

Percolation and convergence properties of graphs related to minimal spanning forests

Christian Hirsch* Tim Brereton† Volker Schmidt†

Abstract

Lyons, Peres and Schramm have shown that minimal spanning forests on randomly weighted lattices exhibit a critical geometry in the sense that adding or deleting only a small number of edges results in a radical change of percolation properties. We show that these results can be extended to a Euclidean setting by considering families of stationary super- and subgraphs that approximate the Euclidean minimal spanning forest arbitrarily closely, but whose percolation properties differ decisively from those of the minimal spanning forest. Since these families can be seen as generalizations of the relative neighborhood graph and the nearest-neighbor graph, respectively, our results provide a new perspective on known percolation results from literature. We argue that the rates at which the approximating families converge to the minimal spanning forest are closely related to geometric characteristics of clusters in critical continuum percolation, and we show that convergence occurs at a polynomial rate.

Keywords: Euclidean minimal spanning forest; percolation; nearest-neighbor graph; relative neighborhood graph; rate of convergence.

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1 Introduction

Euclidean minimal spanning forests were introduced by Aldous and Steele in order to study the first-order asymptotics of certain functionals of the minimal spanning tree on n independent points that are uniformly distributed in the unit cube $[-1/2, 1/2]^d$ in \mathbb{R}^d , $d \geq 2$; see [3]. Subsequently, this idea has been successfully refined to derive central limit theorems [5] and conduct perturbation analysis [1]. Although initially introduced as auxiliary objects for the investigation of large-scale minimal spanning trees, minimal spanning forests themselves exhibit a rich geometric structure on a global scale and have therefore attracted further research interest [4, 23]. In a discrete setting, Lyons, Peres and Schramm [23] showed that for independent edge weights the

*Mathematisches Institut, Ludwig-Maximilians-Universität München, Theresienstraße 39, 80333 München, Germany; E-mail: hirsch@math.lmu.de

†Institute of Stochastics, Ulm University, Helmholtzstr. 18, 89069 Ulm, Germany

minimal spanning forest exhibits a critical percolation behavior: perturbing the forest by removing only an arbitrarily small proportion of edges results in a graph with no infinite connected components, whereas adding only an arbitrarily small proportion of edges results in a graph with a unique infinite connected component – in fact, the entire graph is connected. Returning to the Euclidean origins of the minimal spanning forest, in the present paper we extend the percolation analysis to minimal spanning forests based on point processes in \mathbb{R}^d by considering a family of perturbations that are inspired by graphs in computational geometry. First, this provides a new perspective on percolation results for the relative neighborhood graph and the nearest-neighbor graph. Second, our analysis gives rise to a non-trivial example of the locality of the critical threshold for Bernoulli percolation on stationary random geometric graphs. Third, we show that the perturbations converge to the minimal spanning forest at polynomial velocity, thereby providing a connection to the order of shortest-path scaling in critical continuum percolation.

As perturbations, we consider two families of graphs that approximate the Euclidean minimal spanning forest arbitrarily closely but exhibit decisively different percolation behaviors. We consider the family of creek-crossing graphs, $\{G_n\}_{n \geq 2}$, (see [20]) that approximate the spanning forest from above and introduce a new family of graphs, the minimal-separator graphs, $\{H_n\}_{n \geq 1}$, that approximate the minimal spanning forest from below. Both families are constructed from an underlying stationary point process using deterministic connection rules and are closely related to well-studied graphs in computational geometry, including the *relative neighborhood graph* [28] and the *nearest-neighbor graph* [13]. Here, both the relative neighborhood graph and the nearest-neighbor graph are geometric graphs on a locally finite subset of the Euclidean space. In the nearest-neighbor graph, each node is connected by an edge to the node that is closest in Euclidean distance. In the relative neighborhood graph, two nodes x, y are connected by an edge if there does not exist a third node z whose distance to both x and y is smaller than the distance between x and y , i.e., $\max\{|x - z|, |z - y|\} < |x - y|$ with $|\cdot|$ denoting the Euclidean norm.

We compare percolation on the Euclidean minimal spanning forest to percolation on the approximating families of creek-crossing and minimal-separator graphs. First, we show that, although every connected component is unbounded, there is no Bernoulli percolation on the minimal spanning forest constructed on stationary and ergodic point processes satisfying a weak condition on void probabilities. In contrast, the critical thresholds for Bernoulli percolation on Poisson-based creek-crossing graphs are strictly smaller than 1. As special cases, our general result implies non-triviality of Bernoulli percolation on the Gabriel graph and the relative neighborhood graph, which have been considered separately in literature [9, 10]. In the Gabriel graph, two nodes x, y are connected by an edge if there does not exist a third node z such that $|z - (x + y)/2| < |x - y|/2$. Hence, the relative neighborhood graph is a sub-graph of the Gabriel graph.

Although the creek-crossing graphs do exhibit non-trivial Bernoulli percolation, this becomes increasingly difficult with growing n in the sense that the critical percolation thresholds tend to 1, the percolation threshold of the minimal spanning forest. In particular, this result provides a point-process based example for the locality of critical percolation thresholds. In the setting of discrete transitive graphs, the problem of finding a non-trivial example for locality with limiting threshold equal to 1 has been advertised as an open problem [24]. Recently, Beringer, Pete and Timár have derived a locality criterion for the class of uniformly good unimodular random graphs [8]. However, the minimum spanning tree does not appear to be uniformly good, so that the techniques developed in [8] do not apply in the present setting. Second, we show that the expected

cluster sizes of minimal-separator graphs on a Poisson point process are finite. This result generalizes, and provides a new perspective on the classical result of the absence of percolation in the nearest-neighbor graph [18]. Nevertheless, with growing n the expected cluster sizes tend to infinity, thereby reflecting that in the minimal spanning forest, every connected component is unbounded.

In the final part of the paper, we investigate the relationship between the rates at which the approximating graphs converge to the minimal spanning forest and the failure of the minimal spanning forest to inherit geometric properties from its approximations. For instance, if the creek-crossing graphs converged to the minimal spanning forest too quickly, not only would the minimal spanning forest be connected but it would also have cycles. As the minimal spanning forest does not contain cycles with probability 1, this yields an upper bound on the rate of convergence. We argue that the rates of convergence are related to the tail behavior of chemical distances and cluster sizes in critical continuum percolation. Thus, one would expect these rates to be of polynomial order. We show that the convergence rates have a polynomial lower bound in dimension 2 and polynomial upper bounds in all dimension.

The paper is organized as follows. In Section 2, we present our main results on minimal spanning forests and their approximations, see Theorems 2.5–2.9. In Sections 3, 4 and 5, we provide proofs of Theorems 2.5, 2.6 and 2.7, where we investigate percolation properties of spanning forests, creek-crossing graphs and minimal-separator graphs, respectively. Finally, in Section 6, we prove Theorems 2.8 and 2.9, where we derive polynomial upper bounds (Section 6.1) and polynomial lower bounds (Section 6.2) on the rates at which the approximating families converge to the minimal spanning forest.

2 Definitions and main results

2.1 Approximating Euclidean minimal spanning forests

On any finite subset $\varphi \subset \mathbb{R}^d$ we can define a *minimal spanning tree* as a tree with vertex set φ and minimal total edge length. There is a unique minimal spanning tree provided that φ is ambiguity free, in the sense that there do not exist $x_1, y_1, x_2, y_2 \in \varphi$ with $|x_1 - y_1| = |x_2 - y_2| > 0$ and $\{x_1, y_1\} \neq \{x_2, y_2\}$, see e.g. [4].

Minimal spanning forests are analogues of minimal spanning trees in cases where $\varphi \subset \mathbb{R}^d$ is locally finite, rather than finite. There are a number of equivalent definitions of minimal spanning trees that can be easily extended to locally finite $\varphi \subset \mathbb{R}^d$. However, in the locally finite case, these definitions cease to be equivalent. This means that there is more than one possible definition of a minimal spanning forest. Additionally, the resulting graphs may not be connected, so they are forests rather than trees.

Following [23], we consider, separately, two versions of the minimal spanning forest introduced in [3]: the free minimal spanning forest and the wired minimal spanning forest. Both graphs have the same vertex set, but their edges may be different. The connection rule for the free minimal spanning forest is based on the concept of a creek-crossing path.

Definition 2.1. *Let φ be a locally finite subset of \mathbb{R}^d . The free minimal spanning forest on φ , $\text{FMSF}(\varphi)$, is a geometric graph with vertex set φ and edge set defined by drawing an edge between $x, y \in \varphi$ if and only if there is no creek-crossing path connecting x and y . That is, there does not exist an integer $n \geq 2$ and pairwise distinct vertices $x = x_0, x_1, \dots, x_n = y$ such that $\max_{i \in \{0, \dots, n-1\}} |x_i - x_{i+1}| \leq |x - y|$.*

The connection rule for wired minimal spanning forests is based on the notion of *minimal separators*. We say that $\{x, y\}$ forms a minimal separator of φ and ψ , disjoint locally finite subsets of \mathbb{R}^d , if $x \in \varphi$, $y \in \psi$ and $|x - y| < \inf_{(x', y') \in (\varphi \times \psi) \setminus \{(x, y)\}} |x' - y'|$. In particular, $\{x, y\}$ is the unique minimizer of distances between φ and ψ . On the other

hand, if φ and ψ are both infinite, then $\{x, y\}$ can be the unique minimizer even if the inequality does not hold.

Definition 2.2. Let φ be a locally finite subset of \mathbb{R}^d . The wired minimal spanning forest on φ , $\text{WMSF}(\varphi)$, is a geometric graph with vertex set φ and edge set determined as follows. Two points $x, y \in \varphi$ are connected by an edge in $\text{WMSF}(\varphi)$ if and only if there exists a finite $\psi \subset \varphi$ such that $\{x, y\}$ forms a minimal separator of ψ and $\varphi \setminus \psi$.

In general, $\text{WMSF}(\varphi) \subset \text{FMSF}(\varphi)$, but the free minimal spanning forest and wired minimal spanning forest need not coincide.

In this paper, we consider approximations of both the FMSF and the WMSF. The creek-crossing graphs, $\{G_n\}_{n \geq 2}$, approximate the free minimal spanning forest from above in the sense that $\text{FMSF}(\varphi) = \bigcap_{n \geq 2} G_n(\varphi)$ if φ is ambiguity-free. The minimal-separator graphs, $\{H_n\}_{n \geq 1}$, approximate the wired minimal spanning forest from below, with $\text{WMSF}(\varphi) = \bigcup_{n \geq 1} H_n(\varphi)$.

These graphs are defined using ‘finite’ analogues of the connection rules for the free and wired minimal spanning forests as follows

Definition 2.3. Let φ be a locally finite subset of \mathbb{R}^d . The creek-crossing graphs $\{G_n(\varphi)\}_{n \geq 2}$ are a family of graphs with vertex set φ and the connection rule that $x, y \in \varphi$ are connected by an edge in $G_n(\varphi)$ if and only if there do not exist $m \leq n$ and $x = x_0, x_1, \dots, x_m = y \in \varphi$ such that $\max_{i \in \{0, \dots, m-1\}} |x_i - x_{i+1}| < |x - y|$.

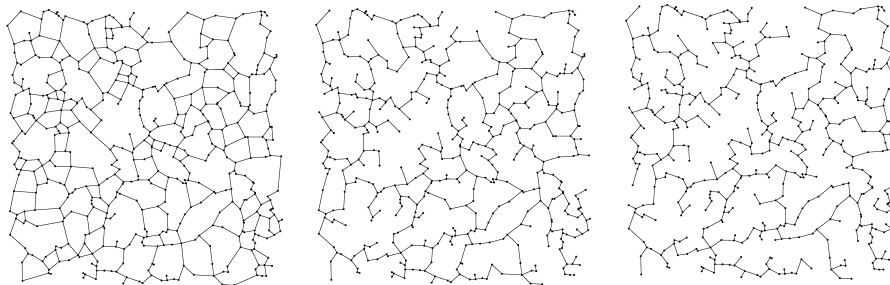


Figure 1: Creek-crossing graphs $G_n(\varphi)$ for $n \in \{2, 5, 10\}$ (from left to right)

Definition 2.4. Let φ be a locally finite subset of \mathbb{R}^d . The minimal separator graphs $\{H_n\}_{n \geq 1}$ are a family of graphs with vertex set φ and the connection rule that $x, y \in \varphi$ are connected by an edge in $H_n(\varphi)$ if there exists $\psi \subset \varphi$ with $\#\psi \leq n$ such that $\{x, y\}$ forms a minimal separator of ψ and $\varphi \setminus \psi$, where $\#\psi$ denotes cardinality of the set ψ .

As mentioned in the introduction these families of graphs are closely related to graphs arising in computational geometry. In particular, $G_2(\varphi)$ is the relative neighborhood graph on φ and $H_1(\varphi)$ is the nearest-neighbor graph on φ . Figures 1 and 2 illustrate the graphs $G_n(\varphi)$ and $H_n(\varphi)$ for a variety of values of n .

2.2 Percolation

First, we consider percolation properties of $\{G_n\}_{n \geq 2}$ and $\{H_n\}_{n \geq 1}$ and analyze how these properties behave when passing to the limiting objects. In general, it is unclear how global properties behave under local graph limits. For instance, it was shown in [20] that, if X is a homogeneous Poisson point process in \mathbb{R}^d , then the graphs $\{G_n(X)\}_{n \geq 2}$ are a.s. connected regardless of the dimension, d . In contrast, this connectedness property is not expected to hold for the minimal spanning forest in sufficiently high dimensions [23, Question 6.8].

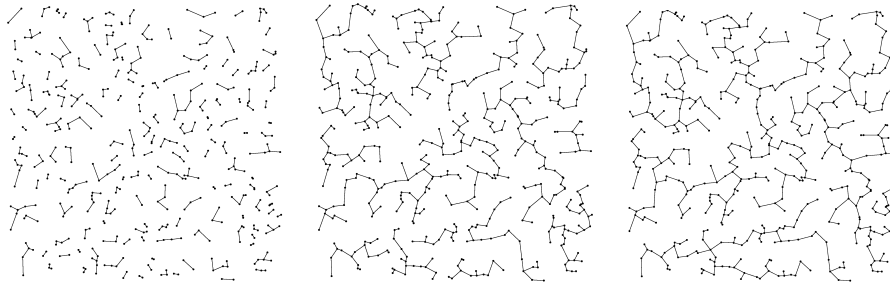


Figure 2: Minimal-separator graphs $H_n(\varphi)$ for $n \in \{1, 10, 50\}$ (from left to right)

In this paper, we investigate Bernoulli percolation on Euclidean minimal spanning forests and their approximations. Recall that in Bernoulli percolation, edges are removed independently with a certain fixed probability. That is, we consider the family of graphs defined as follows. Let $G \subset \mathbb{R}^d$ denote a stationary random geometric graph with vertex set given by a stationary point process X . We attach to each $x \in X$ an iid sequence $\{U_{x,i}\}_{i \geq 1}$ of random variables that are uniformly distributed in $[0, 1]$. Now, consider $x, y \in X$ such that x is lexicographically smaller than y . We say that the link $\{x, y\}$ is p -open if $U_{x,i} \leq p$, where i is chosen such that in the set X the point y is the i th closest point to x . Then, G^p denotes the graph on X whose edge set consists of those pairs of points that are both p -open and form an edge in G . Finally, we say that the graph G^p percolates if there exists an infinite self-avoiding path in G^p . The critical percolation probability of the graph G is given by

$$p_c(G) = \inf\{p \in [0, 1] : \mathbb{P}(G^p \text{ percolates}) > 0\}.$$

Similar to the lattice setting [23], we show that all connected components of $\text{WMSF}(X)$ are infinite under general assumptions on the underlying point process. However, if even an arbitrarily small proportion of edges is removed, then all components are finite. This continues to be true when passing from $\text{WMSF}(X)$ to the larger graph $\text{FMSF}(X)$.

Theorem 2.5. *Let X be a stationary point process with positive and finite intensity.*

- (i) *If $\mathbb{P}(X \cap [-\frac{s}{2}, \frac{s}{2}]^d = \emptyset) \in O(s^{-2d})$, then $p_c(\text{FMSF}(X)) = 1$.*
- (ii) *If X is ambiguity free, then all connected components of $\text{WMSF}(X)$ are infinite.*

Part (ii) of Theorem 2.5 is a direct consequence of the definition, see also [3, Lemma 1]. For part (i) an immediate adaptation of the arguments in [23, Theorem 1.2] does not seem possible. As pointed out by an anonymous referee, if X is ambiguity-free, then part (i) remains valid even without assumptions on the void probabilities, see [4, Theorem 2.5 (i)].

From now on, we assume that X is a Poisson point process, where throughout the manuscript a Poisson point process in \mathbb{R}^d is always assumed to be homogeneous with positive and finite intensity. In this important special case, we have $\text{FMSF}(X) = \text{WMSF}(X)$; see [4, Proposition 2.1]. Although free minimal spanning forests do not admit Bernoulli percolation in the sense that $p_c(\text{FMSF}(X)) = 1$, the following result shows that this changes when $\text{FMSF}(X)$ is replaced by any of the approximating creek-crossing graphs.

Theorem 2.6. *Let X be a Poisson point process. Then,*

- (i) *$p_c(G_n(X)) < 1$ for all $n \geq 2$, and*
- (ii) *$\lim_{n \rightarrow \infty} p_c(G_n(X)) = 1$.*

In other words, Theorem 2.6 shows that the critical probability for Bernoulli percolation is strictly smaller than 1 in any of the creek-crossing graphs G_n , but as $n \rightarrow \infty$, the critical probabilities approach 1.

Part (i) of Theorem 2.6 yields an example of a class of supergraphs whose critical percolation probability is strictly less than that of the original graph. Lattice models with this property are discussed in [16, Sections 3.2, 3.3] and the references given there. Related results in point-process based percolation are given in [14]. Theorem 2.6 generalizes the results for Bernoulli percolation on the Gabriel graph obtained in [9] and the results on the relative neighborhood graph announced in [10].

Part (ii) of Theorem 2.6 provides evidence to the heuristic that for a large class of graphs, the critical probability for Bernoulli percolation should be *local* in the sense that it is continuous with respect to local weak convergence of the underlying graphs. For instance, in the setting of (discrete) transitive graphs, this heuristic is made precise by a conjecture of Schramm, see [7, Conjecture 1.2]. Schramm's conjecture has so far only been verified for specific classes of graphs such as Cayley graphs of Abelian groups [24], including as a special case the celebrated result of Grimmett and Marstrand [17]. A finite analogue of the locality is shown for expander graphs in [7, Theorem 1.3]. To the best of the authors' knowledge, parts (i) and (ii) provide the first example of a family of locally weakly convergent stationary random geometric graphs satisfying $p_c(G_n) < 1$ for every $n \geq 2$, but $\sup_{n \geq 2} p_c(G_n) = 1$. According to the remark following [24, Conjecture 1.1], it is an open problem whether this is possible for discrete transitive graphs.

So far, we have seen that adding an arbitrarily small proportion of edges is sufficient to turn the minimal spanning forest into a graph exhibiting non-trivial Bernoulli percolation. On the other hand, removing only a small proportion of edges in the minimal spanning forest immediately destroys all of the infinite connected components. More precisely, writing $C_{n,H}(X)$ for the connected component of $H_n(X \cup \{o\})$ at the origin, we show that $\mathbb{E} \#C_{n,H}(X)$ is finite for every n but tends to infinite as $n \rightarrow \infty$.

Theorem 2.7. *Let X be a Poisson point process. Then,*

- (i) $\mathbb{E} \#C_{n,H}(X) < \infty$ for all $n \geq 1$ and
- (ii) $\lim_{n \rightarrow \infty} \mathbb{E} \#C_{n,H}(X) = \infty$.

2.3 Rates of convergence

The approximating families $\{G_n\}_{n \geq 2}$ and $\{H_n\}_{n \geq 1}$ can get arbitrarily close to the free and wired minimal spanning forests. However, we have not yet discussed the rates at which they converge. The convergence is quantified using the expected total difference of degrees for vertices inside the unit cube. More precisely,

$$a(n) = \mathbb{E} \# \{ (x, y) \in (X \cap [-\frac{1}{2}, \frac{1}{2}]^d) \times X : \{x, y\} \text{ is an edge in } G_n(X) \text{ but not in } \text{FMSF}(X) \},$$

and

$$b(n) = \mathbb{E} \# \{ (x, y) \in (X \cap [-\frac{1}{2}, \frac{1}{2}]^d) \times X : \{x, y\} \text{ is an edge in } \text{WMSF}(X) \text{ but not in } H_n(X) \}.$$

We note that the rates of convergence are linked to distributional properties of connected components in continuum percolation. For instance, the rate of convergence of the creek-crossing graphs G_n is related to the scaling behavior of the chemical distance in continuum percolation, where the chemical distance between two points is the minimal number of edges in a path connecting them if the points are in the same connected component and ∞ otherwise. More precisely, the link between continuum percolation and Euclidean minimal spanning forests is based on the following observation. Recall that, for $r > 0$, the Gilbert graph $G(\varphi, r)$ on a locally finite vertex set $\varphi \subset \mathbb{R}^d$ is defined by

imposing that $x, y \in \varphi$ are connected by an edge if and only if $|x - y| < r$. Then, the pair $\{x, y\}$ is an edge in $G_n(X)$ and not in $\text{FMSF}(X)$ if and only if x and y can be connected by a path in $G(X, |x - y|)$ but any such path consists of more than n edges. That is, the chemical distance between x and y in $G(X, |x - y|)$ must be finite but larger than n . Likewise, the rate of convergence of the minimal-separator graphs H_n depends on the size of the connected component. This is because, if $\{x, y\}$ forms an edge in $\text{WMSF}(X)$ but not in $H_n(X)$, then the connected components of $G(X, |x - y|)$ containing x and y must be disjoint and both consist of more than n vertices.

When X is a Poisson point process and the Euclidean distance between two points $x, y \in X$ is close to the critical distance in continuum percolation, then the tail behavior of chemical distance between x and y should be of polynomial order provided that the deviation of $|x - y|$ from the critical distance is small when compared to the inverse of the parameter n . Numerical evidence is provided in [11, 15]. The same should hold for the tail behavior of the sizes of the connected component containing x and y . This suggests that $a(n)$ and $b(n)$ should also be of polynomial order. We make this rigorous by showing that $a(n)$ and $b(n)$ lie between polynomial lower and upper bounds. Our proof of the lower bound is based on the Russo-Seymour-Welsh (RSW) theorem in continuum percolation and is therefore only derived in dimension $d = 2$.

Theorem 2.8. *If X is a Poisson point process, then*

$$\limsup_{n \rightarrow \infty} \frac{-\log \min\{a(n), b(n)\}}{\log n} \leq 2d^2 + 6d.$$

Theorem 2.9. *If X is a Poisson point process in \mathbb{R}^2 , then*

$$\liminf_{n \rightarrow \infty} \frac{-\log \max\{a(n), b(n)\}}{\log n} > 0.$$

3 Proof of Theorem 2.5

First, we note that part (ii) of Theorem 2.5 follows as in [23]: For any finite set of points, the minimal separator to the complementary set of the point process is an edge in the wired minimal spanning forest. In the rest of this section, we prove part (i) of Theorem 2.5, i.e., the absence of Bernoulli percolation.

3.1 Absence of Bernoulli percolation on Euclidean minimal spanning forests

To begin with, we provide an auxiliary result on the maximal length of edges of the graph $G_2(X)$ in a bounded sampling window that is used frequently throughout the manuscript. Let $m_s(G)$ be the length of the longest edge in the geometric graph G having at least one vertex in the cube $Q_s(o) = [-s/2, s/2]^d$. That is,

$$m_s(G) = \max\{|x - y| : x \in Q_s(o), \{x, y\} \text{ is an edge in } G\}.$$

If G does not contain any vertices in $Q_s(o)$, we put $m_s(G) = 0$.

We show that, for $G_2(X)$, and by extension for $\text{FMSF}(X)$, $m_s(G)$ grows much slower than s with high probability (whp). That is, with a probability tending to 1 as $s \rightarrow \infty$.

Lemma 3.1. *Let $\alpha \in (0, 2)$ and X be a stationary point process. Then, for $s \rightarrow \infty$,*

$$\mathbb{P}(m_s(G_2(X)) > s^\alpha) \in o(s^d \mathbb{P}(X \cap Q_{2s^{\alpha/2}}(o) = \emptyset)).$$

Proof. The idea of proof is to show that long edges can exist only if there are large areas not containing any points of X . First, assume that $m_s(G_2(X)) > s^\alpha$. Then there exist $x \in X \cap Q_s(o)$ and $y \in X$ such that $|x - y| \geq s^\alpha$ and $\{x, y\}$ is an edge in $G_2(X)$. Now, put

$s' = s/\lceil s^{1-\alpha/2} \rceil$ and subdivide $Q_{3s}(o)$ into $k(s) = 3^d(s/s')^d$ subcubes $Q^1, \dots, Q^{k(s)}$ of side length s' . Then, we define $P = x + \sqrt{d}s' \frac{y-x}{|y-x|}$ and let Q^i denote the subcube containing P . By construction, every point $P' \in Q^i$ satisfies $\max\{|x - P'|, |P' - y|\} < |x - y|$, so that the definition of $G_2(X)$ gives that $Q^i \cap X = \emptyset$. Therefore,

$$\begin{aligned} \mathbb{P}(m_s(G_2(X)) > s^\alpha) &\leq \mathbb{P}(X \cap Q^i = \emptyset \text{ for some } i \in \{1, \dots, k(s)\}) \\ &\leq \sum_{i=1}^{k(s)} \mathbb{P}(X \cap Q^i = \emptyset) \\ &= k(s)\mathbb{P}(X \cap Q_{s'}(o) = \emptyset). \end{aligned}$$

Since $k(s) \in o(s^d)$, we conclude the proof. □

For $s > 0$ and $\varphi \subset \mathbb{R}^d$ locally finite, $F_s(\varphi)$ denotes the set of edges $\{x, y\}$ of $G_2(\varphi)$ having at least one endpoint in $Q_s(o)$. We show that, under suitable assumptions on the point process X , the size of $F_s(X)$ grows at most polynomially in s with high probability.

Lemma 3.2. *Let X be a stationary point process with positive and finite intensity such that $\mathbb{P}(X \cap Q_s(o) = \emptyset) \in O(s^{-2d})$. Then, $\lim_{s \rightarrow \infty} \mathbb{P}(\#F_s(X) \geq s^{2d+2}) = 0$.*

Proof. Lemma 3.1 shows that

$$\lim_{s \rightarrow \infty} \mathbb{P}(\#F_s(X) \leq (\#(X \cap Q_{3s}(o)))^2) \geq \lim_{s \rightarrow \infty} \mathbb{P}(m_s(G_2(X)) \leq s) = 1$$

and the Markov inequality implies that $\lim_{s \rightarrow \infty} \mathbb{P}(\#(X \cap Q_{3s}(o)) \geq s^{d+1}) = 0$. □

The proof of Theorem 2.5 is based on the observation that, due to the absence of cycles, the number of paths in $FMSF(X)$ starting close to the origin and leaving a large cube centered at the origin grows polynomially in the size of the cube, whereas the probability that any such path is open decays at an exponential rate.

Proof of Theorem 2.5. Let $p \in (0, 1)$ be arbitrary and consider Bernoulli bond percolation on $FMSF(X)$. In the following, we fix a labeling $\{E_n\}$ of the edges of $FMSF(X)$. Let O_s denote the family of open self-avoiding paths in $FMSF(X)$ starting in $Q_1(o)$ and leaving $Q_s(o)$. By stationarity, it suffices to show that $\lim_{s \rightarrow \infty} \mathbb{P}(\#O_s > 0) = 0$.

Let T_s denote the set of all self-avoiding paths of the form $\Gamma = (E_{n_1}, \dots, E_{n_k})$, in $FMSF(X)$ such that

1. at least one endpoint of E_{n_1} is contained in $Q_1(o)$,
2. precisely one endpoint of E_{n_k} is contained in $\mathbb{R}^d \setminus Q_s(o)$,
3. all other endpoints of the edges E_{n_1}, \dots, E_{n_k} are contained in $Q_s(o)$.

Note that O_s consists of all self-avoiding paths in T_s of the form $\Gamma = (E_{n_1}, \dots, E_{n_k})$ such that each edge in Γ is p -open. Thus,

$$\#O_s = \sum_{\Gamma=(E_{n_1}, \dots, E_{n_k}) \in T_s} \prod_{j=1}^k 1_{E_{n_j} \text{ is } p\text{-open}},$$

where 1_A denotes the indicator of the event A . If $m_s(G_2(X)) \leq s^{3/4}$ and s is sufficiently large, then every $\Gamma \in T_s$ consists of at least $s^{1/8}$ edges. Therefore,

$$\begin{aligned} \mathbb{P}(\#O_s > 0) &\leq \mathbb{P}(m_s(G_2(X)) \geq s^{3/4}) + \mathbb{P}(\#T_s \geq s^{2d+3}) \\ &\quad + \mathbb{P}\left(\sum_{\Gamma \in T_s} 1_{m_s(G_2(X)) \leq s^{3/4}} 1_{\#T_s \leq s^{2d+3}} \prod_{n \geq 1: E_n \in \Gamma} 1_{E_n \text{ is } p\text{-open}} > 0\right) \\ &\leq \mathbb{P}(m_s(G_2(X)) \geq s^{3/4}) + \mathbb{P}(\#T_s \geq s^{2d+3}) + s^{2d+3} p^{s^{1/8}}. \end{aligned}$$

By Lemma 3.1, $\mathbb{P}(m_s(G_2(X)) \geq s^{3/4})$ tends to 0 as $s \rightarrow \infty$. Thus, it is sufficient to show that $\mathbb{P}(\#T_s \geq s^{2d+3})$ tends to 0 as $s \rightarrow \infty$. Because $\text{FMSF}(X)$ does not contain cycles, $\#T_s$ is bounded from above by the product of $\#(X \cap Q_1(o))$ and the number of edges of $\text{FMSF}(X)$ leaving $Q_s(o)$. In other words, $\#T_s \leq \#(X \cap Q_1(o)) \cdot \#F_s(X)$, so that

$$\mathbb{P}(\#T_s(X) \geq s^{2d+3}) \leq \mathbb{P}(\#(X \cap Q_1(o)) \geq s) + \mathbb{P}(\#F_s(X) \geq s^{2d+2}).$$

By Lemma 3.2, the right-hand side tends to 0 as $s \rightarrow \infty$. \square

4 Proof of Theorem 2.6

4.1 Bernoulli percolation on creek-crossing graphs

First, we prove part (i) of Theorem 2.6, i.e., we show that if X is a homogeneous Poisson point process in \mathbb{R}^d , then $p_c(G_n(X)) < 1$ for every $n \geq 2$. Without loss of generality, the intensity of X is assumed to be 1. The idea of proof is to define a discretized site-percolation model, which exhibits finite range of dependence and whose percolation implies existence of an infinite open path in the graph $G_n(X)$.

Now, the proof of Theorem 2.6 proceeds roughly as follows. For $z \in \mathbb{Z}^d$ we call the site z open if, for all $z' \in \{\pm e_1, \dots, \pm e_d\}$, the points $q(sz)$ and $q(s(z+z'))$ can be connected by a p -open path in the graph $G_n^p(X)$, where for $i \in \{1, \dots, d\}$ we denote by e_i the i th standard unit vector in \mathbb{R}^d and for $x \in \mathbb{R}^d$ we write $q(x)$ for the closest element of the graph $G_n(X)$ to the point x . In order to apply standard results from percolation theory, we need to ensure that this bond percolation process is m -dependent, in the sense that events defined on lattice regions of d_∞ -distance at least m from one another are independent.

As an important tool in the proof of Theorem 2.6, we use the fact that shortest-path lengths in the graphs $G_n(X)$, $n \geq 2$ grow at most linearly in the Euclidean distance except for events of rapidly decaying probability. More precisely, we make use of the following strengthening of the standard concept of convergence with high probability.

Definition 4.1. A family of events $\{A_s\}_{s \geq 1}$ occurs with very high probability (wvhp) if there exist $c_1, c_2 > 0$ such that $1 - \mathbb{P}(A_s) \leq c_1 \exp(-s^{c_2})$ for all $s \geq 1$.

To quantify the growth of shortest-path length, for $x, y \in \mathbb{R}^d$, we write $\ell(x, y)$ for the minimum Euclidean length amongst all paths in $G_n(X)$ connecting $q(x)$ and $q(y)$. In [19, Theorem 1], a linear growth result is shown under assumptions that are verified in [19, Section 3.1] for creek-crossing graphs on the homogeneous Poisson point process. Hence, in the setting of the present paper, we have the following linear growth result.

Proposition 4.2. Let $n \geq 2$ and consider paths in the graph $G_n(X)$. Then, there exists $u_0 \geq 1$, depending only on d, n and the intensity of the Poisson point process X , such that the events $\{\ell(o, se_1) \leq u_0 s\}$ occur wvhp.

Proposition 4.2 is an extension of earlier growth and shape results for planar graphs; see [2, Theorem 1]. The proof is based on a renormalization argument showing that regions of good connectivity dominate a supercritical percolation process, with the construction of [6] used to extract macroscopic paths of at most linearly growing length.

Now, we leverage Proposition 4.2 to prove Theorem 2.6.

Proof of Theorem 2.6, part (i). The proof has two major parts. In the first step, we show that percolation in the discretized site-percolation model implies existence of an infinite open path in the graph $G_n(X)$. In the second step, we show that the site-percolation process is m -dependent. This allows us to use a result from [22, Theorem 1.3] to infer percolation. We begin by formally defining the approximating site-percolation process. For $s > 0$ and $p \in (0, 1)$ we say that $z \in \mathbb{Z}^d$ is (p, s) -good if and only if the following conditions are satisfied.

1. $X \cap Q_{\sqrt{s}}(sz) \neq \emptyset$ and $\#(X \cap Q_{4u_0s}(sz)) \leq s^{d+1}$,
2. $\ell(sz, s(z + z')) \leq u_0s$ for all $z' \in \{\pm e_1, \dots, \pm e_d\}$,
3. every edge E_m in $G_n(X)$ with $E_m \cap Q_{3u_0s}(sz) \neq \emptyset$ is p -open, and
4. $G_n(X) \cap Q_{3u_0s}(sz) = G_n((X \cap Q_{4u_0s}(sz)) \cup \psi) \cap Q_{3u_0s}(sz)$ for all locally finite $\psi \subset \mathbb{Q}^d \setminus Q_{4u_0s}(sz)$.

The first step of the proof is straightforward, as conditions 1, 2 and 3 imply that, if the (p, s) -good sites percolate, then Bernoulli bond percolation occurs on the graph $G_n(X)$ at parameter p . That is, if $\{z_i\}_{i \geq 1}$ forms an infinite self-avoiding path of (p, s) -good sites, then there exists an infinite open path in $G_n(X)$ connecting the points $\{q(sz_i)\}_{i \geq 1}$. In order to complete the second step, the percolation of (p, s) -good sites, we apply the standard m -dependent percolation technique from [22, Theorem 1.3]. Observe that condition 4 implies that the site-percolation process of (p, s) -good sites exhibits finite range of dependence, and, additionally, the range of dependence can be bounded from above by quantities that do not depend on p and s . We can choose both $s > 0$ and $p \in (0, 1)$ sufficiently large so that the probability that conditions 1, 2, 3 and 4 are satisfied becomes arbitrarily close to one. For condition 1 this follows from elementary properties of the Poisson distribution. Condition 2 follows from Proposition 4.2 and condition 4 follows from [19, Lemma 4]. Using conditions 1 and 4, we deduce that condition 3 is satisfied with probability at least $p^{s^{2d+2}}$. Therefore, by first choosing $s > 0$ and then $p \in (0, 1)$ large, the probability of (p, s) -goodness can be chosen arbitrarily close to 1, so that an application of [22, Theorem 1.3] completes the proof. \square

4.2 Locality of the critical probability in Bernoulli percolation

In this section, we prove part (ii) of Theorem 2.6. That is, if X is a homogeneous Poisson point process, then $\lim_{n \rightarrow \infty} p_c(G_n(X)) = 1$. For this purpose, we fix $p \in (0, 1)$ in the entire section and show that, for $n = n(p)$ sufficiently large, there is no Bernoulli percolation in $G_n(X)$ at the level p . The proof is based on a renormalization argument. First, we choose a discretization of \mathbb{R}^d into cubes, such that whp when constructed on the local configuration in the cube, the graph G_n agrees with the minimal spanning forest. By a monotonicity argument, any p -open path of $G_n(X)$ crossing such a cube is also a p -open paths in the graph constructed on the local configuration. In particular, we obtain a large number of cubes exhibiting long p -open path in the minimal spanning forest. This will lead to a contradiction to the behavior identified in Theorem 2.5.

First, we show that, if the approximation is sufficiently fine, then whp we do not see percolation in the cubes used in the renormalization. Additionally, whp there are no very long edges crossing such cubes.

Lemma 4.3. *Let C_n denote the event that*

1. *there does not exist a p -open path in $G_{n^{d+1}}(X \cap Q_{5n}(o))$ from $\partial Q_n(o)$ to $\partial Q_{3n}(o)$,*
2. *$|x - y| \leq n/2$ holds for every locally finite $\varphi \subset \mathbb{Q}^d \setminus Q_{5n}(o)$ and $x, y \in (X \cap Q_{5n}(o)) \cup \varphi$ such that $\{x, y\}$ forms an edge in $G_{n^{d+1}}((X \cap Q_{5n}(o)) \cup \varphi)$ intersecting $Q_{3n}(o)$.*

Then $\lim_{n \rightarrow \infty} \mathbb{P}(X \cap Q_{5n}(o) \in C_n) = 1$.

Proof. To begin with, we proceed similarly as in Lemma 3.1 to verify the second requirement. Subdivide $Q_{5n}(o)$ into subcubes $Q^1, \dots, Q^{(20d)^d}$ of side length $n/(4d)$. Assume that there is a locally finite $\varphi \subset \mathbb{Q}^d \setminus Q_{5n}(o)$ and $x, y \in (X \cap Q_{5n}(o)) \cup \varphi$ such that $\{x, y\}$ forms an edge of length at least $n/2$ in $G_2((X \cap Q_{5n}(o)) \cup \varphi)$ intersecting $Q_{3n}(o)$. Let $P_0 \in Q_{3n}(o) \cap [x, y]$ be arbitrary and put $z = (x + y)/2$. Then, we define $P = P_0 + \frac{\sqrt{dn}}{4d} \frac{z - P_0}{|z - P_0|}$

and let Q^i denote the subcube containing P . By construction, every point $P' \in Q^i$ satisfies $\max\{|x - P'|, |P' - y|\} < |x - y|$, so that the definition of $G_2((X \cap Q_{5n}(o)) \cup \varphi)$ gives that $Q^i \cap X = \emptyset$. Now, we conclude as in Lemma 3.1. It remains to verify absence of long p -open path. We observe that the relation

$$G_{\#(X \cap Q_{5n}(o))}(X \cap Q_{5n}(o)) = \text{FMSF}(X \cap Q_{5n}(o))$$

implies that the probability that $G_{n^{d+1}}(X \cap Q_{5n}(o))$ and $\text{FMSF}(X \cap Q_{5n}(o))$ coincide tends to 1 as $n \rightarrow \infty$. Hence, it suffices to show that whp there are no p -open paths in $\text{FMSF}(X \cap Q_{5n}(o))$ from $\partial Q_n(o)$ to $\partial Q_{3n}(o)$. This can be achieved using similar arguments as in the proof of Theorem 2.5. More precisely, since $\text{FMSF}(X \cap Q_{5n}(o))$ is a tree, the number of edges in $\text{FMSF}(X \cap Q_{5n}(o))$ is given by $\#(X \cap Q_{5n}(o)) - 1$ so that whp there exist at most $6^d n^d$ edges intersecting $\partial Q_n(o)$ or $\partial Q_{3n}(o)$. Again, using the tree property we conclude that whp the number of paths between $\partial Q_n(o)$ and $\partial Q_{3n}(o)$ is at most $6^{2d} n^{2d}$ and that whp each of these paths consists of at least $\frac{1}{2} \sqrt{n}$ hops. Therefore, the expected number of p -open paths connecting $\partial Q_n(o)$ and $\partial Q_{3n}(o)$ is at most $6^{2d} n^{2d} p^{\sqrt{n}/2}$. Since this expression tends to 0 as $n \rightarrow \infty$, we conclude the proof. \square

Now, we complete the proof of Theorem 2.6 using m -dependent percolation theory.

Proof of Theorem 2.6, part (ii). We start by defining a renormalized site percolation process of good sites in \mathbb{Z}^d , where a site $z \in \mathbb{Z}^d$ is *good* if $(X - nz) \cap Q_{5n}(o) \in C_n$. In particular, the process of good sites is a 5-dependent site percolation process and by Lemma 4.3 the probability that a given site is good becomes arbitrarily close to 1 if n is chosen sufficiently large. In particular, by [22, Theorem 1.3], the percolation process of bad sites is stochastically dominated by a subcritical Bernoulli site percolation process, even if we allow bonds of d_∞ -distance 1. Finally, we establish the connection between bad sites and Bernoulli bond percolation on $G_{n^{d+1}}(X)$, by showing that if the p -open bonds in $G_{n^{d+1}}(X)$ percolated, then so would the process of bad sites. If Γ is an infinite p -open path in $G_{n^{d+1}}(X)$, and $Q_n(nz_1), Q_n(nz_2), \dots$ denotes the sequence of n -cubes intersected by Γ , then we claim that $\{z_1, z_2, \dots\}$ is a path of bad sites such that the d_∞ -distance of successive sites equals 1. This will result in the desired contradiction to the subcriticality of the bad sites. To prove the claim, we note that if z_i was good, then $(X - nz_i) \cap Q_{5n}(o) \in C_n$, so that every edge in Γ intersecting $Q_{3n}(nz_i)$ is of length at most $n/2$. Moreover, by monotonicity every such edge in $G_{n^{d+1}}(X)$ is also an edge in $G_{n^{d+1}}(X \cap Q_{5n}(nz_i))$. In particular, taking a suitable subpath of Γ , we obtain a p -open path in $G_{n^{d+1}}(X \cap Q_{5n}(nz_i))$ connecting $\partial Q_n(nz_i)$ and $\partial Q_{3n}(nz_i)$. But this is possible only if z_i is bad. \square

5 Proof of Theorem 2.7

In this section, we prove Theorem 2.7. That is, we show the absence of percolation in minimal-separator graphs. We make extensive use of generalized descending chains, which were introduced in [20] as a modification of the concept of descending chains considered in [12, 18].

Definition 5.1. Let $b > 0$ and φ be a locally finite subset of \mathbb{R}^d . A (possibly finite) sequence $x_1, x_2, \dots \in \varphi$ forms a b -bounded generalized descending chain if there exists an ordered set $I = \{i_1, i_2, \dots\} \subset \{1, 2, \dots\}$ with the following properties.

1. $|i_{j+1} - i_j| \leq 2$ for all $j \geq 0$,
2. $0 < |x_i - x_{i+1}| \leq b$ for all $i \geq 1$,
3. $|x_{i_{j+1}} - x_{i_j}| < |x_{i_{j-1}+1} - x_{i_{j-1}}|$ for all $j \geq 2$,

where we use the convention $i_0 = 0$.

Definition 5.1 is illustrated in Figure 3, where segments corresponding to elements of I are drawn thicker.

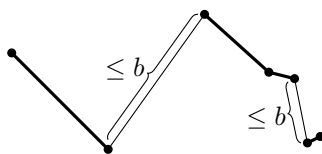


Figure 3: Generalized descending chain

The following deterministic result highlights an essential connection between existence of long paths in $H_n(\varphi)$ and occurrence of long generalized descending chains in φ . Informally speaking, Lemma 5.2 allows us to produce long generalized descending chains from long paths in $H_n(\varphi)$. We recall that $m_s(H_n(\varphi))$ denotes the length of the longest edge in $H_n(\varphi)$ having at least one vertex in $Q_s(o)$.

Lemma 5.2. *Let $n \geq 1$, $s > 64$, $\varphi \subset \mathbb{R}^d$ be a locally finite set containing o and assume $128nm_s(H_n(\varphi)) \leq s$. If $C_{n,H}(\varphi) \not\subset Q_s(o)$, then there is an $nm_s(H_n(\varphi))$ -bounded generalized descending chain starting at $x \in \varphi \cap Q_s(o)$ and leaving $Q_{s/16}(x)$.*

Proof. Let $\gamma = (x_1, \dots, x_l)$ be a self-avoiding path in $H_n(\varphi)$ consisting of $l \geq n + 2$ hops and satisfying $x_1 = o$, $x_2, \dots, x_{l-1} \in Q_s(o)$ and $x_l \notin Q_s(o)$. We say that x_i forms a peak in γ if $i \geq n + 2$ and $\max_{j \in \{i-n, \dots, i-1\}} |x_{j-1} - x_j| \leq |x_{i-1} - x_i|$, see Figure 4.

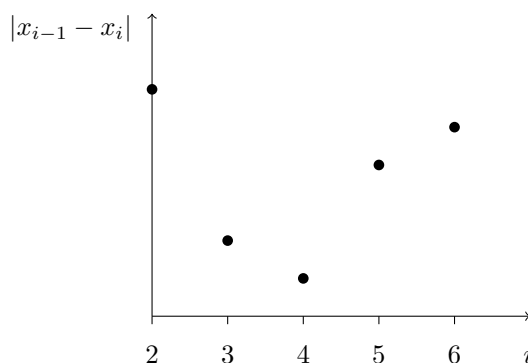


Figure 4: Peak at $i = 6$ when $n = 3$. Vertical axis shows hop lengths.

We claim that for every peak x_i with $i+n < l$ there exists an index $j \in \{i+1, \dots, i+n\}$ such that x_j also constitutes a peak. Let G' be the graph obtained by removing the edge $\{x_{i-1}, x_i\}$ from $G(\varphi, |x_{i-1} - x_i|)$ and write $C(x)$ for the connected component of G' containing $x \in \varphi$. Since $\{x_{i-1}, x_i\}$ forms an edge in $H_n(\varphi)$ there exist $x \in \{x_{i-1}, x_i\}$ and a finite $\psi \subset \varphi$ such that $x \in \psi$, $\#\psi \leq n$ and $\{x_{i-1}, x_i\}$ is a minimal separator between ψ and $\varphi \setminus \psi$. As $\{x_{i-1}, x_i\}$ is a minimal separator, we obtain that $C(x) \subset \psi$. In particular,

$$\#C(x) \leq \#\psi \leq n. \tag{5.1}$$

This forces $x = x_i$ since the contrary would imply

$$x_{i-n-1}, x_{i-n}, \dots, x_{i-1} \in C(x),$$

contradicting (5.1). If none of x_{i+1}, \dots, x_{i+n} is a peak, then $\max_{j \in \{i+1, \dots, i+n\}} |x_j - x_{j-1}| \leq |x_i - x_{i-1}|$, so that $x_j \in C(x_i)$ for all $j \in \{i, \dots, i+n\}$ contradicting (5.1), again. Hence,

if there exists a peak x_i with $i \in \{n + 2, \dots, j_0\}$, then there exists a sequence $x_i = x_{i_1}, x_{i_2}, \dots, x_{i_k}$ of peaks with $i_1 < i_2 < \dots < i_k$, $|i_j - i_{j-1}| \leq n$ and such that $|i_k - l| \leq n$, where $j_0 = \min\{i \in \{1, \dots, l\} : x_i \notin Q_{s/4}(o)\}$ is the index of the first node in the path γ that does not lie in $Q_{s/4}(o)$. In particular, $x_{i_k}, x_{i_{k-1}}, \dots, x_{i_1}, x_{i_1-1}$ forms an $nm_s(H_n(\varphi))$ -bounded generalized descending chain that starts in x_{i_k} and leaves $Q_{s/4}(x_{i_k})$.

It remains to consider the case, where x_i does not form a peak for all $i \in \{n + 2, \dots, j_0\}$. Define $f : \{1, \dots, j_0\} \rightarrow \{1, \dots, j_0\}$ by $i \mapsto \operatorname{argmax}_{j \in \{i-n, \dots, i-1\}} |x_j - x_{j-1}|$ for $i \geq n + 2$ and $i \mapsto 1$ for $i \in \{1, \dots, n + 1\}$. Due to our assumption that none of x_{n+2}, \dots, x_{j_0} forms a peak, we conclude that $|x_j - x_{j-1}| \leq |x_{f(j)} - x_{f(j)-1}|$ for all $j \in \{n + 2, \dots, j_0\}$. We set $k = \inf_{m \geq 1} \{x_{f^{(m)}(j_0)} \in Q_{s/16}(o)\}$, where $f^{(m)}$ denotes the m -fold composition of f . Observe that due to the assumption $nm_s(H_n(\varphi)) \leq s/128$ we have $f^{(k)}(j_0) \geq n + 2$. Therefore, $x_{f^{(k)}(j_0)-1}, x_{f^{(k)}(j_0)}, \dots, x_{j_0-1}, x_{j_0}$ forms an $nm_s(H_n(\varphi))$ -bounded generalized descending chain starting in $x_{f^{(k)}(j_0)-1}$ and leaving $Q_{s/16}(x_{f^{(k)}(j_0)-1})$. \square

In order to prove the absence of percolation with the help of Lemma 5.2, it is useful to consider bounds on the probability for the existence of long generalized descending chains. We make use of the following result from [19, Lemma 5].

Lemma 5.3. *Let X be a homogeneous Poisson point process in \mathbb{R}^d . For each $s > 1$, consider the event that there is no s -bounded generalized descending chain in $X \cup \{o\}$ starting at o and leaving $Q_{8ds^{2d+3}}(o)$. These events occur wvhp.*

For the convenience of the reader, we provide a brief sketch of proof for Lemma 5.3, referring to [19, Lemma 5] for details. We consider the sequence of decreasing segment lengths embedded in a generalized descending chain and note that the occurrence of a long descending chain means that for some sub-interval of $[0, s]$ of length s^{-2d} there are a large number of consecutive segment lengths falling inside this subinterval. Via appropriate renormalization, this event produces a long open path in a suitable subcritical m -dependent percolation process. Similar to the proof of Theorem 2.6, it is then possible to conclude via the finite-dependence approach from [22, Theorem 1.3].

Now we can proceed with the proof of Theorem 2.7.

Proof of Theorem 2.7. Applying the monotone convergence theorem, the second part of Theorem 2.7 becomes an immediate consequence to part (ii) of Theorem 2.5. In order to prove part (i) of Theorem 2.7, note that

$$\begin{aligned} \mathbb{E} \#C_{n,H}(X) &= \int_0^\infty \mathbb{P}(\#C_{n,H}(X) > s) ds \\ &\leq \int_0^\infty \mathbb{P}(\#(X \cap Q_{s^{1/(2d)}}(o)) > s) + \mathbb{P}(C_{n,H}(X) \not\subset Q_{s^{1/(2d)}}(o)) ds. \end{aligned}$$

Hence, it suffices to show that the event $\{C_{n,H}(X) \subset Q_s(o)\}$ occurs with very high probability. Put $A_s^{(1)} = \{m_s(H_n(X \cup \{o\})) \geq s^{1/(4d+8)}\}$. Then, Lemma 3.1 implies that the complements of the events $A_s^{(1)}$ occur wvhp. Furthermore, denote by $A_s^{(2)}$ the event that there exists $\xi \in X \cap Q_s(o)$ and an $ns^{1/(4d+8)}$ -bounded generalized descending chain starting in ξ and leaving $Q_{\sqrt{s}}(\xi)$. Then, Lemma 5.3 implies that the complements of the events $A_s^{(2)}$ occur wvhp. The proof of Theorem 2.7 is completed by noting that the event $\{C_{n,H}(X) \not\subset Q_{s^{1/(2d)}}(o)\}$ implies that at least one of the events $A_s^{(1)}$ or $A_s^{(2)}$ occurs. \square

6 Proofs of Theorems 2.8 and 2.9

In this section, we prove Theorems 2.8 and 2.9. That is, we show that the rates of convergence of the families $\{G_n\}_{n \geq 2}$ and $\{H_n\}_{n \geq 1}$ to the spanning forests are of

polynomial order. The upper bound is established in Section 6.1 and the lower bound in Section 6.2.

6.1 Polynomial upper bounds

The idea for the proof of the upper bound considered in Theorem 2.8 is to show that if the rates of convergence were faster, then the minimal spanning forests would inherit certain geometric properties from their approximations. Specifically, faster decay of $a(n)$ would imply that $\text{FMSF}(X)$ contains cycles and faster decay of $b(n)$ would imply that $\text{WMSF}(X)$ does not percolate. As we know that the minimal spanning forests do not possess these geometric properties, we are able to obtain upper bounds on the rates of convergence.

The upper bounds for $a(n)$ and for $b(n)$ are established separately in Propositions 6.2 and 6.3, respectively. First, we recall from [19, Lemma 6] that there is a close link between the absence of long generalized descending chains in X and the existence of short connections in $G_n(X)$.

Lemma 6.1. *Let $a > 1$, $n \geq 1$ and $\varphi \subset \mathbb{R}^d$ be locally finite. Furthermore, let $\eta, \eta' \in \varphi$ be such that $4n|\eta - \eta'| \leq a$, where η, η' are contained in different connected components of $G_n(\varphi) \cap Q_a(\eta)$. Then, there is an $n|\eta - \eta'|$ -bounded generalized descending chain in φ starting at η and leaving the cube $Q_{a/2}(\eta)$.*

Combining Lemma 6.1 with the observation that long generalized descending chains occur only with a small probability (Lemma 5.3), we can now deduce an upper bound for the decay rate of $a(n)$.

Proposition 6.2. *If X is a Poisson point process, then*

$$\limsup_{n \rightarrow \infty} \frac{-\log a(n)}{\log n} \leq 2d^2 + 6d.$$

Proof. For $n \geq 2$, let $A_n^{(1)}$ denote the event that there exists $z \in \mathbb{Z}^d$ with $|z|_\infty = 4n^{2d+4}$ and such that $|z - q'(z)|_\infty > n^{1/(2d+5)}$, where $q'(z)$ denotes the closest point of X to z . Since X is a Poisson point process, the complements of the events $A_n^{(1)}$ occur whp. Moreover, let $A_n^{(2)}$ denote the event that there exists $x \in X \cap Q_{n^{2d+5}}(o)$ such that there is an $n^{1+1/(2d+4)}$ -bounded generalized descending chain in X starting at x and leaving the cube $Q_{n^{2d+4}}(x)$. Then, we conclude from Lemma 5.3 that the complements of the events $A_n^{(2)}$ occur whp. If neither $A_n^{(1)}$ nor $A_n^{(2)}$ occur, then by Lemma 6.1 the points $q'(z)$ and $q'(z')$ can be connected by a path in $G_n(X)$ which is contained in $[z, z'] \oplus Q_{3n^{2d+4}}(o)$ whenever $z, z' \in \mathbb{Z}^d$ are adjacent sites with $|z|_\infty = |z'|_\infty = 4n^{2d+4}$. Next, put $z_1 = 4n^{2d+4}(-e_1 - e_2)$, $z_2 = 4n^{2d+4}(e_1 - e_2)$, $z_3 = 4n^{2d+4}(e_1 + e_2)$ and $z_4 = 4n^{2d+4}(-e_1 + e_2)$. We conclude that $q'(z_1)$ and $q'(z_3)$ can be connected by a path in

$$G_n(X) \cap (([z_1, z_2] \cup [z_2, z_3]) \oplus Q_{3n^{2d+4}}(o))$$

and also by a path in

$$G_n(X) \cap (([z_1, z_4] \cup [z_4, z_3]) \oplus Q_{3n^{2d+4}}(o)),$$

see Figure 5. Hence, inside $Q_{n^{2d+5}}(o)$ there exists a cycle of edges in $G_n(X)$. Since $\text{FMSF}(X)$ does not contain cycles, we conclude that there exist $x, y \in X \cap Q_{n^{2d+5}}(o)$ such that $\{x, y\}$ is an edge in $G_n(X)$ but not in $\text{FMSF}(X)$. Let $Q^1, \dots, Q^{n^{(2d+5)d}}$ be a subdivision of $Q_{n^{2d+5}}(o)$ into cubes of side length 1. Then,

$$1 - \mathbb{P}(A_n^{(1)}) - \mathbb{P}(A_n^{(2)}) \leq \mathbb{E} \sum_{x \in X \cap Q_{n^{2d+5}}(o)} \sum_{y \in X} 1_{\{x, y\} \text{ is an edge in } G_n(X) \text{ but not in } \text{FMSF}(X)}$$

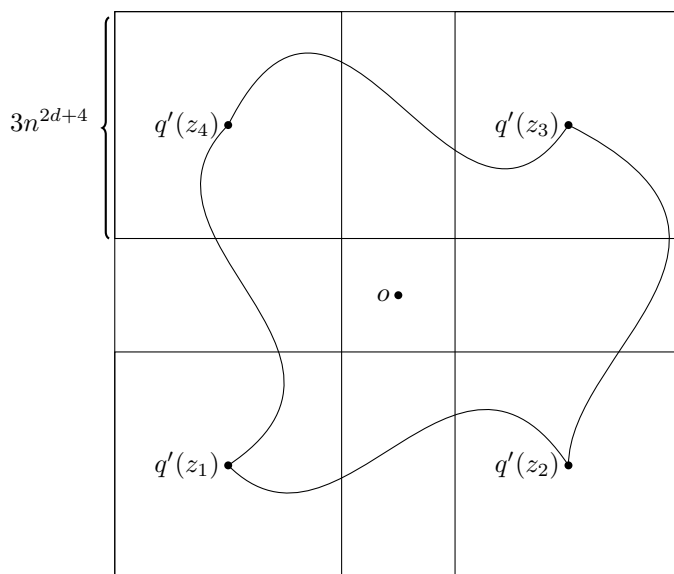


Figure 5: A cycle in $G_n(X) \cap Q_{n^{2d+5}}(o)$

$$\begin{aligned}
 &= \sum_{j=1}^{n^{(2d+5)d}} \mathbb{E} \sum_{x \in X \cap Q^j} \sum_{y \in X} \mathbf{1}_{\{x,y\} \text{ is an edge in } G_n(X) \text{ but not in FMSF}(X)} \\
 &= n^{(2d+5)d} a(n),
 \end{aligned}$$

which completes the proof of Proposition 6.2. □

The reader may have noticed that there still is some room to optimize the above arguments in order to obtain a better exponent for the upper bounds derived in Proposition 6.2. However, even after such improvements the exponent seems to be rather far away from the true exponent in the polynomial rate and the rigorous determination (if possible) of its precise value will certainly need more advanced techniques. Nevertheless, following the arguments and computations in [15, 29], one can provide at least a heuristically motivated conjecture for the true rate in dimension $d = 2$; see also [11] for related numerical results. For two points that are precisely at the critical distance of continuum percolation, the probability that these two points are connected by an edge in $G_n(X)$, but not in $\text{FMSF}(X)$ decays asymptotically as $n^{-\mu}$, where $\mu \approx 1.1056$. This event only depends on points of the Poisson point process that are at chemical distance at most n from the originating two points. Thus, even if the two originating points are not precisely at the critical distance, but still inside an interval of near-criticality of size n^{-1/ν_t} , then the scaling should agree with that at criticality, where $1/\nu_t \approx 0.6634$. This leads to the following conjecture.

Conjecture. The true rate at which $a(n)$ decays is given by $\mu + 1/\nu_t \approx 1.769$.

In order to derive the analogue of Proposition 6.2 for $b(n)$, we make use of the percolation result derived in Lemma 5.2.

Proposition 6.3. *If X is a Poisson point process, then*

$$\limsup_{n \rightarrow \infty} \frac{-\log b(n)}{\log n} \leq 2d^2 + 6d.$$

Proof. For $n \geq 1$ put $A_n^{(1)} = \{m_{n^{2d+5}}(G_2(X)) \geq n^{1/(2d+4)}\}$. Lemma 3.1 implies that the complements of the events $A_n^{(1)}$ occur whp. Furthermore, for $n \geq 1$ let $A_n^{(2)}$ denote the event that there is a $n^{1+1/(2d+4)}$ -bounded generalized descending chain in X starting at $x \in X$ and leaving the cube $Q_{n^{2d+4}}(x)$. Again, Lemma 5.3 implies that the complements of the events $A_n^{(2)}$ occur whp. Now we apply Lemma 5.2 with $s = 16n^{2d+4}$ and to suitably shifted unit cubes inside $Q_n(o)$. Hence, if neither $A_n^{(1)}$ nor $A_n^{(2)}$ occur, then there does not exist a path in $H_n(X)$ starting in $Q_n(o)$ and leaving $Q_{n^{2d+5}}(o)$. However, since each connected component of $\text{WMSF}(X)$ is unbounded, we conclude that if $X \cap Q_n(o) \neq \emptyset$, then there exist $x, y \in X \cap Q_{n^{2d+5}}(o)$ such that $\{x, y\}$ forms an edge in $\text{WMSF}(X)$ but not in $H_n(X)$. Thus,

$$\begin{aligned} 1 - \sum_{i=1}^2 \mathbb{P}(A_n^{(i)}) - \mathbb{P}(X \cap Q_n(o) = \emptyset) &\leq \mathbb{E} \sum_{x \in X \cap Q_{n^{2d+5}}(o)} \sum_{y \in X} 1_{\{x,y\} \text{ is an edge in } \text{WMSF}(X) \setminus H_n(X)} \\ &= \sum_{j=1}^{n^{(2d+5)d}} \mathbb{E} \sum_{x \in X \cap Q_j} \sum_{y \in X} 1_{\{x,y\} \text{ is an edge in } \text{WMSF}(X) \setminus H_n(X)} \\ &= n^{(2d+5)d} b(n), \end{aligned}$$

where as before $Q^1, \dots, Q^{n^{(2d+5)d}}$ denotes a subdivision of $Q_{n^{2d+5}}(o)$ into congruent sub-cubes of side length 1. □

6.2 Polynomial lower bounds

We now derive polynomial lower bounds. Because the derivation of these bounds requires more refined percolation results than the derivation of the upper bounds in Section 6.1, we restrict our attention to the case where X is a unit-intensity Poisson point process in the plane. We make use of the close relationship between the minimal spanning forest and critical percolation, which implies that long-range dependencies can only occur for edges whose lengths are close to the critical radius of continuum percolation. To make this precise, we let

$$r_c = \inf\{r > 0 : \mathbb{P}(G(X, r) \text{ percolates}) > 0\}$$

denote the *critical radius for continuum percolation* associated with the Poisson point process X . Now, we make use of a sophisticated result from two-dimensional continuum percolation stating that at criticality the laws of the sizes of the occupied and vacant connected components admit a power law decay. For the convenience of the reader, we state this result in a form that is most convenient for our purposes. Let $E_{\text{occ},s}$ denote the event that the occupied connected component at the origin of the Boolean model with radius $r_c/2$ is contained in $Q_s(o)$. We also write $E_{\text{vac},s}$ for the corresponding event involving the vacant component.

Lemma 6.4. *It holds that*

$$\liminf_{s \rightarrow \infty} \frac{-\log \max\{\mathbb{P}(E_{\text{occ},s}), \mathbb{P}(E_{\text{vac},s})\}}{\log s} > 0.$$

Lemma 6.4 can be shown using typical arguments from continuum percolation, but as it is not stated explicitly in the standard textbook [25], we include a short proof at the end of this section.

As we take X to be a Poisson point process, we are able to use the Slivnyak-Mecke theorem to obtain alternative representations for $a(n)$ and $b(n)$. For the convenience of the reader, we restate this result and refer the reader e.g. to [26] for further details. We

write \mathbf{N} for the space of all locally finite subsets of \mathbb{R}^d and \mathbf{N} with the smallest σ -algebra such that the evaluation functions $\text{ev}_B : \mathbf{N} \rightarrow \{0, 1, \dots\}$, $\varphi \mapsto \#(\varphi \cap B)$ are measurable for every bounded Borel set $B \subset \mathbb{R}^d$.

Proposition 6.5. *Let X be a unit-intensity Poisson point process and $f : \mathbb{R}^d \times \mathbb{R}^d \times \mathbf{N} \rightarrow [0, \infty)$ be an arbitrary measurable function. Then,*

$$\mathbb{E} \sum_{\substack{x, y \in X \\ x \neq y}} f(x, y, X) = \int_{\mathbb{R}^{2d}} \mathbb{E} f(x, y, X \cup \{x, y\}) \, dx \, dy.$$

In particular,

$$\begin{aligned} a(n) &= \mathbb{E} \# \{(x, y) \in (X \cap Q_1(o)) \times X : \{x, y\} \text{ is an edge in } G_n(X) \setminus \text{FMSF}(X)\} \\ &= \int_{\mathbb{R}^d} \mathbb{P}(\{o, x\} \text{ is an edge in } G_n(X \cup \{o, x\}) \setminus \text{FMSF}(X \cup \{o, x\})) \, dx \\ &= 2\pi \int_0^\infty r \mathbb{P}(\{o, re_1\} \text{ is an edge in } G_n(X \cup \{o, re_1\}) \setminus \text{FMSF}(X \cup \{o, re_1\})) \, dr, \end{aligned}$$

and

$$b(n) = 2\pi \int_0^\infty r \mathbb{P}(\{o, re_1\} \text{ is an edge in } \text{WMSF}(X \cup \{o, re_1\}) \setminus H_n(X \cup \{o, re_1\})) \, dr.$$

As mentioned in Section 2.3, the rate of convergence of the creek-crossing graphs is closely related to the tail behavior of the chemical distance between points that are near to one another in a Euclidean sense. We introduce the random variable R_r as the chemical distance between the vertices $-re_1/2$ and $re_1/2$ in the graph $G(X \cup \{-re_1/2, re_1/2\}, r)$. Then,

$$a(n) = 2\pi \int_0^\infty r \mathbb{P}(n < R_r < \infty) \, dr. \tag{6.1}$$

Likewise, the convergence rates of the minimal separator graphs are related to the tail behavior of cluster sizes. We introduce the random variable S_r as the minimum of the numbers of vertices in the connected components containing $-re_1/2$ and $re_1/2$ if $-re_1/2$ and $re_1/2$ are in separate components, and ∞ otherwise. Then,

$$b(n) = 2\pi \int_0^\infty r \mathbb{P}(n < S_r < \infty) \, dr. \tag{6.2}$$

In order to derive lower bounds on the decay of $a(n)$ and $b(n)$, we subdivide the domains of integration in (6.1) and (6.2) into several parts and consider them separately. We first give a general bound on both $\mathbb{P}(n < R_r < \infty)$ and $\mathbb{P}(n < S_r < \infty)$ which we will use in the case of large r . We write $B_s(x)$ for the ball of radius $s > 0$ centered at $x \in \mathbb{R}^2$.

Lemma 6.6. *Let $r \geq 0$ be arbitrary. Then,*

$$\max\{\mathbb{P}(2 < R_r), \mathbb{P}(S_r < \infty)\} \leq \exp(-\frac{\pi}{4}r^2).$$

Proof. If one of the events $\{R_r > 2\}$ or $\{S_r < \infty\}$ occurs, then $X \cap B_{r/2}(o) = \emptyset$. Thus,

$$\max\{\mathbb{P}(n < R_r < \infty), \mathbb{P}(n < S_r < \infty)\} \leq \exp(-\frac{\pi}{4}r^2). \quad \square$$

Next, we deal with the case of sub-critical r , writing α for the lower limit in Lemma 6.4.

Lemma 6.7. *It holds that*

$$\liminf_{n \rightarrow \infty} \frac{\inf_{r \leq r_c} -\log \max\{\mathbb{P}(n < R_r < \infty), \mathbb{P}(n < S_r < \infty)\}}{\log n} \geq \frac{\alpha}{8}.$$

Proof. We consider the cases involving $a(n)$ and $b(n)$ separately. If $n < R_r < \infty$, then there exists a path in $G(X \cup \{-re_1/2, re_1/2\}, r_c)$ connecting $-re_1/2$ and $re_1/2$, and consisting of at least $n + 1$ edges. In particular, if $\#(X \cap B_{n^{1/4}}(o)) \leq n - 2$, then there exists $x' \in (\frac{r_c}{5}\mathbb{Z}^2) \cap Q_{5r_c}(o)$ such that the connected component of $X \oplus B_{r_c/2}(o)$ at x' leaves $B_{n^{1/4}}(o)$. Hence, by Lemma 6.4,

$$\mathbb{P}(n < R_r < \infty) \leq 5^4 \mathbb{P}(\#(X \cap B_{n^{1/4}}(o)) \geq n - 1) + c_1 n^{-\alpha/4},$$

for some $c_1 > 0$. Since the Poisson concentration inequality [26, Lemma 1.2] shows that the probability of the event $\{\#(X \cap B_{n^{1/4}}(o)) \geq n - 1\}$ decays to 0 exponentially fast in n , we have proven the assertion concerning $a(n)$. Similarly, if $n < S_r < \infty$, then there exists a connected component of $G(X, r_c)$, which intersects $Q_{2r_c}(o)$ and consists of at least n vertices. Again, if $\#(X \cap B_{n^{1/4}}(o)) \leq n - 2$, then there exists $x' \in (\frac{r_c}{5}\mathbb{Z}^2) \cap Q_{5r_c}(o)$ such that the connected component of $X \oplus B_{r_c/2}(o)$ at x' leaves $B_{n^{1/4}}(o)$, and we conclude as before. \square

In the remaining case, r is in the supercritical regime but still not too large.

Lemma 6.8. *It holds that*

$$\liminf_{n \rightarrow \infty} \frac{\inf_{r_c \leq r \leq n^{\alpha/16}} -\log \max\{\mathbb{P}(n < R_r < \infty), \mathbb{P}(n < S_r < \infty)\}}{\log n} \geq \frac{\alpha}{4}.$$

Proof. As in Lemma 6.7, we deal with the assertions involving $a(n)$ and $b(n)$ separately. We first consider the bound for $a(n)$. Assuming that $r < n^{\alpha/16}$ and $n < R_r < \infty$, the connected component of $\mathbb{R}^2 \setminus (X \oplus B_{r_c/2}(o))$ containing the origin leaves $B_{n^{1/4}/2}(o)$ if $\#(X \cap B_{n^{1/4}}(o)) \leq n - 2$. Hence, by Lemma 6.4

$$\mathbb{P}(n < R_r < \infty) \leq \mathbb{P}(\#(X \cap B_{n^{1/4}}(o)) \geq n - 1) + 2^{-\alpha} n^{-\alpha/4}.$$

Since the events $\{\#(X \cap B_{n^{1/4}}(o)) \leq n - 2\}$ occur wvhp, this proves the first claim. For $b(n)$, we claim that if $n < S_r < \infty$, then either $\#(X \cap B_{n^{1/4}}(o)) \leq n - 2$ or the connected component of $\mathbb{R}^2 \setminus (X \oplus B_{r_c/2}(o))$ containing the origin leaves $B_{n^{1/4}/2}(o)$. Once this claim is proven, we conclude the proof as we did for $a(n)$. To show the claim, we note that if the connected component of $\mathbb{R}^2 \setminus (X \oplus B_{r_c/2}(o))$ containing the origin lies within $B_{n^{1/4}/2}(o)$, then there is a closed path in $G(X, r_c)$ that surrounds the origin and is contained in $B_{n^{1/4}}(o)$. In particular, since $\#(X \cap B_{n^{1/4}}(o)) \leq n - 2$ and the connected components of the graph $G(X \cup \{-re_1/2, re_1/2\}, r)$ containing $-re_1/2$ and $re_1/2$ both consist of at least $n + 1$ points, we see that both connected components must intersect the closed path. However, this is a contradiction, since $S_r < \infty$ implies that $-re_1/2$ and $re_1/2$ are contained in different connected components. \square

Using these auxiliary results, we now complete the proof of Theorem 2.9.

Proof of Theorem 2.9. Since the proofs for the cases of $a(n)$ and $b(n)$ are essentially identical, we only consider $a(n)$. The domain of integration in (6.1) is decomposed into three regions: $(0, r_c]$, $[r_c, n^{\alpha/16}]$ and $[n^{\alpha/16}, \infty)$. For the first region, we have

$$\int_0^{r_c} r \mathbb{P}(n < R_r < \infty) dr \leq r_c^2 \sup_{r \in [0, r_c]} \mathbb{P}(n < R_r < \infty),$$

By Lemma 6.7, the right-hand side is of order $O(n^{-\alpha/4})$. For the second region, we have

$$\int_{r_c}^{n^{\alpha/16}} r \mathbb{P}(n < R_r < \infty) dr \leq n^{\alpha/8} \sup_{r \in [r_c, n^{\alpha/16}]} \mathbb{P}(n < R_r < \infty).$$

By Lemma 6.8, the right-hand side is of order $O(n^{-\alpha/8})$. Finally, by Lemma 6.6,

$$\int_{n^{\alpha/16}}^{\infty} r\mathbb{P}(n < R_r < \infty) dr \leq \int_{n^{\alpha/16}}^{\infty} r \exp(-\frac{\pi}{4}r^2) dr = 2\pi^{-1} \exp(-\frac{\pi}{4}n^{\alpha/8}),$$

which completes the proof. □

It remains to provide a proof for Lemma 6.4. As in the classical lattice model [16, Theorem 11.89], the key to proving the desired polynomial rate is a RSW-theorem. Since continuum analogues of the classical theorem were established in [5, 27], Lemma 6.4 can be deduced using standard methods from continuum percolation theory [25]. Still, for the convenience of the reader, we include a proof. For $a, b > 0$, let $\mathcal{H}_{\text{occ}}(a, b)$ denote the event describing the existence of an occupied horizontal crossing of the rectangle $[0, a] \times [0, b]$, i.e., the existence of a connected component of $([0, a] \times [0, b]) \cap (X \oplus B_{r_c/2}(o))$ intersecting both $\{0\} \times [0, b]$ and $\{a\} \times [0, b]$. Furthermore, $\mathcal{H}_{\text{vac}}(a, b)$ denotes the event describing the existence of a vacant horizontal crossing of $[0, a] \times [0, b]$, i.e., the existence of a connected component of $([0, a] \times [0, b]) \cap \mathbb{R}^2 \setminus (X \oplus B_{r_c/2}(o))$ intersecting $\{0\} \times [0, b]$ and $\{a\} \times [0, b]$. Using this notation, we now recall the following two RSW-type theorems for occupied and vacant percolation derived in [5, Lemma 3.3] and [27, Theorem 2.3], respectively. Since it is sufficient for our purposes, we only consider the case, where the radius of the Boolean model is equal to $r_c/2$.

Lemma 6.9. *Let $\varepsilon \in (0, 1/363)$ be arbitrary. Then,*

$$\inf_{s > 0} \inf_{\mathbb{P}(\mathcal{H}_{\text{occ}}(s, 3s)) \geq \varepsilon} \inf_{s' \in (45r_c/2, s/3 - 5r_c/2)} \mathbb{P}(\mathcal{H}_{\text{occ}}(3s', s')) \geq (K\varepsilon^6)^{27}$$

for some constant $K > 0$.

Lemma 6.10. *Let $k \geq 1$ and $\delta_1, \delta_2 > 0$. Then there exist constants $D^{(1)}(k)$ and $D^{(2)}(k, \delta_1, \delta_2)$ for which the following implication holds. If $\ell_1, \ell_2 > 2r_c$ are such that $\mathbb{P}(\mathcal{H}_{\text{vac}}(\ell_1, \ell_2)) \geq \delta_1$ and $\mathbb{P}(\mathcal{H}_{\text{vac}}(\ell_2, 3\ell_1/2)) \geq \delta_2$, then*

$$\mathbb{P}(\mathcal{H}_{\text{vac}}(k\ell_1, \ell_2)) \geq D^{(1)}(k)D^{(2)}(k, \delta_1, \delta_2).$$

We also make use of the following auxiliary result from [5, Lemma 3.2(i)].

Lemma 6.11. *Let $\delta \geq 1/7$ and $\varepsilon \in (0, 1/363)$. If $\mathbb{P}(\mathcal{H}_{\text{vac}}(s, 3s)) \geq \varepsilon$, then*

$$\inf_{\ell \in (r_c, s/3)} \mathbb{P}(\mathcal{H}_{\text{vac}}(\ell, \ell + \delta\ell)) \geq \varepsilon.$$

Finally, we recall an immediate corollary to [5, Theorem 3.4 and 3.5].

Lemma 6.12. *It holds that*

$$\limsup_{s \rightarrow \infty} \max\{\mathbb{P}(\mathcal{H}_{\text{occ}}(3s, s)), \mathbb{P}(\mathcal{H}_{\text{vac}}(3s, s))\} < 1.$$

Using Lemmas 6.9 and 6.10, we now obtain the following standard corollaries which form the basis for the proof of Lemma 6.4.

Corollary 6.13. *It holds that*

$$\liminf_{n \rightarrow \infty} \min\{\mathbb{P}(\mathcal{H}_{\text{vac}}(3^{n+1}, 3^n)), \mathbb{P}(\mathcal{H}_{\text{occ}}(3^{n+1}, 3^n))\} > 0.$$

Proof. For the vacant part, Lemma 6.12 and $\mathbb{P}(\mathcal{H}_{\text{vac}}(s, 3s)) = 1 - \mathbb{P}(\mathcal{H}_{\text{occ}}(3s, s))$ allow us to apply Lemma 6.11, so that $\mathbb{P}(\mathcal{H}_{\text{vac}}(s, 8s/7)) \geq \varepsilon_0$ for all sufficiently large $s \geq 1$. In particular, applying Lemma 6.10 with $\ell_1 = 7 \cdot 3^n/8$, $\ell_2 = 3^n$ and $k = 4$ yields that

$$\liminf_{n \rightarrow \infty} \mathbb{P}(\mathcal{H}_{\text{vac}}(3^{n+1}, 3^n)) \geq D^{(1)}(4)D^{(2)}(4, \varepsilon_0, \varepsilon_0).$$

For the occupied part, using Lemma 6.12 and $\mathbb{P}(\mathcal{H}_{\text{occ}}(s, 3s)) = 1 - \mathbb{P}(\mathcal{H}_{\text{vac}}(3s, s))$ we see that the assumption of Lemma 6.9 is satisfied. Thus,

$$\liminf_{n \rightarrow \infty} \mathbb{P}(\mathcal{H}_{\text{occ}}(3^{n+1}, 3^n)) \geq (K\varepsilon^6)^{27}. \quad \square$$

Using Corollary 6.13, we now to complete the proof of Lemma 6.4.

Proof of Lemma 6.4. We only present a proof of the first claim, since the second one can be shown using similar arguments. Let $g : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be the rotation by $\pi/2$. For every $n \geq 1$ consider the event A_n defined as the joint occurrence of vacant horizontal crossings in the rectangles R_n and $g^2(R_n)$, and of vacant vertical crossings in the rectangles $g(R_n)$ and $g^3(R_n)$, where $R_n = [-3^{2n+1}/2, 3^{2n+1}/2] \times [-3^{2n+1}/2, -3^{2n}/2]$. Put $\varepsilon = \liminf_{n \rightarrow \infty} \mathbb{P}(A_n)$ and note that from the Harris inequality (see e.g. [21, Theorem 1.4]) and Corollary 6.13 we obtain that

$$\varepsilon \geq \liminf_{n \rightarrow \infty} \mathbb{P}(\mathcal{H}_{\text{vac}}(3^{2n+1}, 3^{2n}))^4 > 0.$$

For $s > 0$ let A'_s denote the event that there exists a connected component of $X \oplus B_{r_c/2}(o)$ intersecting both $Q_1(o)$ and $\mathbb{R}^2 \setminus Q_s(o)$ and put $n(s) = \lfloor (\log s / \log 3 - 1) / 2 \rfloor$. Since the events $\{A_n\}_{n \geq n_0}$ are independent provided that $n_0 \geq 1$ is sufficiently large, we arrive at

$$\mathbb{P}(A'_s) \leq \prod_{i=n_0}^{n(s)} \mathbb{P}(A_i^c) \leq (1 - \varepsilon)^{n(s) - n_0 - 1} \leq (1 - \varepsilon)^{-n_0 - 1} s^{\log(1 - \varepsilon) / (4 \log 3)},$$

as asserted. □

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