv a - i \pmod n\Big) = 1.\end{aligned}$$

Writing

$$u_i = u_i(A) := \mathbb{P}\big(i + w_i \pmod n \in A\big), \tag{2.10}$$

we get that

$$\sum_{i=1}^{n} u_i = \sum_{a\in A}\left(\sum_{i=1}^{n} \mathbb{P}\Big(i + w_i \equiv a \pmod n\Big)\right) = |A| = \alpha n. \tag{2.11}$$

We now solve the following maximization problem:

$$\text{maximize } U(A) := \prod_{i=1}^{n} u_i, \text{ under the constraints } u_i \in [0,1], \ \sum_{i=1}^{n} u_i = \alpha n. \tag{2.12}$$

By applying Jensen's inequality to the $\log$ function, we get

$$\log U(A) = \sum_{i=1}^{n} \log u_i \leq n \cdot \log\left(\frac{\sum_{i=1}^{n} u_i}{n}\right) \leq n \log \alpha,$$

and so

$$U(A) \leq \alpha^n, \tag{2.13}$$

as required. □

2.3.2 Proof of item (b)

To show part (b) of Lemma 2.6 it would suffice to show that

$$\exists \delta, \varepsilon > 0, \exists n_0 \in \mathbb{N} \text{ s.t. every } \alpha \in (1/2 - \delta, 1/2 + \delta), \ n > n_0 \text{ satisfy } \mathbb{P}\big(W = \alpha n\big) < e^{-\varepsilon n}. \tag{2.14}$$

To do so we shall use concentration arguments. We begin by showing that

$$\mathbb{E}[W] \geq (\tfrac{1}{2} + \eta)n \tag{2.15}$$

for some $\eta > 0$. To this end write

$$W = \sum_{i=1}^{n} \mathbb{1}\{\forall j < i : w_i + i \neq w_j + j\},$$

and observe that

$$\begin{aligned}\mathbb{P}\big(\forall j < i : w_i + i \neq w_j + j\big) &= \sum_{k \in \mathbb{Z}_+} \mathbb{P}(w_i = k) \prod_{j=1}^{i-1} \mathbb{P}(w_j \neq k + i - j) \\ &\geq \sum_{k \in \mathbb{Z}_+} \mathbb{P}(w_1 = k) \prod_{j=1}^{\infty} \mathbb{P}(w_1 \neq k + j).\end{aligned}$$

Letting $p_i := \mathbb{P}(w_1 = i)$, we have

$$\begin{aligned}\mathbb{E}[W] &\geq n \sum_{k \in \mathbb{Z}_+} \mathbb{P}(w_1 = k) \prod_{j=1}^{\infty} \mathbb{P}(w_1 \neq k + j) \geq n \sum_{k \in \mathbb{Z}_+} p_k \Big(1 - \sum_{j \in \mathbb{N}} p_{k+j}\Big) \\ &= n\Big(1 - \sum_{k < \ell \in \mathbb{Z}_+} p_k p_\ell\Big) = \frac{n}{2}\Big(1 + \sum_{k \in \mathbb{Z}_+} p_k^2\Big).\end{aligned} \tag{2.16}$$

Thus (2.15) is satisfied with $\eta := \frac{1}{2}\sum_{k \in \mathbb{N}} p_k^2$.

Next, we show that W is concentrated around its expectation. To this end we use the concentration properties of self-bounding functions of independent variables. We write $f(w_1, \ldots, w_n) := W$, $g_i(w_1, \ldots, w_{i-1}, w_{i+1}, \ldots, w_n) := |\{w_j + j \ : \ 1 \leq j \leq n, \ j \neq i\}|$, and observe that for all $i \leq n$ we have,

$$f(w_1, \ldots, w_n) - g_i(w_1, \ldots, w_{i-1}, w_{i+1}, \ldots, w_n) \leq 1.$$

$$\sum_{i=1}^{n} \Big(f(w_1, \ldots, w_n) - g_i(w_1, \ldots, w_{i-1}, w_{i+1}, \ldots, w_n)\Big) \leq f(w_1, \ldots, w_n).$$

We then apply [3, Theorem 1 & 7], to obtain that for every $\beta > 0$

$$\mathbb{P}\Big(W \leq \mathbb{E}[W] - n\beta\Big) \leq e^{-\beta^2 \frac{n^2}{2\mathbb{E}[W]}} \leq e^{-n\frac{\beta^2}{2}}.$$

Setting $\delta = \frac{\eta}{2}$ and $\varepsilon = \frac{\eta^2}{8}$ and using (2.15) we observe that for every $\alpha < \frac{1}{2} + \delta \leq \frac{\mathbb{E}[W]}{n} - \frac{\eta}{2}$, we have

$$\mathbb{P}\big(W = \alpha n\big) < \mathbb{P}\Big(W \leq \mathbb{E}[W] - \tfrac{\eta n}{2}\Big) \leq e^{-n\frac{\eta^2}{8}} \leq 2^{-\varepsilon n}.$$

This proves (2.14) and hence completes the proof of part (b) of Lemma 2.6. □

3 High algebraic degree

This section is dedicated to the proof of the following proposition, of which Lemma 1.3 is a straightforward consequence.

Proposition 3.1. *For any constant $B > 0$, there exist constants $c, C, C' > 0$, depending on M, such that for any $1 \leq d \leq n$,*

$$\mathbb{P}(P \text{ has a double root } \alpha \text{ with } \deg(\alpha) \geq d) \leq Cn^{C'} \exp(-cd) + Cn^{-B}.$$

The proof of the proposition relies on the following consequence of Theorem 1.2.

Lemma 3.2. *Let P be the random polynomial as in* (1.1)*. Then there exist constants $C, \varepsilon > 0$ such that for any positive integer k and for $a \in \{-2, 2\}$ we have*

$$\mathbb{P}\left(P(a) \text{ is divisible by } k^2\right) \leq Ck^{-(1+\varepsilon)}.$$

Proof. Fix $a \in \{-2, 2\}$. Let $k \geq 1$ be an integer and let r be the integer satisfying $M2^r \leq k^2 < M2^{r+1}$. By conditioning on $\xi_r, \xi_{r+1}, \ldots, \xi_n$ we have

$$\mathbb{P}\left(P(a) \bmod k^2 = 0\right) \leq \max_{m\in\mathbb{Z}} \mathbb{P}\left(\sum_{j=0}^{r-1} \xi_j a^j \bmod k^2 = m\right) = \max_{m\in\mathbb{Z}} \mathbb{P}\left(\sum_{j=0}^{r-1} \xi_j a^j = m\right), \quad (3.1)$$

where the last equality follows from the fact that $\left|\sum_{j=0}^{r-1} \xi_j a^j\right| \leq M(2^r - 1)$ deterministically and $k^2 \geq M2^r$ by the definition of r. From Theorem 1.2, it follows that there exists a constant $\varepsilon \in (0, 1)$ such that

$$\max_{m\in\mathbb{Z}} \mathbb{P}\left(\sum_{j=0}^{r-1} \xi_j a^j = m\right) \leq \left(\frac{1}{\sqrt{2}}\right)^{r(1+\varepsilon)}. \quad (3.2)$$

Combining (3.1) and (3.2) with the fact that $r > 2\log_2 k - \log_2 M - 1$, we conclude that

$$\mathbb{P}\left(P(a) \bmod k^2 = 0\right) \leq 2^{1+\log_2 M}\left(\frac{1}{\sqrt{2}}\right)^{2(1+\varepsilon)\log_2 k} = 2^{1+\log_2 M} k^{-(1+\varepsilon)}. \qquad \square$$

We shall also use the following bound on the probability that P has a root in close proximity to ± 2, whose proof we postpone to Section 3.1.

Denote by $B(z_0, r)$ the closed ball in $\mathbb{C}$ with center at z_0 and radius r.

Lemma 3.3. *For any constant $B > 0$, there exists $K > 0$ such that*

$$\mathbb{P}\Big(P \text{ has a zero in } B(2, n^{-K}) \cup B(-2, n^{-K})\Big) = O(n^{-B}).$$

Finally, we need a preliminary claim, bounding the number of roots far away from the unit circle.

Claim 3.4. *Let $M \in \mathbb{N}$. For any $n \geq 1$ and any non-zero polynomial f in $\mathbb{Z}[x]$ of the form $f(z) = \sum_{i=0}^{n} a_i z^i$ with $|a_i| \leq M$ for all $0 \leq i \leq n$, the number of zeros of f with modulus at least $\frac{3}{2}$ is at most $64M$.*

Proof. Assume, without loss of generality, that $|a_n| \neq 0$. Let $\tilde{f}(z) = z^n f(z^{-1}) = \sum_{i=0}^{n} a_i z^{n-i}$ be the reciprocal polynomial of f. Denote by $N(f)$ the number of $z \in \mathbb{C}$ for which $f(z) = 0$ and $|z| \geq \frac{3}{2}$. Then $N(f)$ is also the number of $z \in \mathbb{C}$ for which $\tilde{f}(z) = 0$ and $|z| \leq \frac{2}{3}$. Noting that $|\tilde{f}(0)| = |a_n| \geq 1$ we may apply Jensen's formula (see, e.g., [1, Chapter 5.3.1]) and obtain for any $r > \frac{2}{3}$ that

$$\begin{aligned}\max_{0\leq\theta\leq 2\pi} \log|\tilde{f}(re^{i\theta})| &\geq \frac{1}{2\pi}\int_0^{2\pi} \log|\tilde{f}(re^{i\theta})|d\theta \\ &= \log|\tilde{f}(0)| + \sum_{\substack{z:\ \tilde{f}(z)=0,\\ |z|\leq r}} \log\left(\frac{r}{|z|}\right) \geq N(f)\log\left(\frac{r}{2/3}\right).\end{aligned}$$

Observe that when $r < 1$ we have $|\tilde{f}(re^{i\theta})| \leq \frac{M}{1-r}$ for all θ. Thus

$$N(f) \leq \frac{M}{(1-r)\log(3r/2)}, \quad \frac{2}{3} < r < 1$$

and substituting $r = 0.8$, say, we obtain that $N(f) \leq 64M$, as required. $\square$

Proof of Proposition 3.1. Fix $1 \le d \le n$. Let α be an algebraic number of degree $\deg(\alpha) = d$ and let f_α be the associated minimal polynomial of α in $\mathbb{Z}[x]$. Suppose that α is a double root of P. Note that α cannot be a multiple root of f_α, since, otherwise, α is also a root of the polynomial f'_α whose degree is strictly smaller than d, violating the definition of $\deg(\alpha)$. This implies that f_α^2 divides P in $\mathbb{Z}[x]$ (by Gauss's lemma). In particular,

$$\text{the integer } P(a) \text{ is divisible by } f_\alpha(a)^2, \quad \text{for } a = \pm 2. \tag{3.3}$$

Next we obtain a suitable lower bound for $\max\{|f_\alpha(2)|, |f_\alpha(-2)|\}$. Denote by $C(\alpha)$ the set of algebraic conjugates of α (i.e., the set of roots of f_α). Each of these conjugates of α must also be a root of P. So, by Claim 3.4, all but at most $64M$ of the $\beta \in C(\alpha)$ satisfy $|\beta| \ge \frac{3}{2}$. Therefore, we have

$$\begin{aligned} |f_\alpha(-2)| \cdot |f_\alpha(2)| &= \prod_{\beta \in C(\alpha)} |\beta + 2| \cdot |\beta - 2| \\ &\ge \left(\prod_{\beta \in C(\alpha), |\beta| \le 3/2} |\beta^2 - 4| \right) \left(\min_{\beta \in C(\alpha)} |\beta + 2| \wedge 1 \right)^{64M} \left(\min_{\beta \in C(\alpha)} |\beta - 2| \wedge 1 \right)^{64M}. \end{aligned}$$

Let $B > 0$ be given as in Proposition 3.1 and let $K = K(B) > 0$ be as given by Lemma 3.3. Let $\mathcal{E}$ be the event that there is at least one root of P within a distance of n^{-K} from either -2 or 2. Note that the event $\mathcal{E}$ does not depend on α. On the event $\mathcal{E}^c$,

$$\min_{\beta \in C(\alpha)} |\beta - a| \ge \min_{z : P(z) = 0} |z - a| \ge n^{-K} \quad \text{for any } a \in \{-2, 2\}.$$

On the other hand, $|\beta^2 - 4| \ge \frac{7}{4}$ for any $|\beta| \le \frac{3}{2}$. Putting these ingredients together, we conclude that on the event $\mathcal{E}^c$,

$$|f_\alpha(-2)| \cdot |f_\alpha(2)| \ge \left(\tfrac{7}{4}\right)^{d - 64M} n^{-128KM}.$$

Consequently, we obtain the following lower bound

$$\max\left\{|f_\alpha(2)|, |f_\alpha(-2)|\right\} \ge c_1 \exp(c_2 d) n^{-C_1}, \quad \text{on } \mathcal{E}^c, \tag{3.4}$$

where $c_1 := (\frac{7}{4})^{-32M} > 0, c_2 := \frac{1}{2}\log(\frac{7}{4})$ and $C_1 := 64KM$. From (3.3) and (3.4), we arrive at the inclusion of events

$$\{\alpha \text{ is a double root of } P\} \subseteq \mathcal{E} \cup G_2 \cup G_{-2},$$

where $G_a = \left\{P(a) \text{ is divisible by } k^2 \text{ for some integer } k \ge c_1 e^{c_2 d} n^{-C_1}\right\}$ for $a = -2$ or 2. By Lemma 3.3, $\mathbb{P}(\mathcal{E}) = O(n^{-B})$. On other hand, by Lemma 3.2, we deduce that

$$\mathbb{P}(G_a) \le C_2 (e^{c_2 d} n^{-C_1})^{-\varepsilon} = C_2 e^{-c_3 d} n^{C_3},$$

for suitable constants $c_3, C_2, C_3 > 0$. Proposition 3.1 follows. □

3.1 Roots near ± 2

In this section we prove Lemma 3.3. We shall require the following.

Lemma 3.5. *For any constant $B > 0$, there exists $C > 0$ such that for $a \in \{-2, 2\}$,*

$$\mathbb{P}(|P(a)| \le n^{-C} 2^n) = O(n^{-B}).$$

Proof. We prove the lemma for the case $a = 2$, as the argument for the case $a = -2$ is nearly identical. Set $C_1 = \lceil \log_3 M \rceil$. Define a subset of indices J as

$$J = \{j \in \{0, 1, \dots, n\} : j \ge n - C_1 \log_2 n, \text{ and } j \text{ is divisible by } \lceil \log_2(2M+1) \rceil\}.$$

By conditioning on the random variables $\xi_j, j \notin J$, we deduce that

$$\mathbb{P}(|P(2)| \le n^{-C}2^n) \le \sup_{z\in\mathbb{R}} \mathbb{P}\left(\Big| \sum_{j\in J} \xi_j 2^j - z \Big| \le n^{-C}2^n \right).$$

Note that for any two different values of the random vector $(\xi_j)_{j\in J}$ in $\{0, \pm 1, \dots, \pm M\}^{|J|}$, the values of the sum $\sum_{j\in J} \xi_j 2^j$ differ by at least $\frac{1}{2}2^{n-C_1 \log_2 n} = \frac{1}{2}n^{-C_1}2^n$. Thus if we choose $C = C_1 + 1$, then for any fixed $z \in \mathbb{R}$, there exists at most one value of the random vector $(\xi_j)_{j\in J}$ in $\{0, \pm 1, \dots, \pm M\}^{|J|}$ such that $|\sum_{j\in J} \xi_j 2^j - z| \le n^{-C}2^n$. Now by Assumption 1.3, we conclude that

$$\mathbb{P}\left(\Big| \sum_{j\in J} \xi_j 2^j - z \Big| \le n^{-C}2^n \right) \le 2^{-|J|} \le 2^{-\lfloor (4M)^{-1} C_1 \log_2 n - 1 \rfloor} = O(n^{-B}). \qquad \square$$

Proof of Lemma 3.3. By Lemma 3.5, there exists a constant $C > 0$ such that

$$\mathbb{P}(|P(\pm 2)| \ge n^{-C}2^n) = 1 - O(n^{-B}). \tag{3.5}$$

By the Mean Value Theorem and the triangle inequality, for any $z \in \mathbb{C}$ such that $|z| \le n^{-1}$,

$$|P(2+z)| \ge |P(2)| - \sup_{w\in B(2,n^{-1})} |P'(w)| \cdot |z|. \tag{3.6}$$

We can now bound the derivative of the polynomial P' in $B(2, n^{-1})$ by

$$\sup_{w\in B(2,n^{-1})} |P'(w)| \le \sum_{i=0}^{n} Mi(2+n^{-1})^{i-1} \le 3Mn2^n. \tag{3.7}$$

Plugging in the bound (3.5) and (3.7) in (3.6), we deduce that, for any $|z| \le n^{-(C+2)}$ and for sufficiently large n,

$$|P(2+z)| \ge n^{-C}2^n - 3Mn2^n \cdot n^{-(C+2)} > 0,$$

with probability $1 - O(n^{-B})$. The lemma is then obtained by taking $K = C + 2$. $\square$

4 Roots far from the unit circle

In this section we prove Lemma 1.4. We begin by obtaining the following bound on the probability that P has a particular root α far from the unit circle.

Lemma 4.1. *For each algebraic number $\alpha \in \mathbb{A}$, we have*

$$\mathbb{P}\Big(P(\alpha) = 0\Big) \le \exp\Big(-\frac{n \log 2}{\lceil \log(M+1)/|\log|\alpha|| \rceil}\Big).$$

Proof. Assume that $|\alpha| > 1$. Let ℓ be the minimal positive integer for which

$$|\alpha|^\ell > M + 1. \tag{4.1}$$

Write $P(z) = P_1(z) + P_2(z)$ with

$$P_1(z) := \sum_{k=0}^{\lfloor n/\ell \rfloor} \xi_{k\ell} z^{k\ell} \quad \text{and} \quad P_2(z) := P(z) - P_1(z).$$

Since $|\xi_i| \le M$ for all i, the map

$$\big(\xi_0, \xi_\ell, \xi_{2\ell}, \dots, \xi_{\lfloor n/\ell \rfloor \ell}\big) \mapsto P_1(\alpha) : \{-M, \dots, M\}^{\lfloor n/\ell \rfloor + 1} \to \mathbb{C}$$

is one-to-one. Thus, as $P_1(\alpha)$ and $P_2(\alpha)$ are independent, we have

$$\mathbb{P}\Big(P(\alpha)=0\Big) = \mathbb{E}\left[\mathbb{P}\Big(P(\alpha)=0\Big) \mid P_2(\alpha))\right] = \\ = \mathbb{E}\left[\mathbb{P}\Big(P_1(\alpha)=-P_2(\alpha) \mid P_2(\alpha)\Big)\right] \le \left(\max_{x\in\{-M,\dots,M\}} \mathbb{P}\Big(\xi_0=x\Big)\right)^{\lfloor n/\ell\rfloor+1}.$$

Assumption (1.3) and the definition of ℓ imply that

$$\mathbb{P}\Big(P(\alpha)=0\Big) < \left(\tfrac{1}{2}\right)^{\lfloor n/\ell\rfloor+1} \le e^{-\frac{n\log 2}{\ell}} = e^{-\frac{n\log 2}{\lceil \log(M+1)/\log|\alpha|\rceil}}.$$

The case when $|\alpha|<1$ can be handled similarly. This proves the lemma. □

We also make the following simple observation.

Observation 4.1. Let α be a root of P. Then

1. The leading coefficient of the *associated minimal polynomial* of α in $\mathbb{Z}[x]$ is at most M,
2. $\Lambda(\alpha)\le M+1$.

Proof. Write f_α for the associated minimal polynomial of α in $\mathbb{Z}[x]$ and denote the leading coefficient of f_α by m_α. By Gauss's lemma (see, e.g., [2, Proposition 11.3.4]), $f_\alpha | P$ in $\mathbb{Z}[x]$ and, in particular, m_α, the leading coefficient of P, divides ξ_n. Since $|\xi_n|\le M$, we get that $m_\alpha \le M$.

The fact that $\Lambda(\alpha)\le M+1$ is a direct consequence of Rouché Theorem. □

Proof of Lemma 1.4. Let us first estimate the number of algebraic numbers α such that α is a root of some random polynomial P of degree n and $\deg(\alpha)\le C_0\log n$. Write f_α for the associated minimal polynomial of α in $\mathbb{Z}[x]$ and denote, as usual, the leading coefficient of f_α by m_α. If $\alpha_1=\alpha,\alpha_2,\cdots,\alpha_{\deg(\alpha)}$ are conjugates of α, we can express f_α as

$$f_\alpha(x) = m_\alpha(x-\alpha_1)\cdots(x-\alpha_{\deg(\alpha)}) = \sum_{j=0}^{\deg(\alpha)} a_j x^{n-j}.$$

Therefore, by Observation 4.1, we have the following crude bound on the coefficients of f_α,

$$|a_j| = |m_\alpha \sum_{i_1<\cdots<i_j} \alpha_{i_1}\cdots\alpha_{i_j}| \le \left|m_\alpha \binom{\deg(\alpha)}{j}\Lambda(\alpha)^j\right| \le e^{C'\log n}$$

for some $C'>0$ depending on C_0 and M. Since a_j has to be an integer, there are at most $e^{C\log^2 n}$ possibilities for $f_\alpha(x)$ for some constant $C>0$.

Now, by Lemma 4.1, if $\alpha\in\mathbb{A}$ with $\Lambda(\alpha)>1+\frac{C_1}{\log n}$, then

$$\max_{\alpha\in\mathbb{A}} \mathbb{P}\Big(P(\alpha)=0\Big) = e^{-\Omega(n/\log n)}, \tag{4.2}$$

A simple union bound over such α (or, more precisely, over the minimal polynomials f_α) yields the lemma. □

5 Roots near the unit circle

In this section we prove Lemma 1.5. Recall that for $\alpha \in \mathbb{A}$, m_α denotes the leading coefficient of the associated minimal polynomial of α in $\mathbb{Z}[x]$. Fix $C_0, C_1 > 0$. By Observation 4.1, we need to consider for Lemma 1.5 the following set of potential roots of P.

$$A := \Big\{\alpha \in \mathbb{A} : \ 4 < \deg(\alpha) \le C_0 \log n \text{ and } \Lambda(\alpha) \le 1 + \tfrac{C_1}{\log n} \text{ and } m_\alpha \le M\Big\}.$$

To prove Lemma 1.5, we employ the following union bound

$$\mathbb{P}\Big(\exists \alpha \in A : P(\alpha) = 0\Big) \le |A| \cdot \max_{\alpha \in A} \mathbb{P}\Big(P(\alpha) = 0\Big) \tag{5.1}$$

and then proceed to provide upper bounds on $\max_{\alpha\in A} \mathbb{P}\big(P(\alpha) = 0\big)$ and on the cardinality of the set A. This is done using Lemma 5.1 and Lemma 5.2 below, whose proofs are presented in Sections 5.1 and 5.2 respectively.

Lemma 5.1. *Let α be an algebraic number of degree at least 5. Then for every $\epsilon > 0$ there exists $C > 0$ such that*

$$\mathbb{P}(P(\alpha) = 0) \le C n^{-\frac{5}{2}+\epsilon}. \tag{5.2}$$

For any polynomial $f \in Z[x]$, let $\Lambda(f)$ be the maximum moduli of the roots of f.

Lemma 5.2 (counting integral polynomials with small houses)**.** *Let $b > 0$ and let $a \in \mathbb{N}$. Then, for all d sufficiently large, the number of polynomials $f \in \mathbb{Z}[x]$ of degree d with leading coefficient a such that*

$$\Lambda(f) < 1 + \frac{b \log d}{ad}$$

is less than $\exp((ad)^{2/3+b})$.

Note that for every algebraic number $\alpha \in A$, its associated minimal polynomial in $\mathbb{Z}[x]$ has degree at most $C_0 \log n$ and its leading coefficient is bounded by M. Applying Lemma 5.2 to each $a \in \{1, 2, \dots, M\}$ and each degree $1 \le d \le C_1 \log n$ with $b = \frac{1}{6}$, we obtain that for every $\epsilon > 0$,

$$|A| = o(n^\epsilon). \tag{5.3}$$

Plugging (5.2) and (5.3) into (5.1), the Lemma 1.5 follows. □

5.1 Each root of low degree is unlikely

In this section we prove Lemma 5.1. The proof follows closely the proof of [10, Lemma 1] adapted for our case (that is, when the random variables ξ_i's are not Bernoulli random variables). The main ingredient of the proof is the 'inverse Littlewood-Offord type theorem" of Tao and Vu [15, Theorem 1.9], whose specialization for our case is the following.

Theorem 5.3 (Tao and Vu (2010))**.** *Let $(\eta_i)_{0\le i\le n}$ be i.i.d. $\mathrm{Ber}(\frac{1}{2})$ random variables. Let $A, \delta > 0$ and let $(z_i)_{0\le i\le n}$ be complex numbers such that*

$$\max_{z\in\mathbb{C}} \mathbb{P}\Big(\sum_{i=0}^{n} \eta_i z_i = z\Big) \ge n^{-A}.$$

Then there exists a symmetric generalized arithmetic progression (GAP), all of whose elements are distinct, of rank $r \le 2A$ which contains all but $O_{A,\delta}(n^{1-\delta})$ of z_i's (counting multiplicities).

Recall that in our context a *symmetric GAP* Q of rank r is a set of the form

$$Q = \left\{ \sum_{i=1}^{r} n_i u_i : \ n_i \in [-N_i, N_i] \cap \mathbb{Z}, \ \forall i = 1, \ldots, r \right\}, \tag{5.4}$$

where the dimensions $N = (N_1, N_2, \ldots, N_r)$ are r-tuple of positive integers and the steps $u = (u_1, u_2, \ldots, u_r)$ are r-tuple of elements in $\mathbb{C}$. In particular, if $v_1 \ldots, v_{r+1}$ are elements of a GAP of rank r, then there exist nontrivial integer coefficients $(q_1, \ldots, q_{r+1}) \in \mathbb{Z}^{r+1}, (q_1, \ldots, q_{r+1}) \neq \mathbf{0}$ such that $q_1 v_1 + q_2 v_2 + \ldots + q_{r+1} v_{r+1} = 0$.

Proof of Lemma 5.1. Let α be an algebraic number of degree $d \geq 5$ and let $\epsilon > 0$. Assume towards obtaining a contradiction that

$$\mathbb{P}\Big(P(\alpha) = 0 \Big) > n^{-\frac{5}{2}+\epsilon}. \tag{5.5}$$

Proposition 2.1 allows us to represent the random variable ξ_j as $\xi_j = I_j + \Delta_j \eta_j$ where $(I_j, \Delta_j)_{0 \leq j \leq n}$ are i.i.d. random vectors taking values in $\mathbb{Z} \times \mathbb{N}$ and $(\eta_j)_{0 \leq j \leq n}$ are i.i.d. $\mathrm{Ber}(\frac{1}{2})$, independent of $(I_j, \Delta_j)_{0 \leq j \leq n}$. Conditioning on $(I_j, \Delta_j)_{0 \leq j \leq n}$ yields

$$\begin{aligned} \mathbb{P}(P(\alpha) = 0) &= \mathbb{E}\mathbb{P}\big(\sum_{j=0}^{n} \Delta_j \alpha^j \eta_j = -\sum_{j=0}^{n} I_j \alpha^j | (I_j, \Delta_j)_{0 \leq j \leq n} \big) \\ &\leq \mathbb{E} \sup_{z \in \mathbb{C}} \mathbb{P}\big(\sum_{j=0}^{n} \Delta_j \alpha^j \eta_j = z | (\Delta_j)_{0 \leq j \leq n} \big). \end{aligned} \tag{5.6}$$

From (5.5) and (5.6), it follows that there exists a vector $(d_0, d_1, \ldots, d_n) \in \mathbb{N}^{n+1}$ such that

$$\sup_{z \in \mathbb{C}} \mathbb{P}\big(\sum_{j=0}^{n} d_j \alpha^j \eta_j = z \big) > n^{-\frac{5}{2}+\epsilon}.$$

We now apply Theorem 5.3 with $A = \frac{5}{2} - \varepsilon$ and $\delta = \frac{1}{2}$ and $z_j = d_j \alpha^j$ to obtain a symmetric GAP Q of rank $B \leq 2A < 5$ such that all but $O(\sqrt{n})$ many of the coefficients $d_j \alpha^j$ belong to Q. Therefore, for large enough n, there exists $j_0 \in \{0, 1, \ldots, n\}$ for which $d_{j_0+k} \alpha^{j_0+k} \in Q$ for all $k = 0, 1, \ldots, 4$. Since the rank of Q is at most 4, there exists a nontrivial integer linear combination that annihilates the vector $(d_{j_0+k} \alpha^{j_0+k})_{0 \leq k \leq 4}$. Hence the algebraic degree of α is at most 4, in contradiction with our assumption. Hence, the lemma follows. □

5.2 There are not many low degree polynomials with small house

This section is dedicated to the proof of Lemma 5.2.

Proof of Lemma 5.2. The proof of the lemma is an adaptation of the proof of [6, Theorem 1] of Dubickas to the case of non-monic polynomials. Fix $b > 0$, and write $F_{a,d}$ for the collection of polynomials $f \in \mathbb{Z}[x]$ of degree d with leading coefficient a which satisfy

$$\Lambda(f) < 1 + \frac{b \log d}{ad}.$$

In the proof we also make use of the classical Newton identities, known also as Newton—Girard formulae (See [9] for a modern proof). Let $f(x) = \sum_{i=0}^{d} a_i x^{d-i}$ be a polynomial of degree d in $\mathbb{Z}[x]$ with roots $\alpha_1(f), \alpha_2(f), \ldots, \alpha_d(f)$. Define, for $k \geq 1$,

$$S_k = S_k(f) = \sum_{j=1}^{d} \alpha_j(f)^k.$$

Lemma 5.4 (Girard, 1629)**.** *for each* $1 \le k \le d$*,*

$$a_0 S_k + a_1 S_{k-1} + ... + a_{k-1} S_1 + k a_k = 0. \tag{5.7}$$

We further observe that if $g(x) = \sum_{i=0}^{k-1} a_i x^{d-k} + \sum_{i=k}^{d} b_i x^{d-i}$ with $b_k \neq a_k$ then for all $i < k$ we have $S_i(f) = S_i(g)$ while

$$|S_k(f) - S_k(g)| = \frac{k|a_k - b_k|}{a_0} \ge \frac{k}{a_0}. \tag{5.8}$$

With a slight abuse of notation we write $S_k(w) = w_1^k + w_2^k + \cdots + w_d^k$, for $w = (w_1, \ldots, w_d) \in \mathbb{C}^d$.

Let $b > 0$ be given. We say that a set $W \subset \mathbb{C}^d$ is (a, d) *admissible* if the following two conditions are satisfied:

- (Boundedness) We have $\max_{1 \le i \le d} |w_i| < 1 + \frac{b \log d}{ad}$ for all $w \in W$.
- (Separation) For any two distinct $u, v \in W$ we have

$$\max_{1 \le k \le d} \frac{a}{k} \Big| \mathrm{Re}(S_k(u)) - \mathrm{Re}(S_k(v)) \Big| \ge 1.$$

Let

$$S = \{(\alpha_1(f), \ldots, \alpha_d(f)) \ : \ f \in F_{a,d}\}.$$

From (5.8) and from the definition of $F_{a,d}$, we deduce that the set S is (a, d) admissible. To conclude the proof we bound the maximal size of any (a, d) admissible set. In [6, Theorem 2], Dubickas obtained such a bound for the special case when $a = 1$ using an elementary but clever application of volume formulas of polytopes, and classical estimates on the number of Gauss integers in a circle.

Theorem 5.5 (Dubickas, 1999)**.** *The size of any* $(1, \ell)$ *admissible set is* $O_b(\exp(\ell^{2/3+b}))$.

We now use Dubickas' result as a blackbox to bound the cardinality of (a, d) admissible set. Let W be an (a, d) admissible set, and write $\widehat{W}$ for the image of W in $\mathbb{C}^{ad}$ under the repetition map

$$w \in \mathbb{C}^d \mapsto \hat{w} := (\underbrace{w, w, \ldots, w}_{a \text{ times}}) \in \mathbb{C}^{ad}.$$

Clearly, $||\hat{w}||_\infty = ||w||_\infty < 1 + \frac{b \log d}{ad} \le 1 + \frac{b \log(ad)}{ad}$ for all $w \in W$. Furthermore, for every distinct $\hat{u}, \hat{v} \in \widehat{W}$ we have

$$\begin{aligned} \max_{1 \le k \le ad} \frac{1}{k} \Big| \mathrm{Re}(S_k(\hat{u})) - \mathrm{Re}(S_k(\hat{v})) \Big| &\ge \max_{1 \le k \le d} \frac{1}{k} \Big| \mathrm{Re}(S_k(\hat{u})) - \mathrm{Re}(S_k(\hat{v})) \Big| \\ &= \max_{1 \le k \le d} \frac{a}{k} \Big| \mathrm{Re}(S_k(u)) - \mathrm{Re}(S_k(v)) \Big| \ge 1, \end{aligned}$$

where the last inequality follows from the fact $u \neq v \in W$ and W is an (a, d) admissible set. Hence $\widehat{W}$ is $(1, ad)$ admissible. Therefore applying Theorem 5.5 with $\ell = ad$ implies

$$|F_{a,d}| = |S| = O_b(\exp((ad)^{2/3+b})),$$

as required. □

6 Unimodular roots with bounded degree

In this section we will prove Lemma 1.6. Note that there are only finitely many irreducible polynomials in $\mathbb{Z}[x]$ of degree at most 4 with leading coefficients bounded by M in absolute value whose roots are all within distance $M+1$ from the origin. In particular, for large enough n, all such polynomials whose roots are in distance $1+\frac{C_1}{\log n}$ from the origin, have, in fact, all roots on the unit circle. Thus to prove the lemma it would suffice to show that for any fixed $\alpha \in \mathbb{A}$ on the unit circle such that $\deg(\alpha) \le 4$ and $\alpha \neq \pm 1$,

$$\mathbb{P}(\alpha \text{ is a double root of } P) = o(n^{-2}), \tag{6.1}$$

Lemma 1.6 will then follow by applying a simple union bound. If α is an algebraic *integer*, then it has to be a root of unity if α and all of its conjugates lie on the unit circle. However, in general, there are examples of algebraic numbers such that all of their conjugates are on the unit circle yet they are not roots of unity. For example, consider the quadratic polynomial $3x^2 - x + 3$ whose roots are $\frac{1\pm\sqrt{-35}}{6}$. In fact, any polynomial $\sum_{i=0}^{m} b_i x^i$ in $\mathbb{Z}[x]$ that is self-reciprocal (i.e., $b_i = b_{m-i}\ \forall\ i$) which satisfies the condition $|b_m| > \frac{1}{2}\sum_{k=1}^{m-1}|b_k|$ has all its root on the unit circle [16]. Thus, first begin by addressing roots which lie on the unit circle but which are not root of unity.

Lemma 6.1 (unimodular roots that are not roots of unity)**.** *Let $\alpha \in \mathbb{A}$ be such that $|\alpha| = 1$ but α is not a root of unity (i.e., $\alpha^m \neq 1$ for all $m \in \mathbb{N}$). Then under Assumption 1.3,*

$$\mathbb{P}\big(P(\alpha) = 0\big) = O(n^{-5/2}).$$

The proof of Lemma 6.1 is a straightforward application of the following well-known result due to Halász (see [17, Corollary 7.16], [19, Corllary 6.3 and Remark 3.5]).

Lemma 6.2 (Halász)**.** *Let G be an infinite Abelian group. Let $m \ge 1$ and $a_1, a_2, \ldots, a_m \in G$ and let $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_m$ be i.i.d. with $\mathbb{P}(\varepsilon_j = 1) = \mathbb{P}(\varepsilon_j = 0) = 1/2$. Fix $\ell \in \mathbb{N}$ and let R_ℓ be the number of solutions of the equation $a_{i_1} + a_{i_2} + \cdots + a_{i_\ell} = a_{j_1} + a_{j_2} + \cdots + a_{j_\ell}$. Then*

$$\sup_x \mathbb{P}\Big(\sum_{i=1}^{m} a_i\varepsilon_i = x\Big) = O(n^{-2\ell-\frac{1}{2}}R_\ell).$$

Proof of Lemma 6.1. By Proposition 2.1, the random variables $(\xi_j)_{0\le j\le n}$ can be represented as $\xi_j = I_j + \Delta_j\varepsilon_j$ where $(I_j, \Delta_j)_{0\le j\le n}$ are i.i.d. random vectors taking values in $\mathbb{Z}\times\mathbb{N}$ and $(\varepsilon_j)_{0\le j\le n}$'s are i.i.d. $\mathrm{Ber}(\frac{1}{2})$, independent of $(I_j, \Delta_j)_{0\le j\le n}$. Now by conditioning on I_j and Δ_j's, we have

$$\begin{aligned}\mathbb{P}\Big(\sum_{j=0}^{n}\xi_j\alpha^j = 0\Big) &\le \max_x \mathbb{E}\mathbb{P}\Big(\sum_{j=0}^{n}(I_j+\Delta_j\varepsilon_j)\alpha^j = x\Big|(I_j,\Delta_j)_{0\le j\le n}\Big)\\ &\le \mathbb{E}\max_x \mathbb{P}\Big(\sum_{j=0}^{n}\Delta_j\varepsilon_j\alpha^j = x\Big|(\Delta_j)_{0\le j\le n}\Big),\end{aligned}$$

where expectations are taken on the vector $(I_j, \Delta_j)_{0\le j\le n}$. Fix an integer q in the support of the random variable Δ_j and let $\eta := \mathbb{P}(\Delta_j = q) > 0$. Let T denote the random set of indices defined by $T = \{0 \le j \le n : \Delta_j = q\}$. Again, conditioning on $(\varepsilon_j)_{j\notin T}$, write

$$\begin{aligned}\mathbb{E}\max_x \mathbb{P}\Big(\sum_{j=0}^{n}\Delta_j\varepsilon_j\alpha^j = x\Big|(\Delta_j)_{0\le j\le n}\Big) &\le \mathbb{E}\max_x \mathbb{P}\Big(\sum_{j\in T}\Delta_j\varepsilon_j\alpha^j = x\Big|T\Big)\\ &\le \mathbb{E}\max_x \mathbb{P}\Big(\sum_{j\in T}\varepsilon_j\alpha^j = x\Big|T\Big).\end{aligned}$$

We are left with showing that for any deterministic set of indices $T \subseteq \{0, 1, \dots, n\}$, we have

$$\max_x \mathbb{P}\Big(\sum_{j \in T} \varepsilon_j \alpha^j = x \Big) = O(|T|^{-5/2}).$$

Applying Halász's result (Lemma 6.2) with coefficients $(\alpha_j)_{j \in T}$ in $\mathbb{C}$ and $\ell = 2$, we count the number of solutions of the equation

$$\alpha^i + \alpha^j = \alpha^k + \alpha^l, \tag{6.2}$$

where i, j, k, l are arbitrary indices in T. Taking the absolute value on the both sides of (6.2) and using the fact that $|\alpha| = 1$, we have $|1 + \alpha^{j-i}| = |1 + \alpha^{l-k}|$, which implies that either $\alpha^{j-i} = \alpha^{l-k}$ or $\alpha^{j-i} = \alpha^{k-l}$, or equivalently $j - i = \pm(l - k)$. In case that $j - i = l - k$, we may write (6.2) as $\alpha^i(1 + \alpha^{j-i}) = \alpha^k(1 + \alpha^{j-i})$, or equivalently as $(\alpha^i - \alpha^k)(1 + \alpha^{j-i}) = 0$. Since α is not a root of unity, then $1 + \alpha^{j-1} \neq 0$. So, we deduce that $\alpha^i = \alpha^k$ which implies that $i = k$. Plugging it in back in $j - i = l - k$, we also have $j = l$. Similarly, for the case $j - i = k - l$, we end up with the equation $(\alpha^i - \alpha^l)(1 + \alpha^{j-i}) = 0$, which, in turn, implies that $i = l$ and $j = k$. Hence, we conclude that $R_2 \leq 2|T|^2$ and the claim follows.

From the claim, we obtain that

$$\mathbb{E} \max_x \mathbb{P}\Big(\sum_{j \in T} \varepsilon_j \alpha^j = x \Big| T \Big) \leq \mathbb{E} \min\Big(1, \frac{C}{|T|^{5/2}}\Big) = O(n^{-5/2}) + \mathbb{P}\Big(|T| \leq \tfrac{\eta}{2} n\Big).$$

Note that $|T|$ has a binomial distribution corresponding to $n + 1$ trials and success probability η. From the standard result on the concentration of binomial random variable, we know that there exists a constant $c > 0$, depending on η, such that $\mathbb{P}\Big(|T| \leq \frac{\eta}{2} n\Big) \leq e^{-c(n+1)}$. This completes the proof of the lemma. □

Next, we consider roots of unity.

Lemma 6.3 (roots of unity)**.** *Suppose Assumption 1.3 holds. Then there exist constants $c, C > 0$ such that if α satisfies $\alpha^k = 1$ for some positive integer k, then*

$$\mathbb{P}\Big(P'(\alpha) = 0\Big) \leq \left(\frac{C}{\lfloor \frac{n}{k} \rfloor}\right)^{\frac{3 \deg(\alpha)}{2}} + k \exp(-c \lfloor \tfrac{n}{k} \rfloor).$$

The proof of the above lemma is very similar to that of Lemma 1.4 in [13] where the similar bound holds without the additional $k \exp(-c\lfloor \frac{n}{k} \rfloor)$ term for any non-constant coefficient distribution supported on $\{-1, 0, 1\}$. However, for the sake of completeness we include here a proof of Lemma 6.3. The proof of Lemma 6.3 relies heavily on the following classical anti-concentration bound of Sárközi and Szemerédi [14].

Theorem 6.4 (Sárközi and Szemerédi)**.** *Let $(\varepsilon_j)_{1 \leq j \leq N}$ be i.i.d. $\mathrm{Ber}(\frac{1}{2})$ random variables. There exists a constant $C > 0$ such that for any* distinct *integers (a_j), $1 \leq j \leq N$, we have*

$$\max_{m \in \mathbb{Z}} \mathbb{P}\left(\sum_{j=1}^{N} \varepsilon_j a_j = m \right) \leq \frac{C}{N^{3/2}}.$$

Proof of Lemma 6.3. Since ξ_j is a mixture of Bernoulli distribution, we can proceed along the same way as in the proof of Lemma 6.1 to obtain

$$\mathbb{P}\Big(P'(\alpha) = 0\Big) \leq \mathbb{E} \max_x \mathbb{P}\Big(\sum_{j=1}^{n} \varepsilon_j j \Delta_j \alpha^{j-1} = x \Big| (\Delta_j)_{1 \leq j \leq n} \Big), \tag{6.3}$$

where $(\varepsilon_j)_{1\le j\le n}$ are i.i.d. $\mathrm{Ber}(\frac{1}{2})$ and $(\Delta_j)_{1\le j\le n}$ are i.i.d. on $\mathbb{N}$.

Observe that necessarily $\deg(\alpha)\le k$. Set

$$J := \{j\colon 1\le j\le n \text{ and } 0\le (j-1) \bmod k \le \deg(\alpha)-1\},$$
$$\bar{J} := \{1,\dots,n\}\setminus J.$$

Conditionally on $(\Delta_j)_{1\le j\le n}$, define the random variables (S_r), $0\le r\le \deg(\alpha)-1$, by

$$S_r := \sum_{j-1 \bmod k = r} \varepsilon_j j\Delta_j \alpha^{j-1} = \alpha^r \sum_{j-1 \bmod k = r} \varepsilon_j j\Delta_j$$

and

$$\bar{S} := \sum_{j\in\bar{J}} \varepsilon_j j\Delta_j \alpha^{j-1}.$$

Observe that

$$\sum_{j=1}^{n} \varepsilon_j j\Delta_j\alpha^{j-1} = \sum_{j=1}^{n} \xi_j j\alpha^{j-1} = \sum_{r=0}^{\deg(\alpha)-1} S_r + \bar{S}.$$

Now, conditionally on $(\Delta_j)_{1\le j\le n}$, $S_0, S_1, \dots S_{\deg(\alpha)-1}$ and $\bar{S}$ are independent. In addition, (α^r), $0\le r\le \deg(\alpha)-1$, are linearly independent over the rational numbers, and therefore the equation $\sum_{i=0}^{\deg(\alpha)-1} a_i\alpha^i = z$ has at most one integral solution $(a_0,\dots,a_{\deg(\alpha)-1})$ for a given $z\in\mathbb{C}$. Hence, for any given values of $(\Delta_j)_{1\le j\le n}$ and any given $x\in\mathbb{C}$,

$$\begin{aligned}\mathbb{P}\Big(\sum_{j=1}^{n} \varepsilon_j j\Delta_j\alpha^{j-1} = x\Big) &= \mathbb{E}_{\bar{S}}\mathbb{P}\left(\sum_{r=0}^{\deg(\alpha)-1} S_r = x-\bar{S}\,\Big|\,\bar{S}\right) \le \max_{z\in\mathbb{C}} \mathbb{P}\left(\sum_{r=0}^{\deg(\alpha)-1} S_r = z\right)\\ &= \prod_{r=0}^{\deg(\alpha)-1} \max_{z\in\mathbb{C}} \mathbb{P}(S_r=z) = \prod_{r=0}^{\deg(\alpha)-1} \max_{m\in\mathbb{Z}} \mathbb{P}\left(\sum_{j-1 \bmod k = r} \varepsilon_j j\Delta_j = m\right).\end{aligned} \tag{6.4}$$

Let q be a point in the support of Δ_j and let $\eta := \mathbb{P}(\Delta_j = q) > 0$. Define for $0\le r\le \deg(\alpha)-1$, $T_r := \{1\le j\le n : \Delta_j = q \text{ and } j-1 \bmod k = r\}$. Then

$$\max_{m\in\mathbb{Z}} \mathbb{P}\left(\sum_{j-1 \bmod k = r} \varepsilon_j j\Delta_j = m\right) \le \max_{m'\in\mathbb{Z}} \mathbb{P}\left(\sum_{j\in T_r} \varepsilon_j j = m'\right) \le C|T_r|^{-3/2}, \tag{6.5}$$

where in the last step we apply the Sárközi-Szemerédi bound (Theorem 6.4). Combining (6.3), (6.4) and (6.5), we finally arrive at the inequality,

$$\mathbb{P}\Big(P'(\alpha)=0\Big) \le \mathbb{E}\prod_{r=0}^{\deg(\alpha)-1} \min(1, C|T_r|^{-3/2}) \le \left(\frac{C}{\frac{\eta}{2}\lfloor\frac{n}{k}\rfloor}\right)^{\frac{3\deg(\alpha)}{2}} + \sum_{r=0}^{\deg(\alpha)-1} \mathbb{P}\Big(|T_r| \le \tfrac{\eta}{2}\lfloor\tfrac{n}{k}\rfloor\Big). \tag{6.6}$$

Observe that T_r is a binomial random variable with number of trials at least $\lfloor\frac{n}{k}\rfloor$ and success probability η. This gives us the following bound for the left tail of T_r. There exists a constant $c>0$ such that $\mathbb{P}\big(|T_r|\le \frac{\eta}{2}\lfloor\frac{n}{k}\rfloor\big) \le e^{-c\lfloor\frac{n}{k}\rfloor}$. The lemma now follows from (6.6) and the fact that $\deg(\alpha)\le k$. □

It remains to show (6.1). Let $\alpha\in\mathbb{A}$ such that $|\alpha|=1, \alpha\notin\{-1,1\}$ and $\deg(\alpha)\le 4$. First assume that α is a primitive k^{th} root of unity, that is, $\alpha^k=1$ and $\alpha^l\ne 1$ for all positive integer $l<k$. Recall that $\deg(\alpha)=\varphi(k)$ where φ is Euler's totient function, i.e., $\varphi(k) = |\{1\le j\le k\colon \gcd(j,k)=1\}|$ (see, for example, Lemma 7.6 and Theorem 7.7 of

[12]). By standard estimates (see [11, Theorem 2.9]) there exists some constant $c_1 > 0$ for which

$$\varphi(k) \geq \frac{c_1 k}{\log\log(k+2)}.$$

The above bound along with the fact that $\deg(\alpha) \leq 4$ implies the bound $k \leq C_2$ for some absolute constant C_2. On the other hand, since $\alpha \neq \pm 1$, we have $\deg(\alpha) \geq 2$. Thus, by Lemma 6.3, we deduce that $\mathbb{P}(\alpha \text{ is a double root of } P) = O(n^{-3})$.

Now assume that α is not a root of unity. As a direct consequence of Lemma 6.1, we also have that $\mathbb{P}(\alpha \text{ is a double root of } P) = O(n^{-5/2})$. This finishes the proof of the bound (6.1) and hence the proof of Lemma 1.6.

7 Open problems

We conclude the paper with a couple of open problems.

(a) Define $p_{n+1} := \max_{a\in\mathbb{Z}} \mathbb{P}\Big(P_n(2) = a\Big)$. Then

$$\begin{aligned} p_{n+m} &= \max_{a\in\mathbb{Z}} \mathbb{P}\Big(P_{n+m-1}(2) = a\Big) \\ &\geq \max_{a\in\mathbb{Z}} \mathbb{P}\Big(\sum_{j=0}^{m-1} \xi_j 2^j = a\Big) \max_{a\in\mathbb{Z}} \mathbb{P}\Big(\sum_{j=m}^{m+n-1} \xi_j 2^j = a\Big) = p_m p_n. \end{aligned}$$

So, it follows from the subadditive property that there exists some $\lambda > 0$, depending on the law of ξ_0, such that $p_n = e^{-\lambda n(1+o(1))}$. However, the exact value of λ is completely unknown. In the special case when the maximum of atom of ξ_0 is at most $\frac{1}{2}$, our Theorem 1.2 only gives a one-sided bound $\lambda > \frac{1}{2}\log 2$. It would be very interesting to investigate the dependence of value of λ on the law of ξ_0 or, say, on the maximum atom of ξ_0.

(b) It would be very interesting to investigate the minimum distance between adjacent complex roots of a random polynomial. Note that this problem makes sense even if the coefficient distribution is continuous. To best of our knowledge, precise quantitive bounds on the minimum gap are not available even for the i.i.d. Gaussian polynomials.

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