Electronic Journal of Probability

Electron. J. Probab. **22** (2017), no. 94, 1–21.
ISSN: 1083-6489 DOI: 10.1214/17-EJP104

Long time asymptotics of unbounded additive functionals of Markov processes*

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Abstract

Under hypercontractivity and L_p-integrability of transition density for some $p > 1$, we use the perturbation theory of linear operators to obtain existence of long time asymptotics of exponentials of unbounded additive functionals of Markov processes and establish the moderate deviation principle for the functionals. For stochastic differential equations with multiplicative noise, we show the hypercontractivity and the integrability based on Wang's Harnack inequality. As an application of our general results, we obtain the existence of these asymptotics and the moderate deviation principle of additive functionals with quadratic growth for the stochastic differential equations with multiplicative noise under some explicit conditions on the coefficients and prove that these asymptotics solve the related ergodic Hamilton-Jacobi-Bellman equation with nonsmooth and quadratic growth cost in viscosity sense.

Keywords: additive functional; hypercontractivity; long time asymptotics; moderate deviation; perturbation theory.
AMS MSC 2010: 60F10; 60J55; 60H10; 47A55.
Submitted to EJP on March 29, 2017, final version accepted on September 6, 2017.

1 Introduction

The purpose of this paper is to study the existence of the following limits (1.1) and (1.2) for general Markov processes $\{X_t, t \geq 0\}$ and unbounded functions c:

$$\lambda = \lim_{t \to +\infty} \frac{1}{t} \log E_x \left(\exp \left\{ \gamma \int_0^t c(X_s) ds \right\} \right), \tag{1.1}$$

$$V(x) = \lim_{t \to +\infty} \left\{ \log E_x \left(\exp \left\{ \gamma \int_0^t c(X_s) ds \right\} \right) - \lambda t \right\}. \tag{1.2}$$

The existence of the limits (1.1) and (1.2) is closely related to spectral theory, large deviations and functional inequalities. For bounded functions c, Wu [27] proved the limits (1.1) and (1.2) exist under a logarithmic Sobolev inequality, and Kontoyiannis and

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Meyn ([13]) studied the existence of the two limits by the spectral theory under the geometric ergodicity assumption. In [14], the results were extended to a large class of functions whose growth at infinity are strictly less than quadratic when their results are applied to diffusion processes. Cattiaux, Dai Pra and Rœlly ([3]) proposed an approach based on cluster expansion (a method in statistical mechanics, see [15]) to establish the existence of (1.2) for a class of unbounded and nonsmooth functions. Their result can be applied to a class of ergodic HJB equations with constant diffusion coefficient and quadratic growth cost. But the condition

$$\|p(t,\cdot,\cdot)\|_{L_p(\mu\times\mu)} < \infty \text{ for some } p > 2$$

in [3] restricts its applications, where $p(t,x,\cdot)$ is a transition density. One motivation of this paper is to improve the condition to

$$\|p(t,\cdot,\cdot)\|_{L_p(\mu\times\mu)} < \infty \text{ for some } p > 1.$$

Using the perturbation theory of linear operators, we establish existence of the pair (λ, V) of limits (1.1) and (1.2) of unbounded additive functionals for general Markov processes under hypercontractivity and L_p-integrability of transition density $p(t,x,\cdot)$ for some $p > 1$. This improves the condition $\|p(t,\cdot,\cdot)\|_{L_p(\mu\times\mu)}$ for some $p > 2$ in [3]. The improvement plays an important role in application to stochastic differential equations with multiplicative noise. This method can give a precise representation of V by the projection operator. The second motivation is to study moderate deviations of additive functional $\int_0^t c(X_s)ds$ with unbounded function c. Wu ([26]) studied the problem by the perturbation theory, a key step ((2.11) in [26]) is to check when $t \to \infty$,

$$\Lambda_t(z) := \frac{1}{t}\log E_x\left(\exp\left\{z\int_0^t c(X_s)ds\right\}\right)$$

converges to a holomorphic function in a neighborhood of $0 \in \mathbb{C}$. But this is far from trivial because it is very difficult to verify that $\{\Lambda_t(z), t \geq t_0\}$ is a family of holomorphic functions in a common complex neighborhood of $0 \in \mathbb{C}$ for some $t_0 > 0$. To overcome these difficulties, we restrict the logarithmic function on $\mathbb{R}$ to avoid the logarithmic function of complex Feynman-Kac operator in [26] and use the Taylor expansion of the largest modulus eigenvalue of the Feynman-Kac semigroup to replace C^2-regularity in [26]. For stochastic differential equations with multiplicative noise, we show the hypercontractivity and the integrability based on Wang's Harnack inequality ([25]). As an application of our general results, we establish the existence of the pair (λ, V) for stochastic differential equations with multiplicative noise under some explicit conditions on the coefficients and the moderate deviation principle for additive functionals with quadratic growth of the solutions and prove that the pair (λ, V) solves an ergodic HJB equation with nonsmooth and quadratic growth cost in viscosity sense.

Let us first introduce some notations. Let $(E, \mathcal{E})$ be a Polish space and let $\{P(t,x,A), t \in [0,\infty), x \in E, A \in \mathcal{E}\}$ be a transition probability function. P_x denotes the probability measure on $\mathcal{F}$ such that under P_x the coordinate process $\{X_t,\ t \in [0,\infty)\}$ is a right-continuous Markov process with the transition probability functions $P(t,x,A)$ and starting from x. We assume that $P(t,x,A)$ has a stationary distribution μ. Set

$$P_t f(x) = \int f(y)P(t,x,dy).$$

Let $b\mathcal{E}^{\mathbb{C}}$, and $C_b^{\mathbb{C}}(E)$ denote the spaces of bounded measurable and bounded continuous complex functions $f : E \to \mathbb{C}$, respectively. Set

$$L_p^{\mathbb{C}}(E,\mu) = \{f : E \to \mathbb{C};\ \text{measurable and } \|f\|_p < \infty\},$$

where $\|f\|_p = (\int |f(x)|^p \mu(dx))^{1/p}$, $p \geq 1$. For any probability ν on E, set $E_\nu(g) = \int_E \nu(dx) E_x(g)$. We also denote by

$$b\mathcal{E} = \{f : E \to \mathbb{R}; f \in b\mathcal{E}^{\mathbb{C}}\}, \quad C_b(E) = \{f : E \to \mathbb{R}; f \in C_b^{\mathbb{C}}(E)\},$$
$$L_p(E,\mu) = \{f : E \to \mathbb{R}; f \in L_p^{\mathbb{C}}(E,\mu)\}.$$

We introduce the following conditions:

(C1). $\{P_t, t > 0\}$ is μ-hypercontractive, i.e., there exists $\tilde{t}_0 > 0$ such that

$$\|P_{\tilde{t}_0}\|_{2\to 4} := \sup_{\|f\|_2 \leq 1} \|P_{\tilde{t}_0} f\|_4 = 1. \tag{1.3}$$

(C2). $\{P_t, t > 0\}$ has the strong Feller property, i.e., $P_t f \in C_b(E)$ for any $f \in b\mathcal{E}$.

(C3). There exists transition density function $p(t,x,y)$, i.e.,

$$P_t f = \int_E f(y) p(t,x,y) \mu(dy), \quad f \in C_b(E).$$

and there exist $\tilde{t}_1 > 0$ and $p > 1$ such that for any $t \geq \tilde{t}_1$, any compact set $K \subset E$,

$$\sup_{x \in K} \|p(t,x,\cdot)\|_p < \infty. \tag{1.4}$$

(C4). Let $c : E \to \mathbb{R}$ be a mesurable function which satisfies that for any $a > 0$, there exist $\delta_a > 0$ and a locally bounded function $M_a(x)$ such that

$$E_x\left(\exp\{\delta_a |c(X_s)|\}\right) \leq M_a(x) \text{ for all } s \in [0,a]. \tag{1.5}$$

Moreover, there exists $\delta > 0$ such that

$$\int_E \exp\{\delta |c(x)|\} \mu(dx) < \infty. \tag{1.6}$$

Our two abstract results are the following two theorems.

Theorem 1.1. Assume that the conditions (C1), (C2),(C3) and (C4) hold. Then there exists $\bar{\gamma} > 0$ such that for all $\gamma \in [-\bar{\gamma}, \bar{\gamma}]$, the pair (λ, V) of limits (1.1) and (1.2) exists and uniformly converges on each compact subset.

Theorem 1.2. Assume that the conditions (C1), (C2),(C3) and (C4) hold. Let $a(t), t > 0$ be a positive function such that as $t \to \infty$,

$$\frac{\sqrt{t}}{a(t)} \to 0, \quad \frac{a(t)}{t} \to 0.$$

Then for any $z \in \mathbb{R}$,

$$\lim_{t\to\infty} \left| \frac{t}{a^2(t)} \log E_\mu \left(\exp\left\{ \frac{a(t)}{t} z \int_0^t (c(X_s) - \mu(c)) ds \right\} \right) - \frac{\sigma^2(c) z^2}{2} \right| = 0; \tag{1.7}$$

and for any compact $K \subset E$,

$$\lim_{t\to\infty} \sup_{x \in K} \left| \frac{t}{a^2(t)} \log E_x \left(\exp\left\{ \frac{a(t)}{t} z \int_0^t (c(X_s) - \mu(c)) ds \right\} \right) - \frac{\sigma^2(c) z^2}{2} \right| = 0, \tag{1.8}$$

where

$$\sigma^2(c) = 2 \int_0^\infty \int_E (c(x) - \mu(c)) P_s(c - \mu(c))(x) \mu(dx) ds. \tag{1.9}$$

In particular, $P_\mu(\frac{1}{a(t)} \int_0^t (c(X_s) - \mu(c)) ds \in \cdot)$ satisfies a large deviation principle in $\mathbb{R}$ with speed $t/a^2(t)$ and rate function $J(y) = \frac{y^2}{2\sigma^2(c)}$, and $P_x(\frac{1}{a(t)} \int_0^t (c(X_s) - \mu(c)) ds \in \cdot)$ satisfies a local uniform large deviation principle in $\mathbb{R}$. Namely, for any compact $K \subset E$, for any closed set F in $\mathbb{R}$,

$$\limsup_{t\to\infty} \frac{t}{a^2(t)} \left\{ \begin{array}{l} \log P_\mu \left(\frac{1}{a(t)} \int_0^t (c(X_s) - \mu(c)) ds \in F \right) \\ \sup_{x\in K} \log P_x \left(\frac{1}{a(t)} \int_0^t (c(X_s) - \mu(c)) ds \in F \right) \end{array} \right\} \leq - \inf_{y\in F} J(y); \tag{1.10}$$

and for any open set G in $\mathbb{R}$,

$$\liminf_{t\to\infty} \frac{t}{a^2(t)} \left\{ \begin{array}{l} \log P_\mu \left(\frac{1}{a(t)} \int_0^t (c(X_s) - \mu(c)) ds \in G \right) \\ \inf_{x\in K} \log P_x \left(\frac{1}{a(t)} \int_0^t (c(X_s) - \mu(c)) ds \in G \right) \end{array} \right\} \geq - \inf_{y\in G} J(y). \tag{1.11}$$

Remark 1.1. (1). The local uniformity in Theorem 1.2 is with respect to initial points rather than initial measures in [26], Theorem 2.4.

(2). Using the method in this paper, that is the Taylor expansion of the largest modulus eigenvalue of the Feynman-Kac semigroup which can avoid the logarithmic function of complex Feynman-Kac operator in [26], we can prove the main results in [26].

Next, we apply Theorem 1.1 and Theorem 1.2 to the following SDEs with multiplicative noise:

$$dX_t = b(X_t)dt + \sigma(X_t)dB_t, \quad X_0 = x \in \mathbb{R}^d, \tag{1.12}$$

where the coefficients of the SDEs (1.12) satisfy the following conditions (see [25]):

(A1). $b : \mathbb{R}^d \to \mathbb{R}^d$, and $\sigma : \mathbb{R}^d \to \mathbb{R}^d \otimes \mathbb{R}^d$ are continuous differentiable functions, and there exists a constant $K_{\sigma,b}$ such that

$$\|\sigma(x) - \sigma(y)\|_{HS}^2 + 2\langle b(x) - b(y), x - y\rangle \leq K_{\sigma,b}|x - y|^2, \ x, y \in \mathbb{R}^d,$$

where $\|A\|_{HS} = \sqrt{\mathrm{tr}(AA^\tau)}$, $\mathrm{tr}(AA^\tau)$ denotes the trace of square matrix AA^τ and A^τ denotes transpose of matrix A, and $\langle y, z\rangle = \sum_{i=1}^d y_i z_i$ for any $y, z \in \mathbb{R}^d$.

(A2). There exist constants $0 < \kappa_1 \leq \kappa_2 < \infty$ such that

$$\kappa_1^2 I \leq a(x) \leq \kappa_2^2 I, \ x \in \mathbb{R}^d,$$

where $a(x) := \sigma(x)\sigma^\tau(x)$.

(A3). There exists a constant $\vartheta > 0$ such that almost surely

$$|(\sigma(x) - \sigma(y))(x - y)| \leq \vartheta|x - y|, \ x, y \in \mathbb{R}^d.$$

Then under (A1) and (A2), there exists a constant L such that

$$\|\sigma(x)\sigma^\tau(x)\|_{HS} \leq L, \quad \langle b(x), x\rangle \leq L(1 + |x|^2),$$

and the fact that a is continuous and uniform positive definite yields the existence and uniqueness for the weak solution of (1.12) (see [21],Theorem 7.2.1 and localization), and the weak solution has the strong Feller property. On the other hand, the assumption (A1) ensures the uniqueness of the solution of (1.12). Thus, by Yamada-Watanabe theorem (see [10]), we obtain the existence and uniqueness for the strong solution of (1.12) and the strong solution has the strong Feller property.

Theorem 1.3. Let (A1)-(A3) hold for constants $K_{\sigma,b}$, κ_1 and ϑ and let the following condition (DC) be valid,

(DC) . there exists $c_b > 0$ and $R \geq 0$ such that for $|x| \geq R$,

$$\langle b(x), x\rangle \leq -c_b|x|^2.$$

Assume $c_b > K_{\sigma,b}^+ \kappa_2^2/\kappa_1^2$, where $K_{\sigma,b}^+ = \max\{K_{\sigma,b}, 0\}$. Then

(1). The solution $\{X_t, t \geq 0\}$ of the SDEs (1.12) has the following properties:

(a) There exist a unique stationary distribution μ and a constant $\beta > K^{+}_{\sigma,b}/\kappa_1^2$ such that

$$\int_{\mathbb{R}^d} e^{\beta|x|^2}\mu(dx) < \infty, \tag{1.13}$$

In particular, $\{P_t, t > 0\}$ is μ-hypercontractive.

(b) There exists a constant M, such that for any $a > 0$, when $\beta \le \frac{c_b}{\kappa_2^2}$, for all $x \in \mathbb{R}^d$, $s > 0$

$$E_x\left(\exp\left\{\beta|X_s|^2\right\}\right) \le e^{\beta|x|^2} + M. \tag{1.14}$$

(c) There exists a transition density function $p(t,x,y)$ with respect to μ, and there exist $\tilde{t}_2 > 0$ and $p > 1$ such that for any $t \ge \tilde{t}_2$,

$$\sup_{x\in K}\|p(t,x,\cdot)\|_p < \infty, \quad \|p(t,\cdot,\cdot)\|_{L_p(\mu\times\mu)} < \infty. \tag{1.15}$$

(2). Let $c : \mathbb{R}^d \to \mathbb{R}$ be a cost function with quadratic growth, i.e., there exists constant $L > 0$ such that

$$|c(x)| \le L(|x|^2 + 1),\ x \in \mathbb{R}^d. \tag{1.16}$$

Then there exists $\bar{\gamma} > 0$ such that

(a) For all $\gamma \in [-\bar{\gamma}, \bar{\gamma}]$, the pair (λ, V) of limits (1.1) and (1.2) exists and uniformly converges on each compact subset.

(b) For $z \in \mathbb{R}$,

$$\lim_{t\to\infty}\left|\frac{t}{a^2(t)}\log E_\mu\left(\exp\left\{\frac{a(t)}{t}z\int_0^t (c(X_s)-\mu(c))ds\right\}\right) - \frac{\sigma^2(c)z^2}{2}\right| = 0; \tag{1.17}$$

and for any compact $K \subset E$,

$$\lim_{t\to\infty}\sup_{x\in K}\left|\frac{t}{a^2(t)}\log E_x\left(\exp\left\{\frac{a(t)}{t}z\int_0^t (c(X_s)-\mu(c))ds\right\}\right) - \frac{\sigma^2(c)z^2}{2}\right| = 0. \tag{1.18}$$

In particular, $P_\mu(\frac{1}{a(t)}\int_0^t(c(X_s)-\mu(c))ds \in \cdot)$ satisfies a large deviation principle in $\mathbb{R}$ with speed $t/a^2(t)$ and rate function $J(y) = \frac{y^2}{2\sigma^2(c)}$, and $P_x(\frac{1}{a(t)}\int_0^t(c(X_s)-\mu(c))ds \in \cdot)$ satisfies a local uniform large deviation principle in $\mathbb{R}$.

Definition 1.1. Let V be a continuous function on $\mathbb{R}^d$ and $\lambda \in \mathbb{R}$. (λ, V) is called a viscosity solution of the equation (1.19):

$$\lambda = \frac{1}{2}\sum_{i,j=1}^d a_{ij}\frac{\partial^2 V}{\partial x_i \partial x_j} + \sum_{i=1}^d b_i\frac{\partial V}{\partial x_i} + \frac{1}{2}\sum_{i,j=1}^d a_{ij}\frac{\partial V}{\partial x_i}\frac{\partial V}{\partial x_j} + \gamma c, \tag{1.19}$$

if V is a viscosity supersolution of (1.19), that is, for any $\psi \in C^2(\mathbb{R}^d)$ and any local maximum point x of $V - \psi$,

$$\begin{aligned}\lambda - \Bigg(&\frac{1}{2}\sum_{i,j=1}^d a_{ij}(x)\frac{\partial^2\psi(x)}{\partial x_i\partial x_j} + \sum_{i=1}^d b_i(x)\frac{\partial\psi(x)}{\partial x_i}\\ &+ \frac{1}{2}\sum_{i,j=1}^d a_{ij}(x)\frac{\partial\psi(x)}{\partial x_i}\frac{\partial\psi(x)}{\partial x_j} + \gamma c^*(x)\Bigg) \le 0;\end{aligned}$$

and V is a viscosity subsolution of (1.19), that is, for any $\psi \in C^2(\mathbb{R}^d)$ and any local minimum point x of $V - \psi$,

$$\lambda - \Bigg(\frac{1}{2}\sum_{i,j=1}^{d} a_{ij}(x)\frac{\partial^2\psi(x)}{\partial x_i \partial x_j} + \sum_{i=1}^{d} b_i(x)\frac{\partial\psi(x)}{\partial x_i} \\ + \frac{1}{2}\sum_{i,j=1}^{d} a_{ij}(x)\frac{\partial\psi(x)}{\partial x_i}\frac{\partial\psi(x)}{\partial x_j} + \gamma c_*(x)\Bigg) \geq 0;$$

where

$$c^*(x) := \limsup_{y\to x} c(y),\ c_*(x) := \liminf_{y\to x} c(y).$$

Theorem 1.4. If (DC) and (A1)-(A3) hold and $c_b > K^+_{\sigma,b}\kappa_2^2/\kappa_1^2$, then for any $\gamma \in [-\bar{\gamma}, \bar{\gamma}]$, (λ, V) is a viscosity solution of (1.19).

Remark 1.2. Theorem 1.3 and Theorem 1.4 extend Theorem 2 and Proposition 2 in [3] to non-constant diffusion coefficients. They can not be obtained by the method in [3] because some rigor conditions for the diffusion coefficient are needed when the method in [3] is applied to general diffusion processes (see Remark 3.1).

The existence of the solution of (1.19) can date back to [7] and [19], where (1.19) takes the form

$$\lambda = \frac{1}{2}\sum_{i}^{d}\frac{\partial^2 V}{\partial x_i^2} + \sum_{i=1}^{d} b_i(x)\frac{\partial V}{\partial x_i} + \frac{1}{2}\sum_{i=1}^{d}\frac{\partial V}{\partial x_i}\frac{\partial V(x)}{\partial x_i} + \gamma c(x). \tag{1.20}$$

Under sufficient ergodicity assumption on $\{X_t, t \geq 0\}$, if c is bounded and sufficient smooth, then (1.20) has a solution. The case of c unbounded was considered in [16]. Recently, under some smooth conditions, Ichihara and Sheu [9] considered the solution of a type of ergodic HJB equation in the classical sense. When $a \in C^{2+\gamma}$, and $b \in C^{1+\gamma}$, and $c \in C^{1+\gamma}$ for some $\gamma \in (0,1]$, Robertson and Xing ([22]) studied the solution of ergodic HJB equation (1.19) in the classical sense. The advantage of our method is that it can be applied to general nonsmooth and quadratic growth cost functions.

The proofs of Theorem 1.1 and Theorem 1.2, and The proof of Theorem 1.3 will be given in Section 2 and Section 3, respectively. The main tool of the proof of Theorem 1.1 is the perturbation theory of linear operators. The proof of Theorem 1.3 is based on Theorem 1.1 and the Harnack inequality of stochastic differential equations with multiplicative noise in [25]. A sketch proof of Theorem 1.4 is in Section 3. We introduce briefly Kato's perturbation theory of linear operators ([12]) in Appendix.

2 Long time asymptotics and moderate deviations

In this section, we prove Theorem 1.1 and Theorem 1.2 using the perturbation theory of linear operators. we restrict the logarithmic function on $\mathbb{R}$ to avoid the logarithmic function of complex Feynman-Kac operator in [26] and use the Taylor expansion of the largest modulus eigenvalue of the Feynman-Kac semigroup to replace C^2-regularity in [26].

2.1 Proof of Theorem 1.1

Step 1. Analyticity of the Feynman-Kac operators in a neighborhood of 0. By Theorem 5.5.12 in [5], under the assumption (C1), for any $f \in b\mathcal{E}$, $t \geq \tilde{t}_0$,

$$\|P_t f - \mu(f)\|_2 \leq 3^{-[t/\tilde{t}_0]/2}\|f\|_2,$$

where $\mu(f) := \int_E f d\mu$. Moreover, if one set $\alpha_0 = \frac{\log(3/2)}{4\tilde{t}_0}$, then for $1 < u < v < \infty$, and $t \geq 4\tilde{t}_0$ with $e^{\alpha_0 t} \geq \frac{v-1}{u-1}$,

$$\|P_t\|_{u\to v} := \sup_{f\in L_u^{\mathbb{C}}(E,\mu),\|f\|_u\leq 1} \|P_t f\|_v = 1. \tag{2.1}$$

Let $t_0 \geq 4\max\{\tilde{t}_0, \tilde{t}_1, 1/4\}$ be given. $P_{t_0}^{zc}$ denotes the Feynman-Kac operator:

$$P_{t_0}^{zc} g(x) = E_x\left(e^{\int_0^{t_0} zc(X_s)ds} g(X_{t_0})\right). \tag{2.2}$$

Then, under the assumptions (C1) and (C4), by Hölder inequality, for any $q > 1$, there exists a positive constant $\varrho > 0$ such that $\{P_{t_0}^{zc}, z \in U := \{z \in \mathbb{C}; |z| < 2\varrho\}\}$ is a bounded-holomorphic family of operators on $(L_q^{\mathbb{C}}(E,\mu), \|\cdot\|_q)$, and

$$P_{t_0}^{\varrho|c|} I_E \in L_q(E,\mu),$$

where $I_E(x) = 1$ for all $x \in E$. In fact (see [26],P.433), by (2.1), there exists $u \in (1,q)$ such that $\|P_{t_0}\|_{u\to q} = 1$. Take $l \geq 1$ with $lu = q$. Set $q' = (1-1/q)^{-1}$ and $l' = (1-1/l)^{-1}$, i.e., the conjugated numbers of q and l. Then for any nonnegative $g \in L_q^{\mathbb{C}}(E,\mu)$,

$$\begin{aligned}
\|P_{t_0}^{|zc|} g\|_q^q &\leq \int_E \left|E_x\left(e^{l' z\int_0^{t_0}|zc(X_s)|ds}\right)\right|^{q/l'} \left|E_x\left(|g(X_{t_0}|^l)\right)\right|^{q/l} \mu(dx) \\
&\leq \left|\int_E e^{l'q'zt_0|zc(x)|ds}\mu(dx)\right|^{q/(q'l')} \left|\int_E P_{t_0}|g^l(x)|^q \mu(dx)\right|^{1/l} \\
&\leq \left|\int_E e^{l'q'zt_0|zc(x)|ds}\mu(dx)\right|^{q/(q'l')} \|g^l\|_u^{q/l} \\
&= \left|\int_E e^{l'q't_0|zc(x)|ds}\mu(dx)\right|^{q/(q'l')} \|g\|_q^q.
\end{aligned}$$

Thus, if one takes $\varrho = \min\{\frac{\delta}{4l'q't_0}, \frac{\delta_{t_0}}{4q}, \frac{\delta}{4q}\}$, then for any $z \in U := \{z \in \mathbb{C}; |z| < 2\varrho\}$, $P_{t_0}^{zc}$ is bounded on $(L_q(E,\mu), \|\cdot\|_q)$.

It is easy to check that $U \ni z \to \int f(x) P_{t_0}^{zc} g(x)\mu(dx)$ is analytic for any $f, g \in b\mathcal{E}$. Thus, by a characterization of bounded-holomorphic family of operators (see Appendix A), $P_{t_0}^{zc}$ is bounded-holomorphic in U.

Step 2. Strong Feller property and its density estimates for the Feynman-Kac operators.

Firstly, let us show that for any $t \in [0, t_0]$, $x \to P_t^{zc} g(x)$ is continuous, where g is a bounded function on E and $z \in (-2\varrho, 2\varrho)$.

By the Markov property, we can write that

$$P_t^{zc} g(x) = E_x\left(e^{z\int_0^\varepsilon c(X_s)ds} P_{t-\varepsilon}^{zc} g(X_\varepsilon)\right), \quad 0 < \varepsilon < t,$$

and so

$$\begin{aligned}
|P_t^{zc} g(x) - P_t^{zc} g(y)| \leq & E_x\left(\left|e^{z\int_0^\varepsilon c(X_s)ds} - 1\right| P_{t-\varepsilon}^{zc}|g(X_\varepsilon)|\right) \\
&+ E_y\left(\left|e^{z\int_0^\varepsilon c(X_s)ds} - 1\right| P_{t-\varepsilon}^{zc}|g(X_\varepsilon)|\right) \\
&+ |P_\varepsilon(P_{t-\varepsilon}^{zc} g)(x) - P_\varepsilon(P_{t-\varepsilon}^{zc} g)(y)|.
\end{aligned} \tag{2.3}$$

For each ε fixed, for each $M > 0$, $x \to P_\varepsilon(P_{t-\varepsilon}^{zc} g I_{\{|P_{t-\varepsilon}^{zc} g| \leq M\}})(x)$ is continuous from the strong Feller property. Thus for any sequence $x_n \to x$, for each $0 < M < \infty$

$$\lim_{n\to\infty} |P_\varepsilon(P_{t-\varepsilon}^{zc} g I_{\{|P_{t-\varepsilon}^{zc} g|\leq M\}})(x_n) - P_\varepsilon(P_{t-\varepsilon}^{zc} g I_{\{|P_{t-\varepsilon}^{zc} g|\leq M\}})(x)| = 0.$$

By (C4), $x \to P_\varepsilon((P_{t-\varepsilon}^{zc} g)^2)(x)$ is locally bounded. Therefore, for each $x \in E$, for any sequence $x_n \to x$,

$$C(x) := \limsup_{n\to\infty}(P_\varepsilon((P^{zc}_{t-\varepsilon}g)^2)(x_n) + P_\varepsilon((P^{zc}_{t-\varepsilon}g)^2)(x)) < \infty,$$

and

$$P_\varepsilon(|P^{zc}_{t-\varepsilon}g|I_{\{|P^{zc}_{t-\varepsilon}g|\geq M\}})(x) \leq \frac{1}{M}P_\varepsilon((P^{zc}_{t-\varepsilon}g)^2)(x).$$

This implies that

$$\lim_{M\to\infty}\limsup_{n\to\infty} P_\varepsilon(|P^{zc}_{t-\varepsilon}g|I_{\{|P^{zc}_{t-\varepsilon}g|\geq M\}})(x_n) + P_\varepsilon(|P^{zc}_{t-\varepsilon}g|I_{\{|P^{zc}_{t-\varepsilon}g|\geq M\}})(x) = 0.$$

Thus,

$$\lim_{n\to\infty}|P_\varepsilon(P^{zc}_{t-\varepsilon}g)(x_n) - P_\varepsilon(P^{zc}_{t-\varepsilon}g)(x)| = 0. \tag{2.4}$$

By the Hölder inequality,

$$\begin{aligned}
&E_x\left(\left|e^{z\int_0^\varepsilon c(X_s)ds} - 1\right| P^{zc}_{t-\varepsilon}|g(X_\varepsilon)|\right)\\
&\leq \left(E_x\left(\left|e^{z\int_0^\varepsilon c(X_s)ds} - 1\right|^2\right)\right)^{\frac{1}{2}}(P^{2\varrho|c|}_{t-\varepsilon}|g|^2(x))^{1/2}
\end{aligned}$$

and

$$\begin{aligned}
&\left(E_x\left(\left|e^{z\int_0^\varepsilon c(X_s)ds} - 1\right|^2\right)\right)^2\\
&\leq z^4\left(E_x\left(\left(\int_0^\varepsilon c(X_s)ds\right)^2 e^{2|z|\int_0^\varepsilon |c(X_s)|ds}\right)\right)^2\\
&\leq \varepsilon^2 z^4\int_0^\varepsilon E_x\left(|c(X_s)|^4\right)ds\int_0^\varepsilon E_x\left(e^{4|z|\varepsilon|c(X_s)|}\right)ds.
\end{aligned}$$

Thus, by (C4), for any compact K,

$$\lim_{\varepsilon\to 0}\sup_{x\in K} E_x\left(\left|e^{z\int_0^\varepsilon c(X_s)ds} - 1\right| P^{zc}_{t-\varepsilon}|g(X_\varepsilon)|\right) = 0. \tag{2.5}$$

Combining (2.3), (2.4) and (2.5), we obtain that $x \to P^{zc}_t g(x)$ is continuous.

Next, let us give some estimates for the density of the Feynman-Kac operators. Let $z \in (-\varrho, \varrho)$ Since for any $A \in \mathcal{E}$, it is shown that $x \to P^{2zc}_{t_0}I_A(x)$ is continuous. Since

$$P^{zc}_{t_0}I_A(x) \leq (P^{2zc}_{t_0}I_E(x))^{1/2}(P_{t_0}I_A(x))^{1/2},\ A \in \mathcal{E},$$

there exists $(x,y) \to p_z(t_0,x,y)$ non-negative measurable such that

$$P^{zc}_{t_0}I_A(x) = \int_A p_z(t_0,x,y)\mu(dy).$$

Let r be the conjugated number of $2p'$, i.e., $r = \frac{2p}{p+1}$. Then

$$\begin{aligned}
\|p_z(t_0,x,\cdot)\|_r &= \sup_{\|g\|_{2p'}\leq 1}|P^{zc}_{t_0}g(x)|\\
&\leq \sup_{\|g\|_{2p'}\leq 1}|P^{2zc}_{t_0}I_E(x)|^{1/2}(P_{t_0}|g|^2(x))^{1/2}\\
&\leq |P^{2zc}_{t_0}I_E(x)|^{1/2}\|p(t_0,x,\cdot)\|_p^{\frac{1}{2}}.
\end{aligned}$$

Thus, by (C3) and (C4), for any compact set $K \subset E$,

$$\sup_{x\in K}\|p_z(t_0,x,\cdot)\|_r < \infty. \tag{2.6}$$

Since $\|p_z(t_0,x,\cdot)\|_1 = |P^{2zc}_{t_0}I_E(x)| > 0$ is continuous, for any compact set $K \subset E$,

$$\inf_{x\in K}\|p_z(t_0,x,\cdot)\|_1 > 0. \tag{2.7}$$

Thus, for any compact $K \subset E$,

$$M(K) := \left\{ \nu_x(dy) = \frac{p_z(t_0, x, y)}{\int_E p_z(t_0, x, y)\mu(dy)} \mu(dy);\ x \in K \right\}$$

is a tight family of probability measures on E.

Step 3. Existence of the Limits (λ, V). Let q be the conjugated number of r, i.e., $q = 2p' = \frac{2p}{p-1}$. By Riesz-Thorin interpolation theorem, P_{t_0} also has a spectral gap in $L_q^{\mathbb{C}}(E, \mu)$, thus, there exists $0 < \varepsilon < 1/4$ such that

$$(\Sigma_{t_0}^{(q)} \setminus \{1\}) \cap \{\lambda \in \mathbb{C}; |\lambda| \geq 1 - \varepsilon\} = \emptyset, \tag{2.8}$$

where $\Sigma_{t_0}^{(q)}$ denotes the spectra of P_{t_0} regarded as an operator on $L_q^{\mathbb{C}}(E, \mu)$.

Similar to [26], set $T(z) = P_{t_0}^{zc}$ and $\mathbb{B} = L_q^{\mathbb{C}}(E, \mu)$. Then $\{T(z) \in C(\mathbb{B}, \mathbb{B}); z \in U\}$ is a bounded-holomorphic family of bounded operators. Set $\Gamma = \{\lambda \in \mathbb{C}; |\lambda - 1| = \varepsilon/2\}$. Then by Theorem A.1 in Appendix, there exists $0 < \delta_1 < 2\varrho$ such that for any $|z| \leq \delta_1$, $\Gamma \subset P(T(z))$ (i.e., the resolvent set of $T(z)$) and the spectra $\Sigma(T(z)) := \Sigma_{t_0}^{(q)}$ is likewise separated by Γ into two parts $\Sigma'(z) = \{\lambda(z)\}$, $\Sigma''(z)$ with the associated decomposition $\mathbb{B} = M'(z) \oplus M''(z)$ of the space. Furthermore, the dimension of $M'(z)$ is 1, i.e., $\dim(M'(z)) = \dim(M'(0)) = 1$.

Let $G_{t_0}(z)$ denote the complex number with the largest modulus in the spectrum of $P_{t_0}^{zc}$ regarded as an operator on $(L_q^{\mathbb{C}}(E, \mu), \|\cdot\|_q)$. Define

$$J(z) = -\frac{1}{2\pi i} \int_\Gamma \frac{1}{P_{t_0}^{zc} - \zeta} d\zeta. \tag{2.9}$$

Since $J(z)$ is the projection on $M'(z)$ along $M''(z)$, by $T(z) \to T(0)$, we see that there exists $0 < \delta_2 < 2\varrho$ such that for any $|z| \leq \delta_2$, $G_{t_0}(z) = \lambda(z)$, and so

$$|G_{t_0}(z)| \geq 1 - \varepsilon/2.$$

Since $I - J(z)$ is the projection on $M''(z)$ along $M'(z)$, we have

$$T(z)^n - G_{t_0}(z)^n J(z) = T(z)^n (I - J(z)).$$

Since the projection $J(z)$ converges to $J(0)$ as $z \to 0$, $T(z)(I - J(z)) \to T(0)(I - J(0))$. Noting that $\|T(0)\|_{M''(0)} = \sup_{g \in M''(0), \|g\|_q = 1} |T(0)g| \leq 1 - \varepsilon$, there exists $0 < \delta_3 < 2\varrho$ such that for any $|z| \leq \delta_3$, $\|T(z)\|_{M''(z)} \leq 1 - 2\varepsilon/3$, and so,

$$\|(T(z)^n (I - J(z))\| \leq (1 - 2\varepsilon/3)^n, \quad n \geq 1,$$

For $z \in [-2\varrho, 2\varrho]$, $t \in [0, t_0]$, Set

$$g_{z,t}(x) = P_t^{zc} I_E(x),$$

Then for $z \in [-2\varrho, 2\varrho]$, $t \in [0, t_0]$,

$$\underline{g}(x) \leq g_{z,t}(x) \leq \bar{g}(x)$$

where

$$\underline{g}(x) = P_{t_0}^{-2\varrho|c|} I_E(x) \in L_q(E, \mu), \quad \bar{g}(x) = P_{t_0}^{2\varrho|c|} I_E(x) \in L_q(E, \mu).$$

Note that $J(0)I_E = 1$ and $J(z) \to J(0)$ as $z \to 0$. Since $M(K)$ is tight, there exists $0 < \delta_4 < 2\varrho$ such that if $|z| \leq \delta_4$,

$$\eta := \inf_{x \in K} \inf_{t \in [0, t_0]} |\int_E (J(z) g_{z,t})(y) \nu_x(dy)| > 0.$$

Since $M'(z)$ is also one dimensional and $J(z)$ is the projection on $M'(z)$ along $M''(z)$, we obtain that for any $t \in [0, t_0]$,

$$\int_E J(z) g_{z,t}(y) \nu_x(dy) = T(z)(J(z) g_{z,t})(x) = G_{t_0}(z)(J(z) g_{z,t})(x).$$

and there exists a constant $\theta_s \in \mathbb{C}$ such that $J(z) g_{z,s}(x) = \theta_s J(z) I_E(x)$. By $T(z) P_t^{zc} = P_t^{zc} T(z)$, we also have that $J(z) P_t^{zc} = P_t^{zc} J(z)$. Thus

$$\begin{aligned}\theta_{t+s} J(z) I_E(x) =& J(z) g_{z,t+s}(x) = P_t^{zc} J(z) P_s^{zc} I_E(x) \\ =& \theta_s P_t^{zc} J(z) I_E(x) = \theta_s \theta_t J(z) I_E(x),\end{aligned}$$

which implies $\theta_{t+s} = \theta_s \theta_t$. It is obvious that $\theta_0 = 1$ and by

$$\theta_{t_0} J(z) I_E(x) = J(z) P_{t_0}^{zc} I_E(x) = P_{t_0}^{zc} J(z) I_E(x) = G_{t_0}(z) J(z) I_E(x),$$

$\theta_{t_0} = G_{t_0}(z)$. Thus $|\theta_t| = e^{\frac{t}{t_0} \log |G_{t_0}(z)|}$, i.e.,

$$|J(z) g_{z,t}(x)| = e^{t \log |G_{t_0}(z)|/t_0} |J(z) I_E(x)|.$$

Note that for any $x \in K$,

$$E_{\nu_x}\left(\exp\left\{z \int_0^{nt_0} c(X_s) ds\right\}\right) = \frac{E_x\left(\exp\left\{z \int_0^{(n+1)t_0} c(X_s) ds\right\}\right)}{\|p_z(t_0, x, \cdot)\|_1},$$

and for any $t \in [(n+1)t_0, (n+2)t_0]$,

$$\begin{aligned}&E_x\left(\exp\left\{z \int_0^t c(X_s) ds\right\}\right) \\ =& E_x\left(\exp\left\{z \int_0^{(n+1)t_0} c(X_s) ds\right\} g_{z,(t-(n+1)t_0)}(X_{(n+1)t_0})\right),\end{aligned}$$

Thus, for any $t \in [(n+1)t_0, (n+2)t_0]$,

$$\begin{aligned}&\log E_x\left(\exp\left\{z \int_0^t c(X_s) ds\right\}\right) - \frac{t}{t_0} \log |G_{t_0}(z)| - \log |J(z) I_E(x)| \\ =& \log E_x\left(\exp\left\{z \int_0^t c(X_s) ds\right\}\right) - (n+1) \log |G_{t_0}(z)| - \log \left|J(z) g_{z,(t-(n+1)t_0)}(x)\right| \\ =& \log E_{\nu_x}\left(\exp\left\{z \int_0^{nt_0} c(X_s) ds\right\} g_{z,(t-(n+1)t_0)}(X_{nt_0})\right) \\ &- n \log |G_{t_0}(z)| - \log \left|\int J(z) g_{z,(t-(n+1)t_0)}(y) \nu_x(dy)\right| \\ =& \log \frac{\int T(z)^n g_{(t-(n+1)t_0)}(y) \nu_x(dy)}{\left|\int G_{t_0}(z)^n J(z) g_{z,(t-(n+1)t_0)}(y) \nu_x(dy)\right|}.\end{aligned}$$

Set $\bar{\gamma} = \min\{\delta_1, \delta_2, \delta_3, \delta_4\}$. Then for n large enough, for any $z \in [-\bar{\gamma}, \bar{\gamma}]$

$$\begin{aligned}&\sup_{x \in K}\left|\log E_x\left(\exp\left\{z \int_0^t c(X_s) ds\right\}\right) - \frac{t}{t_0} \log |G_{t_0}(z)| - \log |J(z) I_E(x)|\right| \\ \le& \log\left(1 + \frac{\sup_{x \in K} \int |T(z)^n (1 - J(z)) g_{z,(t-(n+1)t_0)}(y)| \nu_x(dy)}{(1 - \varepsilon/2)^n \inf_{x \in K} |\int_E J(z) g_{z,(t-(n+1)t_0)}(y) \nu_x(dy)|}\right) \\ \le& \log\left(1 + \frac{2(1 - 2\varepsilon/3)^{n-1} \|\bar{g}\|_q}{(1 - \varepsilon/2)^{n-1} \eta}\right) \to 0 \text{ as } n \to \infty.\end{aligned} \tag{2.10}$$

Set

$$\lambda = \frac{1}{t_0}\log|G_{t_0}(z)|, \text{ and } V(x) = \log|J(z)I_E(x)|.$$

Then the limits (1.1) and (1.2) uniformly converge on each compact subset. The proof of Theorem 1.1 is completed. □

2.2 The proof of Theorem 1.2

Set $T(z) = P_{t_0}^{z(c-\mu(c))}$ and $\mathbb{B} = L_q^{\mathbb{C}}(E,\mu)$. Let $G(z)$ be the complex number with the largest modulus in the spectrum of $P_{t_0}^{z(c-\mu(c))}$ regarded as an operator on $\mathbb{B}$ and

$$J(z) = -\frac{1}{2\pi i}\int_\Gamma \frac{1}{T(z)-\zeta}d\zeta.$$

Then, it is similar to (2.10) that for any $z \in \mathbb{R}$,

$$\lim_{t\to\infty}\Bigg|\log E_\mu\left(\exp\left\{z\int_0^t (c(X_s)-\mu(c))ds\right\}\right) - \frac{t}{t_0}\log|G(z)| - \log|J(z)I_E(x)|\Bigg| = 0;$$

and for any compact set K,

$$\lim_{t\to\infty}\sup_{x\in K}\Bigg|\log E_x\left(\exp\left\{z\int_0^t (c(X_s)-\mu(c))ds\right\}\right) - \frac{t}{t_0}\log|G(z)| - \log|J(z)I_E(x)|\Bigg| = 0.$$

In particular,

$$\lim_{t\to\infty}\Bigg|\frac{t}{a^2(t)}\log E_\mu\left(\exp\left\{\frac{a(t)}{t}z\int_0^t (c(X_s)-\mu(c))ds\right\}\right) - \frac{t^2}{t_0a^2(t)}\log\left|G\left(\frac{a(t)z}{t}\right)\right|\Bigg| = 0; \tag{2.11}$$

and

$$\lim_{t\to\infty}\sup_{x\in K}\Bigg|\frac{t}{a^2(t)}\log E_x\left(\exp\left\{\frac{a(t)}{t}z\int_0^t (c(X_s)-\mu(c))ds\right\}\right) - \frac{t^2}{t_0a^2(t)}\log\left|G\left(\frac{a(t)z}{t}\right)\right|\Bigg| = 0. \tag{2.12}$$

We can write

$$T(z) = P_{t_0} + zT^{(1)} + z^2T^{(2)} + \cdots,$$

where

$$(T^{(l)}g)(x) = \frac{1}{l!}E_x\left(g(X_{t_0})\left(\int_0^{t_0}(c(X_s)-\mu(c))ds\right)^l\right), \quad l \geq 1.$$

By the definition of $S^{(0)}$ (see (A.14) in Appendix A), $S^{(0)} = -J$. Since $M'(0)$ is one dimensional and J is the projection on $M'(0)$, we have

$$-S^{(0)}g = Jg = E_\mu(g), \quad g \in \mathbb{B}.$$

By the definition of S (see (A.6)), for any $g \in \mathbb{B}$ with $\mu(g) = 0$,

$$Sg = \lim_{\zeta\to 1}(\zeta-1)\times\frac{R(\zeta)g}{\zeta-1} = (P_{t_0}-1)^{-1}g = -\sum_{l=0}^{\infty}P_{t_0}^l g.$$

By Corollary A.1, there exists $r_0 > 0$ such that for any $z \in \mathbb{R}$, when t is large enough,

$$\begin{aligned} &\left| G\left(\frac{a(t)}{t}z\right) - 1 + \frac{a(t)z}{t} tr\left(T^{(1)}S^{(0)}\right) \right. \\ &\qquad \left. + \left(\frac{a(t)}{t}\right)^2 z^2 \left(tr\left(T^{(2)}S^{(0)}\right) - tr\left(T^{(1)}ST^{(1)}S^{(0)}\right)\right) \right| \\ \leq & \frac{\varepsilon \left|\frac{a(t)}{t}z\right|^3}{2r_0^2(r_0 - \left|\frac{a(t)}{t}z\right|)}. \end{aligned} \tag{2.13}$$

For $l \geq 1$, let λ be an eigenvalue the operator $T^{(l)}S^{(0)}$ and let g be a eigenvector of λ. Then

$$\lambda g(x) = (T^{(l)}S^{(0)}g)(x) = -\frac{1}{l!}\mu(g)E_x\left(\left(\int_0^{t_0}(c(X_s)-\mu(c))ds\right)^l\right),$$

and so

$$\lambda\mu(g) = -\frac{1}{l!}\mu(g)E_\mu\left(\left(\int_0^{t_0}(c(X_s)-\mu(c))ds\right)^l\right).$$

Thus, if $\mu(g) = 0$, then $\lambda = 0$; and if $\mu(g) \neq 0$, then

$$\lambda = -\frac{1}{l!}E_\mu\left(\left(\int_0^{t_0}(c(X_s)-\mu(c))ds\right)^l\right).$$

Thus, $tr\left(T^{(1)}S^{(0)}\right) = 0$, and

$$\begin{aligned} tr\left(T^{(2)}S^{(0)}\right) &= -\frac{1}{2}E_\mu\left(\left(\int_0^{t_0}(c(X_s)-\mu(c))ds\right)^2\right) \\ &= -\int_0^{t_0}\int_0^u\int_E (c(y)-\mu(c))(y)P_s(c-\mu(c))(y)\mu(dy)dsdu. \end{aligned}$$

Similarly, we can get

$$\begin{aligned} & tr\left(T^{(1)}ST^{(1)}S^{(0)}\right) \\ =& E_\mu\left(\left(\int_0^{t_0}(c(X_s)-\mu(c))ds\right)\sum_{l=0}^{\infty}E_{X_{t_0}}\left(\int_0^{t_0}(c(X_{s+lt_0})-\mu(c))ds\right)\right) \\ =& \int_0^{t_0}\int_u^\infty\int_E (c(y)-\mu(c))(y)P_s(c-\mu(c))(y)\mu(dy)dsdu. \end{aligned}$$

Therefore

$$\begin{aligned} & -2\left(tr\left(T^{(2)}S^{(0)}\right) - tr\left(T^{(1)}ST^{(1)}S^{(0)}\right)\right) \\ =& 2\int_0^{t_0}\int_0^u\int_E (c(y)-\mu(c))(y)P_s(c-\mu(c))(y)\mu(dy)dsdu \\ & + 2\int_0^{t_0}\int_u^\infty\int_E (c(y)-\mu(c))(y)P_s(c-\mu(c))(y)\mu(dy)dsdu \\ =& 2\int_0^\infty\int_E (c(y)-\mu(c))(y)P_s(c-\mu(c))(y)\mu(dy)ds \\ =:& \sigma^2(c). \end{aligned}$$

Now, by (2.13), we obtain

$$\lim_{t\to\infty}\left|\frac{t^2}{a^2(t)}\log\left|G\left(\frac{a(t)}{t}z\right)-\frac{\sigma^2(c)z^2}{2}\right|=0$$

Which yields (1.7) and (1.8) by (2.11) and (2.12). Finally, by the Gärtner-Ellis theorem, (1.10) and (1.11) follow from (1.7) and (1.8). □

3 Applications to SDEs

In this section, we show Theorem 1.3 and Theorem 1.4. By Theorem 1.1, we only need to show Theorem 1.3 (1). Using Theorem 1.3, the proof of Theorem 1.4 is similar to Proposition 2.2 in [3]. We only give a sketch proof in the section.

3.1 Proofs of Theorem 1.3

By the assumption (DC), if we take $g(x)=|x|^2$, then

$$\sup_{|x|\geq R}\mathcal{L}g(x)\leq L^2-\inf_{|x|\geq R}\langle b(x),x\rangle\leq L^2-c_bR^2\to-\infty \text{ as } R\to\infty,$$

where

$$\mathcal{L}=\frac{1}{2}\sum_{i,j=1}^d a_{ij}(x)\frac{\partial^2}{\partial x_i\partial x_j}+\sum_{i=1}^d b_i(x)\frac{\partial}{\partial x_i}.$$

Thus, the existence and uniqueness of the invariant measure μ follow from Theorem 3.7 in [11].

Step 1. The proof of Theorem 1.3 (1)(a). We prove that for any $0<\beta<c_b/\kappa_2^2$, (1.13) holds.

Similar to the Setup 2 in the proof of Theorem 2 in [3]. Let $g_n\in C_b^2(\mathbb{R})$ be a concave function such that $g_n(u)=u$ if $u\leq n-1$ and $g_n(u)=n$ if $u\geq n$. For $0<\beta<c_b/\kappa_2^2$, set $f_n(x)=e^{\beta g_n(|x|^2)}$. Then by Itô's formula,

$$\begin{aligned}f_n(X_t)=&f_n(X_0)+2\beta\int_0^t f_n(X_s)g_n'(|X_s|^2)\langle X_s,\sigma(X_s)dB_s\rangle\\&+2\beta\int_0^t f_n(X_s)g_n'(|X_s|^2)\langle X_s,b(X_s)\rangle\,ds\\&+\int_0^t f_n(X_s)(2\beta^2|g_n'(|X_s|^2)|^2+2\beta g_n''(|X_s|^2))\langle X_s,a(X_s)X_s\rangle\,ds\\&+\beta\int_0^t f_n(X_s)g_n'(|X_s|^2)\|a(X_s)\|_{HS}^2ds.\end{aligned}$$

By the condition (DC), there exists a constant $l>0$ such that $\langle b(x),x\rangle\leq -c_b|x|^2+l$ for all $x\in\mathbb{R}^d$. Note that $g_n''\leq 0$ and $g_n'\in[0,1]$. Then for any $0\leq s\leq t$,

$$\begin{aligned}E_x\left(f_n(X_t)\right)-E_x\left(f_n(X_s)\right)\leq&-2\beta(c_b-\beta\kappa_2^2)\int_s^t E_x\left(f_n(X_u)g_n'(|X_u|^2)|X_u|^2\right)du\\&+\beta(\kappa_2^2d+2l)\int_s^t E_x\left(f_n(X_u)g_n'(|X_u|^2)\right)du.\end{aligned}$$

For n large enough, we can take a constant $1\leq R^*\leq n-1$ such that $\beta(c_b-\beta\kappa_2^2)(R^*)^2>\beta(\kappa_2^2d+2l)$. Then

$$
\begin{aligned}
&E_x\left(f_n(X_t)\right)-E_x\left(f_n(X_s)\right)\\
&\quad\le \beta(\kappa_2^2 d+2l)\int_s^t E_x\left(f_n(X_u)g_n'(|X_u|^2)I_{[-R^*,R^*]^c}(|X_u|)\right)du\\
&\qquad-\beta(c_b-\beta\kappa_2^2)\int_s^t E_x\left(f_n(X_u)g_n'(|X_u|^2)|X_u|^2 I_{[-R^*,R^*]}(|X_u|)\right)du\\
&\quad\le \beta(\kappa_2^2 d+2l)f_n(R^*))(t-s)-\beta(c_b-\beta\kappa_2^2)\int_s^t E_x\left(f_n(X_u)\right)du.
\end{aligned}
$$

Therefore,

$$
\frac{\partial E_x\left(f_n(X_t)\right)}{\partial t}\le \beta(\kappa_2^2 d+2l)f_n(R^*))-\beta(c_b-\beta\kappa_2^2)E_x\left(f_n(X_t)\right)
$$

By Gronwall's inequality, we obtain

$$
E_x\left(f_n(X_t)\right)\le \frac{\beta(\kappa_2^2 d+2l)f_n(R^*))}{\beta(c_b-\beta\kappa_2^2)}+e^{-\beta(c_b-\beta\kappa_2^2)t}f_n(x).
$$

Therefore,

$$
\begin{aligned}
\int_{\mathbb{R}^d} e^{\beta g_n(|x|^2)}\mu(dx)&=\int_{\mathbb{R}^d} E_x\left(f_n(X_t)\right)\mu(dx)\\
&\le\frac{\beta(\kappa_2^2 d+2l)f_n(R^*))}{\beta(c_b-\beta\kappa_2^2)}+e^{-\beta(c_b-\beta\kappa_2^2)t}\int_{\mathbb{R}^d} e^{\beta g_n(|x|^2)}\mu(dx).
\end{aligned}
$$

That is,

$$
\left(1-e^{-\beta(c_b-\beta\kappa_2^2)t}\right)\int_{\mathbb{R}^d} e^{\beta g_n(|x|^2)}\mu(dx)\le \frac{\beta(\kappa_2^2 d+2l)f_n(R^*))}{\beta(c_b-\beta\kappa_2^2)}.
$$

Now, choosing t such that $e^{-\beta(c_b-\beta\kappa_2^2)t}\le\frac{1}{2}$ and then using the monotone convergence theorem , we obtain $\int_{\mathbb{R}^d} e^{\beta|x|^2}\mu(dx)<\infty$.

Since $c_b>K^+_{\sigma,b}\kappa_2^2/\kappa_1^2$, there exists $\beta>K^+_{\sigma,b}/\kappa_1^2$ such that $0<\beta<c_b/\kappa_2^2$ and (1.13) holds. By Corollary 1.3 in [25], we obtain the hypercontractivity of P_t.

Step 2. The proof of Theorem 1.3 (1)(b). Set $\tau_n=\inf\{t\ge 0,|X_t|\ge n\}$. Then by Itô's formula, for $0<\beta<c_b/\kappa_2^2$, we have

$$
\begin{aligned}
&E_x\left(\exp\left\{\beta|X_{t\wedge\tau_n}|^2\right\}\right)-E_x\left(\exp\left\{\beta|X_{s\wedge\tau_n}|^2\right\}\right)\\
&\le-2\beta(c_b-\beta\kappa_2^2)\int_s^t E_x\left(\exp\left\{\beta|X_{u\wedge\tau_n}|^2\right\}|X_u|^2\right)du\\
&\quad+\beta(\kappa_2^2 d+2l)\int_s^t E_x\left(\exp\left\{\beta|X_{u\wedge\tau_n}|^2\right\}\right)du.
\end{aligned}
$$

Similar to the proof of (1.13), when n large enough, we can take a constant $1\le R^*\le n-1$ such that $\beta(c_b-\beta\kappa_2^2)(R^*)^2>\beta(\kappa_2^2 d+2l)$, and

$$
E_x\left(\exp\left\{\beta|X_{t\wedge\tau_n}|^2\right\}\right)\le\frac{\beta(\kappa_2^2 d+2l)f_n(R^*))}{\beta(c_b-\beta\kappa_2^2)}+e^{-\beta(c_b-\beta\kappa_2^2)t}e^{\beta|x|^2}.
$$

Letting $n\to\infty$, and set $M=\frac{\beta(\kappa_2^2 d+2l)f_n(R^*))}{\beta(c_b-\beta\kappa_2^2)}$, then

$$
E_x\left(\exp\left\{\beta|X_t|^2\right\}\right)\le M+e^{\beta|x|^2}.
$$

Step 3. The proof of Theorem 1.3 (1)(c).

The existence of the transition density function $p(t,x,y)$ with respect to μ follows from Malliavin calculus (see [2],[20]).

Finally, let us show (1.15) using the Harnack-inequality in [25]:

$$(P_tf(x))^\alpha \le P_tf^\alpha(y)\exp\left\{\tilde{\alpha}(t)|x-y|^2\right\} \text{ for all } t>0, x,y\in\mathbb{R}^d, f\in\mathcal{B}_b^+(\mathbb{R}^d),$$

where, for $\alpha > \left(1+\vartheta{\kappa_1}^{-1}\right)^2$

$$\tilde{\alpha}(t) = \frac{K_{\sigma,b}\sqrt{\alpha}(\sqrt{\alpha}-1)}{4\vartheta_\alpha\left((\sqrt{\alpha}-1)\kappa_1-\vartheta_\alpha\right)\left(1-e^{-K_{\sigma,b}t}\right)},$$

and $\vartheta_\alpha = \max\left\{\vartheta, \frac{\kappa_1}{2}(\sqrt{\alpha}-1)\right\}$.

Choose t large enough such that $\mu(U(0,t))>0$ where $U(0,t)=\{x\in\mathbb{R}^d; |x|\le t\}$. Integrating the above inequality for P_t with respect to $\mu(dy)$ on $U(0,t)$, and by Hölder's inequality we get

$$\begin{aligned}
P_tf(x) &\le \frac{1}{\mu(U(0,t))}\int_{U(0,t)}(P_tf^\alpha(y))^{1/\alpha}\exp\left\{\tilde{\alpha}(t)|x-y|^2/\alpha\right\}\mu(dy)\\
&\le \frac{1}{\mu(U(0,t))}\|f\|_\alpha\left(\int_{U(0,t)}\exp\left\{\frac{\tilde{\alpha}(t)}{\alpha-1}|x-y|^2\right\}\mu(dy)\right)^{\frac{\alpha-1}{\alpha}}\\
&\le \exp\left\{\frac{2\tilde{\alpha}(t)}{\alpha}(|x|^2+|t|^2)\right\}\|f\|_\alpha.
\end{aligned}$$

Let p be the conjugated number of α. Then,

$$\|p(t,x,\cdot)\|_p \le \sup_{\|f\|_\alpha\le 1}|P_t|f(x)|| \le \exp\left\{\frac{2\tilde{\alpha}(t)}{\alpha}(|x|^2+|t|^2)\right\}.$$

Since for any $t>0$, when $\alpha\to\infty$, $\frac{2\tilde{\alpha}(t)}{\alpha}\to 0$, thus when α large enough,

$$\|p(t,\cdot,\cdot)\|_{L_p(\mu\times\mu)} \le \int_{\mathbb{R}^d}\exp\left\{\frac{2\tilde{\alpha}(t)}{\alpha}(|x|^2+|t|^2)\right\}\mu(dx) < \infty. \tag{3.1}$$

The proof of (1.15) is completed.

Remark 3.1. For the diffusion process (1.12), we only obtain that (3.1) holds for some $p>1$ which is due to the restriction of the Harnack-inequality. In order to obtain that (3.1) holds for some $p>2$, using the method on P.2609 in [3], an additional rigor restriction condition $\vartheta < (\sqrt{5/2}-1)\kappa_1$ is needed. The improved condition (1.4) plays an important role in application to stochastic differential equations with $\sigma\ne I$.

3.2 The proof of Theorem 1.4

We only give a sketch proof, because its proof is similar to Proposition 2.2 in [3].

It is sufficient to show that $v(x)=e^{V(x)}$ is a viscosity solution of the following linear equation

$$\lambda v = \frac{1}{2}\sum_{i,j=1}^d a_{ij}\frac{\partial^2 v}{\partial x_i\partial x_j} + \sum_{i=1}^d b_i\frac{\partial v}{\partial x_i} + \gamma c v. \tag{3.2}$$

Firstly, as the same in [3] or Step 2 in the proof of Theorem 1.1, we can show continuity of the function $\varphi(t,x)$ defined by

$$\varphi(t,x) = E_x\left(\exp\left\{\gamma\int_0^t c(X_s)ds\right\}\right). \tag{3.3}$$

Next, we show that $v_T(t,x) := \varphi(T-t,x)$ is a viscosity solution of the parabolic equation

$$-\left(\partial_t v_T + \frac{1}{2}\sum_{i,j=1}^d a_{ij}\frac{\partial^2 v_T}{\partial x_i\partial x_j} + \sum_{i=1}^d b_i\frac{\partial v_T}{\partial x_i} + \gamma c v_T\right) = 0, \quad t\in[0,T]. \tag{3.4}$$

Let $(t,x) \in [0,T] \times \mathbb{R}^d$ be given. Let $\psi : [0,t) \times \mathbb{R}^d \to \mathbb{R}$ be a smooth function such that $\psi(t,x) = v_T(t,x)$ and $v_T - \psi$ has a local extreme at (t,x). We can assume that $v_T - \psi$ has a strict local extreme in (t,x). We write

$$\varphi(t,x) = E_x(\varphi(t-\varepsilon, X_\varepsilon)) + \gamma E_x\left(\int_0^\varepsilon c(X_s)\varphi(t-s,X_s)ds\right), \ \ \varepsilon > 0.$$

By a change $t \to T-t$ of the time variable, we get

$$v_T(t,x) - E_x(v_T(t+\varepsilon, X_\varepsilon)) = \gamma E_x\left(\int_0^\varepsilon c(X_s)v_T(t+s,X_s)ds\right). \tag{3.5}$$

Now we use (3.5) to prove that v_T has the subsolution property. Without loss of generality, we assume that ψ is a smooth function with compact support such that $v_T - \psi$ has a local minimum at (t,x), and $\psi(t,x) = v_T(t,x)$ at (t,x). In the same way as [3], we can get

$$\limsup_{\varepsilon\to 0} \frac{v_T(t,x) - E_x(v_T(t+\varepsilon,X_\varepsilon))}{\varepsilon} \le \limsup_{\varepsilon\to 0} \frac{\psi(t,x) - E_x(\psi(t+\varepsilon,X_\varepsilon))}{\varepsilon}. \tag{3.6}$$

On the other hand, by Itô's formula, we have that

$$\begin{aligned} &\limsup_{\varepsilon\to 0} \frac{\psi(t,x) - E_x(\psi(t+\varepsilon,X_\varepsilon))}{\varepsilon} \\ &= -\left(\partial_t\psi(t,x) + \frac{1}{2}\sum_{i,j=1}^d a_{ij}(x)\frac{\partial^2\psi(t,x)}{\partial x_i\partial x_j} + \sum_{i=1}^d b_i(x)\frac{\partial\psi(t,x)}{\partial x_i}\right). \end{aligned} \tag{3.7}$$

Thus, from (3.5),(3.6) and (3.7), we get

$$\begin{aligned} c^*(x)v_T(t,x) &\le \limsup_{\varepsilon\to 0}\frac{1}{\varepsilon}E_x\left(\int_0^\varepsilon c(X_s)v_T(t+s,X_s)ds\right) \\ &\le \limsup_{\varepsilon\to 0}\frac{v_T(t,x) - E_x[v_t(t+\varepsilon,X_\varepsilon)]}{\varepsilon} \\ &\le \limsup_{\varepsilon\to 0}\frac{\psi(t,x) - E_x[\psi(t+\varepsilon,X_\varepsilon)]}{\varepsilon} \\ &= -\left(\partial_t\psi(t,x) + \frac{1}{2}\sum_{i,j=1}^d a_{ij}(x)\frac{\partial^2\psi(t,x)}{\partial x_i\partial x_j} + \sum_{i=1}^d b_i(x)\frac{\partial\psi(t,x)}{\partial x_i}\right), \end{aligned}$$

and so, the subsolution property holds. The supersolution property is proved in the same way.

Set $\tilde{v}_T(t,x) := v_T(t,x)e^{-\lambda(T-t)}$. Then, $\tilde{v}_T$ is a viscosity solution of

$$-\left(\partial_t\tilde{v}_T + \frac{1}{2}\sum_{i,j=1}^d a_{ij}\frac{\partial^2\tilde{v}_T}{\partial x_i\partial x_j} + \sum_{i=1}^d b_i\frac{\partial\tilde{v}_T}{\partial x_i} + \gamma c\tilde{v}_T\right) + \lambda\tilde{v}_T = 0 \tag{3.8}$$

Moreover, $\tilde{v}_T(x) \to v(x)$ as $T \to +\infty$ uniformly on compact sets. In particular, v is continuous.

Finally, we show that v is a viscosity solution of (3.2). Let $x \in \mathbb{R}^d$, and $\psi : \mathbb{R}^d \to \mathbb{R}$ be a smooth solution such that $v(x) = \psi(x)$ and $v - \psi$ has a local minimum at x. Fix $t > 0$, and define $\tilde{\psi}(s,y) := \psi(y) - |y-x|^4 - (s-t)^2$. Then $v - \tilde{\psi}$ has a strict minimum at (t,x), and

$$\begin{aligned} &\partial_t\tilde{\psi}(t,x) + \frac{1}{2}\sum_{i,j=1}^d a_{ij}(x)\frac{\partial^2\tilde{\psi}(t,x)}{\partial x_i\partial x_j} + \sum_{i=1}^d b_i(x)\frac{\partial\tilde{\psi}(t,x)}{\partial x_i} \\ &= \frac{1}{2}\sum_{i,j=1}^d a_{ij}(x)\frac{\partial^2\psi(x)}{\partial x_i\partial x_j} + \sum_{i=1}^d b_i(x)\frac{\partial\psi(x)}{\partial x_i}. \end{aligned} \tag{3.9}$$

That shows that there is a sequence $(t_n, x_n) \to (t, x)$ as $n \to +\infty$ such that $\tilde{v}_n - \tilde{\psi}$ has a local minimum at (t_n, x_n). Therefore,

$$
\begin{aligned}
&-\Bigg(\partial_t \tilde{\psi}(t_n, x_n) + \frac{1}{2}\sum_{i,j=1}^d a_{ij}(x_n)\frac{\partial^2 \tilde{\psi}(t_n, x_n)}{\partial x_i \partial x_j} \\
&+ \sum_{i=1}^d b_i(x_n)\frac{\partial \tilde{\psi}(t_n, x_n)}{\partial x_i} + \gamma c_*(x_n)\tilde{v}_n(x_n)\Bigg) + \lambda \tilde{v}_n(x_n) \geq 0.
\end{aligned} \tag{3.10}
$$

Letting $n \to +\infty$ and using (3.9) and lower-semicontinuity of c_*, we obtain

$$
\frac{1}{2}\sum_{i,j=1}^d a_{ij}(x)\frac{\partial^2 \psi(x)}{\partial x_i \partial x_j} + \sum_{i=1}^d b_i(x)\frac{\partial \psi(x)}{\partial x_i} + \gamma c_*(x)\psi(x) + \lambda \psi(x) \geq 0.
$$

The subsolution property is proved. The supersolution property can be shown in the same way.

A Perturbation theory of linear operators

In this section, we introduce briefly Kato's perturbation theory of linear operators (see Chapter 7 in [12]).

Let $\mathbb{B}$ be a Banach space. Let T be an operator on $\mathbb{B}$ and let $D(T)$ and $R(T)$ denote its domain of definition and range, respectively. The set of closed operators on $\mathbb{B}$ will be denoted by $C(\mathbb{B}, \mathbb{B})$.

M, N are called two complementary linear manifolds of $\mathbb{B}$, if $\mathbb{B} = M \oplus N$, namely, each $u \in \mathbb{B}$ can be uniquely expressed in the form $u = u' + u''$ with $u' \in M$ and $u'' \in N$. u' is called the projection of u on M along N.

T is said to be decomposed according to $\mathbb{B} = M \oplus N$, if

$$
JD(T) \subset D(T), \quad TM \subset M, \quad TN \subset N, \tag{A.1}
$$

where J is the projection on M along N. When T is decomposed as above, the parts T_M, T_N of T in M, N, respectively can be defined. T_M is an operator in the Banach space M with $D(T_M) = D(T) \cap M$ such that $T_M u = Tu \in M$. T_N is defined similarly.

The resolvent set of $T \in C(\mathbb{B}, \mathbb{B})$, denoting by $P(T)$, is the set of the complex number ζ such that $T - \zeta$ is invertible with

$$
R(\zeta) = R(\zeta, T) = (T - \zeta)^{-1} \tag{A.2}
$$

a bounded operator. The operator-valued function $R(\zeta)$ defined on the resolvent set is called the resolvent of T. Thus $R(\zeta)$ has domain $\mathbb{B}$ and range $D(T)$. The complementary set $\Sigma(T)$ in the complex plane of the resolvent set is called the spectrum of T.

Let D be a domain of the complex plane with $0 \in D$. Let $\{T(\chi) \in C(\mathbb{B}, \mathbb{B}); \chi \in D\}$ be a family of bounded operators. $\{T(\chi) \in C(\mathbb{B}, \mathbb{B}); \chi \in D\}$ is called bounded-holomorphic if it is differentiable in norm for all $\chi \in D$. A subset S of $\mathbb{B}$ is said to be a fundamental subset if the span of S is everywhere dense. It is known that $T(\chi)$ is bounded-holomorphic if and only if $(T(\chi)u, g)$ is holomorphic for every u in a fundamental subset of $\mathbb{B}$ and every g in a fundamental subset of $\mathbb{B}^*$.

Assumption **(GP)**: There exist $\varepsilon > 0$ and simple eigenvalue λ_0 of $T(0)$ such that

$$
(\Sigma(T(0)) \setminus \{\lambda_0\}) \cap \{\lambda \in \mathbb{C}; |\lambda - \lambda_0| \leq \varepsilon\} = \emptyset. \tag{A.3}
$$

Set $\Gamma = \{\lambda \in \mathbb{C}; |\lambda - \lambda_0| = \varepsilon/2\}$.

From Theorem VII-1.3 and the comment and the proof of Theorem VII-1.7 in [12] (see VII, Sect. 1.2 and Sect. 1.3, and the proof Theorem IV-3.16), we have the following result.

Theorem A.1. Let the family of bounded operators $\{T(\chi) \in C(\mathbb{B},\mathbb{B}); \chi \in D\}$ be bounded-holomorphic and let $P(T(\chi))$ be the resolvent set of $T(\chi)$. Assume that the assumption **(GP)** hold. Then

(1). The resolvent $R(\zeta,\chi) = (T(\chi)-\zeta)^{-1}$ is bounded-holomorphic in the two variables on the set of all ζ,χ such that ζ is in the resolvent set of $T(0)$ and $|\chi|$ is sufficiently small (depending on ζ).

(2). The spectrum $\Sigma(T(0))$ of $T(0)$ can be separated in two parts $\Sigma'(0) = \{\lambda_0\}$, $\Sigma''(0)$ by the curve Γ in the manner previously described.

(3). There exists an open set $U \ni 0$ such that for any $\chi \in U$, $\Gamma \subset P(T(\chi))$ and the spectra $\Sigma(T(\chi))$ is likewise separated by Γ into two parts $\Sigma'(\chi) = \{\lambda(\chi)\}$, $\Sigma''(\chi)$ with the associated decomposition $\mathbb{B} = M'(\chi) \oplus M''(\chi)$ of the space and $M'(\chi)$ and $M'(0)$ have same dimension. The projection $J(\chi)$ on $M'(\chi)$ along $M''(\chi)$ given by

$$J(\chi) = -\frac{1}{2\pi i}\int_\Gamma R(\zeta,\chi)d\zeta. \tag{A.4}$$

converges to $J(0)$ as $\chi \to 0$.

Under the assumption **(GP)**, λ_0 is a simple eigenvalue of $T = T(0)$, then $M' = M'(0)$ is finite-dimensional and

$$R(\zeta) = -(\zeta-\lambda_0)^{-1}J + \sum_{n=0}^{\infty}(\zeta-\lambda_0)^n S^{n+1}, \tag{A.5}$$

where S is the reduced resolvent of T with respect the eigenvalue λ_0,

$$S = \frac{1}{2\pi i}\int_\Gamma \frac{R(\zeta)}{\zeta-\lambda_0}d\zeta. \tag{A.6}$$

Let $\{T(\chi) \in C(\mathbb{B},\mathbb{B}); \chi \in D\}$ be a bounded-holomorphic family in χ near $\chi = 0$ with the form

$$T(\chi) = T + \chi T^{(1)} + \chi^2 T^{(2)} + \cdots, \tag{A.7}$$

where $T^{(k)}$, $k \geq 1$ are bounded.

Note that under the assumption **(GP)**, the resolvent $R(\zeta,\chi) = (T(\chi)-\zeta)^{-1}$ of $T(\chi)$ is holomorphic in the two variables ζ,χ in each domain in which ζ is not equal to any of the eigenvalues of $T(\chi)$. $R(\zeta,\chi)$ can be expanded into a power series in χ with coefficients depending on ζ as follows:

$$\begin{aligned} R(\zeta,\chi) =& R(\zeta)(1 + A(\chi)R(\zeta))^{-1} \\ =& R(\zeta)\sum_{k=}^{\infty}(-A(\chi)R(\zeta))^k = \quad R(\zeta) + \sum_{n=1}^{\infty}\chi^n R^{(n)}(\zeta), \end{aligned} \tag{A.8}$$

where $A(\chi) = T(\chi) - T$, and each $R^{(n)}$ is an operator-valued function defined by

$$R^{(n)} = \sum_{\substack{\nu_1+\cdots+\nu_p=n \\ 1\leq p\leq n, \nu_j\geq 1}} (-1)^p R(\zeta)T^{(\nu_1)}R(\zeta)T^{(\nu_2)}\cdots T^{(\nu_p)}R(\zeta). \tag{A.9}$$

(A.9) is uniformly convergent for sufficient small χ and $\zeta \in \Gamma$. It is called the second Neumann series for resolvent.

Assume that there exist constants $a, c \in (0,\infty)$ such that

$$\|T^{(n)}\| \leq ac^{n-1},\ n \geq 1. \tag{A.10}$$

Set

$$r_0 = \min_{\zeta\in\Gamma}\{(a\|R(\zeta)\| + c)^{-1}\}. \tag{A.11}$$

Then for any $\chi \in \{\chi; |\chi| \le r_0/2\}$,

$$\sum_{n=1}^{\infty} |\chi|^n \|T^{(n)} R(\zeta)\| < 1$$

which yields that

$$\|A(\chi)R(\zeta)\| = \left| \left(\sum_{n=1}^{\infty} |\chi|^n T^{(n)} \right) R(\zeta) \right| < 1.$$

Therefore, (A.8) is uniformly convergent for $\chi \in \{\chi; |\chi| \le r_0/2\}$ and $\zeta \in \Gamma$,

Under the assumption **(GP)**, the only eigenvalues of the operator $T_r(\chi) := T(\chi)J(\chi)$ are 0 and $\lambda(\chi)$, thus

$$\lambda(\chi) - \lambda_0 = tr((T(\chi) - \lambda_0)J(\chi)).$$

Since $(T(\chi) - \lambda_0)R(\zeta, \chi) = 1 + (\zeta - \lambda_0)R(\zeta, \chi)$, we have

$$\begin{aligned}
&(T(\chi) - \lambda_0)J(\chi) \\
&= -\frac{1}{2\pi i} \int_\Gamma (\zeta - \lambda_0) R(\zeta, \chi) d\zeta \\
&= -\frac{1}{2\pi i} \int_\Gamma (\zeta - \lambda_0) \left(R(\zeta) + \sum_{n=1}^{\infty} \chi^n R^{(n)}(\zeta) \right) d\zeta \\
&= -\frac{1}{2\pi i} \sum_{n=1}^{\infty} \chi^n \int_\Gamma (\zeta - \lambda_0) \sum_{\substack{\nu_1+\cdots+\nu_p=n \\ 1\le p\le n, \nu_j \ge 1}} (-1)^p R(\zeta) T^{(\nu_1)} R(\zeta) T^{(\nu_2)} \cdots T^{(\nu_p)} R(\zeta) d\zeta
\end{aligned}$$

Now, by (A.5), we obtain that

$$\lambda(\chi) - \lambda_0 = \sum_{n=1}^{\infty} \chi^n \lambda^{(n)} \tag{A.12}$$

where

$$\lambda^{(n)} = \sum_{p=1}^{n} \frac{(-1)^p}{p} \sum_{\substack{\nu_1+\cdots+\nu_p=n, \\ k_1+\cdots+k_p=p-1, \nu_i\ge 1, k_j \ge 0}} tr\left(T^{(\nu_1)} S^{(k_1)} \cdots T^{(\nu_p)} S^{(k_p)} \right) \tag{A.13}$$

with

$$S^{(0)} = -J,\ S^{(n)} = S^n,\ n \ge 1. \tag{A.14}$$

It is obvious that if (A.10) holds, and the function $\lambda(\chi) - \lambda$ is holomorphic and bounded by $\varepsilon/2$. By Cauchy's inequality for Taylor coefficients, we have

$$|\lambda^{(n)}| \le \varepsilon r_0^{-n}/2, \quad n \ge 1.$$

Thus, r_0 is a lower bound for the convergence radius of (A.12), and the following result holds.

Corollary A.1. Let the assumption **(GP)** and (A.10) hold. Then there exists t $0 < \eta \le r_0/2$ such that the conclusion in (3) of Theorem A.1 holds and for any $\chi \in U := \{\chi; |\chi| \le \eta\}$, and $m \ge 1$,

$$\left| \lambda(\chi) - \lambda_0 - \sum_{n=1}^{m} \chi^n \lambda^{(n)} \right| \le \frac{\varepsilon |\chi|^{m+1}}{2 r_0^m (r_0 - |\chi|)}. \tag{A.15}$$

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Acknowledgments. The author would like to thank Professor Fengyu Wang and the referee for their careful reading of the manuscript and their valuable comments and suggestions.

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