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A functional limit theorem for excited random walks

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Abstract

We consider the limit behavior of an excited random walk (ERW), i.e., a random walk whose transition probabilities depend on the number of times the walk has visited to the current state. We prove that an ERW being naturally scaled converges in distribution to an excited Brownian motion that satisfies an SDE, where the drift of the unknown process depends on its local time. Similar result was obtained by Raimond and Schapira, their proof was based on the Ray-Knight type theorems. We propose a new method based on a study of the Radon-Nikodym density of the ERW distribution with respect to the distribution of a symmetric random walk.

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1 Introduction and results

Let $\{X(k), k \geq 0\}$ be a sequence of $\mathbb{Z}$-valued random variables such that $|X(k+1) - X(k)| = 1, k \geq 0$. Denote by $\mathcal{F}_n := \sigma(X(0), X(1), \ldots, X(n))$ the filtration generated by $\{X(k)\}$.

Definition 1.1. *A random walk (RW) $\{X(k)\}$ is called an excited random walk (ERW) associated to a (possibly) random sequence $\{\varepsilon_i, i \geq 0\} \subset (-1, 1)$ if*

$$\mathrm{P}\left(X(k+1) - X(k) = 1 | \mathcal{F}_k\right) = 1 - \mathrm{P}\left(X(k+1) - X(k) = -1 | \mathcal{F}_k\right) = p_i, \tag{1.1}$$

where $i = |\{j \leq k \;:\; X(j) = X(k)\}|,\; p_i = \frac{1}{2}(1 + \varepsilon_i)$.

Note that $\{X(k)\}$ is not a Markov chain, generally, and the study of traditional topics of the theory of stochastic processes such as recurrence, invariance principles, etc., is a non-trivial one for ERW. It demands new ideas and approaches, see for example [1, 5, 9, 11, 13, 15, 18] and references therein.

It was proved by Raimond and Schapira [15] that if $\varepsilon_i = \varepsilon_i^{(n)} = \frac{1}{\sqrt{n}}\varphi(\frac{i}{\sqrt{n}})$, where φ is a bounded Lipschitz function, then the sequence of processes $\{X_n(t) := \frac{X^{(n)}([nt])}{\sqrt{n}},\; t \geq 0\}_{n\geq 1}$ converges in distribution in $D([0, \infty))$ to excited Brownian motion that is a solution to the following SDE

$$dY(t) = \varphi(L_Y(t, Y_t))dt + dW(t), \tag{1.2}$$

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where W is a Wiener process, $L_Y(t,x)$ is the local time of Y at x up to time t. Note that the solution of (1.2) is not a Markov process also. Construction and properties of such type equations were considered in [12, 16]. For example, questions on recurrence, transience, positive speed for the solution to (1.2), and a central limit theorem for the solution were studied in [16].

Raimond and Schapira studied the process $\nu(i,k) = |\{j \le i \ : \ X(j) = k\}|$ as a function of the spatial coordinate k. It was proved that some scaling of ν taken at some Markov moments converges to a solution of a Bessel type SDE that appears in a spirit of the Ray-Knight theorem, see also [12]. Then the sequence $X(k)$ (and the process Y_t) were reconstructed from ν (and the local time L_Y, respectively). The corresponding proofs used the neat martingale technique. However the number of details they had to check was really large.

We propose a different method of proving a functional limit theorem for $\{X_n(t), t \in [0,T]\}_{n\ge 1}$. We study the Radon-Nikodym density of $\{X(k), 0 \le k \le n\}$ with respect to the distribution of a symmetric random walk (RW). Then we use Gikhman and Skorokhod result [7] on absolute continuity of the limit process together with the Skorokhod theorem on a single probability space, and invariance principle for the local times of random walks [3].

This method was used in [14] for studying the limit behavior of an RW with modifications upon each visit to 0 whose transition probabilities are defined as in (1.1), where

$$i = |\{j \le k \ : \ X(j) = 0\}|, \ p_i = (\frac{1}{2} + i\Delta) \wedge 1,$$

$\Delta > 0$ is a size of modifications. In this paper a functional limit theorem was proved in a series scheme, where $\Delta_n = cn^{-\alpha}, \ c > 0, \ \alpha > 0$.

It is natural to obtain the drift in integral form in the limit, see (1.2) or (2.2) below, because probabilities $p_i = p_i^{(n)}$ in (1.1) are small perturbations of $1/2$ having the order $n^{-1/2}$. If these perturbations were independent of n, then the limit process might be different from (1.2). For example, assume that there are few "cookies" at each state. At the instant of the i-th visit of the current state the process "eats" the i-th cookie and changes its transition probability to p_i. If the total impact of cookies at each state is "large", then a process is transient and may have a positive speed [1, 8]. If the impact of cookies is "small", for example, if the average total excitation per site $\alpha = \sum_i (2p_i - 1)$ lies in $(-1,1)$, then the natural scaling of ERW may converge to a Brownian motion perturbed at its maximum and minimum [5, 6]. The equation for the limit process is the following

$$Y(t) = \alpha \max_{s\in[0,t]} Y(s) - \alpha \min_{s\in[0,t]} Y(s) + W(t), t \ge 0.$$

Existence of the path-wise solution to this equation with $\alpha \in (-1,1)$ was proved in [4].

2 Main result and proofs

Let $\omega = \{\omega_k\}$ be a fixed sequence of numbers, $\varphi : \mathbb{R}^4 \to \mathbb{R}$ be a bounded measurable function, $n \in \mathbb{N}$.

Set $X^{(n)}(0) = 0$. Define an integer-valued random sequence $\{X^{(n)}(k), k \ge 0\}$ with a unit jump and the distribution defined by

$$\mathrm{P}_\omega\left(X^{(n)}(k+1) - X^{(n)}(k) = 1 | \mathcal{F}_k^{(n)}\right) = 1 - \mathrm{P}_\omega\left(X^{(n)}(k+1) - X^{(n)}(k) = -1 | \mathcal{F}_k^{(n)}\right) = p_{i,k}^{(n)}, \tag{2.1}$$

where

$$\mathcal{F}_k^{(n)} := \sigma(X^{(n)}(0), X^{(n)}(1), \dots, X^{(n)}(k)),\ i = |\{j \le k\ :\ X^{(n)}(j) = X^{(n)}(k)\}|,$$

$$p_{i,k}^{(n)} = \frac{1}{2}(1 + \varepsilon_{k,X^{(n)}(k),i}^{(n)}),\ \varepsilon_{k,x,i}^{(n)} = n^{-1/2}\varphi(\frac{k}{n}, \frac{x}{\sqrt{n}}, \frac{i}{\sqrt{n}}, \omega_k).$$

We will only consider n such that $n^{-1/2}\|\varphi\|_\infty < 1$.

Equation (2.1) means that for any $k \in \mathbb{N}$ and any $i_0 = 0, i_1, \dots, i_k$ such that $|i_{j+1} - i_j| = 1$ we have

$$\mathrm{P}_\omega(X_1^{(n)} = i_1, X_2^{(n)} = i_2, \dots, X_k^{(n)} = i_k) = \Pi_{l=0}^{k-1}\left(p_{i_l,l}^{(n)}\mathbb{1}_{i_{l+1}-i_l=1} + (1 - p_{i_l,l}^{(n)})\mathbb{1}_{i_{l+1}-i_l=-1}\right),$$

where $p_{il}^{(n)}$ are defined above.

Let now $\{\omega_k\}$ be a stationary ergodic sequence of random variables defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. For a fixed realization of $\omega = \{\omega_k\}$ the measure P_ω is called the quenched distribution of the ERW $\{X^{(n)}(k), k \ge 0\}$ with environment ω.

The annealed, or averaged, probability is defined by averaging of P_ω over the environment, i.e.,

$$\mathrm{P}(\cdot) := \mathbb{E}(\mathrm{P}_\omega(\cdot)) = \int_\Omega \mathrm{P}_\omega(\cdot)\mathbb{P}(d\omega).$$

Set $X_n(t) = \frac{X^{(n)}([nt])}{\sqrt{n}},\ n \in \mathbb{N},\ t \ge 0.$

Let $D([0,\infty))$ be the space of cadlag functions equipped with the Skorokhod J_1 topology, see [2].

Theorem 2.1. *Assume that the function* $\varphi : [0,\infty) \times \mathbb{R} \times [0,\infty) \times \mathbb{R} \to \mathbb{R}$ *is bounded and uniformly continuous. Then the sequence* $\{X_n(\cdot),\ n \ge 1\}$ *converges in distribution in* $D([0,\infty))$ *with respect to almost every quenched measure* P_ω*, and also with respect to the averaged measure* P*, to a solution of the SDE*

$$Y_t = \int_0^t \bar{\varphi}(s, Y_s, L_Y(s, Y_s))ds + W(t),\ t \ge 0, \tag{2.2}$$

where $\bar{\varphi}(t,x,l) = \mathrm{E}\varphi(t,x,l,\omega_k)$, W *is a Wiener process.*

Remark 2.2. There is a unique weak solution to (2.2) by Girsanov's theorem (see [10, Chapter IV §4] for the general stochastic equations or [12, §3] for a Brownian motion with a local time drift).

Proof of Theorem 2.1. In order to explain the idea of the proof and to avoid cumbersome calculations, at first we prove the theorem for φ that depends only on the first three of its coordinates, i.e., $\varphi(t,x,l,\omega) = \varphi(t,x,l)$. Then we explain how to handle the general case.

Denote by $\{S(k), k \ge 0\}$ a symmetric random walk, $S(k) = \xi_1 + \dots + \xi_k,\ S(0) = 0$, where $\{\xi_i\}$ are i.i.d., $\mathrm{P}(\xi_i = \pm 1) = 1/2$.

Let $P_{X^{(n)}}$ be a distribution of $\{X^{(n)}(k)\}_{k=0}^n$, $P_{S^{(n)}}$ be a distribution of $\{S(k)\}_{k=0}^n$,

Then $P_{X^{(n)}} \ll P_{S^{(n)}}$ and the Radon-Nikodym density equals:

$$\forall i_0 = 0, i_1, ..., i_n \in \mathbb{Z},\ |i_{k+1} - i_k| = 1,$$

$$\frac{dP_{X^{(n)}}}{dP_{S^{(n)}}}(i_0, i_1, ..., i_n) = \prod_{k=0}^{n-1}\frac{\frac{1}{2}(1 + \varepsilon_{k,i_k,l(k,i_k)}^{(n)}\mathbb{1}_{i_{k+1}-i_k=1} - \varepsilon_{k,i_k,l(k,i_k)}^{(n)}\mathbb{1}_{i_{k+1}-i_k=-1})}{\frac{1}{2}} = \tag{2.3}$$

$$\prod_{k=0}^{n-1}(1 + \varepsilon_{k,i_k,l(k,i_k)}^{(n)}\mathbb{1}_{i_{k+1}-i_k=1} - \varepsilon_{k,i_k,l(k,i_k)}^{(n)}\mathbb{1}_{i_{k+1}-i_k=-1}) =$$

$$\prod_{k=0}^{n-1}\left(1+\frac{1}{\sqrt{n}}\varphi(\frac{k}{n},\frac{i_k}{\sqrt{n}},\frac{l(k,i_k)}{\sqrt{n}})\mathbb{1}_{i_{k+1}-i_k=1}-\frac{1}{\sqrt{n}}\varphi(\frac{k}{n},\frac{i_k}{\sqrt{n}},\frac{l(k,i_k)}{\sqrt{n}})\mathbb{1}_{i_{k+1}-i_k=-1}\right)=$$

$$\prod_{k=0}^{n-1}\left(1+\frac{1}{\sqrt{n}}\varphi(\frac{k}{n},\frac{i_k}{\sqrt{n}},\frac{l(k,i_k)}{\sqrt{n}})(i_{k+1}-i_k)\right),$$

where $l(k,i)=|\{j\leq k\,:\,X^{(n)}(j)=i\}|$.

Hence

$$\frac{dP_{X^{(n)}}}{dP_{S^{(n)}}}(S(0),S(1),...,S(n))=$$

$$\prod_{k=0}^{n-1}\left(1+\frac{1}{\sqrt{n}}\varphi(\frac{k}{n},\frac{S(k)}{\sqrt{n}},\frac{\nu(k,S(k))}{\sqrt{n}})\xi_{k+1}\right), \tag{2.4}$$

where $\nu(k,i)=|\{j\leq k\,:\,S(j)=i\}|$.

Lemma 2.3. *Let $\{X^n,n\geq 1\}$ and $\{Y^n,n\geq 1\}$ be sequences of random elements given on the same probability space and taking values in a complete separable metric space $\mathcal{E}$.*

Assume that

1) $Y_n\overset{\mathrm{P}}{\to}Y_0, n\to\infty$;

2) for each $n\geq 1$ we have the absolute continuity of the distributions

$$P_{X_n}\ll P_{Y_n};$$

3) the sequence $\{\rho_n(Y_n),n\geq 1\}$ converges in probability to a random variable p, where $\rho_n=\frac{dP_{X_n}}{dP_{Y_n}}$ is the Radon-Nikodym density;

4) $\mathrm{E}p=1$.

Then the sequence of distributions $\{P_{X_n}\}$ converges weakly as $n\to\infty$ to the probability measure $\mathrm{E}(p\,|\,Y_0=y)P_{Y_0}(dy)$.

The idea of the proof of the lemma is due to Gikhman and Skorokhod [7]. Since $\{\rho_n(Y_n),n\geq 1\}$ are non-negative random variables with $\mathrm{E}\rho_n(Y_n)=1$, the condition $\mathrm{E}p=1$ yields the uniform integrability of $\{\rho_n(Y_n),n\geq 1\}$. The following calculation implies the conclusion of Lemma 2.3

$$\forall f\in C_b(E):\quad \lim_{n\to\infty}\int_E f\,dP_{X_n}=\lim_{n\to\infty}\mathrm{E}f(X_n)=\lim_{n\to\infty}\mathrm{E}f(Y_n)\rho_n(Y_n)=\mathrm{E}f(Y_0)p=$$

$$\mathrm{E}\left(f(Y_0)\,\mathrm{E}(p\,|\,Y_0)\right)=\int_E f(y)\mathrm{E}(p\,|\,Y_0=y)P_{Y_0}(dy). \tag{2.5}$$

Let us continue the proof of Theorem 2.1. It is sufficient to prove convergence in distribution $\frac{X^{(n)}([n\cdot])}{\sqrt{n}}\Rightarrow Y$ in $D([0,1])$.

We need the following invariance principle for RWs and the local times of RWs.

Theorem 2.4. *[3] There is a probability space and copies $\{S^{(n)}(k),k=0,...,n\}\overset{d}{=}\{S(k),k=0,...,n\}$ defined on this space, and a Wiener process $W(t),t\in[0,1]$, such that*

$$\lim_{n\to\infty}\sup_{t\in[0,1]}|\frac{S^{(n)}([nt])}{\sqrt{n}}-W(t)|=0, \tag{2.6}$$

$$\lim_{n\to\infty}\sup_{t\in[0,1]}\sup_{x\in\mathbb{R}}|\frac{\nu^{(n)}([nt],[x\sqrt{n}])}{\sqrt{n}}-L_W(t,x)|=0, \tag{2.7}$$

with probability 1, where $\nu^{(n)}(k,i)=|\{j\leq k\,:\,S^{(n)}(j)=i\}|$, L_W is the local time of the Wiener process (we consider a modification of L_W that is continuous in t,x).

Let us apply Lemma 2.3, where

$$X_n = X_n(t) = \frac{X^{(n)}([nt])}{\sqrt{n}}, \;\; Y_n = S_n(t) = \frac{S^{(n)}([nt])}{\sqrt{n}}, \; t \in [0,1].$$

It follows from (2.4) that for all sufficiently large n

$$\log \frac{dP_{X_n}}{dP_{S_n}}(S_n) =$$

$$\sum_{k=0}^{n-1} \log \left(1 + \frac{1}{\sqrt{n}} \varphi(\frac{k}{n}, \frac{S^{(n)}(k)}{\sqrt{n}}, \frac{\nu^{(n)}(k, S^{(n)}(k))}{\sqrt{n}}) \xi_{k+1}^{(n)}\right) =$$

$$\frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} \varphi(\frac{k}{n}, \frac{S^{(n)}(k)}{\sqrt{n}}, \frac{\nu^{(n)}(k, S^{(n)}(k))}{\sqrt{n}}) \xi_{k+1}^{(n)} - \frac{1}{2n} \sum_{k=0}^{n-1} \varphi^2(\frac{k}{n}, \frac{S^{(n)}(k)}{\sqrt{n}}, \frac{\nu^{(n)}(k, S^{(n)}(k))}{\sqrt{n}}) +$$

$$\frac{\theta}{3n^{3/2}} \sum_{k=0}^{n-1} |\varphi^3(\frac{k}{n}, \frac{S(k)}{\sqrt{n}}, \frac{\nu^{(n)}(k, S^{(n)}(k))}{\sqrt{n}})| = I_1^n + I_2^n + I_3^n,$$

where $\theta \in (-2, 1)$. Since φ is bounded, $\lim_{n\to\infty} I_3^n = 0$ for all ω.

By (2.6), (2.7), continuity of $L_W(t,x)$ in both of its arguments, and dominated convergence theorem we have convergence with probability 1

$$\lim_{n\to\infty} \frac{1}{2n} \sum_{k=0}^{n-1} \varphi^2(\frac{k}{n}, \frac{S^{(n)}(k)}{\sqrt{n}}, \frac{\nu^{(n)}(k, S^{(n)}(k))}{\sqrt{n}}) = \tag{2.8}$$

$$\lim_{n\to\infty} \frac{1}{2n} \int_0^1 \varphi^2(\frac{[nt]}{n}, \frac{S^{(n)}(\frac{[nt]}{n})}{\sqrt{n}}, \frac{\nu^{(n)}([nt], S^{(n)}([nt]))}{\sqrt{n}}) dt =$$

$$\frac{1}{2} \int_0^1 \varphi^2(t, W(t), L_W(t, W(t)) dt.$$

Lemma 2.5. *We have convergence in probability*

$$\frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} \varphi(\frac{k}{n}, \frac{S^{(n)}(k)}{\sqrt{n}}, \frac{\nu^{(n)}(k, S^{(n)}(k))}{\sqrt{n}}) \xi_{k+1}^{(n)} \xrightarrow{\mathrm{P}} \int_0^1 \varphi(t, W(t), L_W(t, W(t)) dW(t), \; n \to \infty.$$

Proof. We use the idea of Skorokhod [17, Chapter 3, §3]. Let $m \in \mathbb{N}$ be fixed. Then

$$|\frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} \varphi(\frac{k}{n}, \frac{S^{(n)}(k)}{\sqrt{n}}, \frac{\nu^{(n)}(k, S^{(n)}(k))}{\sqrt{n}}) \xi_{k+1}^{(n)} - \int_0^1 \varphi(t, W(t), L_W(t, W(t)) dW(t)| \le$$

$$\Bigg| \sum_{j=0}^{m-1} \sum_{[\frac{jn}{m}] \le k < [\frac{(j+1)n}{m}]} \Bigg(\varphi(\frac{k}{n}, \frac{S^{(n)}(k)}{\sqrt{n}}, \frac{\nu^{(n)}(k, S^{(n)}(k))}{\sqrt{n}}) -$$

$$\varphi(\frac{[jn/m]}{n}, \frac{S^{(n)}([\frac{jn}{m}])}{\sqrt{n}}, \frac{\nu^{(n)}([\frac{jn}{m}], S^{(n)}([\frac{jn}{m}]))}{\sqrt{n}}) \Bigg) \frac{\xi_{k+1}^n}{\sqrt{n}} \Bigg| +$$

$$\Bigg| \sum_{j=0}^{m-1} \Bigg(\varphi(\frac{[jn/m]}{n}, \frac{S^{(n)}([\frac{jn}{m}])}{\sqrt{n}}, \frac{\nu^{(n)}([\frac{jn}{m}], S^{(n)}([\frac{jn}{m}]))}{\sqrt{n}}) \Bigg)$$

$$\Bigg(\Big(\sum_{[\frac{jn}{m}] \le k < [\frac{(j+1)n}{m}]} \frac{\xi_{k+1}^n}{\sqrt{n}} \Big) - \Big(W(\frac{[(j+1)n/m]}{n}) - W(\frac{[jn/m]}{n}) \Big) \Bigg) \Bigg| +$$

$$\Big|\sum_{j=0}^{m-1}\Big(\varphi(\frac{[jn/m]}{n},\frac{S^{(n)}([\frac{jn}{m}])}{\sqrt{n}},\frac{\nu^{(n)}([\frac{jn}{m}],S^{(n)}([\frac{jn}{m}]))}{\sqrt{n}})-$$
$$\varphi(\frac{[jn/m]}{n},W(\frac{[jn/m]}{n}),L_W(\frac{[jn/m]}{n},W(\frac{[jn/m]}{n}))\Big)$$
$$\Big(W(\frac{[(j+1)n/m]}{n})-W(\frac{[jn/m]}{n})\Big)\Big|+$$

$$\Big|\sum_{j=0}^{m-1}\int_{\frac{[jn/m]}{n}}^{\frac{[(j+1)n/m]}{n}}\Big(\varphi(\frac{[jn/m]}{n},W(\frac{[jn/m]}{n}),L_W(\frac{[jn/m]}{n},W(\frac{[jn/m]}{n}))-$$
$$\varphi(t,W(t),L_W(t,W(t))\Big)dW(t)\Big|=$$
$$=I_1^{n,m}+I_2^{n,m}+I_3^{n,m}+I_4^{n,m}.$$

It follows from Theorem 2.4, Lebesgue dominated convergence theorem, and continuity of $L_W(t,x)$ in both of its arguments that

$$\lim_{n\to\infty}\mathrm{E}(I_1^{n,m})^2= \tag{2.9}$$

$$\lim_{n\to\infty}\frac{1}{n}\,\mathrm{E}\sum_{j=0}^{m-1}\sum_{[\frac{jn}{m}]\le k<[\frac{(j+1)n}{m}]}\Big(\varphi(\frac{k}{n},\frac{S^{(n)}(k)}{\sqrt{n}},\frac{\nu^{(n)}(k,S^{(n)}(k))}{\sqrt{n}})$$
$$-\varphi(\frac{[jn/m]}{n},\frac{S^{(n)}([\frac{jn}{m}])}{\sqrt{n}},\frac{\nu^{(n)}([\frac{jn}{m}],S^{(n)}([\frac{jn}{m}]))}{\sqrt{n}})\Big)^2=$$

$$\mathrm{E}\sum_{j=0}^{m-1}\int_{\frac{j}{m}}^{\frac{(j+1)}{m}}\Big(\varphi(t,W(t),L_W(t,W(t)))-\varphi(\frac{j}{m},W(\frac{j}{m}),L_W(\frac{j}{m},W(\frac{j}{m})))\Big)^2dt.$$
$$=\lim_{n\to\infty}\mathrm{E}(I_4^{m,n})^2$$

It follows from Theorem 2.4 that $\lim_{n\to\infty}I_2^{n,m}=\lim_{n\to\infty}I_3^{n,m}=0$ a.s. for each fixed m. So, by dominated convergence theorem

$$\forall m\ge 1\ \ \lim_{n\to\infty}\mathrm{E}(I_2^{n,m})^2=\lim_{n\to\infty}\mathrm{E}(I_3^{n,m})^2=0.$$

So for any $m\ge 1$

$$\limsup_{n\to\infty}\mathrm{E}\Bigg(\sum_{k=0}^{n-1}\frac{1}{\sqrt{n}}\sum_{k=0}^{n-1}\varphi(\frac{k}{n},\frac{S^{(n)}(k)}{\sqrt{n}},\frac{\nu^{(n)}(k,S^{(n)}(k))}{\sqrt{n}})\xi_{k+1}^{(n)}$$
$$-\int_0^1\varphi(t,W(t),L_W(t,W(t))dW(t)\Bigg)^2$$
$$\le 4\mathrm{E}\sum_{j=0}^{m-1}\int_{\frac{j}{m}}^{\frac{(j+1)}{m}}\Big(\varphi(t,W(t),L_W(t,W(t)))-\varphi(\frac{j}{m},W(\frac{j}{m}),L_W(\frac{j}{m},W(\frac{j}{m})))\Big)^2dt. \tag{2.10}$$

Letting $m\to\infty$ we complete the proof of the lemma. □

Since φ is bounded,

$$\mathrm{E}\exp\left\{\int_0^1 \varphi(t,W(t),L_W(t,W(t))dW(t) - \frac{1}{2}\int_0^1 \varphi^2(t,W(t),L_W(t,W(t))dt\right\} = 1 \quad (2.11)$$

by Novikov's theorem.

Therefore, by Lemma 2.3 we have convergence $X_n \Rightarrow Y$, where the distribution of Y has a density $\exp\{\int_0^1 \varphi(t,W(t),L_W(t,W(t))dW(t) - \frac{1}{2}\int_0^1 \varphi^2(t,W(t),L_W(t,W(t))dt\}$ with respect to the Wiener measure. Note that the local time and the integrals are measurable functions with respect to the σ-algebra generated by W. So there was no necessity for calculations of the conditional expectation in Lemma 2.3. By Girsanov's theorem, the process Y is a weak solution to the equation (2.2). The theorem is proved if $\varphi(t,x,l,\omega) = \varphi(t,x,l)$.

Consider the general case.

We prove the theorem if we find the corresponding limits in (2.8), (2.9), and (2.10), where the general summand is replaced by

$$\varphi(\frac{k}{n}, \frac{S^{(n)}(k)}{\sqrt{n}}, \frac{\nu^{(n)}(k,S^{(n)}(k))}{\sqrt{n}}, \omega_k),$$

and the sequence $\{\omega_k, k\geq 0\}$ is independent of $\{S^{(n)}(k)\}$.

The next statement completes the proof of the theorem.

Lemma 2.6. *Let $f:\mathbb{R}^{d+1}\to\mathbb{R}$ be a uniformly continuous and bounded function, $\{\eta_k, k\geq 0\}_{n\geq 1}$ be a stationary ergodic sequence of random variables, $\{\xi_n(t),\ t\geq 0\}_{n\geq 1}$ be a sequence of continuous $\mathbb{R}^d$-valued processes that locally uniformly converges to a process $\xi(t), t\geq 0$, almost surely,*

$$\forall T>0 \quad \lim_{n\to\infty}\sup_{t\in[0,T]}|\xi_n(t)-\xi(t)| = 0 \ \text{ a.s.}$$

Then we have the following almost sure convergence, i.e.,

$$\forall T>0 \quad \frac{1}{n}\sum_{k\leq nT} f(\xi_n(\frac{k}{n}),\eta_k) \stackrel{a.s.}{\to} \int_0^T \bar{f}(\xi(t))dt,\ n\to\infty,$$

where $\bar{f}(x) = \mathrm{E}f(x,\eta_k)$.

Proof. For simplicity let us prove the lemma for $T=1$ and $d=1$ only. The proof of the general case is the same.

Let $\varepsilon>0$ be arbitrary. Choose $\delta>0$ such that

$$\forall x,y,\ |x-y|<\delta,\ \ \forall z\ \ |f(x,z)-f(y,z)|<\varepsilon.$$

Let $M>0, N\in\mathbb{N}$ be such that

$$\mathrm{P}(\forall n\geq N \ \ \sup_{t\in[0,1]}|\xi_n(t)-\xi(t)|<\delta,\ \sup_{t\in[0,1]}|\xi_n(t)|\leq M) > 1-\varepsilon.$$

Set $\Omega_\varepsilon := \{\forall n\geq N \ \ \sup_{t\in[0,1]}|\xi_n(t)-\xi(t)|<\delta,\ \sup_{t\in[0,1]}|\xi_n(t)|\leq M\}$.

Then for each $\omega\in\Omega_\varepsilon$ and any $m>1/\delta$

$$\left|\frac{1}{n}\sum_{k=1}^{n} f(\xi_n(k/n),\eta_k) - \frac{1}{n}\sum_{j=0}^{m-1}\sum_{j/m\leq k/n<(j+1)/m} f(\xi_n(j/m),\eta_k)\right| < \varepsilon.$$

Observe that for each $\omega \in \Omega_\varepsilon$ and any $m > 1/\delta$

$$\frac{1}{n}\left|\sum_{j=0}^{m-1}\sum_{j/m\le k/n<(j+1)/m}\big(f(\xi_n(j/m),\eta_k)-\bar f(\xi_n(j/m))\big)\right|\le$$

$$\frac{2}{n}\sum_{j=0}^{m-1}\max_{|p|\le M\delta}\left|\sum_{j/m\le k/n<(j+1)/m}\big(f([p\delta],\eta_k)-\bar f([p\delta])\big)\right|+2\varepsilon.$$

Since f is bounded, by the ergodic theorem, for any fixed m we have the convergence

$$\frac{2}{n}\sum_{j=0}^{m-1}\max_{|p|\le M\delta}\left|\sum_{j/m\le k/n<(j+1)/m}\big(f([p\delta],\eta_k)-\bar f([p\delta])\big)\right|\overset{a.s.}{\to}0, n\to\infty.$$

It follows from the previous estimates that for a.a. $\omega \in \Omega_\varepsilon$ and all $m > 1/\delta$

$$\limsup_{n\to\infty}\left|\frac{1}{n}\sum_{k=1}^{n}f(\xi_n(k/n),\eta_k)-\int_0^1\bar f(\xi(t))dt\right|\le$$

$$\limsup_{n\to\infty}\left|\frac{1}{n}\sum_{k=1}^{n}f(\xi_n(k/n),\eta_k)-\frac{1}{n}\sum_{j=0}^{m-1}\sum_{j/m\le k/n<(j+1)/m}f(\xi_n(j/m),\eta_k)\right|+$$

$$\limsup_{n\to\infty}\left|\frac{1}{n}\sum_{j=0}^{m-1}\sum_{j/m\le k/n<(j+1)/m}f(\xi_n(j/m),\eta_k)-\frac{1}{n}\sum_{j=0}^{m-1}\sum_{j/m\le k/n<(j+1)/m}\bar f(\xi_n(j/m))\right|+$$

$$\limsup_{n\to\infty}\left|\frac{1}{n}\sum_{j=0}^{m-1}\sum_{j/m\le k/n<(j+1)/m}\bar f(\xi_n(j/m))-\frac{1}{n}\sum_{j=0}^{m-1}\sum_{j/m\le k/n<(j+1)/m}\bar f(\xi(j/m))\right|+$$

$$\limsup_{n\to\infty}\left|\frac{1}{n}\sum_{j=0}^{m-1}\sum_{j/m\le k/n<(j+1)/m}\bar f(\xi(j/m))-\int_0^T\bar f(\xi(t))dt\right|\le$$

$$4\varepsilon+\left|\frac{1}{m}\sum_{j=0}^{m-1}\bar f(\xi(j/m))-\int_0^1\bar f(\xi(t))dt\right|.$$

Passing $m\to\infty$ we get for a.a. $\omega\in\Omega_\varepsilon$

$$\limsup_{n\to\infty}\left|\frac{1}{n}\sum_{k=1}^{n}f(\xi_n(k/n),\eta_k)-\int_0^1\bar f(\xi(t))dt\right|\le 4\varepsilon.$$

Since $\varepsilon > 0$ were arbitrary, this completes the proof of Lemma 2.6 and hence Theorem 2.1. □ □

Remark 2.7. Assumption of boundedness and uniform continuity of φ may be relaxed.

We used boundedness of φ when we applied dominated convergence theorem in Lemma 2.5, and also when we applied Novikov's theorem to (2.11), and when we applied the ergodic theorem in Lemma 2.6.

Using truncation arguments it can be proved that the boundedness assumption on φ can be replaced by the linear growth condition with respect to the second argument. To guarantee that $p_{i,k}^{(n)}$ in (2.1) is a probability we have to define it by $p_{i,k}^{(n)} = (\frac{1}{2}(1+\varepsilon^{(n)}_{k,X^{(n)}(k),i,\omega_k}))\vee 0)\wedge 1$.

If φ depends only on the first three of its coordinates, i.e., $\varphi(t,x,l,\omega)=\varphi(t,x,l)$, we used only the continuity of φ, so the uniform continuity condition is not needed.

If ω_k are bounded random variables, the uniform continuity condition can be replaced by only continuity assumption too.

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