Electron. Commun. Probab. **22** (2017), no. 46, 1–10.
DOI: 10.1214/17-ECP78
ISSN: 1083-589X

ELECTRONIC COMMUNICATIONS in PROBABILITY

Some properties for Itô processes driven by G-Brownian motion*

Xinpeng Li[†] Falei Wang[‡]

Abstract

In this paper, we use partial differential equation (PDE) techniques and probabilistic approaches to study the lower capacity of the ball for the Itô process driven by G-Brownian motion (G-Itô process). In particular, the lower bound on the lower capacity of certain balls is obtained. As an application, we prove a strict comparison theorem in G-expectation framework.

Keywords: G-Brownian motion; lower capacity; strict comparison theorem.
AMS MSC 2010: 60H10; 60H30.
Submitted to ECP on January 24, 2017, final version accepted on August 9, 2017.

1 Introduction

The G-Brownian motion is a continuous process with independent and stationary increments in a nonlinear expectation space called G-expectation space. These notions and related Itô's calculus have been systemically established by Peng [15, 16, 17]. In particular, Gao [5] and Peng [17] have obtained the existence and uniqueness theorem for stochastic differential equations driven by G-Brownian motion. For more research on this field, we refer the reader to [1, 6, 7, 13] and the references therein.

The G-expectation can be also viewed as a upper expectation. Indeed, Denis et al. [2] have proved that G-expectation is the supremum of linear expectations over a weakly compact subset of the probability measures of the stochastic integrals with respect to a standard Brownian motion (see Remark 2.2). This representation theorem allows us to introduce upper and lower capacities related to G-expectation. Moreover, it provides an alternative approach for the study of the probability measures of the classical Itô processes. Note that these measures may be mutually singular and do not have probability density functions (see Fabes and Kenig [4]).

In the classical probability space, Martini [14] has applied a tricky probabilistic approach to get that the marginal law of the stochastic integral with uniformly elliptic and bounded integrand does not weight points. In his paper, the main idea is to study

*Supported by the National Natural Science Foundation of China (No. 11601281, No. 11601282 and No. 11526205), the Natural Science Foundation of Shandong Province (No. ZR2016AQ10 and No. ZR2016AQ11) and the 111 Project (No. B12023).

[†]School of Mathematics and Institute for Advanced Research, Shandong University, 250100 Jinan, China.
E-mail: lixinpeng@sdu.edu.cn

[‡]Zhongtai Securities Institute for Financial Studies and Institute for Advanced Research, Shandong University, 250100 Jinan, China and IRMAR, Université Rennes 1, Campus de Beaulieu, 35042 Rennes Cedex, France.
E-mail: flwang2011@gmail.com (corresponding author)

the aforementioned upper capacity instead of the probability measure of the stochastic integral (see the operator $\hat{C}$ in [14]). Indeed, he has obtained that the upper capacity of G-Brownian motion does not weight points. In sprit of G-stochastic analysis methods and PDE techniques, Hu et al. [9] have studied the upper capacity of general G-Itô process. In their paper, the authors have got some upper bound on the upper capacity of the ball for the G-Itô process, which implies that the upper capacity of the G-Itô process also does not weight points (see Remark 3.9). Moreover, it provides a tool for the research of the path properties of the classical Itô integral (see Examples 3.9 and 3.13 in [9]).

The present paper is devoted to investigating the lower capacity for the G-Itô processes. The key ingredient of our approach is based on certain estimates of viscosity solutions of PDE introduced in [9], from which we show that the ball for G-Itô process has a positive lower capacity. Moreover, we obtain a lower bound on the lower capacity of some balls for G-Brownian motion. In particular, it also provides a lower bound on the marginal law of the balls for the Itô integral (see Remark 3.8). In contrast to the one in [9], the G-Itô process considered in this paper are more general. Indeed, we do not assume the boundedness of the integrands for dt and $d\langle B\rangle$ parts. Moreover, the diffusion term can be unbounded from above.

A direct application of this paper is to discuss the strict comparison theorem in G-expectation framework, which is non-trivial due to the nonlinearity. A strict comparison theorem for G-Brownian motion has been proved by Li [12] based on the Krylov and Safonov estimates [11]. In this paper, we establish a strict comparison theorem for the G-Itô process through the above estimate of lower capacity, which extends the one in [12] to general case. In particular, the necessary and sufficient conditions for the strict comparison theorem are also stated.

The rest of this paper is organized as follows. In Section 2, we present some preliminaries for G-expectation, which are needed in the sequel. In Section 3, we state our main results: The ball for G-Itô process has a positive lower capacity. Some estimates on the lower capacity of the ball for G-Brownian motion are also obtained. As an application, Section 4 is devoted to tackling the strict comparison theorem in G-expectation framework.

2 Preliminaries

In this section, we shall recall some basic notions and results about G-expectation and related capacities. The readers may refer to [17] for more details.

For convenience, every element $x \in \mathbb{R}^n$ is identified to a column vector with i-th component x^i and the corresponding Euclidian norm is denoted by $|x|$. Let Ω be the space of all $\mathbb{R}^d$-valued continuous paths $(\omega_t)_{t\geq 0}$ starting from origin, equipped with the distance

$$\rho(\omega_1, \omega_2) := \sum_{N=1}^{\infty} 2^{-N}((\max_{t\in[0,N]} |\omega_1(t) - \omega_2(t)|) \wedge 1).$$

Denote by $\mathcal{B}(\Omega)$ the Borel σ-algebra of Ω. Let $B_t(\omega) := \omega_t$ be the canonical process and set

$$L_{ip}(\Omega) := \{\varphi(B_{t_1}, \cdots, B_{t_k}) : k \geq 1,\ 0 \leq t_1 < \cdots < t_k < \infty,\ \varphi \in C_{b,Lip}(\mathbb{R}^{k\times d})\},$$

where $C_{b,Lip}(\mathbb{R}^{k\times d})$ is the collection of all bounded and Lipschitz functions defined on $\mathbb{R}^{k\times d}$. Let $\Omega_t = \{\omega_{\cdot\wedge t} : \omega \in \Omega\}$, we can define $L_{ip}(\Omega_t)$ analogously.

For each given monotonic and sublinear function $G : \mathbb{S}_d \to \mathbb{R}$, we could construct a sublinear expectation space $(\Omega, L_{ip}(\Omega), \hat{\mathbb{E}}, (\hat{\mathbb{E}}_t)_{t\geq 0})$ called G-expectation space, where $\mathbb{S}_d$ is the space of all $d \times d$ symmetric matrices. Indeed, for each $\xi \in L_{ip}(\Omega)$ with the form of

$$\xi = \varphi(B_{t_1}, B_{t_2}, \cdots, B_{t_k}),\quad 0 = t_0 < t_1 < \cdots < t_k < \infty,$$

we define the conditional G-expectation by

$$\hat{\mathbb{E}}_t[\xi] := u_i(t, B_t; B_{t_1}, \cdots, B_{t_{i-1}})$$

for each $t \in [t_{i-1}, t_i)$, $i = 1, \ldots, k$. Here, the function $u_i(t, x; x_1, \cdots, x_{i-1})$ parameterized by $(x_1, \cdots, x_{i-1}) \in \mathbb{R}^{(i-1)\times d}$ is the viscosity solution of the following G-heat equation:

$$\partial_t u_i(t, x; x_1, \cdots, x_{i-1}) + G(D^2_{xx} u_i(t, x; x_1, \cdots, x_{i-1})) = 0, \ \ (t, x) \in [t_{i-1}, t_i) \times \mathbb{R}^d$$

with terminal conditions

$$u_i(t_i, x; x_1, \cdots, x_{i-1}) = u_{i+1}(t_i, x; x_1, \cdots, x_{i-1}, x), \text{ for } i < k$$

and $u_k(t_k, x; x_1, \cdots, x_{k-1}) = \varphi(x_1, \cdots, x_{k-1}, x)$. The G-expectation of ξ is defined by $\hat{\mathbb{E}}[\xi] = \hat{\mathbb{E}}_0[\xi]$. In this space the corresponding canonical process B_t is called G-Brownian motion.

For each $p \geq 1$, the completion of $L_{ip}(\Omega)$ (resp. $L_{ip}(\Omega_t)$) under the norm $||X||_p := (\hat{\mathbb{E}}[|X|^p])^{1/p}$ is denoted by $L^p_G(\Omega)$ (resp. $L^p_G(\Omega_t)$). In this paper, we always assume that G is non-degenerate, i.e., there exist two constants $0 < \underline{\sigma} \leq \bar{\sigma} < \infty$ such that

$$\frac{1}{2}\underline{\sigma}^2 \mathrm{tr}[A - B] \leq G(A) - G(B) \leq \frac{1}{2}\bar{\sigma}^2 \mathrm{tr}[A - B], \ \forall A \geq B.$$

Then there exists a bounded and closed subset $\Gamma \subset \mathbb{S}^+_d$ such that

$$G(A) = \frac{1}{2} \sup_{Q \in \Gamma} \mathrm{tr}[AQ],$$

where $\mathbb{S}^+_d$ denotes the space of all $d \times d$ symmetric positive definite matrices. Moreover, we have $|G(A)| \leq \frac{1}{2}\bar{\sigma}^2\sqrt{d}\sqrt{\mathrm{tr}[AA^\top]}$ for each $A \in \mathbb{S}_d$.

The following representation theorem has been proved firstly in Denis et al. [2] (see also [8]).

Theorem 2.1. *There exists a weakly compact set of probability measures $\mathcal{P}$ on $(\Omega, \mathcal{B}(\Omega))$ such that*

$$\hat{\mathbb{E}}[\xi] = \sup_{P \in \mathcal{P}} E_P[\xi], \ \forall \xi \in L^1_G(\Omega).$$

Remark 2.2. Denis et al. [2] have constructed a concrete set $\mathcal{P}_M$ that represents $\hat{\mathbb{E}}$. For simplicity's sake, we consider the 1-dimensional case, thus $G(a) = \frac{1}{2}(\bar{\sigma}^2 a^+ - \underline{\sigma}^2 a^-)$ for each $a \in \mathbb{R}$. Suppose W is a standard Brownian motion defined on Wiener space $(\Omega, (\mathcal{F}_t)_{t\geq 0}, P_0)$, then

$$\mathcal{P}_M := \{P_\theta : P_\theta = P_0 \circ X^{-1}, \ X_t = \int_0^t \theta_s dW_s, \ \theta \in \mathcal{A}_{[\underline{\sigma}, \bar{\sigma}]}\},$$

where $\mathcal{A}_{[\underline{\sigma}, \bar{\sigma}]}$ is the collection of all adapted processes taking values in $[\underline{\sigma}, \bar{\sigma}]$.

Now it is natural to define the upper and lower capacities corresponding to G-expectation (see also [3]),

$$V(A) := \sup_{P \in \mathcal{P}} P(A) \ \text{ and } \ \ v(A) = \inf_{P \in \mathcal{P}} P(A), \ \ \forall A \in \mathcal{B}(\Omega).$$

A set $A \subset \mathcal{B}(\Omega)$ is polar if $V(A) = 0$. A property holds "quasi-surely" (q.s.) if it holds outside a polar set. In the following, we do not distinguish between two random variables X and Y if $X = Y$ q.s.

Definition 2.3 ([17]). *Let $M_G^0(0,T)$ be the collection of processes of the following form: for a given partition $\{t_0,\cdots,t_N\}$ of $[0,T]$,*

$$\eta_t(\omega)=\sum_{i=0}^{N-1}\xi_i(\omega)\mathbf{1}_{[t_i,t_{i+1})}(t),$$

where $\xi_i\in L_{ip}(\Omega_{t_i})$, $i=0,1,2,\cdots,N-1$. For each $p\geq 1$, denote by $M_G^p(0,T)$ the completion of $M_G^0(0,T)$ under the norm $||\eta||_{M_G^p}:=(\hat{\mathbb{E}}[\int_0^T|\eta_t|^p dt])^{1/p}$.

For each $1\leq i,j\leq d$, we denote by $\langle B^i,B^j\rangle$ the cross-variation process. Then for two processes $\eta\in M_G^2(0,T)$ and $\xi\in M_G^1(0,T)$, the G-Itô integrals $\int\eta_s dB_s^i$ and $\int\xi_s d\langle B^i,B^j\rangle_s$ are well defined, see [17].

3 Main results

Now we consider the following $\mathbb{R}^n$-valued G-Itô diffusion process which starts from $\xi\in L_G^2(\Omega_t)$ at time $t\geq 0$ (for convenience, we always use Einstein's summation convention):

$$X_s^{t,\xi}=\xi+\int_t^s b(r,X_r^{t,\xi})dr+\int_t^s h^{jk}(r,X_r^{t,\xi})d\langle B^j,B^k\rangle_r+\int_t^s\sigma(r,X_r^{t,\xi})dB_r,\ \forall s\geq t, \quad (3.1)$$

where $b(t,x),h^{jk}=h^{kj}(t,x):[0,\infty)\times\mathbb{R}^n\to\mathbb{R}^n$ and $\sigma(t,x):[0,\infty)\times\mathbb{R}^n\to\mathbb{R}^{n\times d}$ are continuous deterministic functions. These coefficients are supposed to satisfy the following assumptions:

(H1) There exist two positive constants L and $\tilde{L}$ such that for each $(t,x),(t,y)\in[0,\infty)\times\mathbb{R}^n$,

$$|b(t,x)-b(t,y)|+|h^{jk}(t,x)-h^{jk}(t,y)|\leq L|x-y| \text{ and } |\sigma(t,x)-\sigma(t,y)|\leq\tilde{L}|x-y|.$$

(H2) There exists some constant $\lambda>0$ such that for each $(t,x)\in[0,\infty)\times\mathbb{R}^n$,

$$\lambda I_{n\times n}\leq\sigma(t,x)(\sigma(t,x))^\top,\ \text{if } n\leq d \text{ and } \lambda I_{d\times d}\leq(\sigma(t,x))^\top\sigma(t,x),\ \text{if } n>d.$$

Then the G-SDE (3.1) admits a unique solution $(X_s^{t,\xi})_{t\leq s\leq T}\in M_G^2(t,T)$ for each $T>t$ (see [17]).

Next, we shall use PDE approach to estimate the lower capacity of certain balls for the G-Itô process $X_s^{t,\xi}$. For a fixed real number $T>0$, we consider the following PDE:

$$\begin{cases}\partial_t u+G(\sigma^\top D_{xx}^2u\sigma+2\langle h^{jk},D_xu\rangle)+\langle b,D_xu\rangle=0,\ (t,x)\in[0,T)\times\mathbb{R}^n,\\ u(T,x)=\Phi(x).\end{cases} \quad (3.2)$$

where the function $\Phi\in C_{l,Lip}(\mathbb{R}^n)$. Here, $C_{l,Lip}(\mathbb{R}^n)$ is the space of all real valued functions φ on $\mathbb{R}^n$ such that

$$|\varphi(x)-\varphi(y)|\leq C(1+|x|^K+|y|^K)|x-y|$$

for some positive constants C and K depending on φ. Then it holds that

Theorem 3.1 ([7, 17]). *For each fixed $T>0$, the PDE (3.2) has a unique viscosity solution $u\in C([0,T]\times\mathbb{R}^n)$ with polynomial growth. Moreover, $u(t,x)=\hat{\mathbb{E}}[\Phi(X_T^{t,x})]$ for each $x\in\mathbb{R}^n$. Suppose moreover that $\Phi\in C_{b,Lip}(\mathbb{R}^n)$. Then for each $\xi\in L_G^2(\Omega_t)$, we have*

$$u(t,\xi)=\hat{\mathbb{E}}_t[\Phi(X_T^{t,\xi})].$$

The following lemma is the key point to prove our main results, whose proof is based on [9] with some modifications.

Lemma 3.2. *Given a fixed* $a \in \mathbb{R}^n$. *Suppose that* $\rho = \frac{3}{4}\bar{\sigma}^2\tilde{C}_{a,1}^2((n\wedge d)\underline{\sigma}^2\lambda)^{-1}$, $\theta = (2/3(n\wedge d)\underline{\sigma}^2\lambda)^{-1}$, $\varepsilon = (8\kappa)^{-1}\wedge T$, $m \geq 8\kappa$ *and let* u_m *be the solution of PDE* (3.2) *with the terminal condition* $u_m(T,x) = -\exp(-\frac{m\theta|x-a|^2}{2})$, *where* $n\wedge d = \min\{n,d\}$,

$$\kappa = \bar{\sigma}^2|\tilde{L}|^2 + (\bar{\sigma}^2 d\sqrt{d}+1)L + C_{a,1}^2(\bar{\sigma}^2 d\sqrt{d}+1)^2(4(n\wedge d)\bar{\sigma}^2\lambda)^{-1},$$
$$C_{a,1} = \max_{0\leq t\leq 1}\{|h^{jk}(t,a)|, |b(t,a)| : j,k=1,\dots,d\}, \ \ \tilde{C}_{a,1} = 2\max_{0\leq t\leq 1}|\sigma(t,a)|.$$

Then for any $T\in(0,1]$ *and* $(t,x)\in[T-\varepsilon,T)\times\mathbb{R}^n$, *we have the following estimate*

$$u_m(t,x) \leq -(1+m(T-t))^{-\rho}\exp\left(-\frac{m\theta|x-a|^2}{2(1+m(T-t))}\right). \tag{3.3}$$

Proof. We denote by $\tilde{u}_m$ the function in the right side of inequality (3.3). Thus it suffices to show that $\tilde{u}_m$ is a viscosity supersolution of PDE (3.2) in the interval $[T-\varepsilon,T]$. It is easy to verify that

$$\begin{aligned}
\partial_t\tilde{u}_m &= \frac{\rho m}{1+m(T-t)}\tilde{u}_m - \frac{m^2\theta|x-a|^2}{2(1+m(T-t))^2}\tilde{u}_m, \partial^2_{x_ix_j}\tilde{u}_m\\
&= \frac{m^2\theta^2(x_i-a_i)(x_j-a_j)}{(1+m(T-t))^2}\tilde{u}_m, \ i\neq j,\\
\partial_{x_i}\tilde{u}_m &= -\frac{m\theta(x_i-a_i)}{1+m(T-t)}\tilde{u}_m, \qquad \partial^2_{x_ix_i}\tilde{u}_m = -\frac{m\theta}{1+m(T-t)}\tilde{u}_m + \frac{m^2\theta^2|x_i-a_i|^2}{(1+m(T-t))^2}\tilde{u}_m.
\end{aligned}$$

Note that $\tilde{u}_m\leq 0$. Then for each $(t,x)\in[T-\varepsilon,T)\times\mathbb{R}^n$, we derive that

$$\begin{aligned}
&\partial_t\tilde{u}_m + G(\sigma^\top D^2_{xx}\tilde{u}_m\sigma + (2\langle h^{jk}(t,x), D_x\tilde{u}_m\rangle)_{j,k=1}^d) + \langle b(t,x), D_x\tilde{u}_m\rangle\\
&\leq \frac{m(-\tilde{u}_m)}{1+m(T-t)}[\theta G(\sigma^\top\sigma)-\rho] + \frac{m^2\theta(-\tilde{u}_m)}{(1+m(T-t))^2}[\theta G(-\sigma^\top(x-a)(x-a)^\top\sigma) + \frac{1}{2}|x-a|^2]\\
&\quad + \frac{2m\theta(-\tilde{u}_m)}{1+m(T-t)}G((\langle h^{jk}(t,x), x-a\rangle)_{j,k=1}^d) + \frac{m\theta(-\tilde{u}_m)}{1+m(T-t)}\langle b(t,x), x-a\rangle.
\end{aligned}$$

On the other hand, applying the assumptions (H1)-(H2) yields that

$$\begin{aligned}
G(\sigma^\top\sigma) &\leq \frac{\bar{\sigma}^2}{2}\mathrm{tr}[\sigma^\top\sigma] = \frac{1}{2}\bar{\sigma}^2|\sigma(t,x)-\sigma(t,a)+\sigma(t,a)|^2 \leq \bar{\sigma}^2|\tilde{L}|^2|x-a|^2 + \frac{1}{4}\bar{\sigma}^2\tilde{C}_{a,1}^2,\\
G(-\sigma^\top(x-a)(x-a)^T\sigma) &\leq -\frac{\underline{\sigma}^2}{2}|x-a|^2\mathrm{tr}[\sigma^\top\sigma] \leq -\frac{1}{2}(n\wedge d)\lambda\underline{\sigma}^2|x-a|^2,\\
G((\langle h^{jk}(t,x), x-a\rangle)_{j,k=1}^d) &\leq G((\langle h^{jk}(t,x)-h^{jk}(t,a), x-a\rangle)_{j,k=1}^d)\\
&\quad + G((h^{jk}(t,a), x-a\rangle)_{j,k=1}^d)\\
&\leq \frac{1}{2}\bar{\sigma}^2 d\sqrt{d}(L|x-a|^2 + C_{a,1}|x-a|)\\
\langle b(t,x), x-a\rangle &= \langle b(t,x)-b(t,a), x-a\rangle + \langle b(t,a), x-a\rangle\\
&\leq L|x-a|^2 + C_{a,1}|x-a|.
\end{aligned}$$

By assumption (H2), we have $\tilde{C}_{a,1}^2 \geq 4(n\wedge d)\lambda$, which indicates that

$$C_{a,1}(\bar{\sigma}^2 d\sqrt{d}+1)|x-a| \leq \frac{1}{4}C_{a,1}^2(\bar{\sigma}^2 d\sqrt{d}+1)^2|x-a|^2((n\wedge d)\bar{\sigma}^2\lambda)^{-1} + \frac{1}{4}\bar{\sigma}^2\tilde{C}_{a,1}^2.$$

Consequently, we deduce that

$$\begin{aligned}
&\partial_t \tilde{u}_m + G(\sigma^\top D^2_{xx}\tilde{u}_m\sigma + (2\langle h^{jk}(t,x), D_x\tilde{u}_m\rangle)^d_{j,k=1}) + \langle b(t,x), D_x\tilde{u}_m\rangle \\
&\le \frac{m(-\tilde{u}_m)}{1+m(T-t)}[\frac{1}{2}\theta\bar{\sigma}^2\tilde{C}^2_{a,1} - \rho] + \frac{m^2\theta(\tilde{u}_m)}{4(1+m(T-t))^2}|x-a|^2 \\
&\quad - \frac{m\theta\tilde{u}_m}{1+m(T-t)}(\bar{\sigma}^2|\tilde{L}|^2 + (\bar{\sigma}^2 d\sqrt{d}+1)L + \frac{1}{4}C^2_{a,1}(\bar{\sigma}^2 d\sqrt{d}+1)^2((n\wedge d)\bar{\sigma}^2\lambda)^{-1})|x-a|^2 \\
&\le \frac{m\theta\tilde{u}_m}{1+m(T-t)}|x-a|^2(\frac{m}{4(1+m(T-t))} - \kappa) \\
&\le \frac{m\theta\tilde{u}_m}{1+m(T-t)}|x-a|^2(\frac{m}{4(1+m\varepsilon)} - \kappa) \le 0.
\end{aligned}$$

The proof is complete. □

Remark 3.3. If $b = h^{jk} = 0$ and $|\sigma|$ is bounded by some constant α. From the above proof, we can take $\rho = \alpha^2\bar{\sigma}^2(2(n\wedge d)\lambda\underline{\sigma}^2)^{-1}$, $\theta = ((n\wedge d)\lambda\underline{\sigma}^2)^{-1}$, $\varepsilon = T$ ($\kappa = 0$), $m \ge 0$ and the results remain true.

Now we are ready to state our main results.

Theorem 3.4. *For each $\delta > 0$ and $a \in \mathbb{R}^n$, we have*

$$v(|X^{t,\xi}_T - a| \le \delta) > 0,\ \forall T > t \ge 0.$$

Proof. Without loss of generality, assume that $T \le 1$. In the sequel we shall use the same notations as in Lemma 3.2. Note that $v(A) = -\hat{\mathbb{E}}[-\mathbf{1}_A]$ for each $A \in \mathcal{B}(\Omega)$. Then for each $m \ge 0$, it holds that

$$\begin{aligned}
v(|X^{t,\xi}_T - a| \le \delta) &= -\hat{\mathbb{E}}[-\mathbf{1}_{|X^{t,\xi}_T - a|\le\delta}] \\
&\ge -\hat{\mathbb{E}}[-\exp\{-\frac{m\theta|X^{t,\xi}_T - a|^2}{2}\}\mathbf{1}_{|X^{t,\xi}_T - a|\le\delta}] \\
&= -\hat{\mathbb{E}}[-\exp\{-\frac{m\theta|X^{t,\xi}_T - a|^2}{2}\} + \exp\{-\frac{m\theta|X^{t,\xi}_T - a|^2}{2}\}\mathbf{1}_{|X^{t,\xi}_T - a|>\delta}] \\
&\ge -\hat{\mathbb{E}}[-\exp\{-\frac{m\theta|X^{t,\xi}_T - a|^2}{2}\}] - \exp\{-\frac{m\theta\delta^2}{2}\}.
\end{aligned}$$

Thus we need to estimate the $-\hat{\mathbb{E}}[-\exp\{-\frac{m\theta|X^{t,\xi}_T - a|^2}{2}\}]$ term. The remainder of proof will be divided into two steps.

1. $T - t \le \varepsilon$**.** Recalling Theorem 3.1 and Lemma 3.2, we deduce that for each $m \ge 8\kappa$,

$$\begin{aligned}
\hat{\mathbb{E}}[-\exp\{-\frac{m\theta|X^{t,\xi}_T - a|^2}{2}\}] &= \hat{\mathbb{E}}[\hat{\mathbb{E}}_t[-\exp\{-\frac{m\theta|X^{t,\xi}_T - a|^2}{2}\}]] \\
&\le (1+m(T-t))^{-\rho}\hat{\mathbb{E}}[-\exp\{-\frac{m\theta|\xi - a|^2}{2(1+m(T-t)}\}] \\
&\le (1+m(T-t))^{-\rho}\hat{\mathbb{E}}[-\exp\{-\frac{\theta|\xi - a|^2}{2(T-t)}\}].
\end{aligned}$$

In sprit of Jensen's inequality and note that $\hat{\mathbb{E}}[-\cdot] \ge -\hat{\mathbb{E}}[\cdot]$, we obtain that

$$\hat{\mathbb{E}}[-\exp\{-\frac{m\theta|X^{t,\xi}_T - a|^2}{2}\}] \le -(1+m(T-t))^{-\rho}\exp\{-\frac{\theta}{2(T-t)}\hat{\mathbb{E}}[|\xi - a|^2]\}.$$

Combining the above inequalities indicates that for each $m \ge 8\kappa$,

$$-\hat{\mathbb{E}}[-\mathbf{1}_{|X^{t,\xi}_T - a|\le\delta}] \ge \exp\{-\frac{\theta}{2(T-t)}\hat{\mathbb{E}}[|\xi - a|^2]\}(1+m(T-t))^{-\rho} - \exp\{-\frac{m\theta\delta^2}{2}\}.$$

Note that $\hat{\mathbb{E}}[|\xi-a|^2]<\infty$. Consequently, for large enough m we have

$$\exp\{-\frac{\theta}{2(T-t)}\hat{\mathbb{E}}[|\xi-a|^2]\}(1+m(T-t))^{-\rho}>\exp\{-\frac{m\theta\delta^2}{2}\},$$

which implies that

$$v(|X_T^{t,\xi}-a|\leq\delta)>0.$$

2. $T-t>\varepsilon$**.** We denote

$$\mu:=\begin{cases}\frac{T-t}{\varepsilon}-1, \text{ if } \frac{T-t}{\varepsilon} \text{ is an integer,}\\ [\frac{T-t}{\varepsilon}], \text{ other case.}\end{cases}$$

Note that

$$\begin{aligned}\hat{\mathbb{E}}[-\exp\{-\frac{m\theta|X_T^{t,\xi}-a|^2}{2}\}] =&\hat{\mathbb{E}}[\hat{\mathbb{E}}_{T-\varepsilon}[-\exp\{-\frac{m\theta|X_T^{t,\xi}-a|^2}{2}\}]]\\ =&\hat{\mathbb{E}}[\hat{\mathbb{E}}_{T-\varepsilon}[-\exp\{-\frac{m\theta|X_T^{T-\varepsilon,X_{T-\varepsilon}^{t,\xi}}-a|^2}{2}\}]].\end{aligned}$$

Then it follows from Theorem 3.1 and Lemma 3.2 that for each $m\geq 8\kappa$,

$$\begin{aligned}\hat{\mathbb{E}}[-\exp\{-\frac{m\theta|X_T^{t,\xi}-a|^2}{2}\}] \leq&(1+m\varepsilon)^{-\rho}\hat{\mathbb{E}}[-\exp\{-\frac{m\theta|X_{T-\varepsilon}^{t,\xi}-a|^2}{2(1+m\varepsilon)}\}]\\ \leq&(1+m\varepsilon)^{-\rho}\hat{\mathbb{E}}[-\exp\{-\frac{\frac{m}{2}\theta|X_{T-\varepsilon}^{t,\xi}-a|^2}{2}\}],\end{aligned}$$

where we have used the fact that $m\varepsilon\geq 1$ in the last inequality, since $\varepsilon=(8\kappa)^{-1}$. If $T-t>2\varepsilon$, using Theorem 3.1 and Lemma 3.2 again yields that for each $m\geq 16\kappa$,

$$\hat{\mathbb{E}}[-\exp\{-\frac{m\theta|X_T^{t,\xi}-a|^2}{2}\}]\leq(1+m\varepsilon)^{-\rho}(1+\frac{m}{2}\varepsilon)^{-\rho}\hat{\mathbb{E}}[-\exp\{-\frac{\frac{m}{4}\theta|X_{T-2\varepsilon}^{t,\xi}-a|^2}{2}\}].$$

Iterating the above procedure for μ times implies that for each $m\geq 2^{\mu+2}\kappa$

$$\hat{\mathbb{E}}[-\exp\{-\frac{m\theta|X_T^{t,\xi}-a|^2}{2}\}]\leq\Pi_{i=0}^{\mu-1}(1+\frac{m}{2^i}\varepsilon)^{-\rho}\hat{\mathbb{E}}[-\exp\{-\frac{\frac{m}{2^\mu}\theta|X_{T-\mu\varepsilon}^{t,\xi}-a|^2}{2}\}].$$

Consequently, applying Lemma 3.2 again, we derive that for each $m\geq 2^{\mu+3}\kappa$

$$\hat{\mathbb{E}}[-\exp\{-\frac{m\theta(X_T^{t,\xi}-a)^2}{2}\}]\leq\Pi_{i=0}^{\mu}(1+\frac{m}{2^i}\varepsilon)^{-\rho}\hat{\mathbb{E}}[-\exp\{-\frac{\frac{m}{2^\mu}\theta|\xi-a|^2}{2(1+\frac{m}{2^\mu}(T-t-\mu\varepsilon))}\}].$$

Thus by a similar analysis as in Step 1, we get that for each $m\geq 2^{\mu+3}\kappa$

$$v(|X_T^{t,\xi}-a|\leq\delta)\geq\exp\{-\frac{\theta}{2(T-t-\mu\varepsilon)}\hat{\mathbb{E}}[|\xi-a|^2]\}\Pi_{i=0}^{\mu}(1+\frac{m}{2^i}\varepsilon)^{-\rho}-\exp\{-\frac{m\theta\delta^2}{2}\}.$$

Since $\Pi_{i=0}^{\mu}(1+\frac{m}{2^i}\varepsilon)^{\rho}$ is a function of polynomial growth in m, we conclude that for m large enough

$$\exp\{-\frac{\theta}{2(T-t-\mu\varepsilon)}\hat{\mathbb{E}}[|\xi-a|^2]\}\Pi_{i=0}^{\mu}(1+\frac{m}{2^i}\varepsilon)^{-\rho}>\exp\{-\frac{m\theta\delta^2}{2}\},$$

which ends the proof. □

Remark 3.5. The assumption (H2) is necessary for our method. For example, if $b = x$ and $h^{jk} = \sigma = 0$, then it is easy to check that $v(|X_t^{0,y} - (y+1)e^t| \leq \frac{1}{2}) = 0$ for each $(t, y) \in (0, \infty) \times \mathbb{R}^n$.

Remark 3.6. Note that when the diffusion term is uniformly elliptic and bounded, Theorem 3.4 can be also derived from Krylov and Safanov estimates ([11]) or Harnack's inequality (see Imbert and Silvestre [10]). Indeed, [12] has used this approach to get that the ball for G-Brownian motion has a positive lower capacity. By the method of this paper, we can also study the case that the diffusion term may be unbounded from above and degenerate elliptic ($n > d$). Moreover, we could estimate the lower bound on the lower capacity of the balls for G-Brownian motion.

Theorem 3.7. *For each $a \in \mathbb{R}^d$ and $\delta > 0$, we have*

$$v(|B_t - a| \leq \delta) \geq C_{a,\delta,t}, \quad \forall t > 0,$$

where $C_{a,\delta,t} = \sup_{m\geq 0}(\exp\{-\frac{\theta}{2t}|a|^2\}(1+mt)^{-\rho} - \exp\{-\frac{m\theta\delta^2}{2}\}) > 0$, $\rho = \bar{\sigma}^2(2\underline{\sigma}^2)^{-1}$ *and* $\theta = (d\underline{\sigma}^2)^{-1}$.

Proof. Note that in this case $\sigma = I_{d\times d}$. Then by Remark 3.3, we could take $\rho = \bar{\sigma}^2(2\underline{\sigma}^2)^{-1}$, $\theta = (d\underline{\sigma}^2)^{-1}$, $\varepsilon = T$ ($\kappa = 0$) and $m \geq 0$. Then it follows from the proof of Theorem 3.4 that

$$v(|B_t - a| \leq \delta) \geq \sup_{m\geq 0}(\exp\{-\frac{\theta}{2t}|a|^2\}(1+mt)^{-\rho} - \exp\{-\frac{m\theta\delta^2}{2}\}) > 0,$$

which is the desired result. □

Remark 3.8. By Remark 2.2, for each $a \in \mathbb{R}$ and $\delta > 0$ we have,

$$P_0(|\int_0^t \theta_s dW_s - a| \leq \delta) \geq C_{a,\delta,t}, \quad \forall \theta \in \mathcal{A}_{[\underline{\sigma},\bar{\sigma}]}, \ t > 0,$$

where $C_{a,\delta,t}$ is given in Theorem 3.7.

Remark 3.9. The upper bound on the upper capacity of the ball for G-Brownian motion has been given in [9], which is

$$V(|B_t - a| \leq \delta) \leq \exp(\frac{\theta'}{2})\frac{\delta^{2\rho'}}{t^{\rho'}},$$

where $\rho' = \underline{\sigma}^2(2\bar{\sigma}^2)^{-1}$ and $\theta' = (d\bar{\sigma}^2)^{-1}$. Unfortunately, we cannot obtain such concise estimate for the lower capacity.

4 Applications

As an application, we shall study strict comparison theorem in G-expectation framework. Suppose that $\varphi, \psi \in C_{l,Lip}(\mathbb{R}^n)$. If $\varphi \leq \psi$, then it is easy to check that $\hat{\mathbb{E}}[\varphi(X_T^{t,x})] \leq \hat{\mathbb{E}}[\psi(X_T^{t,x})]$ for each $(t, x) \in [0, T] \times \mathbb{R}^n$. Now we are interested in finding some conditions on φ and ψ under which the strict comparison theorem holds. We first consider the following simple case.

Lemma 4.1. *Let $\varphi \in C_{l,Lip}(\mathbb{R}^n)$ such that $\varphi \leq 0$ and given $(\bar{t}, \bar{x}) \in [0, T) \times \mathbb{R}^n$. Then $\hat{\mathbb{E}}[\varphi(X_T^{\bar{t},\bar{x}})] < 0$ if and only if there exists some point $x_0 \in \mathbb{R}^n$ such that $\varphi(x_0) < 0$.*

Proof. Since $\varphi \leq 0$ and $\hat{\mathbb{E}}[\varphi(X_T^{\bar{t},\bar{x}})] < 0$, we can find some $x_0 \in \mathbb{R}^n$ such that $\varphi(x_0) < 0$. Otherwise, $\varphi(x) = 0$ for all $x \in \mathbb{R}^n$, then $\hat{\mathbb{E}}[\varphi(X_T^{\bar{t},\bar{x}})] = 0$, which is a contradiction.

Conversely, if $\varphi(x_0) < 0$ for some $x_0 \in \mathbb{R}^n$, there exists $\varepsilon > 0$ and $\delta > 0$ such that $\varphi(x) \leq -\varepsilon$ whenever $|x - x_0| < \delta$. Then by Theorem 3.4, we have

$$\hat{\mathbb{E}}[\varphi(X_T^{\bar{t},\bar{x}})] \leq \varepsilon \hat{\mathbb{E}}[-\mathbf{1}_{|X_T^{\bar{t},\bar{x}} - x_0| < \delta}] = -\varepsilon v(|X_T^{\bar{t},\bar{x}} - x_0| < \delta) < 0,$$

which is the desired result. □

Due to the subadditivity of G-expectation, we immediately have the following strict comparison theorem.

Theorem 4.2. *Let $\varphi, \psi \in C_{l,Lip}(\mathbb{R}^n)$ such that $\varphi \leq \psi$ and given $(\bar{t}, \bar{x}) \in [0, T) \times \mathbb{R}^n$. Then $\hat{\mathbb{E}}[\varphi(X_T^{\bar{t},\bar{x}})] < \hat{\mathbb{E}}[\psi(X_T^{\bar{t},\bar{x}})]$ if and only if there exists some point $x_0 \in \mathbb{R}^n$ such that $\varphi(x_0) < \psi(x_0)$.*

The following result is a direct consequence of Theorems 3.1 and 4.2, which can be seen as a strict comparison theorem of PDE (3.2).

Corollary 4.3. *Let $\varphi, \psi \in C_{l,Lip}(\mathbb{R}^n)$ such that $\varphi \leq \psi$. If $\varphi(x_0) < \psi(x_0)$ for some $x_0 \in \mathbb{R}^n$, then*

$$u^{\varphi}(t, x) < u^{\psi}(t, x), \ \forall (t, x) \in [0, T) \times \mathbb{R}^n,$$

where u^f is the viscosity solution to PDE (3.2) with terminal condition $u^f(T, x) = f(x)$, $f = \varphi, \psi$.

In particular, taking $b = h^{jk} = 0$ and $\sigma = I_{d\times d}$, we have the following strict comparison theorem for G-Brownian motion.

Corollary 4.4. *Let $\varphi, \psi \in C_{l,Lip}(\mathbb{R}^d)$ such that $\varphi \leq \psi$ and given $t > 0$. Then $\hat{\mathbb{E}}[\varphi(B_t)] < \hat{\mathbb{E}}[\psi(B_t)]$ if and only if there exists some point $x_0 \in \mathbb{R}^d$ such that $\varphi(x_0) < \psi(x_0)$.*

Remark 4.5. The one-dimensional strict comparison theorem for G-Brownian motion has been obtained firstly in Li [12]. Our method presented in this paper can be also used to study strict comparison theorem for more general G-Itô diffusion process (3.1).

References

[1] Bai, X. and Lin, Y.: On the existence and uniqueness of solutions to stochastic differential equations driven by G-Brownian motion with integral-Lipschitz coefficients. *Acta Mathematicae Applicatae Sinica, English Series* **30**, (2014), 589–610. MR-3285958

[2] Denis, L., Hu, M. and Peng, S.: Function spaces and capacity related to a sublinear expectation: application to G-Brownian motion pathes. *Potential Anal.* **34**, (2011), 139–161. MR-2754968

[3] Denis, L. and Martini, C.: A theoretical framework for the pricing of contingent claims in the presence of model uncertainty. *Annals of Applied Probability* **16**, (2006), 827–852. MR-2244434

[4] Fabes, E. B. and Kenig, C. E.: Examples of singular parabolic measures and singular transition probability densities. *Duke Math. J.* **48**, (1981), 845–856. MR-782580

[5] Gao, F.: Pathwise properties and homomorphic flows for stochastic differential equations driven by G-Brownian motion. *Stochastic Processes and their Applications* 119, (2009), 3356–3382. MR-2568277

[6] Hu, M., Ji, S., Peng, S. and Song, Y.: Backward stochastic differential equations driven by G-Brownian motion. *Stochastic Processes and their Applications* **124**, (2014), 759–784. MR-3131313

[7] Hu, M., Ji, S., Peng, S. and Song, Y.: Comparison theorem, Feynman-Kac formula and Girsanov transformation for BSDEs driven by G-Brownian motion. *Stochastic Processes and their Applications* **124**, (2014), 1170–1195. MR-3138611

[8] Hu, M. and Peng, S.: On representation theorem of G-expectations and paths of G-Brownian motion. *Acta Math. Appl. Sin. Engl. Ser.* **25**, (2009), 539–546. MR-2506990

[9] Hu, M., Wang, F. and Zheng, G.: Quasi-continuous random variables and processes under the G-expectation framework. *Stochastic Processes and their Applications* **126**, (2016), 2367–2387. MR-3505230

[10] Imbert, C. and Silvestre, L.: An introduction to fully nonlinear parabolic equations. *An introduction to the Kähler-Ricci flow*, 7–88, Lecture Notes in Mathematics, 2086, Springer, Cham, 2013. MR-3185332

[11] Krylov, N.V. and Safonov, M.V.: A certain property of solutions of parabolic equations with measurable coefficients. *Izv. Akad. Nauk SSSR Ser. Mat.* **44**, (1980), 161–175; *Math. USSR-Izv.*, **16**, (1981), 151–164. MR-563790

[12] Li, X.: On the strict comparison theorem for G-expectations, arXiv:1002.1765

[13] Lin, Q.: Some properties of stochastic differential equations driven by G-Brownian motion. *Acta Mathematica Sinica, English Series* **29**, (2013), 923–942. MR-3040385

[14] Martini, C.: On the marginal laws of one-dimensional stochastic integrals with uniformly elliptic integrand. *Annales de l'Institut Henri Poincaré, Probabilités et Statistiques* **36**, (2000), 35–43. MR-1743094

[15] Peng, S.: G-expectation, G-Brownian Motion and Related Stochastic Calculus of Itô type. *Stochastic analysis and applications,* 541-567, Abel Symp., 2, Springer, Berlin, 2007. MR-2397805

[16] Peng, S.: Multi-dimensional G-Brownian motion and related stochastic calculus under G-expectation. *Stochastic Processes and their Applications* **118**, (2008), 2223–2253. MR-2474349

[17] Peng, S.: Nonlinear Expectations and Stochastic Calculus under Uncertainty, arXiv:1002.4546

Acknowledgments. The authors would like to thank the Associate Editor and the anonymous referees for their careful reading and helpful suggestions.