Survival asymptotics for branching random walks in IID environments

János Engländer [*]   Yuval Peres[†]

Abstract

We first study a model, introduced recently in [4], of a critical branching random walk in an IID random environment on the d-dimensional integer lattice. The walker performs critical (0-2) branching at a lattice point if and only if there is no 'obstacle' placed there. The obstacles appear at each site with probability $p \in [0, 1)$ independently of each other. We also consider a similar model, where the offspring distribution is subcritical.

Let S_n be the event of survival up to time n. We show that on a set of full $\mathbb{P}_p$-measure, as $n \to \infty$, $P^\omega(S_n) \sim 2/(qn)$ in the critical case, while this probability is asymptotically stretched exponential in the subcritical case.

Hence, the model exhibits 'self-averaging' in the critical case but not in the subcritical one. I.e., in the first case, the asymptotic tail behavior is the same as in a 'toy model' where space is removed, while in the second, the spatial survival probability is larger than in the corresponding toy model, suggesting spatial strategies.

A spine decomposition of the branching process along with known results on random walks are utilized.

Keywords: Branching random walk; catalytic branching; obstacles; critical branching; subcritical branching; random environment; spine; leftmost particle; change of measure; optimal survival strategy.
AMS MSC 2010: Primary: 60J80; Secondary: 60J10.
Submitted to ECP on May 6, 2017, final version accepted on May 15, 2017.

1 Introduction

1.1 Model

We first consider a model, introduced recently in [4], of a critical branching random walk $Z = \{Z_n\}_{n \geq 0}$ in an IID random environment on the d-dimensional integer lattice as follows. The environment is determined by placing *obstacles* on each site, with probability $0 \leq p < 1$, independently of each other. Given an environment, the initial single particle, located at the origin at $n = 0$, first moves according to a nearest neighbor simple random walk, and immediately afterwards, the following happens to it (see Fig. 1.1):

---

[*]Department of Mathematics, University of Colorado, Boulder, CO-80309-0395, USA
  E-mail: janos.englander@colorado.edu
[†]Microsoft Research, One Microsoft Way Redmond, WA 98052, USA
  E-mail: peres@microsoft.com

1. If there is no obstacle at the new location (we call it then a *vacant site*), the particle either vanishes or splits into two offspring particles, with equal probabilities.

2. If there is an obstacle at the new location (and so it is an *occupied site*), then nothing happens to the particle.

The new generation then follows the same rule in the next unit time interval and produces the third generation, etc.

Each lattice point receives an obstacle with probability p, independently.

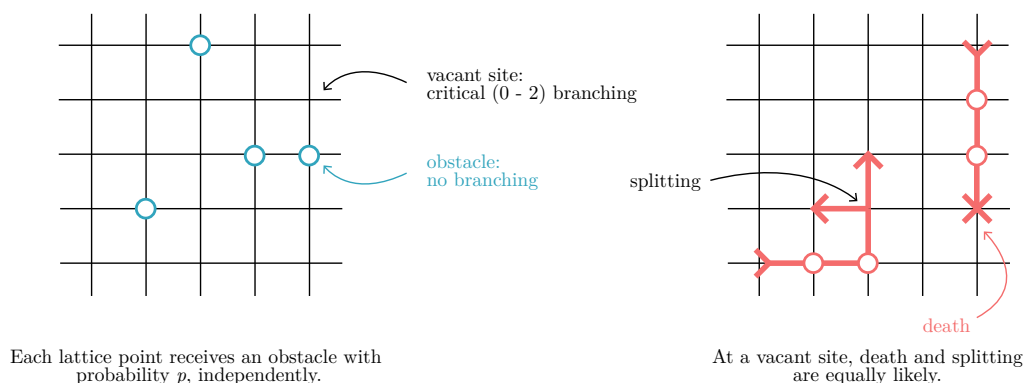At a vacant site, death and splitting are equally likely.

Figure 1.1:

We will also consider the same model when, on vacant sites, critical branching is replaced by a subcritical one with mean $\mu < 1$. (On occupied sites, as before, nothing happens to the particle.) In this latter case we will make the following standard assumption.

Assumption 1.1 ($L \log L$ condition)**.** *In the subcritical case let L denote the random number of offspring, with law $\mathcal{L}$. We assume that*

$$\sum_{k=1}^{\infty} \mathsf{Prob}(L = k)k \log k < \infty.$$

Let $p \in [0, 1)$. In the sequel, $K = K(\omega)$ will denote the set of lattice points with obstacles, $\mathbb{P}_p$ will denote the law of the obstacles and P^ω will denote the law of the BRW given the environment $\omega \in \Omega$. (Here Ω may be identified with $\{0, 1\}^{\mathbb{Z}^d}$.) Define also $\mathbf{P}_p := \mathbb{E}_p \otimes P^\omega$. We will say that a statement holds 'on a set of full $\mathbb{P}_p$-measure,' when it holds under P^ω for $\omega \in \Omega' \subset \Omega$ and $\mathbb{P}_p(\Omega') = 1$.

Finally, $\mathbb{I}_A$ will denote the indicator of the set A, and for $f, g : (0, \infty) \to (0, \infty)$, the notation $f \sim g$ will mean that $\lim_{t \to \infty} \frac{f(t)}{g(t)} = 1$.

1.2 Quenched survival; main result

We are interested in the asymptotic behavior, as time tends to infinity, of the probability that there are surviving particles, and in its possible dependence on the parameters. (Note that in the extreme case, when $p = 0$, the asymptotic behavior is well known.)

We consider the quenched case, and so, we can only talk about the *almost sure* asymptotics, as the probability P^ω itself depends on the realization of the environment.

Let S_n denote the event of survival up to $n \geq 0$. That is, $S_n = \{|Z_n| \geq 1\}$, where $|Z_n|$ is the population size at time n. Our main result will concern the a.s. asymptotic behavior of $P^\omega(S_n)$.

Theorem 1 (Quenched survival probability). *Let $d \geq 1$ and $p \in (0,1)$, and recall that $q := 1 - p$. Then the following holds on a set of full $\mathbb{P}_p$-measure, as $n \to \infty$.*

(i) Critical case:

$$P^\omega(S_n) \sim \frac{2}{qn}; \tag{1.1}$$

(ii) Subcritical case:

$$P^\omega(S_n) = \exp\left[\left(-C_{d,q} \cdot \frac{n}{(\log n)^{2/d}}\right)(1 + o(1))\right], \tag{1.2}$$

where $C_{d,q}$ is a positive constant that does not depend on the branching law.

1.3 Motivation; heuristic interpretation

Consider first the case of critical branching, and recall Kolmogorov's well-known result on the survival of (non-spatial) Galton-Watson processes [8, Formula 10.8]: for critical unit time branching with generating function φ, as $n \to \infty$,

$$\text{Prob (survival up to } n) \sim \frac{2}{n\varphi''(1)}. \tag{1.3}$$

As a particular case, let us consider now a non-spatial *toy model* as follows. Suppose that branching occurs with probability $q \in (0,1)$, and then it is critical binary, that is, consider the generating function

$$\varphi(z) = (1-q)z + \frac{1}{2}q(1+z^2).$$

It then follows that, as $n \to \infty$,

$$\text{Prob (survival up to } n) \sim \frac{2}{qn}. \tag{1.4}$$

Turning back to our spatial model (with critical branching), simulations suggested (see [4]) the *self averaging* property of the model: the asymptotics for the annealed and the quenched case are the same. In fact, this asymptotics is *the same as the one in (1.4)*, where $p = 1 - q$ is the probability that a site has a obstacle. In other words, despite our model being spatial, in an asymptotic sense, the parameter q simply plays the role of the branching probability of the above non-spatial toy model. To put it yet another way, q only introduces a 'time-change.'

In the present paper we would like to establish rigorous results concerning survival.

Our main result will demonstrate that while for critical branching, self-averaging indeed holds, this is not the case for subcritical branching.

For further motivation in mathematics and in mathematical biology, see [4]. For topics related to the quenched and annealed survival of a single particle among obstacles in a continuous setting, see the fundamental monograph [11]. Finally, we mention the excellent current monograph [10] on branching random walks, which also includes the spine method relevant to this paper.

A heuristic interpretation of Theorem 1 is as follows.

(i) Critical case: There is *nothing the 'motion component' could do* to increase the chance of survival, at least as far as the leading order term is concerned (as opposed to the single Brownian particle model in [11]).

Should $|Z_n|$ reduce to one, the probability of that particle staying in the region of obstacles is known to be much less than $\mathcal{O}(1/n)$. So the optimal strategy for this particle to survive would obviously not be an attempt to stay completely in that region; rather, survival will mostly be possible because of the potentially large family trees. Since $|Z|$

is a P^ω-martingale with unit mean for any $\omega \in \Omega$, survival probability is related to the population size:

$$P^\omega(S_n) = \left[E^\omega(|Z_n| \mid S_n) \right]^{-1}. \tag{1.5}$$

In fact, we suspect that on a set of full $\mathbb{P}_p$-measure, under $P^\omega(\cdot \mid S_n)$, the law of $\frac{|Z_n|}{n}$ converges to the exponential distribution with mean $q/2$. (Cf. Theorem C(ii) in [9].)

(ii) Subcritical case: Now the situation is very different, and a spatial strategy *does* make sense, since vacant sites are now 'more lethal.' Unlike in (i), the result now differs from what the toy model suggests, namely, that $\exists \lim_{n \to \infty} \frac{\mathrm{Prob}(\tilde{S}_n)}{\mu^n} > 0$ holds under the LlogL condition (Theorem B in [9]). In our spatial setting, the survival probability has thus improved!

Finally, we note that in [4], in the annealed case with critical branching, the second-order survival asymptotics has also been observed through simulations. Those simulation results[1] suggest that spatial survival strategies do exists, which are not detectable at the logarithmic scale but are visible at the second-order level.

2   Some preliminary results

In this section we present two simple statements concerning our branching random walk model which were proven in [4], and also some a priori bounds.

Lemma A (Monotonicity (Theorem 2.1 in [4]) and its proof). *Let $0 \le p < \widehat{p} \le 1$ and fix $n \ge 0$. Then*

$$\mathbf{P}_p(S_n) \le \mathbf{P}_{\widehat{p}}(S_n).$$

Also, for any $\omega \in \Omega$ and $n \ge 1$, one has $P^\omega(S_n) \ge P^*(S_n)$, where P^* corresponds to the $p = 0$ case.

Although [4] only handles the critical case, the same proof carries through for the subcritical case as well. The proof only uses the fact that if φ is the generating function of the offspring distribution, then $\varphi(z) \ge z$ on $[0,1]$. This remains the case for subcritical branching too, since $\varphi(1) = 1, \varphi'(1) < 1$ and φ is convex from above on the interval.

Lemma B (Extinction (Theorem 2.2 in [4])). *Let $0 \le p < 1$ and let A denote the event that the population survives forever. Then, for $\mathbb{P}_p$-almost every environment, $P^\omega(A) = 0$.*

Again, [4] only handles the critical case, but the same proof carries through for the subcritical case as well. (One then uses that the population size is a supermartingale, instead of a martingale.)

Lemma A yields the following a priori bounds.

Corollary 2 (A priori bounds). *Let $f : \mathbb{Z}_+ \to (0, \infty)$. Then the following holds.*
   (i) Critical case: *On a set of full $\mathbb{P}_p$-measure,*

$$\liminf_{n \to \infty} n P^\omega(S_n) \ge 2, \tag{2.1}$$

and

$$P^\omega(|Z_n| > f(n) \mid S_n) = \mathcal{O}\left(\frac{n}{f(n)} \right), \tag{2.2}$$

as $n \to \infty$.
   (ii) Subcritical case: *On a set of full $\mathbb{P}_p$-measure,*

$$\liminf_{n \to \infty} \mu^{-n} P^\omega(S_n) > 0. \tag{2.3}$$

---

[1]Simulation has indicated [4] that $\mathbf{P}_p(S_n) = \frac{2}{qn} + f(n)$, with $n^{2/3} f(n)$ tending to a positive constant.

Proof. (i) In the critical case, by comparing with the $p = 0$ (no obstacles) case, when survival is less likely, and for which the non-spatial result of Kolmogorov (1.3) is applicable, (2.1) follows by Lemma A. Using that $1 = E^\omega(|Z_n|) = P^\omega(S_n)E^\omega(|Z_n| \mid S_n)$, we infer that

$$\limsup_{n\to\infty} \frac{1}{n} E^\omega(|Z_n| \mid S_n) \leq 1/2.$$

Finally, use the Markov inequality to get (2.2).

(ii) In the subcritical case ($\mu < 1$), the proof is very similar, taking into account the well known result of Heathcote, Seneta and Vere-Jones (see Theorem B in [9]) that under the $L \log L$ condition, (2.3) holds with limit instead of $\liminf$ for $p = 0$. $\qquad\square$

3 Further preparation: size-biasing and spine in the critical case

Consider the critical case in this section. Given (1.5), the asymptotic relation under (1.1) is tantamount to

$$E^\omega(|Z_n| \mid S_n) \sim \frac{qn}{2}, \tag{3.1}$$

as $n \to \infty$. We will actually prove that (3.1) holds on a set of full $\mathbb{P}_p$-measure.

In the particular case when $q = 1$ ($p = 0$, no obstacles) and in a non-spatial setting, this has been shown in [9] (see formula (4.1) and its proof on p. 1132). We will show how to modify the proof in [9] for our case.

(In the subcritical case, we will also reduce the question to the study of the behavior of $E^\omega(|Z_n| \mid S_n)$ as $n \to \infty$.)

3.1 Left-right labeling

At every time of fission, randomly (and independently from everything else in the model) assign 'left' or 'right' labels to the two offspring. So, from now on, every time we write $P^\omega(\cdot \mid S_n)$, we will actually mean, with a slight abuse of notation, $P^\omega(\cdot \mid S_n)$, augmented with the choice of the labels; we will handle $P^\omega(\cdot \cap S_n)$ similarly. Ignoring space, and looking only at the genealogical tree, we say that at time n, a particle is 'to the left' of another one, if, tracing them back to their most recent common ancestor, the first particle is the descendant of the left particle right after the fission. Transitivity is easy to check and thus a total ordering of particles at time n is induced.

3.2 The size-biased critical branching random walk

Recall that if $\{p_k\}_{k\geq 0}$ is a probability distribution on the nonnegative integers with expectation $m \in (0, \infty)$, then the corresponding *size-biased distribution* is defined by $\widehat{p}_k := kp_k/m$ for $k \geq 1$. We will denote the size biased law obtained from $\mathcal{L}$ by $\widehat{\mathcal{L}}$.

Given the environment ω, the *size-biased critical branching random walk* with corresponding law $\widehat{P}^\omega$ is as follows. The motion component is unchanged, that is particles still perform symmetric random walk, however the branching does change according to these rules:

- The initial particle does not branch until the first time it steps on a vacant site, at which moment it splits into a random number offspring according to $\widehat{\mathcal{L}}$.

- One of the offspring is picked uniformly (independently from everything else) to be designated as the 'spine offspring.' The other offspring launch copies of the original branching random walk (with ω being translated according to the position of the site).

- Whenever the 'spine offspring' is situated next time at a vacant site, it splits into a random number offspring according to $\widehat{\mathcal{L}}$, and the above procedure is repeated, etc.

Definition 3 (Spine). The distinguished line of decent formed by the successive spine offspring will be called the *spine.*

Note the following.

(i) **(Survival)** Because of the size biasing, the new process is immortal $\widehat{P}^\omega$-a.s.

(ii) **(Martingale change of measure)** For any given ω, the law of the size-biased critical random walk satisfies that

$$\left.\frac{\mathrm{d}\widehat{P}^\omega}{\mathrm{d}P^\omega}\right|_{\mathcal{F}_n} = |Z_n|,$$

where $\{\mathcal{F}_n\}_{n\geq 0}$ is the natural filtration of the branching random walk, and the lefthand side is a Radon-Nikodym derivative on $\mathcal{F}_n$. This is a change of measure by the nonnegative, unit mean martingale $|Z|$. The proof is essentially the same as in [9]. Even though in that paper the setting is non-spatial, it is easy to check that the proof carries through in our case, because the mean offspring number is always one, irrespective of the site. (See p. 1128 in [9].)

In particular, when the critical law $\mathcal{L}$ is binary (either 0 or 2 offspring, with equal probabilities), the law $\widehat{\mathcal{L}}$ is deterministic, namely it is dyadic (that is, 2 offspring with probability one). In this case, the spine particle always splits into two at vacant sites.

In addition to $\widehat{P}^\omega$, we also define the law $\widehat{P}^\omega_*$ which is the distribution of the size-biased branching random walk, *augmented with* the designation of the spine within it. The corresponding, augmented filtration, $\{\mathcal{G}_n\}_{n\geq 0}$ is richer than $\{\mathcal{F}_n\}_{n\geq 0}$, as it now keeps track of the position of the spine as well.

The significance of the new law $\widehat{P}^\omega$ is as follows. Let us denote the spine's path up to n by $\{X_i\}_{0\leq i\leq n}$. Let

$$A_n := \{\text{The spine particle is the leftmost particle of } Z_n.\}.$$

Then, size biasing and conditioning on A_n has the combined effect of simply conditioning the process on survival up to n. That is, the distribution of Z restricted on $\{\mathcal{F}_n\}$ is the same under $P^\omega(\cdot \mid S_n)$ and under $\widehat{P}^\omega_*(\cdot \mid A_n)$. To see why this is true, let $C_{n,k} := \{|Z_n| = k\}$. One has for $F \in \mathcal{F}_n$ that

$$\widehat{P}^\omega_*(F \mid A_n) = \frac{\widehat{P}^\omega_*(F \cap A_n)}{\widehat{P}^\omega_*(A_n)} = \frac{\sum_{k\geq 1} \widehat{P}^\omega_*(F \cap A_n \cap C_{n,k})}{\sum_{k\geq 1} \widehat{P}^\omega_*(A_n \cap C_{n,k})}$$

$$= \frac{\sum_{k\geq 1} (1/k)\widehat{P}^\omega(F \cap C_{n,k})}{\sum_{k\geq 1} (1/k)\widehat{P}^\omega(C_{n,k})} = \frac{\sum_{k\geq 1} (1/k)E^\omega(|Z_n|; F \cap C_{n,k})}{\sum_{k\geq 1} (1/k)E^\omega(|Z_n|; C_{n,k})}$$

$$= \frac{\sum_{k\geq 1} P^\omega(F \cap C_{n,k})}{\sum_{k\geq 1} P^\omega(C_{n,k})} = \frac{P^\omega(F \cap S_n)}{P^\omega(S_n)} = P^\omega(F \mid S_n).$$

In particular, for $n \geq 1$,

$$E^\omega(|Z_n| \mid S_n) = \widehat{E}^\omega_*(|Z_n| \mid A_n). \tag{3.2}$$

Remark 4 (Optimal survival strategy). It is important to point out the (obvious) interpretation of the equation $P^\omega(\cdot \mid S_n) = \widehat{P}^\omega_*(\cdot \mid A_n)$: it gives the description of the 'optimal survival strategy.'

3.3 Frequency of vacant sites along the spine in the critical case

Let the branching be critical, and let

$$L_n := \sum_{i=1}^{n} \mathbb{I}_{\{X_i \in K^c\}}$$

denote the (random) amount of time spent by X (the spine) on vacant sites between times 1 and n.

Lemma 3.1 (Frequency of visiting vacant sites). *On a set of full $\mathbb{P}_p$-measure,*

$$\lim_{n \to \infty} \widehat{P}_*^{\omega} \left(\left| \frac{L_n}{n} - q \right| > \epsilon \right) = 0, \ \forall \epsilon > 0.$$

Proof. Let

$$F_{n,\epsilon} := \bigcup_{1 \leq i \leq |Z_n|} \left\{ \left| \frac{L_n^i}{n} - q \right| > \epsilon \right\} \in \mathcal{F}_n,$$

where L_n^i is defined similarly to L_n for $Z^{i,n}$, the ith particle in Z_n on S_n; we define $F_{n,\epsilon} := \emptyset$ on S_n^c. Then

$$\widehat{P}_*^{\omega} \left(\left| \frac{L_n}{n} - q \right| > \epsilon \right) = \sum_{k \geq 1} \widehat{P}_*^{\omega} \left(\left| \frac{L_n}{n} - q \right| > \epsilon \mid |Z_n| = k \right) \widehat{P}_*^{\omega} \left(|Z_n| = k \right)$$

$$= \sum_{k \geq 1} \widehat{P}_*^{\omega} \left(\left| \frac{L_n}{n} - q \right| > \epsilon \mid |Z_n| = k, A_n \right) \widehat{P}_*^{\omega} \left(|Z_n| = k \right)$$

$$\leq \sum_{k \geq 1} \widehat{P}_*^{\omega} \left(F_{n,\epsilon} \mid |Z_n| = k, A_n \right) \widehat{P}_*^{\omega} \left(|Z_n| = k \right),$$

where, in the second equality, we used the fact that given $|Z_n| = k$, the event A_n means that out of the k surviving particles, the spine is the 'leftmost' one,[2] and hence, given $|Z_n| = k$, A_n is independent of the event $\left\{ \left| \frac{L_n}{n} - q \right| > \epsilon \right\}$. (Of course, $|Z_n|$ itself is not at all independent of this last event.) Switching back to the original measure now, the righthand side equals

$$\sum_{k \geq 1} P^{\omega} \left(F_{n,\epsilon} \mid |Z_n| = k \right) k P^{\omega} \left(|Z_n| = k \right),$$

and by the union bound, this can be estimated from above by

$$\sum_{k \geq 1} k^2 Q^{\omega} \left(\left| \frac{T_n}{n} - q \right| > \epsilon \right) P^{\omega} \left(|Z_n| = k \right) = Q^{\omega} \left(\left| \frac{T_n}{n} - q \right| > \epsilon \right) E^{\omega}(|Z_n|^2),$$

where T_n denotes the time spent on vacant sites between times 1 and n by a simple random walk in the environment ω, starting at the origin, with corresponding probability Q^{ω}. We will use $\mathbb{Q}_p$ for the law of the environment.

Now, it is easy to show that $E^{\omega}(|Z_n|^2) = \mathtt{Var}^{\omega}(|Z_n|^2) + 1 \leq n + 1$, and so it is sufficient to check (the ridiculously crude bound) that $Q^{\omega} \left(\left| \frac{T_n}{n} - q \right| > \epsilon \right) = o(1/n)$ on a set of full measure.

Next, observe that it is in fact sufficient to verify the *upper tail* large deviations. Indeed, the lower tail large deviations for the time spent in K^c can be handled similarly, since they are exactly the upper tail large deviations for the time spent in K.

[2]Which notion has nothing to do with physical space.

The statement reduces to one about a d-dimensional *random walk in random scenery*.[3] We now have to consider a scenery such that it assigns the value 1 to each lattice point with probability q, and the value 0 otherwise (that is, the scenery is the indicator of vacancy). With a slight abuse of notation, we will still use Q^ω and $\mathbb{Q}_p$ for the corresponding laws.

Since it was easier to locate the corresponding annealed result in the literature (see also the remark at the end of the proof), we will use that one, and then show how one easily gets the quenched statement from the annealed one.

To this end, define the random variable $Y^* := Y - q$, where Y is the 'scenery variable,' that is, $Y = 1$ with probability q and $Y = 0$ otherwise. Then Y^* is centered, and defining

$$T_n^* := \sum_{k=1}^{n} Y^*(X_k),$$

one can apply[4] Theorem 1.3 in [5], yielding that for $\epsilon > 0$,

$$\limsup_{n \to \infty} n^{-\frac{d}{d+2}} \log(E_{\mathbb{Q}_p} \otimes Q^\omega)\left(\frac{T_n}{n} > q + \epsilon\right) =$$
$$\limsup_{n \to \infty} n^{-\frac{d}{d+2}} \log(E_{\mathbb{Q}_p} \otimes Q^\omega)\left(\frac{T_n^*}{n} > \epsilon\right) \le -C_\epsilon,$$

where C_ϵ is finite and positive, and is given in [5] in terms of a variational problem. We now easily obtain the quenched result too, since for any positive sequence $\{a_n\}_{n \ge 0}$, the Markov inequality yields

$$\mathbb{Q}_p\left(Q^\omega\left(\frac{T_n}{n} > q + \epsilon\right) > a_n\right) \le (a_n)^{-1}(E_{\mathbb{Q}_p} \otimes Q^\omega)\left(\frac{T_n}{n} > q + \epsilon\right).$$

Given that $(E_{\mathbb{Q}_p} \otimes Q^\omega)\left(\frac{T_n}{n} > q + \epsilon\right) \le \exp\left(-C_\epsilon n^{\frac{d}{d+2}}(1 + o(1))\right)$, to finish the proof, we can pick any sequence satisfying that

$$\sum_n a_n^{-1} \exp\left(-\frac{1}{2}C_\epsilon n^{\frac{d}{d+2}}\right) < \infty.$$

Then, by the Borel-Cantelli Lemma,

$$\mathbb{Q}_p\left(Q^\omega\left(\frac{T_n}{n} > q + \epsilon\right) > a_n \text{ occurs finitely often}\right) = 1.$$

Clearly, by picking, for example, $a_n := 1/n^{1+\delta}$, $\delta > 0$, we are done. $\qquad\square$

Remark 5. Regarding RWRS, we note that in Theorem 2.3 in [3] the *quenched* large deviations have been studied in a more continuous version, namely for a Brownian motion in a random scenery, where the scenery is constant on blocks in $\mathbb{R}^d$.

4 Proof of Theorem 1 – critical case

Our goal is to verify (3.1). To this end, note that the spine has a (nonnegative) number of 'left bushes' and a (nonnegative) number of 'right bushes' attached to it; each such bush is a branching tree itself. A 'left bush' ('right bush') is formed by particles which are to the left (right) of the spine particle at time n. It is clear that, under conditioning on A_n, each left bush dies out completely by time n (see Fig. 4.1 with $t = n$).

---

[3]Random walks in random scenery (RWRS) were first introduced, in dimension one, by Kesten-Spitzer and also by Borodin, in 1979. (See e.g. [6, 7].)

[4]A formal application of the result in [5] would require that $\epsilon \in \text{int}(\text{supp}(Y^*(0)))$, however this is only used in their lower bound. In fact, we believe that here the lower estimate holds as well, despite the scenery variable being atomic, but we do not need that.
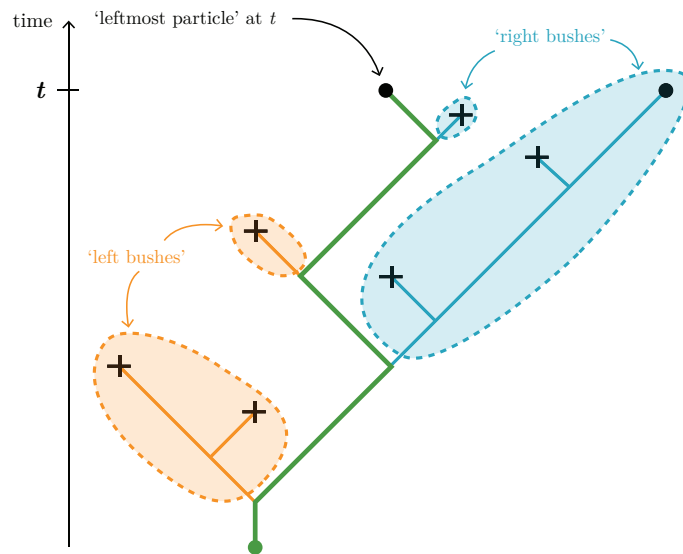
Under A_n, 'left bushes' are doomed to die out by t; 'right bushes' may survive.

Figure 4.1:

Because of (3.2), we are left with the task of showing that

$$\widehat{E}_*^\omega(|Z_n| \mid A_n) \sim \frac{qn}{2}, \text{ as } n \to \infty. \tag{4.1}$$

The proof of this statement is similar to the proof of (4.1) in [9] (with $\sigma^2 = 1$), except that, as we will see, now we also have to show that

$$\widehat{E}_*^\omega(\text{number of all bushes along the spine}) = qn(1 + o(1)). \tag{4.2}$$

The reason is that in [9], the spine particle branched at every unit time, which is not the case now. In our case, the spine $\{X_i\}_{1 \le n}$ splits into two at each vacant site and thus bushes are attached each time (larger than zero and smaller than n) when X is at a vacant site.

For $n \ge 1$ given, define the set of indices (times)

$$J_n := \{1 \le j \le n - 1 \mid X_j \in \mathbb{Z}^d \text{ is a vacant site}\};$$

then (4.2) can be written as $\widehat{E}_*^\omega(|J_n|) = qn(1 + o(1))$ (cf. equation (4.6) in the sequel). Furthermore, let $j \in J_n$ and

- $(LB)_j$ be the event that there is a left bush launched from the space-time point (X_j, j);
- $(RB)_j$ be the event that there is a right bush launched from (X_j, j);
- $(LBE)_j$ be the event that there is a left bush launched from X_j which becomes extinct by n;
- $A_{n,j} := (RB)_j \cup (LBE)_j$.

Then

$$A_n = \bigcap_{j \in J_n} A_{n,j}, \tag{4.3}$$

where the events in the intersection are independent under $\widehat{P}^\omega$. (See again Fig. 4.1.) Conditioning on A_n can be obtained by conditioning successively on $A_{n,j}$, $j \in J_n$.

For $j \in J_n$, let the random variable $R_{n,j}$ be the 'right-contribution' of the jth bush to $|Z_n|$. That is, $R_{n,j} = 0$ on $(LB)_j$, and on $(RB)_j$ it is the contribution of the right bush, stemming from (X_j, j), to $|Z_n|$. The 'left contribution' $S_{n,j}$ is defined similarly, and $Z_{n,j} := R_{n,j} + S_{n,j}$ is the total contribution. Note that $A_{n,j} = \{S_{n,j} = 0\}$.

Let $\{R'_{n,j}\}_{j \in J_n}$ be independent random variables under a law $\widetilde{Q}^\omega$ such that

$$\widetilde{Q}^\omega(R'_{n,j} \in \cdot) = \widehat{P}^\omega_*(R_{n,j} \in \cdot \mid A_{n,j}),$$

and let $\mathsf{Q}^\omega := \widehat{P}^\omega_* \times \widetilde{Q}^\omega$, with expectation E^ω_{Q}. Furthermore, let $R^*_{n,j} := \mathbb{I}_{A_{n,j}} R_{n,j} + \mathbb{I}_{A^c_{n,j}} R'_{n,j}$, and $R^*_n := \sum_{j \in J_n} R^*_{n,j}$. Then, for $j \in J_n$,

$$\widehat{P}^\omega_*(Z_{n,j} \in \cdot \mid A_{n,j}) = \widehat{P}^\omega_*(R_{n,j} + S_{n,j} \in \cdot \mid A_{n,j}) = \mathsf{Q}^\omega(R^*_{n,j} \in \cdot). \qquad (4.4)$$

(The $S_{n,j}$ term in the second probability has zero contribution.) Using (4.3) along with (4.4), it follows that

$$\frac{1}{n} \widehat{E}^\omega_* (|Z_n| \mid A_n) = \frac{1}{n} \widehat{E}^\omega_* \left(|Z_n| \mid \bigcap_{j \in J_n} A_{n,j} \right) =$$

$$\frac{1}{n} \widehat{E}^\omega_* \left(1 + \sum_{j \in J_n} Z_{n,j} \mid \bigcap_{j \in J_n} A_{n,j} \right) = E^\omega_{\mathsf{Q}} \left(\frac{1}{n} + \frac{1}{n} \sum_{j \in J_n} R^*_{n,j} \right) = \frac{1}{n} + E^\omega_{\mathsf{Q}} \left(\frac{R^*_n}{n} \right).$$

Hence, the desired assertion (4.1) will follow once we show that (on a set of full $\mathbb{P}_p$-measure)

$$\lim_{n \to \infty} E^\omega_{\mathsf{Q}} \left(\frac{R^*_n}{n} \right) = \frac{q}{2}.$$

Denoting $R_n := \sum_{j \in J_n} R_{n,j}$, the same proof as in [9] reveals that

$$\lim_{n \to \infty} E^\omega_{\mathsf{Q}} \frac{1}{n} |R_n - R^*_n| = 0. \qquad (4.5)$$

(The intuitive reason is that $A^c_{n,j} = \{S_{n,j} > 0\}$, while the probability of the survival of a bush tends to zero as the height of the bush tends to infinity; thus $A^c_{n,j}$ only occurs rarely. The fact that now we do not have a bush launched at every position of the spine makes the estimated term even smaller.)

In view of (4.5), it is sufficient to show that $\lim_{n \to \infty} \widehat{E}^\omega_* (R_n/n) = q/2$. Since the 'branching' can be considered (degenerate) critical at an obstacle and $\widehat{E}^\omega_* (R_{n,j} \mid J_n) = \frac{1}{2}$, $j \in J_n$ (as each bush is equally likely to be left or right under $\widehat{P}^\omega_*$), one has

$$\widehat{E}^\omega_* (R_n/n) = \widehat{E}^\omega_* \left(\frac{1}{n} \sum_{j \in J_n} R_{n,j} \right) = \frac{1}{2} \widehat{E}^\omega_* (|J_n|/n) = \frac{1}{2} \widehat{E}^\omega_* (|L_{n-1}|/n),$$

where we are using the notation of Lemma 3.1 ($|J_{n+1}| = L_n$). Hence, our goal is to show that

$$\lim_{n \to \infty} \widehat{E}^\omega_* (|L_n|/n) = q. \qquad (4.6)$$

Write

$$\widehat{E}^\omega_* (L_n) = \widehat{E}^\omega_* \left(L_n \mid \frac{L_n}{n} \in (q - \epsilon, q + \epsilon) \right) \widehat{P}^\omega_* \left(\frac{L_n}{n} \in (q - \epsilon, q + \epsilon) \right)$$

$$+ \widehat{E}^\omega_* \left(L_n \mid \frac{L_n}{n} \notin (q - \epsilon, q + \epsilon) \right) \widehat{P}^\omega_* \left(\frac{L_n}{n} \notin (q - \epsilon, q + \epsilon) \right).$$

Now use Lemma 3.1. Since the first probability on the righthand side is $1 - o(1)$ and the second is $o(1)$, and since $0 \le L_n \le n$, and $\epsilon > 0$ is arbitrary, we are done. $\qquad \square$

5 Proof of Theorem 1 – subcritical case

Recall the definition of Q^ω from the proof of Lemma 3.1 and that K is the 'total obstacle configuration.' Just like in Subsection 3.3, let T_n be the time spent in K^c (vacant sites).

Let Y be a simple random walk on $\mathbb{Z}^d$ and let $\mathcal{E}^\omega$ denote the corresponding expectation. Then

$$\mathrm{DV}(n) = \mathrm{DV}(\omega, n, \mu) := \mathcal{E}^\omega \left(\mu^{\sum_1^n 1_{K^c}(Y_i)} \right) = \mathcal{E}^\omega \left(\mu^{T_n} \right)$$

is the quenched probability that Y would survive up to time n, were 'soft' killing being introduced.[5] By 'soft killing' we mean that at each vacant site, independently, the particle is killed with probability $1 - \mu$.

In this discrete setting, q plays the role of the 'intensity' and μ plays the role of the 'shape function'. In fact, it is known that given $q \in (0, 1)$, for $\mathbb{Q}_q$-almost every ω,

$$\mathrm{DV}(n) = \exp \left[-C_{d,q} \cdot \frac{n}{(\log n)^{2/d}} (1 + o(1)) \right], \tag{5.1}$$

as $n \to \infty$, and $C_{d,q} > 0$ does not depend on μ. See formula (0.1) on p.58 of [1] for hard obstacles. The proof for hard obstacles actually extends for soft obstacles. Indeed, it becomes easier, since in the case of soft obstacles one does not have to worry about 'percolation effects,' that is that the starting point of the process is perhaps not in an infinite trap-free region. Clearly, the lower estimate for survival among hard obstacles is still valid for soft obstacles; the method of proving the upper estimate is a discretized version of Sznitman's 'enlargement of obstacles' in both cases. (See also [2] for similar results and for the enlargement technique in the discrete setting.)

Returning to our branching process and the event S_n, we first show that on a set of full $\mathbb{P}_p$-measure,[6] as $n \to \infty$,

$$P^\omega(S_n) = \frac{\mathrm{DV}(n)}{E^\omega(|Z_n| \mid S_n)}. \tag{5.2}$$

The expectation $E^\omega|Z_n|$ can in fact be expressed as a functional of a *single* particle (this is the 'Many-To-One' Lemma [10] for branching random walk):

$$E^\omega|Z_n| = \mathcal{E}^\omega \left(\mu^{T_n} \right) = \mathrm{DV}(n).$$

This follows from the fact that for $u_n(x) = u_n^\omega(x) := E_x^\omega|Z_n|$, one has the recursion

$$u_n(x) = \sum_{y \sim x} u_{n-1}(y)(\mathbb{I}_{\{y \in K\}} + \mu \mathbb{I}_{\{y \in K^c\}})p(x, y),$$

where $y \sim x$ means that y is a neighbor of x, and $p(\cdot, \cdot)$ is the one-step kernel for the walk. (See again [10].) Thus

$$P^\omega(S_n)E^\omega(|Z_n| \mid S_n) = E^\omega|Z_n| = \mathcal{E}^\omega \mu^{T_n}, \tag{5.3}$$

proving (5.2). Since the denominator on the righthand side of (5.2) is at least one, it follows that

$$P^\omega(S_n) \leq \mathrm{DV}^\mu(n), \tag{5.4}$$

where we emphasize the dependence on μ.

---

[5] As the reader has probably guessed already, DV refers to 'Donsker-Varadhan.'

[6] $\mathbb{P}_p$ of course can be identified with $\mathbb{Q}_q$ ($q = 1 - p$). We used the latter one simply to emphasize we were talking about the random walk setting and not the branching RW one.

On the other hand, we claim that

$$P^\omega(S_n) \geq \mathrm{DV}^{\mu^*}(n), \tag{5.5}$$

where $\mu^* := 1 - p_0$, and $p_0 > 0$ is the probability of having zero offspring (i.e. death) for the law $\mathcal{L}$.

Indeed, consider a random walk with 'soft killing' (RWSK), where soft killing means that the walk is killed with probability p_0, independently at each vacant site. Clearly, given an environment ω, $P^\omega(S_n)$ is not less than the probability of survival up to n by RWSK, as we can couple the latter process with our branching random walk Z by embedding it[7] into Z.

Having (5.4) and (5.5) at our disposal, we can now conclude the assertion of Theorem 1(ii), because $\mathrm{DV}^\mu(n)$ and $\mathrm{DV}^{\mu^*}(n)$ both have the asymptotic behavior given in (5.1), despite the fact that $\mu > \mu^*$ in general. □

References

[1] Antal, P., Trapping problems for the simple random walk, PhD dissertation (1994), ETH, Zurich.

[2] Antal, P., Enlargement of obstacles for the simple random walk. Ann. Probab. **23** (1995), no. 3, 1061–1101. MR-1349162

[3] Asselah, A. and Castell, F., Large deviations for Brownian motion in a random scenery. Probab. Theory Related Fields **126** (2003), no. 4, 497–527. MR-2001196

[4] Engländer, J. and Sieben, N., Critical branching random walk in an IID environment. Monte Carlo Methods Appl. **17** (2011), no. 2, 169–193. MR-2819706

[5] Gantert, N., König, W. and Shi, Z., Annealed deviations of random walk in random scenery. Ann. Inst. H. Poincaré Probab. Statist. **43** (2007), no. 1, 47–76. MR-2288269

[6] Guillotin-Plantard, N. and Poisat, J., Quenched central limit theorems for random walks in random scenery. Stochastic Process. Appl. **123** (2013), no. 4, 1348–1367. MR-3016226

[7] Guillotin-Plantard, N., Poisat, J. and dos Santos, R. S., A quenched functional central limit theorem for planar random walks in random sceneries. Electron. Commun. Probab. **19** (2014), no. 3. MR-3164750

[8] Harris, T. E., The theory of branching processes, Dover Phoenix Editions, (Corrected reprint of the 1963 original) (2002) MR-1991122

[9] Lyons, R., Pemantle, R. and Peres, Y., Conceptual Proofs of L Log L Criteria for Mean Behavior of Branching Processes. Ann. Probab. **23** (1995) No. 3, 1125–1138. MR-1349164

[10] Shi, Z., Branching Random Walks: École d'Été de Probabilités de Saint-Flour XLII - 2012, Lecture Notes in Mathematics, Springer (2015). MR-3444654

[11] Sznitman, A-S., Brownian motion, obstacles and random media, Springer Monographs in Mathematics, Springer (1998). MR-1717054

---

[7]Always choose an offspring at random after splitting.