A tightness criterion for random fields, with application to the Ising model

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Abstract

We present a criterion for a family of random distributions to be tight in local Hölder and Besov spaces of possibly negative regularity on general domains. We then apply this criterion to find the sharp regularity of the magnetization field of the two-dimensional Ising model at criticality, answering a question of [8].

1 Introduction

The main goal of this paper is to provide a tightness criterion in local Hölder and Besov spaces of negative regularity. Roughly speaking, for $\alpha < 0$, a distribution f on $\mathbb{R}^d$ is α-Hölder regular if for every $x \in \mathbb{R}^d$ and every smooth, compactly supported test function φ, we have

$$\lambda^{-d} \langle f, \varphi(\lambda^{-1}(\cdot - x)) \rangle \lesssim \lambda^{\alpha} \qquad (\lambda \to 0). \tag{1.1}$$

Random objects taking values in distribution spaces are of interest in several areas of probability theory. The spaces considered here are close to those introduced in [19] in the context of non-linear stochastic PDE's. Another case of recent interest is the scaling limit of the critical two-dimensional Ising model, see [8, 12]. Fluctuations in homogenization of PDE's with random coefficients are also described by random distributions resembling the Gaussian free field, see [25, 24, 16, 2, 3]. More generally, the class of random objects whose scaling limit is the Gaussian free field is wide, see for instance [27, 17, 7] for the $\nabla \varphi$ random interface model, [21] for random domino tilings, or [22, 5] for Coulomb gases.

As in [19], we wish to devise spaces where (1.1) holds *locally* uniformly over x. Such spaces can be thought of as local Besov spaces. We also wish to allow for distributions that are defined on a domain $U \subseteq \mathbb{R}^d$, but not necessarily on the full space $\mathbb{R}^d$. Besov

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spaces defined on domains of $\mathbb{R}^d$ have already been considered, see e.g. [30, Section 1.11] and the references therein. In the standard definition, a distribution f belongs to the Besov space on U if and only if there exists a distribution g in the Besov space on $\mathbb{R}^d$ (with same exponents) such that $g_{|U} = f$; the infimum of the norm of g over all admissible g's then provides with a norm for the Besov space on U.

In applications to the problems of probability theory mentioned above, this definition is often too stringent. Consider the case of homogenization. Let u_ε be the solution to a Dirichlet problem on $U \subseteq \mathbb{R}^d$, for a divergence-form operator with random coefficients varying on scale $\varepsilon \to 0$. While $\varepsilon^{-\frac{d}{2}}(u_\varepsilon - \mathbb{E}[u_\varepsilon])$ is expected to converge to a random field in the bulk of the domain, this is most likely not the case close to the boundary: a comparably very large boundary layer is expected to be present. This boundary layer should become asymptotically thinner and thinner as $\varepsilon \to 0$, but should nevertheless prevent convergence to happen in a function space such as the one alluded to above.

As a consequence, we will define local Besov spaces that are very tolerant to bad behavior close to the boundary. In short, we take the inductive limit of Besov spaces over compact subsets of the domain. Even when $U = \mathbb{R}^d$, the space thus defined will be stricty larger than the usual Besov space on $\mathbb{R}^d$, because of its locality. This locality is convenient for instance when handling stationary processes.

While we did not find previous works where such spaces appear, readers familiar with Besov spaces will not be surprised by the results presented here. On the other hand, we hope that probabilists will appreciate to find here a tightness criterion that is very convenient to work with. In order to convince the reader of the latter, we now state a particular case of our main tightness result, Theorem 2.30, when the domain U is the whole space $\mathbb{R}^d$. For each $\alpha \in \mathbb{R}$, we define a function space $\mathcal{C}^\alpha_{\mathrm{loc}}(\mathbb{R}^d)$ of distributions with "local α-Hölder regularity", and for any given $r \geqslant |\alpha|$, we identify a *finite* family of compactly supported functions $\phi, (\psi^{(i)})_{1 \leqslant i < 2^d}$ of class C^r such that the following holds.

Theorem 1.1. *Let $(f_m)_{m \in \mathbb{N}}$ be a family of random linear forms on $C^r_c(\mathbb{R}^d)$, let $1 \leqslant p < \infty$ and let $\beta \in \mathbb{R}$ be such that $|\beta| < r$. Assume that there exists a constant $C < \infty$ such that for every $m \in \mathbb{N}$, the following two statements hold:*

$$\sup_{x \in \mathbb{R}^d} \mathbb{E}\left[|\langle f_m, \phi(\,\cdot\, - x)\rangle|^p\right]^{1/p} \leqslant C \, ; \tag{1.2}$$

and, for every $i \in \{1, \ldots, 2^d - 1\}$ and $n \in \mathbb{N}$,

$$\sup_{x \in \mathbb{R}^d} 2^{dn} \, \mathbb{E}\left[\left|\langle f_m, \psi^{(i)}(2^n(\,\cdot\, - x))\rangle\right|^p\right]^{1/p} \leqslant C \, 2^{-n\beta}. \tag{1.3}$$

Then the family (f_m) is tight in $\mathcal{C}^\alpha_{\mathrm{loc}}(\mathbb{R}^d)$ for every $\alpha < \beta - \frac{d}{p}$.

Note that the assumption in Theorem 1.1 simplifies when the field under consideration is stationary, since the suprema in (1.2) and (1.3) can be removed. Although we are primarily motivated by applications of this result for negative exponents of regularity, the statements we prove are insensitive to the sign of this exponent. Naturally, such tightness statements can then be lifted to statements of convergence in $\mathcal{C}^\alpha_{\mathrm{loc}}(\mathbb{R}^d)$ provided that one verifies that the sequence (f_m) has a unique possible limit point (and the latter can be accomplished by checking that for each test function $\chi \in C^\infty_c(\mathbb{R}^d)$ the random variable $\langle f_m, \chi \rangle$ converges in law as m tends to infinity).

When $\alpha < 0$, the definition of the space $\mathcal{C}^\alpha$ is easy to state and in agreement with the intuition of (1.1), see Definition 2.1 below. For $\alpha \in (0,1)$, the space $\mathcal{C}^\alpha$ is (the separable version of) the space of α-Hölder regular functions. For any $\alpha \in \mathbb{R}$, the space $\mathcal{C}^\alpha$ is the Besov space with regularity index α and integrability exponents ∞, ∞, which we denote by $\mathcal{B}^\alpha_{\infty,\infty}$. The assumption in Theorem 1.1 is sufficient to establish tightness in $\mathcal{B}^{\alpha,\,\mathrm{loc}}_{p,q}(\mathbb{R}^d)$

for every $\alpha < \beta$ and $q \in [1, \infty]$. A variant of the argument also provides for a result in the spirit of the Kolmogorov continuity theorem, see Proposition 2.32 below.

The proof of Theorem 1.1 relies on showing that the family f_m belongs with high probability to a bounded set in $\mathcal{C}_{\mathrm{loc}}^{\beta - d/p}(\mathbb{R}^d)$, and exploiting the well-known compact embedding result of Proposition 2.27 to obtain tightness for $\alpha < \beta - \frac{d}{p}$. The more general Theorem 2.30 for arbitrary Besov spaces $\mathcal{B}_{p,q}^{\alpha, \mathrm{loc}}(U)$ follows along the same lines, once we come up with a working definition of $\mathcal{B}_{p,q}^{\alpha, \mathrm{loc}}(U)$ on an arbitrary domain $U \subseteq \mathbb{R}^d$. This is done in Proposition 2.22 using the concept of spanning sequence introduced in Definition 2.21.

The functions ϕ, $(\psi^{(i)})_{1 \leq i < 2^d}$ are chosen as wavelets with compact support. We found it interesting to distinguish the treatment of $\mathcal{C}^\alpha$-type spaces from the more general $\mathcal{B}_{p,q}^\alpha$ spaces. Besides allowing for a simpler definition, the $\mathcal{C}^\alpha$-type spaces indeed enable us to give a fully self-contained proof of Theorem 1.1, save for the existence of wavelets with compact support which of course we do not reprove. We borrow more facts from the literature on function spaces to prove the tightness criterion in general Besov spaces.

We then apply the tightness criterion of Theorem 2.30 to study the magnetization field of the two-dimensional Ising model at the critical temperature. Let $U \subseteq \mathbb{R}^2$ be an open set, and for $a > 0$, let $U_a := U \cap (a\mathbb{Z}^2)$. Denote by $(\sigma_y)_{y \in U_a}$ the Ising spin system at the critical temperature, with, say, $+$ boundary condition, and define the magnetization field

$$\widehat{\Phi}_a := a^{\frac{15}{8}} \sum_{y \in U_a} \sigma_y \, \delta_y, \tag{1.4}$$

where δ_y is the Dirac mass at y. Dirac masses do not belong to $\mathcal{B}_{2,2}^{-1}(U)$, and thus prevent the family $(\widehat{\Phi}_a)_{a \in (0,1]}$ from being tight in this space. Following [8], we will thus prefer to work with the piecewise constant random field

$$\Phi_a := a^{-\frac{1}{8}} \sum_{y \in U_a} \sigma_y \, \mathbb{1}_{S_a(y)}, \tag{1.5}$$

where $S_a(y)$ is the square centered at y of side length a. We note however that the set of limit points of $(\widehat{\Phi}_a)_{a \in (0,1]}$ and of $(\Phi_a)_{a \in (0,1]}$ coincide. Indeed, one can check using Definition 2.1 that the difference $\Phi_a - \widehat{\Phi}_a$ converges to zero almost surely in, say, $\mathcal{C}_{\mathrm{loc}}^\alpha(U)$, for every $\alpha < -3$.

In [8], the authors showed that for $U = [0,1]^2$ and every $\varepsilon > 0$, the family $(\Phi_a)_{a \in (0,1]}$ is tight in $\mathcal{B}_{2,2}^{-1-\varepsilon}(U)^1$, and proceeded to discuss similar results in more general domains. They asked in which precise function spaces the family (Φ_a) is tight.

Using the Onsager correlation bounds and the tightness criterion for general domains of Theorem 2.30, we prove the following result.

Theorem 1.2. *Fix an open set $U \subseteq \mathbb{R}^2$. For every $\varepsilon > 0$ and $p, q \in [1, \infty]$, the family of Ising magnetization fields $(\Phi_a)_{a \in (0,1]}$ on U is tight in $\mathcal{B}_{p,q}^{-\frac{1}{8} - \varepsilon, \, \mathrm{loc}}(U)$.*

We also prove that the previous result is essentially sharp, when $U = \mathbb{R}^2$.

Theorem 1.3. *Let $\varepsilon > 0$ and $p, q \in [1, \infty]$. If Φ is a limit point of the family of Ising magnetization fields $(\Phi_a)_{a \in (0,1]}$ on $\mathbb{R}^2$, then $\Phi \notin \mathcal{B}_{p,q}^{-\frac{1}{8} + \varepsilon, \, \mathrm{loc}}(\mathbb{R}^2)$ with positive probability. In particular, the family $(\Phi_a)_{a \in (0,1]}$ is not tight in $\mathcal{B}_{p,q}^{-\frac{1}{8} + \varepsilon, \, \mathrm{loc}}(\mathbb{R}^2)$.*

It was shown recently that there exists a unique limit point to the family $(\Phi_a)_{a \in (0,1]}$, see [8, 12]. Theorem 1.3 makes it clear that this limit is singular (even on compact subsets) with respect to every $P(\varphi)$ Euclidean field theory, since the latter fields take values in $\mathcal{B}_{p,q}^{-\varepsilon, \, \mathrm{loc}}(\mathbb{R}^2)$ for every $\varepsilon > 0$ and $p, q \in [1, \infty]$.

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[1]See for instance [4, Definition 2.68] or [6] for the identification between the Sobolev spaces used in [8] and the spaces $\mathcal{B}_{2,2}^\alpha$ we use in the present paper.

The paper is organized as follows. In Section 2, we review some properties of wavelets and Besov spaces on $\mathbb{R}^d$, define local Besov spaces, and state and prove the tightness criterion in Theorem 2.30, which is a generalization of Theorem 1.1 above. We also provide a version of Kolmogorov's continuity theorem for local Besov spaces in Proposition 2.32. We then turn to the Ising model in Section 3. After recalling some classical facts about this model, we prove Theorems 1.2 and 1.3. Appendix A contains some functional analysis results which we needed to prove the tightness criterion in the general Besov space $\mathcal{B}_{p,q}^{\alpha,\mathrm{loc}}(U)$.

2 Tightness criterion

We begin by introducing some general notation. If $u = (u_n)_{n \in I}$ is a family of real numbers indexed by a countable set I, and $p \in [1, \infty]$, we write

$$\|u\|_{\ell^p} = \left(\sum_{n \in I} |u_n|^p \right)^{1/p},$$

with the usual interpretation as a supremum when $p = \infty$. We write $B(x, R)$ for the open Euclidean ball centred at x and of radius R. For every open set $U \subseteq \mathbb{R}^d$ and $r \in \mathbb{N} \cup \{\infty\}$, we write $C^r(U)$ to denote the set of r times continuously differentiable functions on U, and $C_c^r(U)$ the subset of $C^r(U)$ of functions with compact support. We simply write C^r and C_c^r for $C^r(\mathbb{R}^d)$ and $C_c^r(\mathbb{R}^d)$ respectively. For $f \in C^r$, we write

$$\|f\|_{C^r} := \sum_{|i| \leqslant r} \|\partial_i f\|_{L^\infty},$$

where the sum is over multi-indices $i \in \mathbb{N}^d$.

We define the Hölder space of exponent $\alpha < 0$ very similarly to [19, Definition 3.7].

Definition 2.1 (Besov-Hölder spaces)**.** *Let $\alpha < 0$, $r_0 := -\lfloor \alpha \rfloor$, and*

$$\mathscr{B}^{r_0} := \{\eta \in C^{r_0} \ : \ \|\eta\|_{C^{r_0}} \leqslant 1 \ \text{and} \ \mathrm{Supp}\, \eta \subseteq B(0,1)\}.$$

For every $f \in C_c^\infty$, denote

$$\|f\|_{\mathcal{C}^\alpha} := \sup_{\lambda \in (0,1]} \sup_{x \in \mathbb{R}^d} \sup_{\eta \in \mathscr{B}^{r_0}} \lambda^{-\alpha} \int_{\mathbb{R}^d} f \, \lambda^{-d} \eta \left(\frac{\cdot - x}{\lambda} \right). \tag{2.1}$$

The Hölder space $\mathcal{C}^\alpha$ is the completion of C_c^∞ with respect to the norm $\| \cdot \|_{\mathcal{C}^\alpha}$. For every open set $U \subseteq \mathbb{R}^d$, the local Hölder space $\mathcal{C}_{\mathrm{loc}}^\alpha(U)$ is the completion of C_c^∞ with respect to the family of seminorms

$$f \mapsto \|\chi f\|_{\mathcal{C}^\alpha},$$

where χ ranges in $C_c^\infty(U)$.

Remark 2.2. By definition, an element of $\mathcal{C}^\alpha$ defines a continuous mapping on

$$\{\eta(\cdot - x) \in C^{r_0} \ : \ x \in \mathbb{R}^d, \ \|\eta\|_{C^{r_0}} \leqslant 1 \ \text{and} \ \mathrm{Supp}\, \eta \subseteq B(0,1)\}$$

and taking values in $\mathbb{R}$. It is straightforward to extend this mapping to a linear form on $C_c^{r_0}$. In particular, we may and will think of $\mathcal{C}^\alpha$ as a subset of the dual of C_c^∞. Similarly, the space $\mathcal{C}_{\mathrm{loc}}^\alpha(U)$ can be seen as a subset of the dual of $C_c^\infty(U)$.

Remark 2.3. Our definition of $\mathcal{C}^\alpha$ (and similarly for $\mathcal{C}_{\mathrm{loc}}^\alpha$) departs slightly from the more common one consisting of considering all distributions f such that $\|f\|_{\mathcal{C}^\alpha}$ is finite. The present definition has the advantage of making the space $\mathcal{C}^\alpha$ separable.

Remark 2.4. As will be seen shortly, the topology of $\mathcal{C}_{\text{loc}}^{\alpha}$ is metrisable.

The gist of the tightness criterion we want to prove is that it suffices to check a condition of the form of (2.1) for a *finite* number of test functions. As announced in the introduction, these test functions are chosen as the basis of a wavelet analysis. We now recall this notion.

Definition 2.5. *A* multiresolution analysis *of $L^2(\mathbb{R}^d)$ is an increasing sequence $(V_n)_{n \in \mathbb{Z}}$ of subspaces of $L^2(\mathbb{R}^d)$, together with a* scaling function *$\phi \in L^2(\mathbb{R}^d)$, such that*

- *$\bigcup_{n \in \mathbb{Z}} V_n$ is dense in $L^2(\mathbb{R}^d)$, $\bigcap_{n \in \mathbb{Z}} V_n = \{0\}$;*
- *$f \in V_n$ if and only if $f(2^{-n} \cdot) \in V_0$;*
- *$(\phi(\,\cdot\, - k))_{k \in \mathbb{Z}^d}$ is an orthonormal basis of V_0.*

Definition 2.6. *A multiresolution analysis is called r-regular ($r \in \mathbb{N}$) if its scaling function ϕ can be chosen in such a way that*

$$|\partial^k \phi(x)| \leqslant C_m (1 + |x|)^{-m}$$

for every integer m and for every multi-index $k \in \mathbb{N}^d$ with $|k| \leqslant r$.

While a given sequence (V_n) can be associated with several different scaling functions to form a multiresolution analysis, a multiresolution analysis is entirely determined by the knowledge of its scaling function. We denote by W_n the orthogonal complement of V_n in V_{n+1}.

Theorem 2.7 (compactly supported wavelets)**.** *For every positive integer r, there exist $\phi, (\psi^{(i)})_{1 \leqslant i < 2^d}$ such that*

- *$\phi, (\psi^{(i)})_{i < 2^d}$ all belong to C_c^r;*
- *ϕ is the scaling function of a multiresolution analysis (V_n);*
- *$(\psi^{(i)}(\,\cdot\, - k))_{i < 2^d, k \in \mathbb{Z}^d}$ is an orthonormal basis of W_0.*

This result is due to [13] (see also e.g. [29, Chapter 6]). We recall that a wavelet basis on $\mathbb{R}^d$ can be constructed from one on $\mathbb{R}$ by taking products of wavelet functions for each coordinate. We also recall from [23, Theorem 2.6.4] that for every multi-index $\beta \in \mathbb{N}^d$ such that $|\beta| < r$ and every $i < 2^d$, we have

$$\int x^{\beta} \psi^{(i)}(x) \, \mathrm{d}x = 0. \tag{2.2}$$

Except for Theorem 2.7 and (2.2), we will give a self-contained proof of the tightness criterion in $\mathcal{C}_{\text{loc}}^{\alpha}(U)$. From now on, we fix both $r \in \mathbb{N}$ and a wavelet basis $\phi, (\psi^{(i)})_{i < 2^d} \in C_c^r$, as obtained with Theorem 2.7. Let R be such that

$$\operatorname{Supp} \phi \subseteq B(0, R), \qquad \operatorname{Supp} \psi^{(i)} \subseteq B(0, R) \quad (i < 2^d). \tag{2.3}$$

For any $n \in \mathbb{Z}$ and $x \in \mathbb{R}^d$, if we define

$$\phi_{n,x}(y) := 2^{dn/2} \, \phi(2^n(y - x)) \tag{2.4}$$

and $\Lambda_n = \mathbb{Z}^d / 2^n$, then $(\phi_{n,x})_{x \in \Lambda_n}$ is an orthonormal basis of V_n. Similarly, we define

$$\psi_{n,x}^{(i)}(y) := 2^{dn/2} \, \psi^{(i)}(2^n(y - x)),$$

so that $(\psi_{n,x}^{(i)})_{i < 2^d, x \in \Lambda_n, n \in \mathbb{Z}}$ is an orthonormal basis of $L^2(\mathbb{R}^d)$. For $f \in L^2(\mathbb{R}^d)$, we set

$$v_{n,x} f := (f, \phi_{n,x}), \qquad w_{n,x}^{(i)} f := (f, \psi_{n,x}^{(i)}), \tag{2.5}$$

where $(\cdot, \cdot)$ is the scalar product of $L^2(\mathbb{R}^d)$. Denoting by $\mathscr{V}_n$ and $\mathscr{W}_n$ the orthogonal projections on V_n, W_n respectively, we have

$$\mathscr{V}_n f = \sum_{x \in \Lambda_n} v_{n,x}(f) \, \phi_{n,x}, \qquad \mathscr{W}_n f = \sum_{i < 2^d, x \in \Lambda_n} w_{n,x}^{(i)}(f) \, \psi_{n,x}^{(i)}, \tag{2.6}$$

and for every $k \in \mathbb{Z}$,

$$f = \mathscr{V}_k f + \sum_{n=k}^{+\infty} \mathscr{W}_n f \tag{2.7}$$

in $L^2(\mathbb{R}^d)$.

Definition 2.8 (Besov spaces). *Let $\alpha \in \mathbb{R}$, $|\alpha| < r$ and $p, q \in [1, \infty]$. The Besov space $\mathcal{B}_{p,q}^\alpha$ is the completion of C_c^∞ with respect to the norm*

$$\|f\|_{\mathcal{B}_{p,q}^\alpha} := \|\mathscr{V}_0 f\|_{L^p} + \big\|(2^{\alpha n} \|\mathscr{W}_n f\|_{L^p})_{n \in \mathbb{N}}\big\|_{\ell^q}. \tag{2.8}$$

The local Besov space $\mathcal{B}_{p,q}^{\alpha, \mathrm{loc}}(U)$ *is the completion of $C^\infty(U)$ with respect to the family of semi-norms*

$$f \mapsto \|\chi f\|_{\mathcal{B}_{p,q}^\alpha}$$

indexed by $\chi \in C_c^\infty(U)$.

Remark 2.9. This characterization of Besov spaces is the most useful for us to obtain a tightness result for local domains, as stressed in the Introduction, due to its projection on compactly supported wavelets. However, in Appendix A we outline a proof of the equivalence between Definition 2.1 and one based on the Littlewood-Paley decomposition (see Definition A.4). This second definition is the one used in [4] to prove a large number of results on the spaces $\mathcal{B}_{p,q}^\alpha$, including their relation with other well-known function spaces.

Remark 2.10. Similarly to the observation of Remark 2.3, our definition of $\mathcal{B}_{p,q}^\alpha$ departs slightly from the usual one, which consists in considering the set of distributions such that (2.8) is finite. The two definitions coincide only when both p and q are finite. The present definition has the advantage of making the space separable in every case, by taking the closure of a family of smooth compactly supported functions. On the other hand, for $\alpha \in (0, 1)$, the more standard definition of the space $\mathcal{B}_{\infty,\infty}^\alpha$ would coincide with the Hölder space of regularity α (see Appendix A and [4]), which is not separable.

Remark 2.11. One can check that the space $\mathcal{B}_{p,q}^\alpha$ of Definition 2.8 does not depend on the choice of the multiresolution analysis, in the sense that for any $r > |\alpha|$, any different r-regular multiresolution analysis yields an equivalent norm (see Proposition A.2 of the appendix). In this section, we recall that we fix $r \in \mathbb{N}$, and consider Besov spaces $\mathcal{B}_{p,q}^\alpha$ with $\alpha \in \mathbb{R}$, $|\alpha| < r$.

Remark 2.12. It is clear that if $\alpha_1 \leqslant \alpha_2 \in \mathbb{R}$ and $q_1 \geqslant q_2 \in [1, \infty]$, then

$$\|f\|_{\mathcal{B}_{p,q_1}^{\alpha_1}} \leqslant C \|f\|_{\mathcal{B}_{p,q_2}^{\alpha_2}},$$

where C is independent of $f \in C_c^\infty$. In particular, the space $\mathcal{B}_{p,q_2}^{\alpha_2}$ is continuously embedded in $\mathcal{B}_{p,q_1}^{\alpha_1}$. Similarly, for $p_1 \leqslant p_2$ and for a given $\chi \in C_c^\infty$, there exists a constant $C < \infty$ such that for every $f \in C_c^\infty$,

$$\|\chi f\|_{\mathcal{B}_{p_1,q}^\alpha} \leqslant C \|\chi f\|_{\mathcal{B}_{p_2,q}^\alpha}.$$

Indeed, this is a consequence of Jensen's inequality and the fact that for each $n \in \mathbb{N}$, the support of $\mathscr{W}_n(\chi f)$ is contained in the bounded set $2R + \mathrm{Supp}\,\chi$. Hence, the space $\mathcal{B}_{p_2,q}^{\alpha, \mathrm{loc}}(U)$ is continuously embedded in $\mathcal{B}_{p_1,q}^{\alpha, \mathrm{loc}}(U)$.

Remark 2.13. A different notion of Hölder space on a domain, encoding more precise weighted information on the size of the distribution as one gets closer and closer to the boundary of the domain, has been introduced in the very recent work [11].

The finiteness of $\|f\|_{\mathcal{B}_{p,q}^\alpha}$ can be expressed in terms of the magnitude of the coefficients $v_{n,x}(f)$ and $w_{n,x}^{(i)}(f)$.

Proposition 2.14 (Besov spaces via wavelet coefficients). *For every $p \in [1,\infty]$, there exists $C \in (0,\infty)$ such that for every $f \in C_c^\infty$ and every $n \in \mathbb{Z}$,*

$$C^{-1} \|\mathscr{V}_n f\|_{L^p} \leqslant 2^{dn\left(\frac{1}{2}-\frac{1}{p}\right)} \left\|(v_{n,x}f)_{x\in\Lambda_n}\right\|_{\ell^p} \leqslant C \|\mathscr{V}_n f\|_{L^p}, \tag{2.9}$$

$$C^{-1} \|\mathscr{W}_n f\|_{L^p} \leqslant 2^{dn\left(\frac{1}{2}-\frac{1}{p}\right)} \left\|\left(w_{n,x}^{(i)}f\right)_{i<2^d,x\in\Lambda_n}\right\|_{\ell^p} \leqslant C \|\mathscr{W}_n f\|_{L^p}. \tag{2.10}$$

Proof. We will prove only (2.9) in detail, since (2.10) follows in the same way. (See also [23, Proposition 6.10.7].) Recalling (2.3), we have $\operatorname{Supp} \phi_{n,x} \subseteq B(x, 2^{-n}R)$ and thus, for every $y \in \mathbb{R}$,

$$\mathscr{V}_n f(y) = \sum_{x\in\Lambda_n, x\in B(y,2^{-n}R)} v_{n,x}(f)\phi_{n,x}(y). \tag{2.11}$$

Let $p < +\infty$. Since the sum $\sum_{x\in\Lambda_n, x\in B(y,2^{-n}R)}$ is finite uniformly over n, we can use Jensen's inequality to obtain:

$$\begin{aligned}
\|\mathscr{V}_n f\|_{L^p}^p &= \int \left|\sum_{x\in\Lambda_n, x\in B(y,2^{-n}R)} v_{n,x}(f)\phi_{n,x}(y)\right|^p \, \mathrm{d}y \\
&\lesssim \int \sum_{x\in\Lambda_n, x\in B(y,2^{-n}R)} |v_{n,x}(f)\phi_{n,x}(y)|^p \, \mathrm{d}y \\
&\lesssim \sum_{x\in\Lambda_n} |v_{n,x}(f)|^p \int_{B(x,2^{-n}R)} |\phi_{n,x}(y)|^p \, \mathrm{d}y \\
&\lesssim \left\|(v_{n,x}f)_{x\in\Lambda_n}\right\|_{\ell^p}^p \|\phi_{n,0}\|_{L^p}^p.
\end{aligned}$$

The leftmost inequality of (2.9) follows from the scaling properties of $\phi_{n,0}$, namely:

$$\|\phi_{n,x}\|_{L^p} = 2^{dn\left(\frac{1}{2}-\frac{1}{p}\right)}\|\phi_{0,x}\|_{L^p}. \tag{2.12}$$

For $p = +\infty$ we estimate $\|\mathscr{V}_n f\|_{L^\infty}$ using

$$|\mathscr{V}_n f(y)| \lesssim R^d \sup_{x\in\Lambda_n} |v_{n,x}f| |\phi_{n,x}(y)| \lesssim \|\phi_{n,0}(y)\|_{L^\infty} \sup_{x\in\Lambda_n} |v_{n,x}f|.$$

This yields the upper bound for $\|\mathscr{V}_n f\|_{L^p}$.

As for the rightmost inequality, notice that $v_{n,x}(\mathscr{V}_n f) = v_{n,x}f$, that is, $v_{n,x}f = \int \phi_{n,x}(y)\mathscr{V}_n f(y)\mathrm{d}y$. Let $p < +\infty$ and p' be its conjugate exponent. By Hölder's inequality,

$$|v_{n,x}f| \leqslant \|\phi_{n,x}\|_{L^{p'}} \|\mathscr{V}_n f \mathbb{1}_{B(x,2^{-n}R)}\|_{L^p},$$

and moreover,

$$\sum_{x\in\Lambda_n} \int |\mathscr{V}_n f(y)|^p \mathbb{1}_{B(x,2^{-n}R)}(y)\mathrm{d}y = \int |\mathscr{V}_n f(y)|^p \sum_{x\in\Lambda_n} \mathbb{1}_{B(x,2^{-n}R)}(y)\mathrm{d}y \lesssim \|\mathscr{V}_n f\|_{L^p}^p.$$

By (2.12), we have $\|\phi_{n,x}\|_{L^{p'}} \lesssim 2^{dn\left(\frac{1}{2}-\frac{1}{p'}\right)} = 2^{-dn\left(\frac{1}{2}-\frac{1}{p}\right)}$, and this concludes the proof for the case $p < +\infty$. For $p = +\infty$, we just notice that $|v_{n,x}f| \leqslant \|\mathscr{V}_n f\|_{L^\infty}\|\phi_{n,x}\|_{L^1}$. $\qquad\square$

Remark 2.15. For each $k \in \mathbb{Z}$, the norm

$$\|f\|_{\mathcal{B}_{p,q}^{\alpha,k}} = \left\|(v_{k,x}f)_{x \in \Lambda_k}\right\|_{\ell^p} + \left\|\left(2^{\alpha n}2^{dn\left(\frac{1}{2}-\frac{1}{p}\right)}\left\|\left(w_{n,x}^{(i)}f\right)_{i<2^d,x\in\Lambda_n}\right\|_{\ell^p}\right)_{n\geqslant k}\right\|_{\ell^q}$$

is equivalent to that in (2.8). This is easy to show using Proposition 2.14 and the definition of multiresolution analysis.

As we now show, for $\alpha < 0$, the Besov space $\mathcal{B}_{\infty,\infty}^\alpha$ of Definition 2.8 coincides with the Besov-Hölder space $\mathcal{C}^\alpha$ given by Definition 2.1.

Proposition 2.16. *Let $\alpha < 0$. There exist $C_1, C_2 \in (0,\infty)$ such that for every $f \in C_c^\infty$, we have*

$$C_1\|f\|_{\mathcal{C}^\alpha} \leqslant \|f\|_{\mathcal{B}_{\infty,\infty}^\alpha} \leqslant C_2\|f\|_{\mathcal{C}^\alpha}. \tag{2.13}$$

Proof. The result is classical and proved e.g. in [19, Proposition 3.20]. We recall the proof for the reader's convenience. One can check that there exists $C < \infty$ such that for every $f \in C_c^\infty$, $n \in \mathbb{Z}$ and $x \in \mathbb{R}^d$,

$$2^{\frac{dn}{2}}|w_{n,x}^{(i)}f| \leqslant C\|f\|_{\mathcal{C}^\alpha}, \tag{2.14}$$

and this yields the second inequality in (2.13). Conversely, we let $f \in C^\infty$ satisfy $\|f\|_{\mathcal{B}_{\infty,\infty}^\alpha} \leqslant 1$. We aim to show that there exists a constant $C < \infty$ (independent of f) such that for every $y \in \mathbb{R}^d$, $\eta \in \mathscr{B}^{r_0}$ (with $r_0 = -\lfloor\alpha\rfloor$) and $\lambda \in (0,1]$, we have

$$\lambda^{-\alpha}\int_{\mathbb{R}^d} f\,\lambda^{-d}\eta\left(\frac{\cdot - y}{\lambda}\right) \leqslant C.$$

We write $\eta_{\lambda,y} := \lambda^{-d}\eta((\cdot - y)/\lambda)$, and observe that

$$\int f\,\eta_{\lambda,y} = \sum_{x\in\Lambda_0}(v_{0,x}f)(v_{0,x}\eta_{\lambda,y}) + \sum_{i<2^d}\sum_{n\geqslant 0}\sum_{x\in\Lambda_n}(w_{n,x}^{(i)}f)(w_{n,x}^{(i)}\eta_{\lambda,y}).$$

We consider only the second term of the sum above, as the first one can be obtained with the same technique. By the definition of $\|f\|_{\mathcal{B}_{\infty,\infty}^\alpha}$, for every $n \geqslant 0$, we have

$$2^{\frac{dn}{2}}|w_{n,x}^{(i)}f| \leqslant C2^{-\alpha n}. \tag{2.15}$$

In order for $w_{n,x}^{(i)}\eta_{\lambda,y}$ to be non-zero, we must have $|x - y| \leqslant C(\lambda \vee 2^{-n})$. Moreover, by a Taylor expansion of η around x and (2.2), we have

$$2^{-n} \leqslant \lambda \quad\Longrightarrow\quad 2^{\frac{dn}{2}}|w_{n,x}^{(i)}\eta_{\lambda,y}| \leqslant C2^{-rn}\lambda^{-d-r}, \tag{2.16}$$

while

$$2^{-n} \geqslant \lambda \quad\Longrightarrow\quad 2^{\frac{dn}{2}}|w_{n,x}^{(i)}\eta_{\lambda,y}| \leqslant C2^{dn}. \tag{2.17}$$

and the same bound holds for $2^{\frac{dn}{2}}|v_{0,x}\eta_{\lambda,y}|$. For each $n \geqslant 0$, there exists a compact set $K_n \subseteq \Lambda_n$ independent from f such that the condition $w_{n,x}^{(i)}\eta_{\lambda,y} \neq 0$ implies that $x \in K_n$. Since the sum over $x \in \Lambda_n \cap K_n$ has less than $C2^{nd}$ terms, the result follows. $\qquad\square$

Remark 2.17. Notice that we can replace $r_0 = -\lfloor\alpha\rfloor$ by a generic integer $r > |\alpha|$ in Definition 2.1, obtaining an equivalent norm. Indeed, Proposition 2.16 shows that it suffices to control the behavior of f against shifted and rescaled versions of the wavelet functions ϕ and $\psi^{(i)}$.

Remark 2.18. In view of Proposition 2.16, when $\alpha < 0$, we have $\mathcal{C}^\alpha = \mathcal{B}^\alpha_{\infty,\infty}$, and $\mathcal{C}^\alpha_{\mathrm{loc}}(U) = \mathcal{B}^{\alpha,\,\mathrm{loc}}_{\infty,\infty}(U)$ where $\mathcal{C}^\alpha$ and $\mathcal{C}^\alpha_{\mathrm{loc}}(U)$ are given by Definition 2.1. By extension, we set

$$\mathcal{C}^\alpha := \mathcal{B}^\alpha_{\infty,\infty} \quad \text{and} \quad \mathcal{C}^\alpha_{\mathrm{loc}}(U) := \mathcal{B}^{\alpha,\,\mathrm{loc}}_{\infty,\infty}(U)$$

for every $\alpha \in \mathbb{R}$. Although we will not use this fact here, note that for $\alpha \in (0,1)$, there exists a constant $C < \infty$ such that for every $f \in C^\infty_c$,

$$C^{-1}\|f\|_{\mathcal{C}^\alpha} \leqslant \|f\|_{L^\infty} + \sup_{0 < |x-y| \leqslant 1} \frac{|f(y)-f(x)|}{|y-x|^\alpha} \leqslant C\|f\|_{\mathcal{C}^\alpha}. \tag{2.18}$$

The proof of this fact can be obtained similarly to that of Proposition 2.16 (see also [23, Theorem 6.4.5]). Hence, one can show that for $\alpha \in (0,\infty) \setminus \mathbb{N}$, the space $\mathcal{C}^\alpha$ is the separable version of the space $C^{\lfloor \alpha \rfloor}$ of functions whose derivative of order $\lfloor \alpha \rfloor$ is $(\alpha - \lfloor \alpha \rfloor)$-Hölder continuous. (By "separable version of", we mean that there is a natural norm associated with the space just described, and we take the completion of the space of smooth functions with respect to this norm.) For $\alpha \in \mathbb{N}$, the space $\mathcal{C}^\alpha$ is stricty larger than the (separable version of) the space of C^α functions. We refer to [4] for details.

The following proposition is a weak manifestation of the multiplicative structure of Besov spaces, which is exposed in more details in the appendix.

Proposition 2.19 (multiplication by a smooth function). *Let $r > |\alpha|$ and $p, q \in [1,\infty]$. For every $\chi \in C^r_c$, the mapping $f \mapsto \chi f$ extends to a continuous functional from $\mathcal{B}^\alpha_{p,q}$ to itself.*

Partial proof of Proposition 2.19. We give a proof for the particular case $\alpha < 0$ and $p = q = \infty$. The general case is postponed to the appendix. Let $f \in C^\infty_c$ and consider the integral

$$\lambda^{-d} \int f(y)\chi(y)\eta\left(\frac{y-x}{\lambda}\right) \mathrm{d}y.$$

For every $\lambda > 0$ and $x \in \mathbb{R}^d$, define $\tilde{\eta}$ as: $\tilde{\eta}_{\lambda,x}\left(\frac{y-x}{\lambda}\right) = \chi(y)\eta\left(\frac{y-x}{\lambda}\right)$. Then $\tilde{\eta}_{\lambda,x}(z) = \chi(z\lambda + x)\eta(z)$ for $z \in \mathbb{R}^d$. One can notice that $\tilde{\eta}_{\lambda,x} \in C^{r_0}_c$ and $\mathrm{Supp}\,\tilde{\eta}_{\lambda,x} \subseteq \mathrm{Supp}\,\eta$. Hence, by Proposition 2.16, there exists $C > 0$ (possibly different in every line) such that:

$$\lambda^{-d} \int f(y)\chi(y)\eta\left(\frac{y-x}{\lambda}\right) \mathrm{d}y \leqslant C\lambda^\alpha \|f\|_{\mathcal{B}^\alpha_{\infty,\infty}} \|\tilde{\eta}_{\lambda,x}\|_{C^{r_0}_c}$$

$$\leqslant C\lambda^\alpha \|f\|_{\mathcal{B}^\alpha_{\infty,\infty}} \|\chi(\lambda\,\cdot\,)\|_{C^{r_0}_c}$$

$$\leqslant C\lambda^\alpha \|f\|_{\mathcal{B}^\alpha_{\infty,\infty}} \|\chi\|_{C^{r_0}_c},$$

uniformly over $f \in C^\infty_c$, $\lambda \in (0,1]$, $\eta \in \mathscr{B}^{r_0}$ and $x \in \mathbb{R}^d$. The result follows by the fact that C^∞_c is dense in $\mathcal{B}^\alpha_{\infty,\infty}$. $\qquad\square$

Remark 2.20. The notion of a complete space makes sense for arbitrary topological vector spaces, since a description of neighbourhoods of the origin is sufficient for defining what a Cauchy sequence is. Yet, in our present setting, the topology of $\mathcal{B}^{\alpha,\,\mathrm{loc}}_{p,q}(U)$ is in fact metrisable. To see this, note that there is no loss of generality in restricting the range of χ indexing the semi-norms to a countable subset of $C^\infty_c(U)$, e.g. $\{\chi_n, n \in \mathbb{N}\}$ such that for every compact $K \subseteq U$, there exists n such that $\chi_n = 1$ on K. Indeed, it is then immediate from Proposition 2.19 that if χ has support in K, then $\|\chi f\|_{\mathcal{B}^\alpha_{p,q}} \leqslant C\|\chi_n f\|_{\mathcal{B}^\alpha_{p,q}}$ for some C not depending on f. Hence, we can view $\mathcal{B}^{\alpha,\,\mathrm{loc}}_{p,q}(U)$ as a complete (Fréchet) space equipped with the metric

$$d_{\mathcal{B}^{\alpha,\,\mathrm{loc}}_{p,q}(U)}(f,g) = \sum_{n=0}^{+\infty} 2^{-n}\, \|\chi_n(f-g)\|_{\mathcal{B}^\alpha_{p,q}} \wedge 1. \tag{2.19}$$

We now give an alternative family of semi-norms, based on wavelet coefficients, that is equivalent to the family given in Definition 2.8 or Remark 2.20.

Definition 2.21 (spanning sequence). *Recall that R is such that (2.3) holds. Let $K \subseteq U$ be compact and $k \in \mathbb{N}$. We say that the pair (K, k) is adapted if*

$$2^{-k} R < \mathrm{dist}(K, U^c). \tag{2.20}$$

We say that the set $\mathscr{K}$ is a spanning sequence if it can be written as

$$\mathscr{K} = \{(K_n, k_n), \ n \in \mathbb{N}\},$$

where (K_n) is an increasing sequence of compact subsets of U such that $\bigcup_n K_n = U$ and for every n, the pair (K_n, k_n) is adapted.

For every adapted pair (K, k), $f \in C_c^\infty(U)$ and $n \geqslant k$, we let

$$v_{n,K,p} f = 2^{dn\left(\frac{1}{2} - \frac{1}{p}\right)} \left\| (v_{n,x} f)_{x \in \Lambda_n \cap K} \right\|_{\ell^p}, \tag{2.21}$$

$$w_{n,K,p} f = 2^{dn\left(\frac{1}{2} - \frac{1}{p}\right)} \left\| \left(w_{n,x}^{(i)} f\right)_{i < 2^d, x \in \Lambda_n \cap K} \right\|_{\ell^p}, \tag{2.22}$$

and we define the semi-norm

$$\|f\|_{\mathcal{B}_{p,q}^{\alpha,K,k}} = v_{k,K,p} f + \left\| (2^{\alpha n} w_{n,K,p} f)_{n \geqslant k} \right\|_{\ell^q}. \tag{2.23}$$

Proposition 2.22 (Local Besov spaces via wavelet coefficients). *Let $p, q \in [1, \infty]$.*
(1) For every adapted pair (K, k), the mapping $f \mapsto \|f\|_{\mathcal{B}_{p,q}^{\alpha,K,k}}$ extends to a continuous semi-norm on $\mathcal{B}_{p,q}^{\alpha,\mathrm{loc}}(U)$.
(2) The topology induced by the family of semi-norms $\| \cdot \|_{\mathcal{B}_{p,q}^{\alpha,K,k}}$, indexed by adapted pairs (K, k), is that of $\mathcal{B}_{p,q}^{\alpha,\mathrm{loc}}(U)$.
(3) Let $\mathscr{K}$ be a spanning sequence. Part (2) above remains true when considering only the seminorms indexed by pairs in $\mathscr{K}$.

Remark 2.23. Another metric that is compatible with the topology on $\mathcal{B}_{p,q}^{\alpha,\mathrm{loc}}(U)$ is thus given by

$$d'_{\mathcal{B}_{p,q}^{\alpha,\mathrm{loc}}(U)}(f, g) = \sum_{n=0}^{+\infty} 2^{-n} \|f - g\|_{\mathcal{B}_{p,q}^{\alpha,K_n,k_n}} \wedge 1,$$

where $\mathscr{K} = \{(K_n, k_n), n \in \mathbb{N}\}$ is any given spanning sequence.

Proof of Proposition 2.22. In order to prove parts (1–2) of the proposition, it suffices to show the following two statements.

> For every adapted pair (K, k), there exists $\chi \in C_c^\infty(U)$ and $C < \infty$ s.t.
> $$\forall f \in C^\infty(U), \ \|f\|_{\mathcal{B}_{p,q}^{\alpha,K,k}} \leqslant C \|\chi f\|_{\mathcal{B}_{p,q}^\alpha}; \tag{2.24}$$

> For every $\chi \in C_c^\infty(U)$, there exists (K, k) adapted pair and $C < \infty$ s.t.
> $$\forall f \in C^\infty(U), \ \|\chi f\|_{\mathcal{B}_{p,q}^\alpha} \leqslant C \|f\|_{\mathcal{B}_{p,q}^{\alpha,K,k}}. \tag{2.25}$$

We begin with (2.24). Let (K, k) be an adapted pair, and let $\chi \in C_c^\infty(U)$ be such that $\chi = 1$ on $K + \overline{B}(2^{-k} R)$. For every $n \geqslant k$ and $x \in \Lambda_n \cap K$,

$$v_{n,x} f = v_{n,x}(\chi f), \qquad w_{n,x}^{(i)} f = w_{n,x}^{(i)}(\chi f) \quad (i < 2^d),$$

and as a consequence,

$$v_{n,K,p}(f) \leqslant 2^{dn\left(\frac{1}{2} - \frac{1}{p}\right)} \left\| (|v_{n,x}(\chi f)|)_{x \in \Lambda_n} \right\|_{\ell^p} \leqslant C \|\mathscr{V}_n(\chi f)\|_{L^p}$$

(where we used (2.9) in the last step), and similarly with $v_{n,K,p}, v_{n,x}$ and $\mathscr{V}_n$ replaced by $w_{n,K,p}, w_{n,x}^{(i)}$ and $\mathscr{W}_n$ respectively. We thus get that

$$
\begin{aligned}
\|f\|_{\mathcal{B}_{p,q}^{\alpha,K,k}} &= v_{k,K,p}f + \left\|(2^{\alpha n}w_{n,K,p}f)_{n\geqslant k}\right\|_{\ell^q} \\
&\leqslant C\left(\|\mathscr{V}_k(\chi f)\|_{L^p} + \left\|(2^{\alpha n}\|\mathscr{W}_n(\chi f)\|_{L^p})_{n\geqslant n_0}\right\|_{\ell^q}\right) \leqslant C\|\chi f\|_{\mathcal{B}_{p,q}^{\alpha}}.
\end{aligned}
$$

We now turn to (2.25). In order to also justify part (3), we will show that we can in fact pick the adapted pair in $\mathscr{K} = \{(K_n, k_n), n \in \mathbb{N}\}$.

Let (K, k) be an adapted pair. For every $f \in C^\infty(U)$, we define

$$
f_K = \sum_{x \in \Lambda_k \cap K} v_{k,x}(f)\,\phi_{k,x} + \sum_{\substack{n \geqslant k, i < 2^d \\ x \in \Lambda_n \cap K}} w_{n,x}^{(i)}(f)\,\psi_{n,x}^{(i)}. \tag{2.26}
$$

The functions f and f_K coincide on

$$
K' := \left\{x \in \mathbb{R}^d : \mathsf{dist}(x, K^c) \geqslant 2^{-k}R\right\}. \tag{2.27}
$$

(Although the notation is not explicit in this respect, we warn the reader that f_K and K' are defined in terms of the pair (K, k) rather than in terms of K only.) Let $\chi \in C_c^\infty(U)$ with compact support $L \subseteq U$. Assuming that

$$
\text{there exists } n \in \mathbb{N} \text{ s.t. } L \subseteq K_n', \tag{2.28}
$$

we see that for such an n,

$$
\|\chi f\|_{\mathcal{B}_{p,q}^{\alpha}} = \|\chi f_{K_n}\|_{\mathcal{B}_{p,q}^{\alpha}} \leqslant C\|f_{K_n}\|_{\mathcal{B}_{p,q}^{\alpha}} \leqslant C\|f\|_{\mathcal{B}_{p,q}^{\alpha,K_n,k_n}}
$$

by Proposition 2.19 and (2.23). Hence, it suffices to justify (2.28). Let $\mathsf{d} = \mathsf{dist}(L, U^c)$. Since $x \mapsto \mathsf{dist}(x, U^c)$ is positive and continuous on L, we obtain $\mathsf{d} > 0$. If U is bounded, then there exists $n \in \mathbb{N}$ such that K_n contains the compact set $\{x : \mathsf{dist}(x, U^c) \geqslant \mathsf{d}/2\}$. We must then have $2^{-k_n}R < \mathsf{d}/2$, so that

$$
\begin{aligned}
x \in L &\Rightarrow \mathsf{dist}(x, K_n^c) \geqslant \mathsf{dist}(x, U^c) - \frac{\mathsf{d}}{2} \geqslant \frac{\mathsf{d}}{2} > 2^{-k_n}R \\
&\Rightarrow x \in K_n'.
\end{aligned}
$$

If U is unbounded, we can do the same reasoning with U replaced by

$$
U \cap (L + B(0, R)),
$$

so the proof is complete. $\qquad\square$

Remark 2.24. For any adapted pair (K, k), the quantity $\|f\|_{\mathcal{B}_{p,q}^{\alpha,K,k}}$ is well defined as an element of $[0, +\infty]$ as soon as f is a linear form on $C_c^r(U)$, through the interpretation of $v_{k,x}f$ and $w_{n,x}^{(i)}f$ in (2.5) as a duality pairing.

The characterization of Proposition 2.22 yields a straightforward proof of embedding properties between Besov spaces (see for example [4, Proposition 2.71]).

Proposition 2.25 (Local Besov embedding). *Let* $1 \leqslant p_2 \leqslant p_1 \leqslant +\infty$, $1 \leqslant q_2 \leqslant q_1 \leqslant +\infty$, $\alpha \in \mathbb{R}$ *and*

$$
\beta = \alpha + d\left(\frac{1}{p_2} - \frac{1}{p_1}\right).
$$

If $|\alpha|, |\beta| < r$ *and* (K, k) *is an adapted pair, then there exists* $C < \infty$ *such that for every linear form* f *on* $C_c^r(U)$,

$$
\|f\|_{\mathcal{B}_{p_1,q_1}^{\alpha,K,k}} \leqslant C\|f\|_{\mathcal{B}_{p_2,q_2}^{\beta,K,k}}.
$$

In particular, we have $\mathcal{B}_{p_2,q_2}^{\beta,\mathrm{loc}}(U) \subseteq \mathcal{B}_{p_1,q_1}^{\alpha,\mathrm{loc}}(U)$.

Proof. We write the norm (2.23), recall (2.21)–(2.22), and use the fact that $\|\cdot\|_{\ell^{p_1}} \leqslant \|\cdot\|_{\ell^{p_2}}$ if $p_1 \geqslant p_2$. $\qquad\square$

Due to our definition of the space $\mathcal{B}_{p,q}^{\alpha,\,\mathrm{loc}}(U)$ as a completion of $C^\infty(U)$, the fact that $\|f\|_{\mathcal{B}_{p,q}^{\alpha,K,k}}$ is finite for every adapted pair (K,k) does not necessarily imply that $f \in \mathcal{B}_{p,q}^{\alpha,\,\mathrm{loc}}(U)$. We have nonetheless the following result.

Proposition 2.26 (A criterion for belonging to $\mathcal{B}_{p,q}^{\alpha,\,\mathrm{loc}}(U)$)**.** *Let $|\alpha'| < r$ and let $p,q \in [1,\infty]$. Let f be a linear form on $C_c^r(U)$, and let $\mathscr{K}$ be a spanning sequence. If for every $(K,k) \in \mathscr{K}$,*

$$\|f\|_{\mathcal{B}_{p,q}^{\alpha',K,k}} < \infty,$$

then for every $\alpha < \alpha'$, the form f belongs to $\mathcal{B}_{p,1}^{\alpha,\,\mathrm{loc}}(U)$.

Proof of Proposition 2.26. We first check that for every $(K,k) \in \mathscr{K}$, there exists a sequence $(f_{N,k})_{N \in \mathbb{N}}$ in $C_c^r(U)$ such that $\|f - f_{N,k}\|_{\mathcal{B}_{p,1}^{\alpha,K,k}}$ tends to 0 as N tends to infinity. The functions

$$f_{N,k} := \sum_{x \in \Lambda_k \cap K} v_{k,x}(f)\,\phi_{k,x} + \sum_{\substack{k \leqslant n \leqslant N, i < 2^d \\ x \in \Lambda_n \cap K}} w_{n,x}^{(i)}(f)\,\psi_{n,x}^{(i)}$$

satisfy this property. Now notice that for $(\tilde{k}, \tilde{K}) \in \mathscr{K}$ such that $\tilde{K} \supset K$, the function $f_{N,\tilde{k}}$ coincides with $f_{N,k}$ on the set K' of (2.27). Then defining $f_N = f_{N,N}$, we obtain that for every $\chi \in C_c^\infty(U)$, there exists $n_0, N_0(n_0)$ such that for every $n \geqslant n_0$ and $N \geqslant N_0$,

$$\|(f_N - f)\chi\|_{\mathcal{B}_{p,1}^{\alpha}} = \|(f_{N,k_n} - f)\chi\|_{\mathcal{B}_{p,1}^{\alpha}},$$

where we have indexed the spanning sequence as $\mathscr{K} = (k_n, K_n)_{n \in \mathbb{N}}$. By (2.25), there exist $(k_m, K_m) \in \mathscr{K}$, $C > 0$ with m large enough, such that:

$$\|(f_{N,k_n} - f)\chi\|_{\mathcal{B}_{p,1}^{\alpha}} \leqslant C \|f_{N,k_n} - f\|_{\mathcal{B}_{p,1}^{\alpha,K_m,k_m}}$$

We can eventually choose $m = n$ to obtain $\|(f_N - f)\chi\|_{\mathcal{B}_{p,1}^{\alpha}} \to 0$ for every $\chi \in C_c^\infty(U)$, which by Proposition 2.22 is the needed result. $\qquad\square$

Naturally, tightness criteria rely on the identification of compact subsets of the space of interest.

Proposition 2.27 (Compact embedding)**.** *Let U be an open subset of $\mathbb{R}^d$. For every $\alpha < \alpha'$ and $p,q,s \in [1,+\infty]$, the embedding $\mathcal{B}_{p,q}^{\alpha',\,\mathrm{loc}}(U) \subseteq \mathcal{B}_{p,s}^{\alpha,\,\mathrm{loc}}(U)$ is compact.*

Proof. By Proposition 2.22 and the definition of boundedness in Fréchet spaces, a sequence $(f_m)_{m \in \mathbb{N}}$ of elements of $\mathcal{B}_{p,q}^{\alpha',\,\mathrm{loc}}(U)$ is bounded in $\mathcal{B}_{p,q}^{\alpha',\,\mathrm{loc}}(U)$ if and only if for every adapted pair (K,k), we have

$$\sup_{m \in \mathbb{N}} \|f_m\|_{\mathcal{B}_{p,q}^{\alpha',K,k}} < \infty.$$

We show that for every adapted pair (K,k), there exists a subsequence $(m_{n_k})_{n_k \in \mathbb{N}}$ and $f^{(K)}$ in $\mathcal{B}_{p,s}^{\alpha,\,\mathrm{loc}}(U)$ such that $\|f_{m_{n_k}} - f^{(K)}\|_{\mathcal{B}_{p,s}^{\alpha',K,k}}$ tends to 0 as n tends to infinity. The assumption that $\sup_m \|f_m\|_{\mathcal{B}_{p,q}^{\alpha',K,k}} < \infty$ can be rewritten as

$$\left\|\left(v_{k,x} f_m\right)_{x \in \Lambda_k \cap K}\right\|_{\ell^p} + \left\|\left(2^{n[\alpha' + d(\frac{1}{2} - \frac{1}{p})]} \left\|\left(w_{n,x}^{(i)} f_m\right)_{i < 2^d, x \in \Lambda_n \cap K}\right\|_{\ell^p}\right)_{n \geqslant k}\right\|_{\ell^q} \leqslant C,$$

uniformly over $m \in \mathbb{N}$. By a diagonal extraction argument, there exist a subsequence, which we still denote (f_m) for convenience, and numbers $\tilde{v}_{k,x}$, $\tilde{w}_{n,x}^{(i)}$ such that

$$\left\| (v_{k,x}f_m - \tilde{v}_{k,x})_{x \in \Lambda_k \cap K} \right\|_{\ell^p} +$$

$$\left\| \left(2^{n[\alpha + d(\frac{1}{2} - \frac{1}{p})]} \left\| \left(w_{n,x}^{(i)} f_m - \tilde{w}_{n,x}^{(i)} \right)_{i < 2^d, x \in \Lambda_n \cap K} \right\|_{\ell^p} \right)_{n \geqslant k} \right\|_{\ell^s} \xrightarrow[m \to \infty]{} 0.$$

Defining

$$f^{(K)} = \sum_{x \in \Lambda_k \cap K} \tilde{v}_{k,x} \, \phi_{k,x} + \sum_{\substack{n \geqslant k, i < 2^d \\ x \in \Lambda_n \cap K}} \tilde{w}_{n,x}^{(i)} \, \psi_{n,x}^{(i)},$$

we have $f^{(K)} \in \mathcal{B}_{p,s}^{\alpha,\mathrm{loc}}(U)$ and $\|f_m - f^{(K)}\|_{\mathcal{B}_{p,s}^{\alpha,K,k}} \to 0$ as m tends to infinity. The subsequence (f_m) is Cauchy in $\mathcal{B}_{p,s}^{\alpha,\mathrm{loc}}(U)$. Indeed, for every $(K,k) \in \mathscr{K}$, there exists $n_0(K)$ such that for every $n, m \geqslant n_0$,

$$\|f_n - f_m\|_{\mathcal{B}_{p,s}^{\alpha,K,k}} \leqslant \|f_n - f^{(K)}\|_{\mathcal{B}_{p,s}^{\alpha,K,k}} + \|f^{(K)} - f_m\|_{\mathcal{B}_{p,s}^{\alpha,K,k}} < \varepsilon.$$

This completes the proof. $\qquad\square$

Remark 2.28. Proposition 2.27 would not be true if $\mathcal{B}_{p,q}^{\alpha',\mathrm{loc}}(U)$ and $\mathcal{B}_{p,s}^{\alpha,\mathrm{loc}}(U)$ were replaced by their global counterparts, respectively $\mathcal{B}_{p,q}^{\alpha'}$ and $\mathcal{B}_{p,s}^{\alpha}$. Indeed, one can take for example a non-zero function $f \in C_c^\infty$ and consider the sequence $(f(\cdot - np))_{n \geqslant 1}$, with $p \in \mathbb{R}^d \setminus \{0\}$. This sequence is bounded in every global Besov space $\mathcal{B}_{p,q}^{\alpha}$, but has no convergent subsequence in any of these spaces.

An immediate consequence of Propositions 2.26 and 2.27 is:

Corollary 2.29. *Let $|\alpha'| < r$, $p, q \in [1, \infty]$, let $\mathscr{K}$ be a spanning sequence, and for every $(K, k) \in \mathscr{K}$, let $M_K \in [0, \infty)$. For every $\alpha < \alpha'$, $s \in [1, \infty]$, the set*

$$\left\{ f \text{ linear form on } C_c^r(U) \text{ such that } \forall (K, k) \in \mathscr{K}, \; \|f\|_{\mathcal{B}_{p,q}^{\alpha',K,k}} \leqslant M_K \right\} \tag{2.29}$$

is compact in $\mathcal{B}_{p,s}^{\alpha,\mathrm{loc}}(U)$.

Theorem 2.30 (Tightness criterion). *Recall that $\phi, (\psi^{(i)})_{1 \leqslant i < 2^d}$ are in C_c^r and such that (2.3) holds, and fix $p \in [1, \infty)$, $q \in [1, \infty]$ and $\alpha, \beta \in \mathbb{R}$ satisfying $|\alpha|, |\beta| < r$, $\alpha < \beta$. Let $(f_m)_{m \in \mathbb{N}}$ be a family of random linear forms on $C_c^r(U)$, and let $\mathscr{K}$ be a spanning sequence (see Definition 2.21). Assume that for every $(K, k) \in \mathscr{K}$, there exists $C = C(K, k) < \infty$ such that for every $m \in \mathbb{N}$,*

$$\sup_{x \in \Lambda_k \cap K} \mathbb{E}\left[\left| \langle f_m, \phi(2^k(\cdot - x)) \rangle \right|^p \right]^{1/p} \leqslant C, \tag{2.30}$$

and

$$\sup_{x \in \Lambda_n \cap K} 2^{dn} \, \mathbb{E}\left[\left| \langle f_m, \psi^{(i)}(2^n(\cdot - x)) \rangle \right|^p \right]^{1/p} \leqslant C 2^{-n\beta} \qquad (i < 2^d, n \geqslant k). \tag{2.31}$$

Then the family (f_m) is tight in $\mathcal{B}_{p,q}^{\alpha,\mathrm{loc}}$. If moreover $\alpha < \beta - \frac{d}{p}$, then the family is also tight in $\mathcal{C}_{\mathrm{loc}}^{\alpha}(U)$.

Proof. By (2.4) and (2.5), we have for every $(K, k) \in \mathscr{K}$, uniformly over m that

$$\sup_{x \in \Lambda_k \cap K} \mathbb{E}\left[|v_{k,x} f_m|^p \right] \lesssim 1,$$

$$\sup_{x \in \Lambda_n \cap K} 2^{\frac{dnp}{2}} \, \mathbb{E}\left[\left| w_{n,x}^{(i)} f_m \right|^p \right] \lesssim 2^{-np\beta} \qquad (i < 2^d, n \geqslant k).$$

Recalling the definition of $v_{k,K,p}$ and $w_{n,K,p}$ in (2.21) and (2.22) respectively, we have

$$|v_{k,K,p}f_m|^p \lesssim \sum_{x \in \Lambda_k \cap K} |v_{k,x}f_m|^p,$$

so that

$$\mathbb{E}\left[|v_{k,K,p}f_m|^p\right] \lesssim 1.$$

Similarly,

$$|w_{n,K,p}f_m|^p \lesssim 2^{dn\left(\frac{p}{2}-1\right)} \sum_{i<2^d, x \in \Lambda_k \cap K} \left|w_{n,x}^{(i)}f_m\right|^p,$$

so that

$$\mathbb{E}\left[|w_{n,K,p}f_m|^p\right] \lesssim 2^{-np\beta}.$$

It follows from these two observations and from (2.23) that

$$\sup_{m \in \mathbb{N}} \mathbb{E}\left[\|f_m\|^p_{\mathcal{B}^{\beta,K,k}_{p,\infty}}\right] < \infty. \tag{2.32}$$

By Chebyshev's inequality, for any given $\varepsilon > 0$, there exist (M_K) such that if we set

$$\mathscr{E} := \left\{ f \text{ linear form on } C_c^r(U) \text{ such that } \forall (K,k) \in \mathscr{K}, \ \|f\|_{\mathcal{B}^{\beta,K,k}_{p,\infty}} \leqslant M_K \right\},$$

then for every m,

$$\mathbb{P}[f_m \in \mathscr{E}] \geqslant 1 - \varepsilon.$$

By Corollary 2.29, this implies the tightness result in $\mathcal{B}^{\alpha,\mathrm{loc}}_{p,q}(U)$. For the second statement, we note that (2.32) and Proposition 2.25 imply that

$$\sup_{m \in \mathbb{N}} \mathbb{E}\left[\|f_m\|^p_{\mathcal{B}^{\beta-d/p,K,k}_{\infty,\infty}}\right] < \infty.$$

The conclusion then follows in the same way. $\qquad\square$

Remark 2.31. We can also infer from the proof that for each $\chi \in C_c^\infty(U)$, there exists a constant $\widetilde{C}_\chi$ such that under the assumption of Theorem 2.30, we have

$$\sup_{m \in \mathbb{N}} \mathbb{E}\left[\|\chi f_m\|^p_{\mathcal{B}^\beta_{p,\infty}}\right] < \widetilde{C}_\chi C,$$

as well as

$$\sup_{m \in \mathbb{N}} \mathbb{E}\left[\|\chi f_m\|^p_{\mathcal{C}^{\beta-\frac{d}{p}}}\right] < \widetilde{C}_\chi C.$$

We conclude this section by proving a statement analogous to the Kolmogorov continuity theorem. Recalling from Remark 2.18 the interpretation of the space $\mathcal{C}^\alpha$ as a Hölder space, the satement below can indeed be seen as a generalization of the classical result of Kolmogorov. (The fact that the statement can apply to positive exponents of regularity is due to the cancellation property (2.2).)

Proposition 2.32. *Let $(f(\eta), \eta \in C_c^r(U))$ be a family of random variables such that, for every $\eta, \eta' \in C_c^r(U)$ and every $\mu \in \mathbb{R}$, there exists a measurable set $A = A(\mu, \eta, \eta')$ with $\mathbb{P}(A) = 1$ such that*

$$f(\mu\eta + \eta')(\omega) = \mu f(\eta)(\omega) + f(\eta')(\omega) \qquad \forall \omega \in A. \tag{2.33}$$

Assume also the following weak continuity property: for each compact $K' \subseteq U$ and each sequence $\eta_n, \eta \in C_c^r(U)$ with $\mathrm{Supp}\,\eta_n \subseteq K'$, we have

$$\eta_n \xrightarrow[n\to\infty]{\text{in } C_c^{r-1}} \eta \quad \Longrightarrow \quad f(\eta_n) \xrightarrow[n\to\infty]{\text{prob.}} f(\eta).$$

Let $p \in [1, \infty)$, $q \in [1, \infty]$, and let $\alpha, \beta \in \mathbb{R}$ be such that $|\alpha|, |\beta| < r$ and $\alpha < \beta$. Let $\mathscr{K}$ be a spanning sequence, and assume finally that, for every $(K, k) \in \mathscr{K}$, there exists $C > 0$ such that for every $n \geqslant k$,

$$\sup_{x \in \Lambda_k \cap K} \mathbb{E}\left[\left|f\left(\phi(2^k(\cdot - x)))\right)\right|^p\right]^{\frac{1}{p}} \leqslant C$$

and

$$\sup_{x \in \Lambda_n \cap K} 2^{dn} \mathbb{E}\left[|f\left(\psi(2^n(\cdot - x)))\right|^p\right]^{\frac{1}{p}} \leqslant C 2^{-n\beta}.$$

Then there exists a random distribution $\tilde{f}$ taking values in $\mathcal{B}_{p,q}^{\alpha, \mathrm{loc}}(U)$ such that for every $\eta \in C_c^r(U)$,

$$\left(\tilde{f}, \eta\right) = f(\eta) \qquad \text{a.s.} \tag{2.34}$$

Moreover, if $\alpha < \beta - \frac{d}{p}$, then $\tilde{f}$ takes values in $\mathcal{C}_{\mathrm{loc}}^{\alpha}(U)$ with probability one.

Proof. For every $(K, k) \in \mathscr{K}$ and $N \in \mathbb{N}$, we define

$$\tilde{f}_{N,k} := \sum_{x \in \Lambda_k \cap K} v_{k,x}(f)\, \phi_{k,x} + \sum_{\substack{k \leqslant n \leqslant N, i < 2^d \\ x \in \Lambda_n \cap K}} w_{n,x}^{(i)}(f)\, \psi_{n,x}^{(i)},$$

where we set

$$v_{k,x}(f) := f(\phi_{k,x}) \quad \text{and} \quad w_{n,x}^{(i)}(f) = f(\psi_{n,x}^{(i)}).$$

Cleary, $\tilde{f}_{N,k}$ is almost surely in C_c^r. Following the proof of Theorem 2.30, we get:

$$\mathbb{E}\left[2^{dn(\frac{p}{2}-1)} \sum_{x \in \Lambda_n \cap K, i < 2^d} |w_{n,x}^{(i)}(f)|^p\right] \lesssim 2^{-np\beta},$$

where the implicit constant does not depend on n. Hence, for each $\beta' < \beta$ and each fixed integer k, we deduce by the Chebyshev inequality and the Borel-Cantelli lemma that $(\tilde{f}_{N,k})_N$ is a Cauchy sequence in $\mathcal{B}_{p,\infty}^{\beta'}$ with probability one. We denote the limit by $\tilde{f}_k$. It is clear that $\tilde{f}_k$ converges to some element $\tilde{f}$ of $\mathcal{B}_{p,\infty}^{\beta', \mathrm{loc}}(U)$ as k tends to infinity, since for each $\chi \in C_c^r$ with compact support in U, the sequence $\chi \tilde{f}_k$ is eventually constant as k tends to infinity. By Proposition 2.25, if $\alpha < \beta - \frac{d}{p}$, then $\tilde{f} \in \mathcal{C}_{\mathrm{loc}}^{\alpha}(U)$ with probability one. There remains to check that for every $\eta \in C_c^r(U)$, the identity (2.34) holds. By the orthogonality properties of $(\phi_{k,x}, \psi_{n,x}^{(i)})$ and the fact that η has compact support in U, we have, for k sufficiently large,

$$\eta = \sum_{x \in \Lambda_k \cap K} (\phi_{k,x}, \eta)\, \phi_{k,x} + \lim_{N \to +\infty} \sum_{\substack{k \leqslant n \leqslant N, i < 2^d \\ x \in \Lambda_n \cap K}} (\psi_{n,x}^{(i)}, \eta)\, \psi_{n,x}^{(i)},$$

where we recall that $(\cdot, \cdot)$ denotes the scalar product of $L^2(\mathbb{R}^d)$. We fix such k sufficiently large, and denote

$$\eta_N := \sum_{x \in \Lambda_k \cap K} (\phi_{k,x}, \eta)\, \phi_{k,x} + \sum_{\substack{k \leqslant n \leqslant N, i < 2^d \\ x \in \Lambda_n \cap K}} (\psi_{n,x}^{(i)}, \eta)\, \psi_{n,x}^{(i)}.$$

By a Taylor expansion of η and (2.2), one can check that there exists $C(d, \eta) < \infty$ such that

$$2^{\frac{dn}{2}} \left|(\psi_{n,x}^{(i)}, \eta)\right| \leqslant C 2^{-rn}.$$

From this, together with the expressions for η_N and η above, we obtain that $\exists C(d, \eta) < \infty$ such that for any multi-index $\alpha \leqslant |r|$

$$\|\partial^\alpha \eta - \partial^\alpha \eta_N\|_{L^\infty} < C \sum_{n > N} 2^{-rn} 2^{|\alpha|n}$$

and thus

$$\eta_N \xrightarrow[N \to \infty]{\text{in } C_c^{r-1}} \eta.$$

Therefore by the weak continuity assumption, we deduce that

$$f(\eta_N) \xrightarrow[N \to \infty]{\text{prob.}} f(\eta).$$

In order to conclude, there remains to verify that

$$(\tilde{f}, \eta_N) = f(\eta_N) \qquad \text{a.s.}$$

This follows from the assumption (2.33). $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

3 Application to the critical Ising model

In this section, we apply the tightness criterion presented in Theorem 2.30 to the magnetization field of the two-dimensional Ising model at the critical temperature. We will use extensively some basic notions related to the FK percolation model [15] and its relation to the Ising model via the Edwards-Sokal coupling [14].

3.1 Introduction to the random cluster model

The random cluster model, or FK percolation model, was first introduced in [15]. We refer to [18] for a comprehensive book on the subject.

Let Λ be a finite subset of $\mathbb{Z}^d$, $\Omega = \{0, 1\}^{\mathbb{E}^d}$ with $\mathbb{E}^d$ the set of edges of the graph $\mathbb{Z}^d$, and $\mathcal{F}$ be the σ-algebra generated by cylinder sets. For $\omega \in \Omega$, let ω_e be the component of ω at $e \in \mathbb{E}^d$. Let $E_\Lambda = \{e = \langle x, y \rangle \in \mathbb{E}^d \mid x \in \Lambda, y \in \Lambda\}$ the set of edges with both endpoints in Λ. For $\xi \in \Omega$, define the following finite subset of Ω:

$$\Omega_\Lambda^\xi = \{\omega \in \Omega \mid \omega_e = \xi_e \quad \forall e \in \mathbb{E}^d \setminus E_\Lambda\}.$$

Definition 3.1. *Let $p \in [0, 1]$, $q \in (0, \infty)$. The FK probability measure on $(\Omega, \mathcal{F})$ with boundary condition ξ is*

$$\phi_{\Lambda, p, q}^\xi(\omega) = \begin{cases} \frac{1}{Z_{\xi, \Lambda}} \left[\prod_{e \in E_\Lambda} p^{\omega_e} (1 - p)^{1 - \omega_e} \right] q^{k(\omega)} & \text{if } \omega \in \Omega_\Lambda^\xi \\ 0 & \text{otherwise} \end{cases} \tag{3.1}$$

with $Z_{\xi, \Lambda}(p, q) = \sum_{\omega \in \Omega_\Lambda^\xi} \left[\prod_{e \in E_\Lambda} p^{\omega_e} (1 - p)^{1 - \omega_e} \right] q^{k(\omega)}$ and $k(\omega)$ the number of connected components of the graph $(\mathbb{Z}^d, \eta(\omega))$, with $\eta(\omega) = \{e \in \mathbb{E}^d \mid \omega_e = 1\}$.

We will call the edge e *open* if $\omega_e = 1$, and *closed* otherwise. We call *open clusters* the connected components of $(\mathbb{Z}^d, \eta(\omega))$, and write $x \leftrightarrow y$ if x, y are in the same open cluster, $x \not\leftrightarrow y$ otherwise. An *open path* is a (possibly infinite) sequence (e_i) of edges belonging to $\eta(\omega)$. The boundary condition is *free* if $\xi_e = 0 \ \forall e \in \mathbb{E}^d$ and *wired* if $\xi_e = 1$ $\forall e \in \mathbb{E}^d$.

Remark 3.2. For both free and wired boundary conditions, if the domain Λ is the union of two subsets Λ_1 and Λ_2 such that $E_{\Lambda_1} \cap E_{\Lambda_2} = \emptyset$, then the configurations on Λ_1 and Λ_2 are independent. Indeed, calling $\overline{k(\omega, \mathbb{E}^d \setminus E_\Lambda)}$ the number of open clusters of ω that do not intersect $\mathbb{E}^d \setminus E_\Lambda$, we have $\overline{k(\omega, \mathbb{E}^d \setminus E_\Lambda)} = \overline{k(\omega, \mathbb{E}^d \setminus E_{\Lambda_1})} + \overline{k(\omega, \mathbb{E}^d \setminus E_{\Lambda_2})}$.

Although in general the states on two different edges are not independent, the model exhibits a "domain Markov" [10] or "nesting" [18] property. Let $\mathcal{F}_\Lambda$ (respectively $\mathcal{T}_\Lambda$) be the σ-algebra generated by the states of edges in E_Λ (respectively in $\mathbb{E}^d \setminus E_\Lambda$). We have the following result.

Lemma 3.3 ([18, Lemma 4.13])**.** *Let $p \in [0,1]$, $q \in (0,\infty)$, and let Λ, Δ be finite subsets of $\mathbb{Z}^d$ with $\Lambda \subseteq \Delta$. For every $\xi \in \Omega$, every event $A \in \mathcal{F}_\Lambda$ and every $\omega \in \Omega_\Delta^\xi$,*

$$\phi_{\Delta,p,q}^\xi(A \mid \mathcal{T}_\Lambda)(\omega) = \phi_{\Lambda,p,q}^\omega(A). \tag{3.2}$$

The set $\Omega = \{0,1\}^{\mathbb{E}^d}$ has a partial ordering given by $\omega \leqslant \omega'$ if $\forall e \in \mathbb{E}^d$ $\omega_e \leqslant \omega_e'$. A function $X : \Omega \to \mathbb{R}$ is called *increasing* if $\omega \leqslant \omega' \Rightarrow X(\omega) \leqslant X(\omega')$. Likewise, an event $A \in \mathcal{F}$ is called increasing if the random variable $\mathbb{1}_A$ is increasing. As a direct consequence of the FKG inequality and [18, Lemma 4.14], we have the following monotonicity properties.

Lemma 3.4. *Let $p \in [0,1]$, $q \geqslant 1$ and $\Lambda \subseteq \Delta \subseteq \mathbb{Z}^d$ finite sets. Then:*

- *For every $\eta \leqslant \xi \in \Omega$ and for every increasing event A:*

$$\phi_{\Lambda,p,q}^\eta(A) \leqslant \phi_{\Lambda,p,q}^\xi(A).$$

- *For every increasing event $A \in \mathcal{F}_\Lambda$:*

$$\phi_{\Delta,p,q}^1(A) \leqslant \phi_{\Lambda,p,q}^1(A)$$

For $p \in [0,1]$, $q \geqslant 1$, the random cluster measure $\phi_{\Lambda,p,q}^\xi$ for both free and wired boundary conditions admits a thermodynamic limit as $\Lambda \to \mathbb{Z}^d$ [18, Theorems 4.17 and 4.19], which we call $\phi_{p,q}^\xi$. For every boundary condition ξ such that $\phi_{\Lambda,p,q}^\xi$ admits a limit and every increasing event A, we have

$$\phi_{p,q}^0(A) \leqslant \phi_{p,q}^\xi(A) \leqslant \phi_{p,q}^1(A).$$

3.2 Relation with the 2-d Ising model

Now consider the Ising-Potts model on a finite set $\Lambda \subseteq \mathbb{Z}^d$ as follows. Take a configuration space $\Sigma_\Lambda^0 = \{-1,1\}^\Lambda$.

The Ising probability measure with *free* boundary condition on Λ is defined by

$$\pi_G^0(\sigma) = \frac{1}{Z_I} e^{-\beta H(\sigma)} \qquad H(\sigma) = -\sum_{e \in E_\Lambda} \mathbb{1}_{\sigma_e = 1}, \tag{3.3}$$

with $\beta > 0$, $\sigma_e = \sigma_x \sigma_y$ and $Z_I^0(\beta) = \sum_{\sigma \in \Sigma_\Lambda^0} e^{-\beta H(\sigma)}$.

Similarly, let $\Sigma_\Lambda^1 = \{\sigma \in \{-1,1\}^\Lambda \mid \sigma_x = 1 \; \forall x \in \partial\Lambda\}$. The Ising probability measure with $+$ boundary condition on Λ is defined as

$$\pi_\Lambda^1(\sigma) = \frac{1}{Z_I^1} e^{-\beta H(\sigma)} \mathbb{1}_{\Sigma_\Lambda^1}(\sigma) \tag{3.4}$$

with $Z_I^1(\beta) = \sum_{\sigma \in \Sigma_\Lambda^1} e^{-\beta H(\sigma)}$. Random variables σ_x for $x \in \mathbb{Z}^d$ are called spins.

Remark 3.5. Traditionally, the Hamiltonian of the Ising model is written as

$$H'(\sigma) = -\sum_{x \sim y} \sigma_x \sigma_y$$

with $x \sim y$ nearest neighbours. Defining $\lambda_\beta(\sigma) \propto e^{-\beta' H'(\sigma)}$ for the usual Ising measure, we recover it as $\lambda_{\beta/2} \sim \pi_\beta$.

The Edwards-Sokal coupling on Λ with boundary condition $\xi \in \{0,1\}$ consists of defining the probability measure on $\Sigma_\Lambda^\xi \times \Omega$

$$\mu_\Lambda^\xi(\sigma, \omega) = \frac{1}{Z_{ES}^\xi} \prod_{e \in E_\Lambda^\xi} [(1-p)\mathbb{1}_{\omega_e=0} + p\mathbb{1}_{\omega_e=1}\mathbb{1}_{\sigma_e=1}] \, \mathbb{1}_{\Omega_\Lambda^\xi}(\omega) \tag{3.5}$$

with Z_{ES}^ξ such that $\sum_{(\sigma,\omega)\in\Sigma_\Lambda^\xi\times\Omega} \mu_\Lambda^\xi(\sigma,\omega) = 1$. From now on we fix

$$e^{-\beta} = 1 - p \quad \text{and} \quad q = 2. \tag{3.6}$$

It is easy to obtain the following lemma (see [18]).

Lemma 3.6. *Let $p \in [0,1]$, $e^{-\beta} = (1-p)$, $q = 2$ and $\xi \in \{0,1\}$. Let μ_Λ^ξ be defined as in (3.5). Then:*

- *The marginal of μ_Λ^ξ on Σ_Λ^ξ is π_Λ^ξ.*
- *The marginal of μ_Λ^ξ on Ω is $\phi_{\Lambda,p,2}^\xi$.*

In order to characterize the regularity of the Ising magnetization field Φ_a on an unbounded domain $U \subseteq \mathbb{R}^2$, in the next sections we will use the well-known FK-Ising coupling for infinite volume measures.

Theorem 3.7 ([18, Theorem 4.91])**.** *Let $p \in [0,1]$, $q = 2$, $e^{-\beta} = (1-p)$.*

- *Let ω be sampled from $\Omega = \{0,1\}^{\mathbb{E}^2}$ with law $\phi_{p,q}^1$. Conditional on ω, each vertex is assigned a random spin $\sigma_x \in \{-1,+1\}$ such that:*

    1. *$\sigma_x = 1$ if $x \leftrightarrow \infty$*
    2. *σ_x takes values in $\{-1,1\}$ with probability $\frac{1}{2}$ if $x \nleftrightarrow \infty$*
    3. *$\sigma_x = \sigma_y$ if $x \leftrightarrow y$*
    4. *spins in different open clusters are independent.*

    Then the configuration $\sigma = \{\sigma_x\}_{x\in\mathbb{Z}^d}$ is distributed according to the weak limit π^1 of Ising measures with $+$ boundary condition.

- *Let σ be sampled from $\Sigma = \{-1,+1\}^{\mathbb{Z}^d}$ with the Ising limit law π^1. Conditional on σ, each edge is assigned a random state $\omega_e \in \{0,1\}$ such that:*

    1. *the states of different edges are independent*
    2. *$\omega_e = 0$ if $\sigma_x \neq \sigma_y$*
    3. *if $\sigma_x = \sigma_y$, then $\omega_e = 1$ with probability p and 0 otherwise.*

    Then the edge configuration $\omega = \{\omega_e\}_{e\in\mathbb{E}^2}$ has law $\phi_{p,q}^1$.

A similar argument is valid for $\phi_{p,q}^0$ and the infinite-volume Ising measure π^0, with the difference that no fixed value is assigned to σ_x in the case $x \leftrightarrow \infty$.

3.3 Tightness of the Ising magnetization field

We now consider the planar Ising magnetization field at critical temperature β_c, on an open set $U \subseteq \mathbb{R}^2$ (possibly unbounded or equal to $\mathbb{R}^2$). Call $U_a = U \cap a\mathbb{Z}^2$ for $a > 0, a \in \mathbb{R}$. As in [8] we define an approximation of the Ising magnetization field at scale $a > 0$ as

$$\Phi_a := a^{-\frac{1}{8}} \sum_{y \in U_a} \sigma_y \mathbb{1}_{S_a(y)}, \tag{3.7}$$

where $S_a(y)$ is the (open) square centered at y of side-length a, and σ_y is the Ising spin at y.

We investigate this quantity at critical temperature, with either $+$ or free boundary condition on U_a. Our aim is to establish its tightness in $\mathcal{B}_{p,q}^{\alpha,\mathrm{loc}}(U)$. In order to do that, we will choose a spanning sequence $\mathscr{K}$ of U and bound (2.30), (2.31) for Φ_a, which if p is even become

$$a^{-\frac{1}{8}} \sup_{x \in \Lambda_k \cap K} \left[\sum_{y_1 \dots y_p \in U_a} \mathbb{E}_{U_a}^{\xi}(\sigma_{y_1} \cdots \sigma_{y_p}) \prod_{j=1}^{p} \int_{S_a(y_j)} \varphi(2^k(z-x)) \mathrm{d}z \right]^{\frac{1}{p}}, \qquad (3.8)$$

$$a^{-\frac{1}{8}} 2^{2n} \sup_{x \in \Lambda_n \cap K} \left[\sum_{y_1 \dots y_p \in U_a} \mathbb{E}_{U_a}^{\xi}(\sigma_{y_1} \cdots \sigma_{y_p}) \prod_{j=1}^{p} \int_{S_a(y_j)} \psi^{(i)}(2^n(z-x)) \mathrm{d}z \right]^{\frac{1}{p}}, \qquad (3.9)$$

with $(K,k) \in \mathscr{K}$. Here $\mathbb{E}_{U_a}^{\xi}(\sigma_{y_1} \cdots \sigma_{y_p})$ is the expectation with respect to the Ising-Potts measure $\pi_{U_a}^{\xi}$ at critical temperature with either free or $+$ boundary condition (see (3.3) and (3.4)).

In the following discussion we will exploit the Ising-FK relation discussed in Subsection 3.2 and introduce some lemmas which are useful to prove Theorem 1.2.

Let $\Lambda \subseteq \mathbb{Z}^2$ be a finite set. Define $A_{y_1 \dots y_n}^1 \subseteq \{0,1\}^{E_\Lambda}$ the event that each open cluster of the FK model on Λ contains an even number of the points $y_1, \dots, y_n$, or is connected to the boundary $\partial \Lambda$. Define also $A_{y_1 \dots y_n}^{1,\infty} \subseteq \{0,1\}^{\mathbb{E}^2}$ the event that each open cluster of the FK model on $\mathbb{Z}^2$ contains an even number of the points $y_1, \dots, y_n$, or is infinite. Finally, let $A_{y_1 \dots y_n}^0$ be the event that each open cluster contains an even number of the points $y_1, \dots, y_n$. It is easy to notice that all these events are increasing, i.e. they are preserved when switching any ω_e from 0 to 1.

Lemma 3.8. *Let ϕ be the FK probability measure with $p \in [0,1]$ and $q = 2$, and take $e^{-\beta} = (1-p)$. Then for any $n \geqslant 1$:*

1. $\mathbb{E}_\Lambda^+(\sigma_{y_1} \cdots \sigma_{y_n}) = \phi_\Lambda^1(A_{y_1 \dots y_n}^1)$.
2. $\mathbb{E}_{\mathbb{Z}^2}^+(\sigma_{y_1} \cdots \sigma_{y_n}) = \phi_{\mathbb{Z}^2}^1(A_{y_1 \dots y_n}^{1,\infty})$.
3. $\mathbb{E}_\Lambda^{\mathrm{free}}(\sigma_{y_1} \cdots \sigma_{y_n}) = \phi_\Lambda(A_{y_1 \dots y_n}^0)$.
4. $\mathbb{E}_{\mathbb{Z}^2}^{\mathrm{free}}(\sigma_{y_1} \cdots \sigma_{y_n}) = \phi_{\mathbb{Z}^2}^0(A_{y_1 \dots y_n}^0)$

Proof. We only prove the first point in this lemma, as the other equalities can be obtained with the same arguments, using Theorem 3.7. Let $f(\sigma) = \sigma_{y_1} \cdots \sigma_{y_n}$, from Lemma 3.6 and (3.5) we can write

$$\mathbb{E}_\Lambda^+[f(\sigma)] = \sum_{\sigma \in \Sigma_\Lambda^1} f(\sigma) \sum_{\omega \in \Omega} \mu_\Lambda^1(\sigma, \omega)$$

$$= \frac{1}{Z_{ES}^1} \sum_{\sigma \in \Sigma_\Lambda^1} f(\sigma) \sum_{\omega \in \Omega_\Lambda^1} \prod_{e \in E_\Lambda} [(1-p)\mathbb{1}_{\omega_e=0} + p\mathbb{1}_{\omega_e=1}\mathbb{1}_{\sigma_e=1}]$$

$$= \frac{1}{Z_{ES}^1} \sum_{\omega \in \Omega_\Lambda^1} (1-p)^{|E_\Lambda \setminus \eta_\Lambda(\omega)|} p^{|\eta_\Lambda(\omega)|} \sum_{\sigma \in \Sigma_\Lambda^1} f(\sigma) \prod_{e \in \eta_\Lambda(\omega)} \mathbb{1}_{\sigma_e=1}$$

with $\eta_\Lambda(\omega) = \{e \in E_\lambda \mid \omega_e = 1\}$.

Now take $\omega \in \Omega_\Lambda^1$ such that one or more of its clusters contain an odd number of points in $y_1 \dots y_n$. The sum $\sum_{\sigma \in \Sigma_\Lambda^1} f(\sigma) \prod_{e \in \eta_\Lambda(\omega)} \mathbb{1}_{\sigma_e=1}$ is zero (indeed, each odd cluster takes the values $+1$ and -1 and all terms cancel out). Conversely, if $\omega \in A_{y_1 \dots y_n}^1$, the

product $\sigma_{y_1} \cdots \sigma_{y_{2k}}$ in the same cluster is equal to 1. We can write then:

$$\mathbb{E}_\Lambda^+[f(\sigma)] = \frac{1}{Z_{ES}^1} \sum_{\omega \in \Omega_\Lambda^1} \mathbb{1}_{A_{y_1 \ldots y_n}^1}(\omega)(1-p)^{|E_\Lambda \setminus \eta_\Lambda(\omega)|} p^{|\eta_\Lambda(\omega)|} \sum_{\sigma \in \Sigma_\Lambda^1} \prod_{e \in \eta_\Lambda(\omega)} \mathbb{1}_{\sigma_e=1}$$

$$= \frac{1}{Z_{ES}^+} \sum_{\omega \in \Omega_\Lambda^1} \mathbb{1}_{A_{y_1 \ldots y_n}^1}(\omega)(1-p)^{|E_\Lambda \setminus \eta_\Lambda(\omega)|} p^{|\eta_\Lambda(\omega)|} 2^{\overline{k(\omega, \mathbb{E}^2 \setminus E_\Lambda)}}$$

Here $\overline{k(\omega, \mathbb{E}^2 \setminus E_\Lambda)}$ is the number of connected clusters of ω that do not intersect $\mathbb{E}^2 \setminus E_\Lambda$. The following equivalence between partition functions yields the result:

$$Z_{ES}^1 = \sum_{\omega \in \Omega_\Lambda^1} (1-p)^{|E_\Lambda \setminus \eta_\Lambda(\omega)|} p^{|\eta(\omega)|} \sum_{\sigma \in \Sigma_\Lambda^1} \prod_{e \in \eta_\Lambda(\omega)} \mathbb{1}_{\sigma_e=1}$$

$$= \frac{1}{2} \sum_{\omega \in \Omega_\Lambda^1} (1-p)^{|E_\Lambda \setminus \eta_\Lambda(\omega)|} p^{|\eta_\Lambda(\omega)|} 2^{k(\omega, E_\Lambda)} = \frac{1}{2} Z_{FK}^{1,\Lambda}(p, 2). \qquad \square$$

We are going to need a well-known inequality for the 2-d Ising model of Onsager, formulated using connection probabilities for the FK model. See also [10, Lemma 5.4].

Lemma 3.9. *Let $m \in \mathbb{N}$ and $B_m = [-m, m]^2 \cap \mathbb{Z}^2$. At critical temperature $p_c = 1 - e^{-\beta_c}$, there exists $C > 0$ such that:*

$$\phi_{B_m, p_c, q=2}^1(0 \leftrightarrow \partial B_m) \leqslant Cm^{-\frac{1}{8}}.$$

The following proposition is known (see [8, Proposition 3.9] for a sketch of the proof), but we give here a different (and complete) proof which employs *the pin and sum argument with hairy cycles* of A. Abdesselam [1].

Proposition 3.10. *Let $p \in \mathbb{N}$. There exists $C > 0$ such that, for every $N \in \mathbb{N}$:*

$$\sum_{y_1, \ldots, y_p \in U_N} \mathbb{E}_{U_N(\mathbb{Z}^2)}^\xi(\sigma_{y_1} \cdots \sigma_{y_p}) \leqslant C(N+1)^{\frac{15}{8}p} \qquad (3.10)$$

with $U_N = [0, N]^2 \cap \mathbb{Z}^2$ and $\mathbb{E}_{U_N(\mathbb{Z}^2)}^\xi$ being the expectation on either U_N or $\mathbb{Z}^2$ at critical temperature β_c.

Proof. The events $A_{y_1 \ldots y_p}$ are increasing, and we have $A_{y_1 \ldots y_p}^0 \subseteq A_{y_1 \ldots y_p}^1$ when the events are on the same domain (finite or infinite). From the coupling of Lemma 3.8, and using the monotonicity properties of Lemma 3.4 it is easy to obtain $\mathbb{E}_{U_N(\mathbb{Z}^2)}^\xi \leqslant \mathbb{E}_{U_N}^+$. We are then left to show the inequality for this term.

We start by showing that

$$\sum_{\substack{y_1, \ldots, y_p \in U_N \\ y_i \neq y_j \forall i \neq j}} \mathbb{E}_{U_N}^+(\sigma_{y_1} \cdots \sigma_{y_p}) \leqslant CN^{\frac{15}{8}p}. \qquad (3.11)$$

The event $A_{y_1 \ldots y_p}^1$ of Lemma 3.8 implies that every point in $\{y_1, \ldots, y_p\}$ is connected by an open path to another point in $\{y_1, \ldots, y_p\}$ or to the boundary ∂U_N, which we call y_0. For every $1 \leqslant i \leqslant p$, call $\ell_i = \min_{j \geqslant 0, j \neq i} d(y_i, y_j)$ where $d(y_i, y_j)$ is the $\mathbb{Z}^2$ distance between y_i and y_j, and define $B_i = y_i + [\![-\ell_i/4, \ell_i/4]\!]^2$, $F = \bigcup_{i=1}^p B_i$. Notice that the graph $F \subseteq \mathbb{Z}^2$ has p disjoint components.

From Lemma 3.3, Remark 3.2 and since $\phi_{U_N}^+(A) = \sum_\omega \phi_{U_N}^+(A \mid \mathcal{T}_F)(\omega)\phi_{U_N}^+(\omega)$, we obtain

$$\mathbb{E}_{U_N}^+(\sigma_{y_1} \cdots \sigma_{y_p}) \leqslant \phi_{U_N}^+\left(\bigcap_{i=1}^p \{y_i \leftrightarrow \partial B_i\}\right) \leqslant \prod_{i=1}^p \phi_{B_i}^+(y_i \leftrightarrow \partial B_i),$$

where we used the monotonicity property of Lemma 3.4 in the second inequality. Lemma 3.9 yields:

$$
\sum_{\substack{y_1,\ldots,y_p\in U_N \\ y_i\neq y_j \forall i\neq j}} \mathbb{E}^+_{U_N}(\sigma_{y_1}\cdots\sigma_{y_p}) \lesssim \sum_{\substack{y_1,\ldots,y_p\in U_N \\ y_i\neq y_j \forall i\neq j}} \prod_{i=1}^{p} \left[\min_{j\geqslant 0, j\neq i} \mathrm{d}(y_i,y_j) \right]^{-\frac{1}{8}}
$$

$$
\lesssim \sum_{\substack{y_1,\ldots,y_p\in U_N \\ y_i\neq y_j \forall i\neq j}} \prod_{i=1}^{p} \sum_{\substack{j=0 \\ j\neq i}}^{p} \mathrm{d}(y_i,y_j)^{-\frac{1}{8}}
$$

$$
\lesssim \sum_{\substack{j_1\ldots j_p=0 \\ j_i\neq i}}^{p} \sum_{\substack{y_1,\ldots,y_p\in U_N \\ y_i\neq y_j \forall i\neq j}} \mathrm{d}(y_1,y_{j_1})^{-\frac{1}{8}}\cdot\ldots\cdot \mathrm{d}(y_p,y_{j_p})^{-\frac{1}{8}}
$$

It is easy to see that for $i\in\{1\ldots p\}$, $j\in\{0\ldots p\}$

$$
\sum_{\substack{y_i\in U_N \\ y_i\neq y_j}} \mathrm{d}(y_i,y_j)^{-\frac{1}{8}} \lesssim N^{\frac{15}{8}}, \tag{3.12}
$$

there are indeed $\sim k$ points at distance k from y_j.

To estimate the term

$$
\sum_{\substack{y_1,\ldots,y_p\in U_N \\ y_i\neq y_j \forall i\neq j}} \mathrm{d}(y_1,y_{j_1})^{-\frac{1}{8}}\cdots \mathrm{d}(y_p,y_{j_p})^{-\frac{1}{8}} \qquad (0\leqslant j_i\leqslant p, j_i\neq i) \tag{3.13}
$$

we need to find the right order in which to compute the sums $\sum_{y_i}$. We associate then (3.13) to a graph with $p+1$ vertices $\{0,1,\ldots,p\}$ and p directed edges, such that to $\mathrm{d}(y_i,y_{j_i})$ corresponds an edge going from i to j_i.

Notice that every vertex in $\{1,\ldots,p\}$ has exactly one edge going to a vertex in $\{0,1,\ldots,p\}$ and the vertex 0 has no outgoing edges. Therefore, following the directed edges starting from any vertex in $\{1,\ldots,p\}$ one either ends up at the vertex 0, or enters a cycle (because every vertex except 0 has an outgoing edge). This cycle cannot be escaped, again because vertices in $\{1,\ldots,p\}$ have only one outgoing edge (indeed, to every y_i there is only one y_{j_i} associated to it).

This said, we can conclude that our graph has one or more connected components, each of which can be of two distinct types:

- a tree with root in the vertex 0
- a cycle, possibly with branches attached to it (i.e. each point of the cycle can be the root of a tree).

We can then proceed to estimate every sum in (3.13) in the order given by the oriented graph, starting from the leaves. This is just a repeated application of (3.12), until we reach the root (0) or a circle. Hence every connected component with root in 0 and k edges gives a term of order $N^{\frac{15}{8}k}$. For example we can estimate the following term as follows (starting from the leaves y_1 and y_3):

$$
\sum_{\substack{y_1,y_2,y_3,\in U_N \\ y_i\neq y_j \forall i\neq j}} \mathrm{d}(y_1,y_2)^{-\frac{1}{8}}\mathrm{d}(y_2,y_0)^{-\frac{1}{8}}\mathrm{d}(y_3,y_2)^{-\frac{1}{8}}
$$

$$
\leqslant \sum_{y_2\in U_N} \mathrm{d}(y_2,y_0)^{-\frac{1}{8}} \sum_{\substack{y_1\in U_N \\ y_1\neq y_2}} \mathrm{d}(y_1,y_2)^{-\frac{1}{8}} \sum_{\substack{y_3\in U_N \\ y_3\neq y_2}} \mathrm{d}(y_2,y_3)^{-\frac{1}{8}} \lesssim N^{\frac{45}{8}}
$$

Summing on circles does not pose any additional problem: indeed one can just choose a point within the circle (call it $\widehat{y}_2$) and sum keeping fixed both the "inbound" point $\widehat{y}_1$ and the "outbound" point $\widehat{y}_3$:

$$\sum_{\substack{\widehat{y}_2 \in U_N \\ \widehat{y}_2 \neq \widehat{y}_1, \widehat{y}_2 \neq \widehat{y}_3}} \mathrm{d}(\widehat{y}_1, \widehat{y}_2)^{-\frac{1}{8}} \mathrm{d}(\widehat{y}_2, \widehat{y}_3)^{-\frac{1}{8}} \leqslant \sum_{\substack{\widehat{y}_2 \in U_N \\ \widehat{y}_2 \neq \widehat{y}_1}} \frac{\mathrm{d}(\widehat{x}_1, \widehat{x}_2)^{-\frac{1}{4}}}{2} + \sum_{\substack{\widehat{y}_2 \in U_N \\ \widehat{y}_2 \neq \widehat{y}_3}} \frac{\mathrm{d}(\widehat{y}_2, \widehat{y}_3)^{-\frac{1}{4}}}{2}$$
$$\lesssim N^{2-\frac{1}{4}}$$

where we used Young inequality. Then (for a circle with k edges) the sum over the remaining vertices $\widehat{y}_3 \ldots \widehat{y}_k$ gives an estimation of order $N^{\frac{15}{8}(k-2)}$. This proves (3.11).

Now consider the general case in which two or more points concide. At the price of a factor $p!$ we can reorder the points, and take the last $p-k$ points to be all different from each other (with $2 \leqslant k \leqslant p$). Conversely, $\{y_1, \ldots, y_k\}$ can be partitioned in m subsets such that all the points in the same subset are equal: we call k_i the number of points in the i-th subset with $k = k_1 + \ldots + k_m$, and therefore $m \leqslant k/2$. We want to show that:

$$\sum_{\substack{\overline{y}_1, \ldots, \overline{y}_m \in U_N, \\ \overline{y}_i \neq y_j, i \leqslant m, j \in [k+1,p]}} \sum_{\substack{y_{k+1}, \ldots, y_p \in U_N \\ y_i \neq y_j}} \mathbb{E}^+_{U_N}(\sigma_{\overline{y}_1}^{k_1} \cdots \sigma_{\overline{y}_m}^{k_m} \sigma_{y_{k+1}} \cdots \sigma_{y_p}) \leqslant C N^{\frac{15}{8}p}.$$

As before we define $\ell_i = \min_{j \geqslant 0, j \neq i} \mathrm{d}(y_i, y_j)$ for every $k+1 \leqslant i \leqslant p$ and $B_i = y_i + [\![-\ell_i/4, \ell_i/4]\!]^2$. Notice that the event $A^1_{y_1 \ldots y_p}$ implies that every y_i with $i \geqslant k+1$ is connected by an open path to the boundary of B_i. Then using the results already obtained:

$$\sum_{\substack{\overline{y}_1, \ldots, \overline{y}_m \in U_N \\ \overline{y}_i \neq y_j, i \leqslant m, k+1 \leqslant j \leqslant p}} \sum_{\substack{y_{k+1}, \ldots, y_p \in U_N \\ y_i \neq y_j}} \mathbb{E}^+_{U_N}(\sigma_{\overline{y}_1}^{k_1} \cdots \sigma_{\overline{y}_m}^{k_m} \sigma_{y_{k+1}} \cdots \sigma_{y_p})$$
$$\lesssim N^{2m} \phi^+_{U_N}\left(\bigcap_{i=k+1}^{p} \{y_i \leftrightarrow \partial B_i\}\right) \lesssim N^{2m} N^{\frac{15}{8}(p-k)} \leqslant N^{\frac{15}{8}p}. \qquad \square$$

We are now ready to prove Theorem 1.2.

Proof of Theorem 1.2. By Theorem 2.30, the result is proved as soon as we can bound (3.8) and (3.9) for any even $p \geqslant 2$. If the domain U is bounded, we choose $\mathscr{K} = (K_n, n)_{n \in \mathbb{N}}$ as its spanning sequence, with:

$$K_n = \{x \in \mathbb{R}^2 \mid \mathrm{dist}(x, U^c) \geq (2 + \delta) R 2^{-n}\} \tag{3.14}$$

for $\delta > 0$ and R such that (2.3) holds. If U is unbounded, it suffices to take $\widehat{K}_n = K_n \cap \overline{B}(0, n)$: in both cases we have a valid spanning sequence according to Definition 2.21.

We first consider (3.9). From the support properties of $\psi^{(i)}(2^n(\cdot - x))$ (2.3) we can restrict the sum over y_j to the set

$$\Omega_{n,x} = \{y \in U_a \mid \mathrm{d}(y, x) < 2^{-n} R + a/\sqrt{2}\}.$$

Now we bound (3.9) separately for small and large values of n.

If $2^n \geqslant R a^{-1}$ we have

$$\sum_{y_1 \ldots y_p \in U_a} \mathbb{E}^\xi_{U_a}(\sigma_{y_1} \cdots \sigma_{y_p}) \prod_{j=1}^{p} \int_{S_a(y_j)} \psi^{(i)}(2^n(z-x)) \mathrm{d}z$$
$$\leqslant \sum_{y_1 \ldots y_p \in \Omega_{n,x}} \prod_{j=1}^{p} \int_{S_a(y_j)} \left|\psi^{(i)}(2^n(z-x))\right| \mathrm{d}z \leqslant \sum_{y_1 \ldots y_p \in \Omega_{n,x}} 2^{-2pn} \lesssim 2^{-2pn}.$$

This gives the estimation

$$a^{-\frac{1}{8}}2^{2n} \sup_{x \in \Lambda_n \cap K} \left[\sum_{y_1 \dots y_p \in U_a} \mathbb{E}_{U_a}^{\xi}(\sigma_{y_1} \cdots \sigma_{y_p}) \prod_{j=1}^{p} \int_{S_a(y_j)} \psi^{(i)}(2^n(z-x)) \mathrm{d}z \right]^{\frac{1}{p}} \lesssim 2^{\frac{1}{8}n}.$$

Conversely, if $2^n < Ra^{-1}$ we first notice that

$$\Omega_{n,x} \subseteq \tilde{U}_{a,x} = [x - 2R2^{-n}, x + 2R2^{-n}]^2 \cap a\mathbb{Z}^2$$

and then using Lemma 3.4:

$$\sum_{y_1 \dots y_p \in \tilde{U}_{a,x}} \mathbb{E}_{U_a}^{\xi}(\sigma_{y_1} \cdots \sigma_{y_p}) \lesssim \sum_{y_1 \dots y_p \in \tilde{U}_{a,x}} \mathbb{E}_{U_a}^{+}(\sigma_{y_1} \cdots \sigma_{y_p})$$

$$\lesssim \sum_{y_1 \dots y_p \in \tilde{U}_{a,x}} \mathbb{E}_{\tilde{U}_{a,x}}^{+}(\sigma_{y_1} \cdots \sigma_{y_p}) \lesssim \sum_{y_1 \dots y_p \in [\![-N,N]\!]^2} \mathbb{E}_{[\![-N,N]\!]^2}^{+}(\sigma_{y_1} \cdots \sigma_{y_p})$$

with $N = \lfloor \frac{2R2^{-n}}{a} \rfloor$. By Proposition 3.10, we finally obtain

$$\sum_{y_1 \dots y_p \in \tilde{U}_{a,x}} \mathbb{E}_{U_a}^{\xi}(\sigma_{y_1} \cdots \sigma_{y_p}) \lesssim a^{-\frac{15}{8}p} 2^{-\frac{15}{8}pn},$$

uniformly over x. As a result, (3.9) can be bound from above by $C2^{\frac{1}{8}n}$ for some $C > 0$. Using the same techniques it is easy to obtain a bound for (3.8):

$$a^{-\frac{1}{8}} \sup_{x \in \Lambda_k \cap K} \left[\sum_{y_1 \dots y_p \in U_a} \mathbb{E}_{U_a}^{\xi}(\sigma_{y_1} \cdots \sigma_{y_p}) \prod_{j=1}^{p} \int_{S_a(y_j)} \varphi(2^k(z-x)) \mathrm{d}z \right]^{\frac{1}{p}} \lesssim 1.$$

Therefore, by the tightness criterion of Theorem 2.30 we have shown that Φ_a is tight in $\mathcal{B}_{p,q}^{-\frac{1}{8}-\varepsilon,\mathrm{loc}}(U)$ for $p \geqslant 2$ and even. The embedding described in Remark 2.12 yields the result for all $p \in [1, \infty]$. $\qquad \square$

3.4 Absence of tightness in higher-order spaces

In this subsection, we prove Theorem 1.3. The proof is based on the following lemma, which is a consequence of the RSW-type bounds for the FK model obtained in [10].

Lemma 3.11 ([10, Proposition 27])**.** *There exists $c > 0$ such that for any $y_1, y_2 \in \mathbb{Z}^2$ with* $\mathrm{d}(y_1, y_2) > 0$*:*

$$\mathbb{E}_{\mathbb{Z}^2}^{\xi}(\sigma_{y_1} \sigma_{y_2}) \geqslant c\, \mathrm{d}(y_1, y_2)^{-\frac{1}{4}}$$

for any boundary condition ξ.

In order to show the absence of tightness we only need the following partial converse to Proposition 3.10 for two-points correlations.

Lemma 3.12. *There exists $c > 0$ such that, for every $N \in \mathbb{N}$:*

$$\sum_{y_1, y_2 \in U_N} \mathbb{E}_{\mathbb{Z}^2}^{\xi}(\sigma_{y_1} \sigma_{y_2}) \geqslant c(N+1)^{\frac{15}{4}}$$

with $U_N = [0, N]^2 \cap \mathbb{Z}^2$ and $\mathbb{E}_{\mathbb{Z}^2}^{\xi}$ being the expectation on $\mathbb{Z}^2$ with arbitrary boundary conditions.

Proof. The result is immediate since there are $(N+1)^4$ terms in the sum, each being larger than $c(N+1)^{-\frac{1}{4}}$ for some fixed constant $c > 0$. $\qquad \square$

We now present an equivalent norm $\mathscr{E}_p^\alpha$ for Besov spaces, which reduces to Definition 2.1 in the case $p = \infty$.

Definition 3.13 ([20, Definition 2.5]). *Let $f \in C_c^\infty$. For every $\alpha < 0$ and $p \in [1, \infty]$ we introduce the norm*

$$\|f\|_{\mathscr{E}_p^\alpha} := \sup_{\lambda \in (0,1]} \lambda^{-\alpha} \left\| \sup_{\eta \in \mathscr{B}_{r_0}} |\langle f, \eta_{\lambda,x} \rangle| \right\|_{L^p(\mathrm{d}x)}$$

with $\eta_{\lambda,x} := \lambda^{-d} \eta\left(\lambda^{-1}(\cdot - x)\right)$ and $\mathscr{B}_{r_0}$ as in Definition 2.1.

The following is a straightforward generalization of Proposition 2.16.

Lemma 3.14 ([20, Proposition 2.6]). *Let $\alpha < 0$. There exist $C_1, C_2 \in (0, \infty)$ such that for every $f \in C_c^\infty$, we have*

$$C_1 \|f\|_{\mathscr{E}_p^\alpha} \leqslant \|f\|_{\mathcal{B}_{p,\infty}^\alpha} \leqslant C_2 \|f\|_{\mathscr{E}_p^\alpha}. \tag{3.15}$$

The advantage of Definition 3.13 is that it allows us to easily obtain lower bounds on the Besov norm of some distribution by testing against a non-negative function. We can now proceed to prove Theorem 1.3.

Proof of Theorem 1.3. We decompose the proof into three steps.

Step 1. In this first step, we recall that for a non-negative random variable X, we have

$$\mathbb{P}\left[X > \frac{\mathbb{E}[X]}{2}\right] \geqslant \frac{(\mathbb{E}[X])^2}{4\mathbb{E}[X^2]}. \tag{3.16}$$

Indeed, this follows from

$$\mathbb{E}[X] = \mathbb{E}[X \mathbb{1}_{X \leqslant \mathbb{E}[X]/2}] + \mathbb{E}[X \mathbb{1}_{X > \mathbb{E}[X]/2}] \leqslant \frac{\mathbb{E}[X]}{2} + \mathbb{E}\left[X^2\right]^{\frac{1}{2}} \mathbb{P}\left[X > \frac{\mathbb{E}[X]}{2}\right]^{\frac{1}{2}},$$

by the Cauchy-Schwarz inequality.

Step 2. Let η be a smooth non-negative function supported on the ball $B(0,1)$ and such that $\eta \equiv 1$ on $B(0, 1/2)$. We set $\eta_{\lambda,x} := \lambda^{-2} \eta(\lambda^{-1}(\cdot - x))$ and

$$X_{a,\lambda} := \int_{B(0,1)} |\langle \Phi_a, \eta_{\lambda,x} \rangle| \, \mathrm{d}x.$$

In this step, we show that there exists a constant $c > 0$ such that for every $a < \lambda \in (0, 1]$,

$$\mathbb{P}\left[X_{a,\lambda} \geqslant c \lambda^{-\frac{1}{8}}\right] \geqslant c. \tag{3.17}$$

As in the proof of Theorem 1.2, we can use Proposition 3.10 to show that there exists a constant $C < \infty$ such that for every $p \in \{2, 4\}$, $a < \lambda \in (0, 1]$ and $x \in \mathbb{R}^2$,

$$\mathbb{E}\left[(\langle \Phi_a, \eta_{\lambda,x} \rangle)^p\right] \leqslant C \lambda^{-\frac{p}{8}}. \tag{3.18}$$

By a similar reasoning, we obtain from Lemma 3.12 that there exists a constant $c > 0$ such that for every $a < \lambda \in (0, 1]$ and $x \in \mathbb{R}^2$,

$$\mathbb{E}\left[(\langle \Phi_a, \eta_{\lambda,x} \rangle)^2\right] \geqslant c \lambda^{-\frac{1}{4}}. \tag{3.19}$$

Combining (3.16), (3.18) with $p = 4$ and (3.19), we deduce that for every $a < \lambda \in (0, 1]$ and $x \in \mathbb{R}^2$,

$$\mathbb{P}\left[|\langle \Phi_a, \eta_{\lambda,x} \rangle| \geqslant \frac{\sqrt{c}}{2} \lambda^{-\frac{1}{8}}\right] \geqslant \frac{c}{C}.$$

In particular, after reducing the constant $c > 0$ as necessary, we obtain that for every $a < \lambda \in (0, 1]$ and $x \in \mathbb{R}^2$,

$$\mathbb{E}\left[|\langle \Phi_a, \eta_{\lambda, x}\rangle|\right] \geqslant c\lambda^{-\frac{1}{8}}, \tag{3.20}$$

and thus that

$$\mathbb{E}[X_{a,\lambda}] \geqslant c\lambda^{-\frac{1}{8}}.$$

Using (3.18) with $p = 2$ and Jensen's inequality, we also have, for every $a < \lambda \in (0, 1]$,

$$\mathbb{E}[X_{a,\lambda}^2] \leqslant C\lambda^{-\frac{1}{4}}.$$

We therefore obtain (3.17) by another application of (3.16).

Step 3. Let $\alpha > -\frac{1}{8}$, and let $\overline{\Phi}$ be a possible limit point of the family $(\Phi_a)_{a \in (0,1]}$. Passing to the limit along a subsequence in (3.17), we get that for every $\lambda \in (0, 1]$,

$$\mathbb{P}\left[\int_{B(0,1)} \left|\langle \overline{\Phi}, \eta_{\lambda, x}\rangle\right| \, \mathrm{d}x \geqslant c\lambda^{-\frac{1}{8}}\right] \geqslant c. \tag{3.21}$$

By Remark 2.12, in order to prove Theorem 1.3, it suffices to show that $\overline{\Phi} \notin \mathcal{B}_{1,\infty}^{\alpha, \mathrm{loc}}(\mathbb{R}^2)$ with positive probability. Let χ be a non-negative smooth function of compact support such that $\chi \equiv 1$ on $B(0, 2)$. By Lemma 3.14, there exists a constant $c' > 0$ such that for every $\lambda \in (0, 1]$,

$$\|\chi\overline{\Phi}\|_{\mathcal{B}_{1,\infty}^\alpha} \geqslant c'\lambda^{-\alpha} \int_{\mathbb{R}^2} \left|\langle \chi\overline{\Phi}, \eta_{\lambda, x}\rangle\right| \, \mathrm{d}x \geqslant c'\lambda^{-\alpha} \int_{B(0,1)} \left|\langle \overline{\Phi}, \eta_{\lambda, x}\rangle\right| \, \mathrm{d}x.$$

Combining this with (3.21) yields

$$\mathbb{P}\left[\|\chi\overline{\Phi}\|_{\mathcal{B}_{1,\infty}^\alpha} \geqslant cc'\lambda^{-\alpha-\frac{1}{8}}\right] \geqslant c.$$

Since $\alpha > -\frac{1}{8}$, letting λ tend to 0 gives

$$\mathbb{P}\left[\|\chi\overline{\Phi}\|_{\mathcal{B}_{1,\infty}^\alpha} = +\infty\right] \geqslant c > 0,$$

which completes the proof. $\qquad\qquad\square$

A

In Section 2 we left behind some details for the sake of self-containedness: in particular the proof of Proposition 2.19 with Besov spaces of the type $\mathcal{B}_{p,q}^{\alpha, \mathrm{loc}}$ for any $p, q \geqslant 1$. In order to show that this statement is true in the general case (and not only for $\mathcal{B}_{\infty,\infty}^{\alpha, \mathrm{loc}}$) we need some results about the product of elements of Besov spaces. We obtain these by relating the Besov spaces as defined in this paper with those in [4], defined via the Littlewood-Paley decomposition.

Theorem A.1 ([23, Proposition 2.9.4]). *Let $\alpha > 0$, $p, q \in [1, \infty]$ and $f \in L^p(\mathbb{R}^d)$. The following two properties are equivalent.*

1. *Let $r > \alpha$ be an integer and $\phi, (V_n)_{n \in \mathbb{Z}}$ be a r-regular multiresolution analysis as of Definition 2.5. Then the sequence $2^{n\alpha}\|\mathscr{W}_n f\|_{L^p}$ belongs to $\ell^q(\mathbb{N})$ and $\mathscr{V}_0 f$ belongs to $L^p(\mathbb{R}^d)$.*

2. *There exists a sequence of positive numbers $\varepsilon_n \in \ell^q(\mathbb{N})$ and a sequence of functions $f_0, g_0, g_1, \ldots \in L^p(\mathbb{R}^d)$ such that $f = f_0 + \sum_{n \geqslant 0} g_n$, $\|g_n\|_{L^p} \leqslant \varepsilon_n 2^{-n\alpha}$ for $n \geqslant 0$ and $\|\partial^k g_n\|_{L^p} \leqslant \varepsilon_n 2^{(m-\alpha)n}$ for some integer $m > \alpha$ and every multi-index $k \in \mathbb{N}^d$ such that $|k| = m$.*

In particular, the functions $f_0 = \mathscr{V}_0 f$, $g_n = \mathscr{W}_n f$ verify (2). Moreover, the norms $\|f_0\|_{L^p} + \|2^{n\alpha}\|g_n\|_{L^p}\|_{\ell^q}$ and $\|\mathscr{V}_0 f\|_{L^p} + \|2^{n\alpha}\|\mathscr{W}_n f\|_{L^p}\|_{\ell^q}$ are equivalent.

A first consequence of this result is the fact that the Besov spaces defined in Section 2 are independent from the choice of a particular wavelet basis or multiresolution analysis.

Proposition A.2 (Equivalence of multiresolution analyses). *For any $\alpha \in \mathbb{R}$ and any positive integer r such that $r > |\alpha|$, the norm $\|\cdot\|_{\mathcal{B}_{p,q}^\alpha}$ of Definition 2.8 does not depend on the given r-regular multiresolution analysis, i.e. every r-regular multiresolution analysis yields an equivalent norm.*

Proof. Theorem A.1 gives the equivalence of norms for $\alpha > 0$.

For $\alpha < 0$, $p, q \in [1, \infty]$, define $\alpha' = -\alpha$, $1/p + 1/p' = 1$ and $1/q + 1/q' = 1$. We introduce the following norm which is clearly independent from the choice of multiresolution analysis:

$$\|f\|_{\tilde{\mathcal{B}}_{p,q}^\alpha} = \sup_{\substack{g \in L^{p'} \\ \|g\|_{\mathcal{B}_{p',q'}^{\alpha'}} \leqslant 1}} \langle f, g \rangle$$

(notice that this norm is slightly different from the norm of the dual of $\mathcal{B}_{p',q'}^{\alpha'}$, because we choose $\mathcal{B}_{p',q'}^{\alpha'}$ to be the complection of C_c^∞ with respect to the norm $\|\cdot\|_{\mathcal{B}_{p',q'}^{\alpha'}}$).

We want to show that $\|\cdot\|_{\tilde{\mathcal{B}}_{p,q}^\alpha}$ and $\|\cdot\|_{\mathcal{B}_{p,q}^\alpha}$ are equivalent. Let $f \in \mathcal{C}_c^\infty$. The bound $\|f\|_{\tilde{\mathcal{B}}_{p,q}^\alpha} \lesssim \|f\|_{\mathcal{B}_{p,q}^\alpha}$ is straightforward: by Theorem A.1 we can write $g = \mathscr{V}_0 g + \sum_n \mathscr{W}_n g$ and obtain

$$\langle f, g \rangle = \langle \mathscr{V}_0 f, \mathscr{V}_0 g \rangle + \sum_{n \geqslant 0} \langle \mathscr{W}_n f, \mathscr{W}_n g \rangle \leqslant \|f\|_{\mathcal{B}_{p,q}^\alpha} \|g\|_{\mathcal{B}_{p',q'}^{\alpha'}}$$

thanks to the orthogonality in L^2 between spaces W_n and Hölder's inequality.

To show that $\|f\|_{\mathcal{B}_{p,q}^\alpha} \lesssim \|f\|_{\tilde{\mathcal{B}}_{p,q}^\alpha}$, recall that if $f \in L^p(\mu)$ then

$$\|f\|_{L^p(\mu)} = \sup_{g \in L^{p'}(\mu), \|g\|_{L^{p'}} \leqslant 1} \int f(x) g(x) \mu(\mathrm{d}x)$$

(see e.g. Lemma 1.2 of [4]). Then for every $\delta > 0$ there exists $h_0 \in L^{p'}$ such that $\|h_0\|_{L^{p'}} \leqslant 1$ and $\|\mathscr{V}_0 f\|_{L^p} \leqslant \int \mathscr{V}_0 f(x) h_0(x) \mathrm{d}x + \delta$. Take $Q_N^{q'} = \{(a_n)_{n \geqslant 0} \in \ell^{q'} \mid \|a_n\|_{\ell^{q'}} \leqslant 1, a_n = 0 \text{ for } n > N\}$. We have

$$\|f\|_{\mathcal{B}_{p,q}^\alpha} = \|\mathscr{V}_0 f\|_{L^p} + \sup_{N \in \mathbb{N}} \sup_{(a_n) \in Q_N^{q'}} \sum_{n=0}^{N} a_n 2^{\alpha n} \|\mathscr{W}_n f\|_{L^p}.$$

As above, for every $n \geqslant 0$ there exist $g_n \in L^{p'}$ such that $\|g_n\|_{L^{p'}} \leqslant 1$ and $\|\mathscr{W}_n f\|_{L^p} \leqslant \int \mathscr{W}_n f(x) g_n(x) \mathrm{d}x + \varepsilon_n$. Now we can estimate the norm

$$\|f\|_{\mathcal{B}_{p,q}^\alpha} \leqslant \langle \mathscr{V}_0 f, \mathscr{V}_0 h_0 \rangle + \sup_{N \in \mathbb{N}} \sup_{(a_n) \in Q_N^{q'}} \sum_{n=0}^{N} \langle \mathscr{W}_n f, 2^{n\alpha} a_n \mathscr{W}_n g_n \rangle + \varepsilon$$

$$\varepsilon = \delta + \sup_{N \in \mathbb{N}} \sup_{(a_n) \in Q_N^{q'}} \sum_{n=0}^{N} 2^{n\alpha} a_n \varepsilon_n$$

where we used the fact that the spaces W_n are orthogonal in L^2. The remainder ε can be made arbitrarily small: indeed $\sum_{n=0}^{N} 2^{n\alpha} a_n \varepsilon_n \leqslant \|2^{\alpha n}\|_{\ell^q} \sup_{n \geqslant 0} \varepsilon_n$ (recall that $\alpha < 0$). Define

$$g_N = \mathscr{V}_0 h_0 + \sum_{n=0}^{N} 2^{n\alpha} a_n \mathscr{W}_n g_n.$$

The operators $\mathscr{V}_n : L^p \to L^p$ and $\mathscr{W}_n : L^p \to L^p$ are uniformly bounded: we can estimate the norm of g_N as

$$\|g_N\|_{\mathcal{B}_{p',q'}^{\alpha'}} \leqslant \|h_0\|_{L^{p'}} + \|2^{n\alpha'}2^{n\alpha}a_n\|g_n\|_{L^{p'}}\|_{\ell^{q'}} \leqslant C$$

and then

$$\|f\|_{\mathcal{B}_{p,q}^{\alpha}} \leqslant \sup_{N\in\mathbb{N}} \sup_{(a_n)\in Q_N^{q'}} \langle f, g_N\rangle + \varepsilon = \sup_{\substack{g_N\in L^{p'}\\ \|g_N\|_{\mathcal{B}_{p',q'}^{\alpha'}}\leqslant C}} \langle f, g_N\rangle + \varepsilon \lesssim \|f\|_{\tilde{\mathcal{B}}_{p,q}^{\alpha}} + \varepsilon.$$

This completes the proof of the result for $\alpha \neq 0$. The case $\alpha = 0$ can then be recovered by interpolation. $\qquad\square$

We now introduce the Littlewood-Paley decomposition. We refer to [4, Chapter 2] for this definition.

Proposition A.3 (Dyadic partition of unity). *There exist $\chi \in \mathcal{C}_c^{\infty}(\mathbb{R}^d)$ with values in $[0,1]$ and support cointained in the ball $\mathcal{B} = \{x \in \mathbb{R}^d \mid |x| \leqslant 3/4\}$, and $\rho \in C_c^{\infty}(\mathbb{R}^d)$ with values in $[0,1]$ and support contained in the annulus $\mathcal{A} = \{x \in \mathbb{R}^d \mid 3/4 \leqslant |x| \leqslant 8/3\}$, such that for every $x \in \mathbb{R}^d$:*

$$1 = \chi(x) + \sum_{n\geqslant 0}\rho(2^{-n}x)$$

and the sum is finite. We have also that, if $|n - n'| \geqslant 2$:

$$\operatorname{Supp}\rho(2^{-n}\cdot)\cap\operatorname{Supp}\rho(2^{-n'}\cdot) = \emptyset \tag{A.1}$$

and if $n \geqslant 1$:

$$\operatorname{Supp}\chi\cap\operatorname{Supp}\rho(2^{-n}\cdot) = \emptyset$$

Definition A.4 (Littlewood-Paley-Besov space). *Let $f \in C_c^{\infty}$, for every $n \geqslant -1$ the dyadic Littlewood-Paley blocks are defined as*

$$\Delta_{-1}u = \mathcal{F}^{-1}(\chi\widehat{f})$$
$$\Delta_n u = \mathcal{F}^{-1}(\rho(2^{-n}\cdot)\widehat{f}) \quad \text{for every } n \geqslant 0$$

where $\mathcal{F}(f) = \widehat{f}$ is the Fourier transform of f (and $\mathcal{F}^{-1}$ its inverse).

Define the norm $\|\cdot\|_{\mathcal{B}_{p,q}^{\alpha,\mathrm{LP}}}$ as

$$\|f\|_{\mathcal{B}_{p,q}^{\alpha,\mathrm{LP}}} = \left\|(2^{\alpha n}\|\Delta_n f\|_{L^p})_{n\geqslant -1}\right\|_{\ell^q}$$

and the Littlewood-Paley-Besov space $\mathcal{B}_{p,q}^{\alpha,\mathrm{LP}}$ as the closure of C_c^{∞} with respect to this norm.

Remark A.5. It is easy to check that the space $\mathcal{B}_{p,q}^{\alpha,\mathrm{LP}}$ does not depend on the choice of a dyadic partition of unity $\chi,\rho \in C_c^{\infty}$, and that the operators $\Delta_n : L^p \to L^p$, $\Delta_{-1} : L^p \to L^p$ are uniformly bounded for every $p \in [1,\infty]$ (see [4, Section 2.2]).

Remark A.6 (Equivalence of LP-wavelet Besov spaces). The space $\mathcal{B}_{p,q}^{\alpha,\mathrm{LP}}$ defined above coincides with the Besov space $\mathcal{B}_{p,q}^{\alpha}$ that we used throughout these notes (Definition 2.8): i.e. for $f \in C_c^{\infty}$ their respective norms are equivalent. Indeed, the functions $\Delta_{-1}f$ and $\Delta_n f$ verify the conditions within point (2) of Theorem A.1. The property

$$\|\partial^k\Delta_n f\|_{L^p} \leqslant \varepsilon_n 2^{(m-\alpha)n}$$

for $\varepsilon_n \in \ell^q$ is obtained by *Bernstein estimates* [4, Lemma 2.1], while the other two conditions are easily checked directly.

Now we can use Theorems 2.82 and 2.85 of [4], which yield a general proof of Proposition 2.19.

Theorem A.7 (Multiplicative inequalities). *Let $p, p_1, p_2, q, q_1, q_2 \in [1, \infty]$ be such that*

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} \quad and \quad \frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}.$$

(1) If $\alpha > 0$, then the mapping $(f, g) \mapsto fg$ extends to a bilinear continuous functional from $\mathcal{B}_{p_1, q_1}^{\alpha} \times \mathcal{B}_{p_2, q_2}^{\alpha}$ to $\mathcal{B}_{p, q}^{\alpha}$.
(2) If $\alpha < 0 < \beta$ with $\alpha + \beta > 0$, then the mapping $(f, g) \mapsto fg$ extends to a bilinear continuous functional from $\mathcal{B}_{p_1, q_1}^{\alpha} \times \mathcal{B}_{p_2, q_2}^{\beta}$ to $\mathcal{B}_{p, q}^{\alpha}$.

Remark A.8. Theorem A.7 yields, as announced, a complete proof of Proposition 2.19. In Section 2 we proved that for any $\alpha < 0$ and $\chi \in C_c^{r_0}$, $r_0 = -\lfloor \alpha \rfloor$, the mapping $f \mapsto \chi f$ extends to a continuos functional on $\mathcal{C}^{\alpha}$. This result can be extended to $\mathcal{B}_{p, q}^{\alpha}$ observing that $C_c^{r_0} \subseteq \mathcal{B}_{\infty, \infty}^{r_0}$ (see [4, Section 2.7]) and applying the theorem above.

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