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On the Semi-classical Brownian Bridge Measure

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Abstract

We prove an integration by parts formula for the probability measure on the pinned path space induced by the Semi-classical Riemmanian Brownian Bridge, over a manifold with a pole, followed by a discussion on its equivalence with the Brownian Bridge measure.

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1 Introduction

Let M be a complete smooth Riemannian manifold of dimension n. We assume that M is connected and stochastic complete, by the latter we mean that its minimal heat kernel satisfies $\int p_t(x,y)dy = 1$. Let us denote by r the Riemannian distance and by $C([0,1];M)$ the space of continuous curves: $\sigma : [0,1] \to M$. This is a Banach manifold modelled on the Wiener space, a chart containing a path σ is given by a tubular neighbourhood of σ and the coordinate map is induced from the exponential map given by the Levi-Civita connection on the underlying finite dimensional manifold. For $x_0, y_0 \in M$ we denote by $C_{x_0}M$ and $C_{x_0,y_0}M$, respectively, the based and the pinned space of continuous paths over M:

$$\begin{aligned} C_{x_0}M &= \{\sigma \in C([0,1];M) : \quad \sigma(0) = x_0\}, \\ C_{x_0,y_0}M &= \{\sigma \in C([0,1];M) : \quad \sigma(0) = x_0, \quad \sigma(1) = y_0\}. \end{aligned}$$

The pullback tangent bundle of $C_{x_0}M$ consisting of continuous $v : [0,1] \to TM$ with $v(0) = 0$ and $v(t) \in T_{\sigma(t)}M$ where $\sigma \in C([0,1];M)$ which for each σ can be identified by parallel translation with continuous paths on $T_{x_0}M$, the latter is identified with $\mathbb{R}^n$ with a frame u_0. To define gradient operators we make a choice of a family of L^2 subspaces together with an Hilbert space structure, and so we have a family of continuously embedded L^2 subspaces $\mathcal{H}_\sigma$ which defines the L^2 sub-bundle $\mathcal{H} := \cup_\sigma \mathcal{H}_\sigma$. Firstly we denote by H the Cameron-Martin space over $\mathbb{R}^n$,

$$H := \left\{ h \in C([0,1];\mathbb{R}^n) : h(0) = 0, |h|_{H^1} := \left(\int_0^1 |\dot{h}_s|^2 ds \right)^{\frac{1}{2}} < \infty \right\},$$

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and by H^0 its subset consisting of h with $h(1)=0$. If $/\!/.(\sigma)$ denotes stochastic parallel translation along a path σ we denote by $\mathcal{H}_\sigma$ and $\mathcal{H}_\sigma^0$ the Bismut tangent spaces:

$$\mathcal{H}_\sigma = \{/\!/.(\sigma)h : h \in H\}, \qquad \mathcal{H}_\sigma^0 = \{/\!/.(\sigma)h : h \in H, h(1)=0\},$$

specifying respectively the 'admissible' tangent vectors at $\sigma \in C_{x_0}M$ and vectors at $\sigma \in C_{x_0,y_0}M$. These vector spaces are given the inner product inherited from the Cameron-Martin space H.

There are many questions concerning the path space $C_{x_0}M$ and its topologically less trivial subspaces $C_{x_0,x_0}M$. Some questions are concerned with the topology, some others are concerned with the 'Laplacian' d^*d, where d^* is the L^2 dual of the differential d, and with the properties of the probability measure defining the L^2 spaces. These include the questions whether a Poincaré inequality (spectral gap) holds or whether the stronger Logarithmic Sobolev inequality holds for the relevant probability measure on a particular manifold with a particular Riemannian metric. One programme for the topology is to follow the Hodge-de Rham theory for smooth compact manifolds which in particular observes that the L^2 de Rham cohomology groups defined by the complex of exterior differential forms are topological invariants. The fundamental questions are then concerned with whether there is a complex of L^2 exterior differentials on the path spaces which plays the role of the natural L^2 de Rham complex of finite dimensional manifolds. Observing that these manifolds and their tensor tangent spaces are locally Banach spaces, it is non-trivial to choose a suitable family of Hilbert sub-spaces [20]. These call for the Cameron-Martin theory of Gaussian measures when the space is $\mathbb{R}^n$ with the trivial Euclidean metric. For non-linear spaces, including $\mathbb{R}^n$ with a non-trivial Riemannian metric, there is no longer a suitable Gaussian measure. These call for Malliavin calculus in which the integration by parts formula is the basis for all studies and which implies in particular the closability of the fundamental operator d.

As we mentioned earlier, for an L^2 analysis on $C_{x_0,y_0}M$ we need a probability measure which is usually taken to be the probability distribution of the conditioned Brownian motion. The heat kernel measure, the distribution of a Brownian sheet, offers also an alternative measure, see [35, 12, 38]. See also [30] for a study of the measure induced by a conditioned hypoelliptic stochastic process. The Brownian bridge measures are notoriously difficult to analyse, in particular the Poincaré inequality are known for very few classes of Riemannian manifolds, [3, 2, 7]. If we suppose that M has a pole y_0, by which we mean that the exponential map $\exp_{y_0} : T_{y_0}M \to M$ is a diffeomorphism, another probability measure, the probability distribution of the Semi-classical Riemannian bridge, becomes available to us. For a simply connected Riemannian manifold with non-positive sectional curvature, every point is a pole. The Semi-classical Riemannian bridge measures are easier to handle, Poincaré inequalities for these measures could lead to Poincaré inequalities for the Brownian bridge measures, especially if they are equivalent.

We denote by $\nu = \nu_{x_0,y_0}$ the probability distribution of the Semi-classical Brownian bridge and by $L^2(C_{x_0,y_0}M;\mathbb{R})$ the corresponding L^2 space. A Semi-classical Riemannian Brownian bridge $(\tilde{x}_s, s \le 1)$ is a time dependent strong Markov process with generator $\frac{1}{2}\triangle + \nabla \log k_{1-s}(\cdot, y_0)$ where,

$$k_t(x_0,y_0) := (2\pi t)^{-\frac{n}{2}} e^{-\frac{r^2(x_0,y_0)}{2t}} J^{-\frac{1}{2}}(x_0),$$

and $J(y) = |\det D_{\exp_{y_0}^{-1}(y)} \exp_{y_0}|$ is the Jacobian determinant of the exponential map at y_0. Semi-classical Riemannian Brownian bridges (*Semi-classical bridge* for short) were introduced by K. D. Elworthy and A. Truman in [14, 15], over thirty years ago, in their semi-classical analysis for Schrödinger operators and was inspired by Classical

Mechanics. They also gave a heat kernel formula in terms of the Semi-classical bridge, which was further explored in [16] and [37]. If p_t is the heat kernel, the Brownian bridge is a Markov process with generator $\frac{1}{2}\Delta + \nabla \log p_{1-t}(\cdot, y_0)$. Let us discuss briefly the two time dependent potential functions that drives the Brownian motion to the terminal value. They are close to each other as $t \to 1$, by Varadhan's asymptotic relations [41]: $(1-t)\log p_{1-t}(x, y_0) \sim -\frac{1}{2}r^2(x, y_0)$. There is also the relation $\lim_{t\to 1}(1-t)\nabla \log p_{1-t}(x, y_0) = -\dot\gamma(0)$ where γ is normal geodesic from y_0 to x. The two drift vector fields $\nabla \log p_{1-t}(\cdot, y_0)$ and $\nabla \log k_{1-t}(\cdot, y_0)$ differ by $-\frac{1}{2}\nabla \log J$ near the terminal time.

Let us consider the unbounded linear differential operator d on $L^2(C_{x_0,y_0}M; \mathbb{R})$ taking values in $L^2(\cup_\sigma \mathcal{H}^*_\sigma)$ where for $v \in \cup_\sigma \mathcal{H}^*_\sigma$, its L^2 norm is

$$\|v\| := \left(\int_{C_{x_0,y_0}M} \left(\left| /\!/_{\cdot}^{-1} v_{\cdot}(\sigma) \right|_H \right)^2 d\nu(\sigma) \right)^{\frac{1}{2}}.$$

Another norm can be given, taking into accounts of the damping effects of the Ricci curvature, which will be discussed later. As the distance function from the Semi-classical bridge to the pole is precisely the n-dimensional Bessel bridge where $n = \dim(M)$, the advantage of the Semi-classical Brownian bridge measure is that it is easier to handle, which we demonstrate by studying the elementary property of the divergence operator. Our main result is an integration by parts formula for d. First order Feyman-Kac type formulas together with estimates for the gradient of the Feynman-Kac kernel using the Semi-classical bridge process and the damped stochastic parallel translation can be found in [34].

Denote by OM the space of orthonormal frames over M and $\{H_i\}$ the canonical horizontal vector fields on OM associated to an orthonormal basis $\{e_i\}$ of $\mathbb{R}^n$ so that H_i is the horizontal lift of ue_i. For a tangent vector v on M, we will denote by $\tilde v$ the horizontal lift of v to TOM. Let $\{\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P}\}$ be a filtered probability space on which is given a family of independent one-dimensional Brownian motions $\{B^i\}$. We define $B_t = (B^1_t, \dots, B^n_t)$. Let $u_0 \in \pi^{-1}(x)$ be a frame at x, u_t and $\tilde u_t$ be the solution to the stochastic differential equations,

$$du_s = \sum_{i=1}^n H_i(u_s) \circ dB^i_s, \quad d\tilde u_s = \sum_{i=1}^n H_i(\tilde u_s) \circ dB^i_s + \tilde A_s(\tilde u_s) ds, \quad \tilde u_0 = u_0, \tag{1.1}$$

where $\circ$ denote Stratonovich integration and $A_s = \nabla \log k_{1-s}(\cdot, y_0)$ and $\tilde A_s$ its horizontal lift to the orthonormal frame bundle. Then $\tilde x_s := \pi(\tilde u_s)$ is a Semi-classical bridge from x_0 to y_0 in time 1. Let Ric_x denote the Ricci curvature at $x \in M$, by $\mathrm{Ric}^\sharp_x : T_xM \to T_xM$ we mean the linear map given by the relation $\langle \mathrm{Ric}^\sharp_x u, v\rangle = \mathrm{Ric}_x(u, v)$.

Denote $r = r(\cdot, y_0)$ for simplicity. We will need the following geometric conditions. Set

$$\Phi = \frac{1}{2} J^{\frac{1}{2}} \triangle J^{-\frac{1}{2}} = \frac{1}{8}|\nabla \log J|^2 - \frac{1}{4}\Delta(\log J). \tag{1.2}$$

C1: The Ricci curvature is bounded.
C2: $|\nabla\Phi| + |\nabla(\log J)| \le c(e^{ar^2} + 1)$ for some $c > 0$ and a is an explicit constant to be given later.
C3: Φ is bounded from below.
C4: For each t, k_t and $|\nabla k_t|$ are bounded, $|\nabla\Phi|$ is bounded.

The condition that the Ricci curvature is bounded ensures that the solution to the canonical SDE is differentiable in the sense of Malliavin calculus. It also implies that $|W_t|$ is bounded where W_t is the solution to the stochastic damped parallel translation equation for the Brownian motion (i.e. with the right hand side $-\frac{1}{2}\mathrm{Ric}^\sharp$), and that the integration by parts formula holds for the Brownian motion measure. Observe that k_t and $|\nabla k_t|$ are bounded if $rJ^{-\frac{1}{2}}$ and $J^{-\frac{1}{2}}\nabla \log J^{-\frac{1}{2}}$ grow at most exponentially. Here we do not strive for the best possible conditions, as the optimal conditions will manifest themselves when Clark-Ocone formula and Poincaré inequalities are studied.

Our main results is the following integration by parts theorem, whose proof is given in §2. In §3 we discuss the equivalence of the Semi-classical bridge measures and the Brownian bridge measures, see also [32] for further studies in this direction.

Theorem 1.1. *Assume* **C1–C4** *hold. Then for any* $F, G \in \mathrm{Cyl}$ *and* $h \in H^0$ *the following integration by parts formula holds,*

$$\int_{C_{x_0,y_0}M} G(\sigma) dF\left(/\!/_{.}(\sigma)h_.\right)\nu(d\sigma) + \int_{C_{x_0,y_0}M} F(\sigma) dG\left(/\!/_{.}(\sigma)h_.\right)\nu(d\sigma)$$
$$= \mathbf{E}\left[(FG)(\tilde{x}_.)\int_0^1 \langle \dot{h}_s + \frac{1}{2}\tilde{u}_s^{-1}\mathrm{Ric}^\sharp(\tilde{u}_s h_s), d\tilde{B}_s\rangle\right] + \mathbf{E}\left[(FG)(\tilde{x}_.)\int_0^1 d\Phi(\tilde{u}_s h_s)ds\right].$$

Here $d\tilde{B}_s = dB_s + \tilde{u}_s^{-1}\nabla \log k_{1-s}(\tilde{x}_s)\, ds$. *In particular* $d : \mathrm{Cyl} \subset L^2(C_{x_0,y_0}M) \to L^2(\cup_\sigma \mathcal{H}_\sigma)$ *is closable and*

$$\mathrm{div}(G\,\tilde{u}_.h) = dG\,(\tilde{u}_.h_.) - G\int_0^1 \langle \dot{h}_s + \frac{1}{2}\tilde{u}_s^{-1}\mathrm{Ric}^\sharp(\tilde{u}_s h_s), d\tilde{B}_s\rangle - G\int_0^1 d\Phi(\tilde{u}_s h_s)ds.$$

For based path space over a compact manifold, with Brownian motion measure (the Wiener measure), this was proved in [9], for non-compact manifolds see [18, 20], [22], [40], and [6]. For pinned manifolds with measure coming from the classical Brownian bridge measure, this was proved in [10] and [29], for a local formula see [8].

Let us now clarify the definition of d. A common definition for d, which we use, is to take its initial domain to be Cyl, the set of cylindrical functions of the form $F(\sigma) = f(\sigma_{t_1}, \dots, \sigma_{t_m})$ where $m \in \mathbb{N}$, $0 < t_1 < t_2 < \dots < t_m < 1$, and f is a BC^1 function on the m-fold product space of M, or Cyl_0 the subset containing $f(\sigma_{t_1}, \dots, \sigma_{t_m})$ where f is compactly supported. The H-derivative (Malliavin derivative) of F in the direction of $u_.(\sigma)h_. \in T_\sigma C_{x_0}M$ is:

$$(dF)(/\!/_.(\sigma)h_.) = \sum_{k=1}^m \partial_k f\big(/\!/_{t_k}(\sigma)h_{t_k}\big),$$

where $\partial_k f$ denotes the derivative of f in its kth component and $/\!/$ denotes parallel translation and identified with u in the sequel. Denote by $G(s,t)$ and $G^0(s,t)$, respectively, the Green's functions of $\frac{d}{ds}$ on $(0,1)$ with suitable Dirichlet conditions: $G(s,t) = s \wedge t$ and $G^0(s,t) = s\wedge t - st$. Then

$$(\nabla F)(\sigma)(t) = \sum_{k=1}^m G(t_k, t)/\!/_{t_k,t}(\sigma)\nabla_k f(\sigma_{t_1}, \sigma_{t_2}, \dots, \sigma_{t_m}),$$
$$(\nabla^0 F)(\sigma)(t) = \sum_{k=1}^m G^0(t_k, t)/\!/_{t_k,t}(\sigma)\nabla_k f(\sigma_{t_1}, \sigma_{t_2}, \dots, \sigma_{t_m}),$$

where $\nabla_k f$ denotes the gradient of f in the kth variable. We have

$$\|\nabla F\|^2 = \sum_{i,j=1}^m G(t_k, t_j)\langle /\!/_{t_k,t_j}\nabla_k f, \nabla_j f\rangle,$$
$$\|\nabla^0 F\|^2 = \sum_{i,j=1}^m G^0(t_k, t_j)\langle /\!/_{t_k,t_j}\nabla_k f, \nabla_j f\rangle.$$

It is an open problem whether the closure of d with initial domain BC^1 agrees wth the closure of d with initial value the cylindrical functions. This is the Markov uniqueness problem, studied in [19] where it was only proved that the latter including BC^2.

Open Question. It remains to study the validity of Poincaré inequality for ν and use it to explore whether a Poincaré inequality holds for the Brownian bridge measure. See §3 for a conjecture.

2 Proof of Theorem 1.1

To clarify the singularities at the terminal time we first prove a lemma concerning the divergence of a suitable vector field on the path space. Let $\tilde u_t$ be as defined by (1.1), $\tilde x_t = \pi(\tilde u_t)$. Recall that $k_t(x_0, y_0) = (2\pi t)^{-\frac{n}{2}} e^{-\frac{r^2(x_0,y_0)}{2t}} J^{-\frac{1}{2}}(x_0)$ and

$$d\tilde B_s \;=\; dB_s + \tilde u_s^{-1} \nabla \log k_{1-s}(\tilde x_s, y_0)\, ds.$$

The reference to y_0 will be dropped from time to time for simplicity. Define $\mathrm{ric}_u = u^{-1}\,\mathrm{Ric}^\sharp u$.

Lemma 2.1. *Assume stochastic completeness,* **C2***, and* $h \in H^0$*. Then,* $\int_0^t \left\langle \dot h_s + \frac{1}{2}\mathrm{ric}_{\tilde u_s}(h_s), d\tilde B_s \right\rangle$ *converges to* $\int_0^1 \left\langle \dot h_s + \frac{1}{2}\mathrm{ric}_{\tilde u_s}(h_s), d\tilde B_s \right\rangle$ *as* $t \to 1$*. Furthermore,*

$$\lim_{t\to 1} \mathbf{E}\left(\langle \nabla\, \mathrm{log} k_{1-t}(\cdot, y_0), \tilde u_t h_t\rangle\right)^2 = 0,$$

$\int_0^t \langle \dot h_s + \frac{1}{2}\mathrm{ric}_{\tilde u_s}(h_s), d\tilde B_s\rangle$ *converges in* $L^2(\Omega, \mathbb{P})$ *as* t *approaches* 1*, and*

$$\begin{aligned}\int_0^1 \left\langle \dot h_s + \frac{1}{2}\mathrm{ric}_{\tilde u_s}(h_s), d\tilde B_s \right\rangle = \int_0^1 \left\langle \dot h_s + \frac{1}{2}\mathrm{ric}_{\tilde u_s}(h_s), dB_s \right\rangle - \int_0^1 d\Phi(\tilde u_s h_s) ds \\ - \int_0^1 \nabla d\left(\mathrm{log} k_{1-s}(\tilde x_s, y_0)\right)(\tilde u_s dB_s, \tilde u_s h_s).\end{aligned}$$

Proof. The singularities in the integral $\int_0^1 \left\langle \dot h_s + \frac{1}{2}\mathrm{ric}_{\tilde u_s}(h_s), d\tilde B_s \right\rangle$ come from the involvement of $\nabla \log k_{1-s}(\tilde x_s, y_0)$ and we only need to worry about

$$\alpha_t := \int_0^t \left\langle \dot h_s + \frac{1}{2}\mathrm{ric}_{\tilde u_s}(h_s), \tilde u_s^{-1} \nabla \log k_{1-s}(\tilde x_s, y_0) \right\rangle ds. \tag{2.1}$$

To deal with $\int_0^t \left\langle \dot h_s, \tilde u_s^{-1} \nabla \log k_{1-s}(\tilde x_s, y_0) \right\rangle ds$, which involves the derivative of h_s, we use integration by parts. Since $\frac{D}{ds}(u_s h_s) = u_s \dot h_s$, by stochastic calculus applied to $d\left(\mathrm{log} k_{1-s}\right)(u_s h_s)$, where d denotes spatial differentiation, we see that

$$\begin{aligned}&\langle \nabla \mathrm{log} k_{1-t}(\tilde x_t), \tilde u_t h_t\rangle \\ &= \int_0^t \left\langle \nabla \mathrm{log} k_{1-s}(\tilde x_s), \tilde u_s \dot h_s \right\rangle ds + \sum_{i=1}^n \int_0^t \nabla d\left(\mathrm{log} k_{1-s}\right)(\tilde u_s e_i, \tilde u_s h_s) dB_s^i \\ &\quad + \int_0^t \nabla d\left(\mathrm{log} k_{1-s}\right)\left(\nabla \mathrm{log} k_{1-s}(\tilde x_s), \tilde u_s h_s\right)\, ds \\ &\quad + \int_0^t \left(\frac{1}{2}\,\mathrm{trace}\, \nabla^2 + \frac{\partial}{\partial s}\right)\left(d\left(\mathrm{log} k_{1-s}(\tilde x_s)\right)\right)(\tilde u_s h_s)\, ds,\end{aligned}$$

the first term on the right hand side being α_t. By the following identities,

$$\nabla \mathrm{log} k_{1-s}(x) = -\frac{r(x)\nabla r(x)}{1-s} + \nabla \mathrm{log}(J^{-\frac{1}{2}}), \quad \Delta r = \frac{n-1}{r} + \langle \nabla r, \nabla \log J\rangle,$$

the following set of formulas are easy to verify.

$$\begin{aligned}
\Delta \mathrm{log} k_{1-s} &= -\frac{n}{1-s} - \frac{r\langle \nabla r, \nabla \log J\rangle}{1-s} - \frac{1}{2}\Delta(\log J),\\
\frac{\partial}{\partial s}\log k_{1-s} &= \frac{n}{2(1-s)} - \frac{r^2}{2(1-s)^2},\\
|\nabla \log k_{1-s}|^2 &= \frac{r^2}{(1-s)^2} + \frac{1}{4}|\nabla \log J|^2 + \frac{r\langle \nabla r, \nabla \log J\rangle}{1-s}.
\end{aligned} \tag{2.2}$$

It follows that

$$\left(\frac{1}{2}\Delta + \frac{\partial}{\partial s}\right)(\mathrm{log} k_{1-s}) + \frac{1}{2}|\nabla \log k_{1-s}|^2 = \frac{1}{8}|\nabla \mathrm{log} J|^2 - \frac{1}{4}\Delta(\mathrm{log} J) = \Phi. \tag{2.3}$$

Let $\Delta^1 := -(dd^* + d^*d)$ denote the Laplace-Beltrami Kodaira operator on differential 1-forms. By the Weitzenböck formula, $\left(\frac{1}{2}\,\mathrm{trace}\,\nabla^2 + \frac{\partial}{\partial s}\right)d = \left(\frac{1}{2}\Delta^1 d + \frac{1}{2}\mathrm{Ric}^\sharp d + \frac{\partial}{\partial s}d\right)$, observing that $\mathrm{trace}\,\nabla^2 = -\nabla^*\nabla$, and consequently,

$$\begin{aligned}
&\left(\frac{1}{2}\,\mathrm{trace}\,\nabla^2 + \frac{\partial}{\partial s}\right) d\left(\mathrm{log} k_{1-s}(\tilde{x}_s)\right)\\
=&d\left(\frac{1}{2}\Delta + \frac{\partial}{\partial s}\right)(\mathrm{log} k_{1-s}(\tilde{x}_s)) + \frac{1}{2}\mathrm{Ric}^\sharp\left(d\mathrm{log} k_{1-s}(\tilde{x}_s)\right)\\
=&-\frac{1}{2}d(|\nabla \mathrm{log}\, k_{1-s}(\cdot)|^2) + d\Phi + \frac{1}{2}\mathrm{Ric}^\sharp\left(d\mathrm{log} k_{1-s}(\tilde{x}_s)\right).
\end{aligned}$$

Thus

$$\begin{aligned}
&\nabla d\left(\mathrm{log} k_{1-s}\right)\left(\nabla \mathrm{log} k_{1-s}(\tilde{x}_s), \tilde{u}_s h_s)\right) + \left(\frac{1}{2}\,\mathrm{trace}\,\nabla^2 + \frac{\partial}{\partial s}\right)\left(d\left(\mathrm{log} k_{1-s}\right)\right)(\tilde{u}_s h_s)\\
&= d\Phi(\tilde{u}_s h_s) + \frac{1}{2}\mathrm{Ric}\left(\nabla \mathrm{log} k_{1-s}(\tilde{x}_s), \tilde{u}_s h_s\right).
\end{aligned}$$

Let us return to $\langle \nabla \mathrm{log} k_{1-t}(\tilde{x}_t), \tilde{u}_t h_t\rangle$:

$$\begin{aligned}
&\langle \nabla \mathrm{log} k_{1-t}(\tilde{x}_t), \tilde{u}_t h_t\rangle\\
=&\int_0^t \left\langle \nabla \mathrm{log} k_{1-s}(\tilde{x}_s), \tilde{u}_s \dot{h}_s\right\rangle ds + \sum_{i=1}^n \int_0^t \nabla d\left(\mathrm{log} k_{1-s}\right)(\tilde{u}_s e_i, \tilde{u}_s h_s) dB_s^i\\
&+ \int_0^t d\Phi(\tilde{u}_s h_s)\, ds + \frac{1}{2}\int_0^t \mathrm{Ric}\left(\nabla \mathrm{log} k_{1-s}(\tilde{x}_s), \tilde{u}_s h_s\right)\, ds.
\end{aligned}$$

We thus obtain the following relation:

$$\begin{aligned}
\alpha_t &= \langle \nabla \mathrm{log} k_{1-t}(\tilde{x}_t), \tilde{u}_t h_t\rangle + \frac{1}{2}\int_0^t D\log k_{1-s}(\mathrm{Ric}^\sharp(\tilde{u}_s h_s)) ds\\
&= \langle \nabla\ \mathrm{log} k_{1-t}(\cdot), \tilde{u}_t h_t\rangle - \int_0^t \langle \nabla\Phi, \tilde{u}_s h_s\rangle\, ds\\
&\quad - \sum_{i=1}^n \int_0^t \nabla d\left(\mathrm{log} k_{1-s}\right)\left(\tilde{u}_s e_i, \tilde{u}_s h_s\right) dB_s^i.
\end{aligned}$$

We will prove that each of the terms on the right hand side converges as t approaches 1. Furthermore $\langle \nabla\ \mathrm{log} k_{1-t}(\cdot), \tilde{u}_t h_t\rangle$ converges to zero. We first observe that there exists a constant C such that $\mathbf{E}[r(\tilde{x}_t)^p] \le C t^{\frac{p}{2}}$. Indeed $r_t := r(\tilde{x}_t, y_0)$ satisfies

$$r_t - r_0 = \beta_t + \int_0^t \frac{1}{2}\Delta r(\tilde{x}_s) ds - \int_0^t \frac{r(\tilde{x}_s)}{1-s} ds - \frac{1}{2}\int_0^t \langle \nabla r, \nabla \log J\rangle_{\tilde{x}_s}\, ds$$

$$=\beta_t + \int_0^t \frac{n-1}{2r_s} ds - \int_0^t \frac{r_s}{1-s} ds,$$

where β_t is a one dimensional Brownian motion and we have used the fact that $\Delta r = \frac{n-1}{r} + \langle \nabla r, \nabla \log J \rangle$. Thus r_s is a Bessel bridge starting at $r(x_0, y_0)$ and ending at 0 at time 1. In particular $\lim_{t\uparrow 1} \tilde{x}_t = y_0$ and $(r_t, t \le 1)$ is a continuous semi-martingale. Furthermore for any $p > 1$, $\mathbf{E}[r(\tilde{x}_t)^p] \le Ct^{\frac{p}{2}}$. If K_t denotes the standard Gaussian kernel on $\mathbb{R}^n$ then for $z_1, z_2 \in \mathbb{R}^n$ with $|z_1 - z_2| = r(x_0, y_0)$,

$$\mathbf{E}[r(\tilde{x}_t, y_0)^p] = \frac{1}{K_1(z_1, z_2)} \int_{\mathbb{R}^n} |z - z_2|^p K_t(z_1, z) K_{1-t}(z, z_2) dz \le C|z_1 - z_2|^{\frac{p}{2}}.$$

We also know that $\mathbf{E}[e^{2ar_t^2}] < \infty$ for some a and $t \le 1$, invoking condition C2.

We show below that (2.1) has a limit as $t \to 1$. Firstly, since $|d\Phi| \le ce^{ar^2}$,

$$\lim_{t\to 1} \mathbf{E} \left[\int_t^1 \langle \nabla \Phi, \tilde{u}_s h_s \rangle \, ds \right]^2 = 0.$$

We work with the first term on the right hand side. By the definition of k_{1-t},

$$\langle \nabla \ \log k_{1-t}(\cdot, y_0), \tilde{u}_t h_t \rangle = \frac{r(\tilde{x}_t) \langle \nabla r(\tilde{x}_t), \tilde{u}_t h_t \rangle}{1-t} + \langle \nabla \log J_{\tilde{x}_t}^{-\frac{1}{2}}, \tilde{u}_t h_t \rangle.$$

Since $|d(\log J_x^{-\frac{1}{2}})| \le ce^{ar(x)^2}$, $\lim_{t\to 1} \langle \nabla \log J_{\tilde{x}_t}^{-\frac{1}{2}}, \tilde{u}_t h_t \rangle$ converges in $L^2(\Omega)$. Thus

$$\lim_{t\uparrow 1} \mathbf{E} \left| \langle \nabla \log J^{-\frac{1}{2}}(\tilde{x}_t), \tilde{u}_t h_t \rangle \right|^2 = 0, \tag{2.4}$$

using the fact that $h_t \to 0$. Also, by the symmetry of the Euclidean bridge, $\mathbf{E}[r^2(\tilde{x}_t, y_0)] \le C\,(t \wedge (1-t))$ and hence

$$\mathbf{E} \left| \frac{r(\tilde{x}_t)\, \langle \nabla r(\tilde{x}_t), \tilde{u}_t h_t \rangle}{1-t} \right|^2 \le C \frac{|h_t|^2}{1-t}.$$

Since $h_1 = 0$, and $h \in H$,

$$\frac{|h_t|^2}{1-t} = \frac{1}{1-t} \left| \int_t^1 \dot{h}_s dr \right|^2 \le \int_t^1 |\dot{h}_s|^2 ds \to 0,$$

as $t \to 1$, using the fact that $h \in H$. We conclude that

$$\lim_{t\to 1} \mathbf{E} \left[\langle \nabla \ \log k_{1-t}(\cdot), \tilde{u}_t h_t \rangle \right]^2 = 0.$$

For the final term we observe that

$$\nabla d\, (\log k_{1-s})\, (\tilde{u}_s e_i, \tilde{u}_s h_s) = -\frac{\nabla r(\tilde{u}_s e_i) \nabla r(\tilde{u}_s h_s)}{1-s} - \frac{r \nabla dr(\tilde{u}_s e_i, \tilde{u}_s h_s)}{1-s}.$$

We further observe that the Frobenius norm of the Hessian of the distance function satisfies:

$$\|\nabla dr\|_F := \left(\sum_{i,j=1}^n \langle \nabla_{E_i} \partial r, E_j \rangle \right)^{\frac{1}{2}} \le \frac{1}{\sqrt{n-1}} \Delta r \le \frac{1}{\sqrt{n-1}} \left(\frac{n-1}{r} + \langle \nabla r, \nabla \log J \rangle \right).$$

Since $|\nabla \log J| \le ce^{ar^2}$, for some constant C, which may depend on n,

$$\mathbf{E} \left[\sum_{i=1}^n \int_0^t \nabla d\, (\log k_{1-s})\, (\tilde{u}_s e_i, \tilde{u}_s h_s)\, dB_s^i \right]^2$$
$$\le C \int_0^t \frac{|h_s|^2}{(1-s)^2} ds \le C \frac{|h_t|^2}{1-t} + 4C \int_0^t |\dot{h}_s|^2 ds.$$

This follows from the following standard computation,

$$\int_0^t \frac{|h_s|^2}{(1-s)^2} ds = \frac{|h_t|^2}{1-t} - \int_0^t \frac{\langle h_s, 2\dot{h}_s \rangle}{1-s} ds \le \frac{|h_t|^2}{1-t} + 2\int_0^t |\dot{h}_s|^2 ds + \frac{1}{2}\int_0^t \frac{|h_s|^2}{(1-s)^2} ds. \quad (2.5)$$

This concludes the proof of the convergence of the integral. The required identity follows from the formula, given earlier, for α_t. □

Let u_t be the solution to the equation $du_t = \sum_{i=1}^n H_i(u_t) \circ dB_t^i$ with initial value $u_0 \in \pi^{-1}(x_0)$. Then $x_t := \pi(u_t)$ is a Brownian motion on M starting at x_0 and the integration by parts formula holds on $L^2(C_{x_0}M; \mu)$. For any $F, G \in \mathrm{Cyl}$, and $h \in H(T_{x_0}M)$ with $h(0) = 0$, d is the differential on $L^2(C_{x_0}M)$ with respect to the Brownian motion measure:

$$\mathbf{E}[dF(u.h.)G] = -\mathbf{E}[FdG(u.h.)] + \mathbf{E}\left[FG\int_0^1 \langle \dot{h}_s + \frac{1}{2}u_s^{-1}\mathrm{Ric}^\sharp(u_s h_s), dB_s\rangle\right]. \quad (2.6)$$

If M is compact, see e.g. B. Driver [9]. This is also known to hold if the Ricci curvature is bounded from below. The divergence of $u.h.$ is

$$\mathrm{div}(u.h.) = \int_0^1 \langle \dot{h}_s + \frac{1}{2}u_s^{-1}\mathrm{Ric}^\sharp_{u_s}(u_s h_s), dB_s\rangle.$$

The following lemma completes the proof of Theorem 1.1.

Lemma 2.2. *Suppose stochastic completeness,* **C2–C4***, and suppose that the integration by parts formula (2.6) holds for the Brownian motion measure.*

Proof. Let $h \in H^0$. Our plan is to pass the integration on the path space to the pinned path space by a Girsanov transform. We first observe that if $F \in \mathrm{Dom}(d)$, adapted to $\mathcal{G}_t$ where $t < 1$, then

$$\mathbf{E}[dF(\tilde{u}h.)] = \mathbf{E}\left[dF(uh.)\frac{k_{1-t}(x_t)}{k_1(x_0)} e^{-\int_0^t \Phi(x_s)ds}\right].$$

In fact, the formula for the probability density between the original probability measure on $\mathcal{G}_t$ and the one for which $B_t - \int_0^t u_s^{-1}\nabla\log k_{1-s}(x_s)\, ds$ is a Brownian motion, is:

$$M_t = \exp\left[\sum_{i=1}^m \int_0^t \langle \nabla\log k_{1-s}(x_s, y_0), u_s e_i\rangle dB_s^i - \frac{1}{2}\int_0^t |\nabla\log k_{1-s}(x_s, y_0)|^2 ds\right].$$

By an application of Itô's formula, and identities (2.2–2.3) in the proof of Lemma 2.1,

$$M_t = \frac{k_{1-t}(x_t, y_0)}{k_1(x_0, y_0)} \exp\left(-\int_0^t \Phi(x_s)ds\right).$$

Since the Brownian motion and the Semi-classical bridge are conservative, then $(M_s, s \le t)$ is a martingale for any $t < 1$.

Since Φ is bounded from below and has bounded derivative, $e^{-\int_0^t \Phi(x_s)ds}$ can be approximated by smooth cylindrical functions in the domain of d. Next we observe that

$$\nabla k_{1-s}(\cdot, y_0) = 2\pi(1-s)^{-\frac{n}{2}} e^{-\frac{r^2}{2(1-s)}} J^{-\frac{1}{2}}\left(-\frac{r\nabla r}{1-s} + \nabla\log J^{-\frac{1}{2}}\right),$$

is bounded and smooth, so $\frac{k_{1-t}(x_t, y_0)}{k_1(x_0, y_0)} e^{-\int_0^t \Phi(x_s)ds}$ belongs to the domain of d. Consequently, for F, G measurable with respect to the canonical filtration up to time $t < 1$, we apply Girsanov transform to the equation $du_t = H(u_t) \circ dB_t$ and the integration by parts formula (2.6) to see

$$\begin{aligned}
&\mathbf{E}[G(\tilde{x}.)\, dF(\tilde{u}.h.)] = \mathbf{E}\left[dF(u.h.)\, G(x.)M_t\right] \\
&= \mathbf{E}\left[(FG)(x.)M_t \operatorname{div}(u.h.)\right] - \mathbf{E}\left[(FG)(x.)\, dM_t(u.h.)\right] - \mathbf{E}[F(x.)\, dG(u.h.)M_t] \\
=&\mathbf{E}\left[(FG)(x.)M_t \int_0^t \left\langle \dot{h}_s + \frac{1}{2}u_s^{-1}\mathrm{Ric}^\sharp_{u_s}(h_s), dB_s\right\rangle\right] - \mathbf{E}[F(x.)dG(u.h.)M_t] \\
&- \mathbf{E}\left[(FG)(x.)M_t\, d\left(\log k_{1-t}(x_t, y_0) - \int_0^t \Phi(x_s)ds\right)(u.h.)\right].
\end{aligned}$$

Using Girsanov transform again, we shift from (u_s, B_s) to $(\tilde{u}_s, \tilde{B}_s)$ and obtain that

$$\begin{aligned}
\mathbf{E}[G(x.)dF(\tilde{u}.h.] =&\mathbf{E}\left[(FG)(\tilde{x}.)\int_0^t \langle \dot{h}_s + \frac{1}{2}\tilde{u}_s^{-1}\mathrm{Ric}^\sharp_{\tilde{u}_s}(h_s), d\tilde{B}_s\rangle\right] - \mathbf{E}[F(\tilde{x}.)\, dG(\tilde{u}.h.)] \\
&- \mathbf{E}\left[(FG)(\tilde{x}.)\langle \nabla \log k_{1-t}(\tilde{x}_t, y_0), \tilde{u}_t h_t\rangle - (FG)(\tilde{x}.)\int_0^t d\Phi(u_s h_s)ds\right].
\end{aligned}$$

We take $t \uparrow 1$, by (2.4) and Lemma 2.1, $\lim_{t\uparrow 1}\langle \nabla \log k_{1-t}(\tilde{x}_t, y_0), \tilde{u}_t h_t\rangle = 0$ in L^2, hence

$$\begin{aligned}
&\mathbf{E}[G(\tilde{x}.)dF(\tilde{u}.h.)] + \mathbf{E}[F(\tilde{x}.)dG(\tilde{u}.h.)] \\
=&\mathbf{E}\left[(FG)(\tilde{x}.)\int_0^1 \left\langle \dot{h}_s + \frac{1}{2}\tilde{u}_s^{-1}\mathrm{Ric}^\sharp(\tilde{u}_s h_s), d\tilde{B}_s\right\rangle\right] + \mathbf{E}\left[(FG)(\tilde{x}.)\left(\int_0^1 d\Phi(\tilde{u}_s h_s)ds\right)\right].
\end{aligned}$$

By a limiting procedure we see that the identity holds for $G = 1$. In particular we conclude that d is a closable operator. Furthermore, $\mathrm{Dom}(\mathrm{div})$ contains $G\tilde{u}h$ where $G \in \mathrm{Cyl}$ and

$$\operatorname{div}(\tilde{u}.h.) = -\int_0^1 \left\langle \dot{h}_s + \frac{1}{2}\tilde{u}_s^{-1}\mathrm{Ric}^\sharp(\tilde{u}_s h_s), d\tilde{B}_s\right\rangle - \left(\int_0^1 d\Phi(\tilde{u}_s h_s)ds\right).$$

The conclusion of Theorem 1.1 follows. Observe that under condition **C1**, the integration by parts formula holds for the Brownian motion measure. This completes the proof of Lemma 2.2. □

2.1 Comment

Let us consider briefly for which manifolds our assumptions on Φ hold. Denote by ∂r the radial curvature which, evaluated at $x \in M$, is the unit vector field tangent to the normal geodesic between x and the pole pointing away from the pole. The Hessian of r describes the change of the Riemannian tensor in the radial directions, while the change of the volume form in the radial direction is associated to the Laplacian of r. More precisely we have:

$$L_{\partial r} g = 2\,\mathrm{Hess}(r), \qquad L_{\partial r} d\mathrm{vol} = \Delta r d\mathrm{vol}, \qquad \Delta r = \frac{n-1}{r} + dr(\nabla \log J),$$

indicating how the Jacobian determinant adjusts the speed of the convergence so that the Semi-classical bridge behaves exactly like the Euclidean Brownian bridge.

For the Hyperbolic space, Φ is bounded from the formula below,

$$\Phi = -\frac{1}{8}(n-1)^2c^2 + \frac{1}{8}(n-1)(n-3)\left(\frac{1}{r^2} - c^2\sinh^{-2}(rc)\right).$$

If (N, o) is a model space, its Riemannian metric in the geodesic polar coordinates takes the form $g = dr^2 + f^2(r)d\theta^2$, then on $N \setminus \{o\}$, $\mathrm{Hess}(r) = \frac{f'(r)}{f(r)}(g - dr \otimes dr)$. For the hyperbolic space of constant sectional curvature $-c^2$, the Riemannian metric is $g = dr^2 + (\frac{1}{c}\sinh(cr))^2 d\theta^2$. Also $\mathrm{Hess}(r^2) = 2dr\otimes dr + 2cr\coth(cr)(g - dr\otimes dr)$. Furthermore its Jacobian determinant is $J = (\frac{\sinh(cr)}{cr})^{(n-1)}$.

For manifolds of non-constant curvature we may use the Hessian comparison theorem. The radial curvature at a point $x \in M$ is the sectional curvature in a plane at T_xM containing the radial vector field ∂_r. Let us recall a comparison theorem from [25, R. E. Greene and H. Wu]: let (N, o) be another Riemannian manifolds with a pole which we denote by o. Suppose that $(\gamma(t), t \in [0, b])$ is a normal geodesic in M with the initial point y_0 and $(\gamma_2(t) : t \in [0, b])$ a normal geodesic in N from o. We suppose that the radial curvature at $\gamma_2(t)$ is greater than or equal to the radial curvatures at $\gamma(t)$. By this we mean the curvature operator $\mathcal{R}$ on M and $\mathcal{R}_2$ on N satisfy the relation $\langle \mathcal{R}(w, \dot\gamma)w, \dot\gamma\rangle \le \langle \mathcal{R}_2(w_2, \dot\gamma_2)w_2, \dot\gamma_2\rangle$ for any unit vectors $w \in ST_{\gamma(t)}M$ and $w_2 \in ST_{\gamma_2(t)}N$, satisfying the relation $\langle w, \partial_r\rangle = \langle w_2, \partial_r\rangle$ where ∂_r denotes the radial vector fields for both manifolds. Then for any nondecreasing function $\alpha : \mathbb{R}_+ \to \mathbb{R}$, $\mathrm{Hess}(\alpha\circ r_2)(\gamma_2(t)) \le \mathrm{Hess}(\alpha \circ r)(\gamma(t))$, where r_2 is the Riemannian distance function on N from o. For a precise Hessian formulas and Hessian estimates for heat kernels on manifolds with a pole please see [31, 1].

3 Conclusion and further questions

We have proved an integration by parts formula on $L^2(C_{x_0,y_0}, \nu)$ where ν is the probability measure induced by the Semi-classical bridge. A probability measure μ on the path space is said to satisfy the Poincaré inequality if there exists a constant c such that

$$\int \left(F - \int F d\mu\right)^2 d\mu \le c\int \left(|\nabla F|_{\mathcal{H}}\right)^2 d\mu$$

for all $F \in \mathcal{D}om(d)$ and the inner product on $\mathcal{H}$ can be defined either by stochastic parallel translation or by damped stochastic parallel translation.

Conjecture. A Poincaré inequality holds for the Semi-classical bridge measure on a class of Cartan-Hadamard manifolds. Of course it is reasonable to assume growth conditions on J, J^{-1} and suitable conditions on the range of the sectional curvature. Observe that the closure of d depends on the measure used.

We remark that, for the Brownian bridge measure the question whether the Poincaré inequality holds is not solved satisfactorily. The spectral gap inequality is known to hold for Gaussian measure on $\mathbb{R}^n$ by L. Gross [26], who also made a conjecture on its validity. The spectral gap inequality has been proven to hold on the hyperbolic space [7], for a class of radially symmetric Riemannian manifolds in [2] where asymptotics estimates for the spectral gap are also given. The latter is based on estimates in [1]. See also [3, 23, 5, 21, 17]. A counter example exists [13], see also [28, 24].

It is interesting to investigate the equivalence of the Semi-classical bridge, ν, and the Brownian bridge measure, ν_1, which might lead to new results /method on the spectral gap problem for the Brownian bridge measure. For the rest of this section we assume that the standard SDEs for the Brownian bridge and for the Semi-classical bridge are *conservative*. Let $(\tilde u_t)$ be defined as in (1.1), which is conservative, and let $\tilde x_s = \pi(\tilde u_s)$, the Semi-classical bridge with initial value x_0.

Remark 3.1. Since the Brownian bridges and the Semi-classical bridges are globally defined, then for any $t < 1$ the Radon Nikodym derivative is: $\frac{d\nu_1}{d\nu} = e^{N_t - \frac{1}{2}\langle N\rangle_t}$ where

$$N_t = \int_0^t \langle \nabla \log p_{1-s}(\tilde x_s, y_0) - \nabla \log k_{1-s}(\tilde x_s, y_0), \tilde u_s dB_s\rangle .$$

Observe that

(a) ν_1 is absolutely continuous w.r.t. ν on $[0,1]$ if and only if $G_t := e^{N_t - \frac{1}{2}\langle N\rangle_t}$ converges in L^1 as $t \to 1$.

(b) Suppose that $G_t \to \bar{G}$ in $L^1(\Omega)$, in which case the convergence holds also a.s.. If furthermore $\bar{G} > 0$ a.s., then the two measures are equivalent.

If $\{N_t - \frac{1}{2}\langle N\rangle_t, t \in [0,1)\}$ is L^1 bounded, it converges almost surely and so does G_t. If furthermore $\lim_{t\to 1}(N_t(\omega) - \frac{1}{2}\langle N\rangle_t(\omega)) \neq -\infty$, then $\bar{G}(\omega) > 0$. If furthermore $N_t - \frac{1}{2}\langle N\rangle_t$ converges in L^1 then its limit is finite almost surely and $\bar{G} > 0$. However, the convergence of G_t in L^1 is a difficult question, which cannot be casually replaced by "almost sure convergence plus $\bar{G} \in L^1$". The standard example is the 1-dimensional Brownian motion stopped when it hits 1.

For the invertibility of $\frac{d\nu_2}{d\nu_1}$, where ν_1, ν_2 are measures on a topological group, quasi-invariant under the action of a subgroup whose action is ergodic for ν_1, see [4]. The authors used this to prove the equivalence of the heat kernel measure, ν_2, and the Brownian bridge measure, ν_1, on a loop space over a simply connected compact Lie group. The quasi-invariance of ν_1, ν_2 were proved in [36] and [10] respectively, the ergodicity of ν_1 in [27], and the fact that $\nu_2 < \nu_1$ in [11]. In the latter a characterisation for the heat kernel measure by a potential from [5] is a key. We, on the other hand, will continue to follow the outline in the early remark. For a Riemannian metric of the form $dr^2 + f(r)^2 d\theta^2$ on $\mathbb{R}^n$, suppose that $\ln J$ has bounded derivatives of all orders, and some other conditions, Aida [1] proved that $\sup |\nabla \log p_t(x,y_0) - \frac{1}{t}\exp_{y_0}^{-1}(x)|$ is uniformly bounded on $[0,1) \times M$, c.f. [25]. Observe that $\nabla \log k_{1-t}(x,y_0) = -\frac{\nabla r^2}{2t} - \frac{1}{2}\nabla \log J$ and hence the assumptions below are reasonable.

Lemma 3.2. *Let C_1, C_2, C_3, and $\delta < 1$ be positive constants s.t. $C_1 + \frac{C_2}{1-\delta} \le \frac{1}{4}$. Suppose that $|\nabla \log \frac{p_{1-t}(x,y_0)}{k_{1-t}(x,y_0)}|^2 \le C_1 r^2(x,y_0) + C_2 \frac{r^2(x,y_0)}{(1-t)^\delta} + C_3$ for all $t < 1$. Then the Brownian bridge measure and the Semi-classical bridge measure are equivalent.*

Proof. Since $r(\tilde{x}_s, y_0)$ is the n-dimensional Bessel bridge, $\mathbf{E} e^{c \sup_{s\in[0,1]} r^2(\tilde{x}_s, y_0)}$ is finite for any $c < \frac{1}{8}$. We simply use the standard representation of the Brownian bridge by a Brownian motion (B_t) from which: $r(\tilde{x}_s, y_0) \le (1-s) r(x_0, y_0) + |B_s - sB_1|$. Since $\langle N, N\rangle_t = \left|\nabla \log \frac{p_{1-s}(\cdot,y_0)}{k_{1-s}(\cdot,y_0)}(\tilde{x}_s)\right|^2$, the following quantity is finite:

$$\sup_{t<1} \mathbf{E} e^{\frac{1}{2}\langle N,N\rangle_t} \le e^{C_3} \mathbf{E} e^{(\frac{1}{2}C_1 + \frac{1}{2}\frac{C_2}{1-\delta}) \sup_{s\in[0,1]} r^2(\tilde{x}_s, y_0)}.$$

By Novikov's criterion, G_t converges in L^1. Since $\{N_t - \frac{1}{2}\langle N,N\rangle_t, t \in [0,1)\}$ is L^2 bounded, it converges in L^2 and so has a finite limit. Thus $\lim_{t\to 1} G_t \neq 0$. □

By a theorem in [15], the elementary formula

$$p_t(x_0,y_0) = k_t(x_0,y_0) \mathbf{E} e^{\int_0^t \Phi(\tilde{x}_s) ds}$$

holds if Φ is bounded from below and more generally if $\mathbf{E}\left[e^{\int_0^t \Phi(\tilde{x}_s)} ds\right] < \infty$, [31]. The latter holds if $\Phi(x) \le C_1 + C_2 r^2(x,y_0)$ where $C_2 < \frac{1}{8}$. Let $\tilde{W}_s$ denote the solution to the equation

$$\frac{D}{dt}\tilde{W}_t = -\frac{1}{2}\mathrm{Ric}^\sharp(\tilde{W}_t) - \frac{1}{2}\,\mathrm{Hess}(\log J)(\tilde{W}_t) - \frac{\mathrm{Hess}\, r^2(\cdot,y_0)(\tilde{W}_t)}{2(1-t)}$$

with initial value the identity. Set

$$\alpha(t,x) = \sup_{|v|=1, v\in T_xM} \left\{-\mathrm{Ric}_x(v,v) - \mathrm{Hess}(\log J)(v,v) - \frac{\mathrm{Hess}\, r^2(\cdot,y_0)(v,v)}{(1-t)}\right\},$$

then $|\tilde{W}_t|$ is controlled by lower bounds on $\alpha(t,x)$. For the hyperbolic space, the Hessian of the distance square is given by the explicit formula $\mathrm{Hess}(r^2) = 2dr\otimes dr + 2cr\coth(cr)(g - dr\otimes dr)$. Suppose that we may differentiate $\log \mathbf{E} e^{\int_0^t \Phi(\tilde{x}_s)ds}$ with respect to the initial point, then

$$d\log \mathbf{E} e^{\int_0^t \Phi(\tilde{x}_s)ds}(v) = \frac{\mathbf{E}\left[e^{\int_0^t \Phi(\tilde{x}_s)ds}\int_0^t d\Phi(\tilde{W}_s(v))]ds\right]}{\mathbf{E} e^{\int_0^t \Phi(\tilde{x}_s)ds}}. \tag{3.1}$$

See [1] for this. Suppose that for each t, $\alpha(t,\cdot)$ is bounded from below we expect that

$$\mathbf{E}\left[df(\tilde{W}_t(v_0))\right] = d(\mathbf{E} f(\tilde{x}_t))(v_0), \quad x_0 \in M,\ v_0 \in T_{x_0}M,$$

for every function $f \in BC^1$, see e.g. [33, 31]. If $\Phi \in BC^1$ then equation (3.1) should follow by discrete time approximation and by induction.

Proposition 3.3. *Suppose that Φ is bounded, then $\frac{p_t(x_0,y_0)}{k_t(x_0,y_0)}$ is bounded. Suppose that furthermore (3.1) holds. Then for any $v \in T_{x_0}M$, where ∇ denotes differentiation w.r.t. the first space variable,*

$$\left|\nabla \log \frac{p_{1-t}(x_0,y_0)}{k_{1-t}(x_0,y_0)}\right|^2 \le c\left|\mathbf{E}\int_0^t d\Phi(\tilde{W}_s(v))ds\right|^2 \le c\mathbf{E}\int_0^{1-t} e^{\int_0^s \alpha(r,\tilde{x}_r)dr}|\nabla\Phi(\tilde{x}_s)|^2 ds.$$

If there exist a constant C such that $|\nabla\Phi| \le C$ and $\sup_{(t,x)\in[0,1)\times M}\alpha(t,x) \le C$ (e.g. if r is convex and if $\mathrm{Ric} + \mathrm{Hess}(\log J) \ge -C$) then the Brownian bridge measure and the Semi-classical bridge measure are equivalent.

Proof. By the assumption, $p_{1-t}(x_0,y_0) = k_{1-t}(x_0,y_0)\mathbf{E}\left[e^{\int_0^{1-t}\Phi(\tilde{x}_s)ds}\right]$, the first statement follows trivially from the estimate $\frac{1}{\mathbf{E} e^{\int_0^{1-t}\Phi(\tilde{x}_s)ds}} \le \mathbf{E} e^{-\int_0^{1-t}\Phi(\tilde{x}_s)ds}$. We differentiate the elementary formula and conclude that there exists a constant c such that for any $v \in T_{x_0}M$ with $|v| = 1$,

$$\begin{aligned}\left|\nabla \log \frac{p_{1-t}(x_0,y_0)}{k_{1-t}(x_0,y_0)}\right|^2 &= \left|\frac{\mathbf{E}\left[e^{\int_0^{1-t}\Phi(\tilde{x}_s)ds}\int_0^{1-t} d\Phi(\tilde{W}_s(v))]ds\right]}{\mathbf{E} e^{\int_0^{1-t}\Phi(\tilde{F}_s(x))ds}}\right|^2 \\ &\le c\mathbf{E}\left|\int_0^{1-t} d\Phi(\tilde{W}_s(v))ds\right|^2.\end{aligned}$$

This follows from Cauchy-Schwartz's inequality and and the boundedness of $|\Phi|$. Since

$$\frac{d}{dt}|\tilde{W}_t|^2 = -(\mathrm{Ric} + \mathrm{Hess}\log J)(\tilde{W}_t, \tilde{W}_t) - \frac{\mathrm{Hess}\, r^2(\tilde{W}_t,\tilde{W}_t)}{1-t},$$

we see that $|\tilde{W}_t|^2 \le e^{\int_0^t \alpha(s,\tilde{x}_s)ds}$ and $\left|\nabla\log\frac{p_{1-t}(x_0,y_0)}{k_{1-t}(x_0,y_0)}\right|^2 \le c\mathbf{E}\left[\int_0^{1-t} e^{\int_0^s \alpha(r,\tilde{x}_r)dr}|\nabla\Phi(\tilde{x}_s)|^2 ds\right]$. The rest of the conclusion follows from Lemma 3.2. □

Finally we discuss the equivalence of the radial parts of the two bridge processes, especially on radially symmetric manifolds. Set

$$\partial_r \log J = \langle \nabla r(\cdot,y_0), \nabla\log J\rangle, \qquad \partial_r\nabla\log p_{1-t}(\cdot,y_0) = \langle\nabla\log p_{1-t}(\cdot,y_0), \nabla r(\cdot,y_0)\rangle.$$

Proposition 3.4. *Suppose that $\partial_r\log J$ and $\partial_r\nabla\log p_{1-t}(\cdot,y_0)$ are functions of $r(\cdot,y_0)$. Let $\delta < \frac{1}{2}$ and c_1,c_2,c_3 be positive constants such that $c_1 + \frac{c_2}{1-2\delta} \le \frac{1}{4}$. Suppose that*

$$\left|\frac{r(\cdot,y_0)}{1-t} + \partial_r\nabla\log p_{1-t}(\cdot,y_0) - \frac{1}{2}\partial_r\log J\right|^2 \le c_1 r^2(\cdot,y_0) + c_2\frac{r(\cdot,y_0)}{(1-t)^\delta} + c_3.$$

Then the radial parts of the Semi-classical bridge and that of the Brownian bridge are equivalent.

Proof. The radial process, $\tilde{r}_t = (\tilde{x}_t, y_0)$, of the Semi-classical bridge satisfies the SDE

$$d\tilde{r}_t = d\beta_t + \frac{1}{2}\frac{n-1}{\tilde{r}_t}dt - \frac{\tilde{r}_t}{1-t}dt,$$

where β_t is a 1-dimensional Brownian motion. Since $\Delta r = \frac{n-1}{r} - \nabla \log J$, the radial process of the Brownian bridge satisfies

$$dr_t = d\beta_t + \frac{1}{2}\frac{n-1}{r_t}dt - \frac{1}{2}\partial_r \nabla \log J dt + \partial_r \log p_{1-t}(\cdot, y_0)dt.$$

Thus (r_s) is absolutely continuous with respect to $(\tilde{r}_s)$ on $[0,t]$ where $t<1$ with Radon Nikodym derivative the exponential martingale of

$$N_t = \int_0^t \left(-\frac{1}{2}\partial_r \log J + \partial_r \log p_{1-s}(\cdot, y_0) + \frac{\tilde{r}_s}{1-s} \right) d\beta_s.$$

Observe that

$$\begin{aligned}\langle N, N\rangle_t &= \int_0^t \left| -\frac{1}{2}\partial_r \log J + \partial_r \log p_{1-s}(\cdot, y_0) + \frac{\tilde{r}_s}{1-s} \right|^2 ds \\ &\le \int_0^1 \left(c_1 \tilde{r}_s^2 + c_2 \frac{\tilde{r}_s}{(1-s)^\delta} + c_3 \right) ds.\end{aligned}$$

Then by the Cauchy-Schwartz inequality applied to the middle term, the exponential integrability of the Bessel bridge, and Novikov's criterion, both $e^{N_t - \frac{1}{2}\langle N,N\rangle_t}$ and $(N_t)^2$ are uniformly integrable. In particular the two measures are equivalent. □

We conclude the discussion on the equivalence problem with the following observation: the probability distribution of the equation $dz_t = dB_t - c\frac{z_t}{1-t}dt$, where $c \neq 1$, is singular with respect to that of the classical Brownian bridge measure on $[0,1]$. This and a comparison of their respective Cameron-Martin spaces are proved in [32], where generalised Brownian bridges are studied.

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