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Kesten's incipient infinite cluster and quasi-multiplicativity of crossing probabilities

Deepan Basu * Artem Sapozhnikov †

Abstract

In this paper we consider Bernoulli percolation on an infinite connected bounded degrees graph G. Assuming the uniqueness of the infinite open cluster and a quasi-multiplicativity of crossing probabilities, we prove the existence of Kesten's incipient infinite cluster. We show that our assumptions are satisfied if G is a slab $\mathbb{Z}^2 \times \{0,\dots,k\}^{d-2}$ ($d \geq 2$, $k \geq 0$). We also argue that the quasi-multiplicativity assumption should hold for $G = \mathbb{Z}^d$ when $d < 6$, but not when $d > 6$.

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1 Introduction

Let G be an infinite connected bounded degrees graph with a vertex set V. Let ρ be the graph metric on V, and define for $v \in V$ and positive integers $m \leq n$,

$$B(v,n) = \{x \in V \ : \ \rho(v,x) \leq n\}, \quad S(v,n) = \{x \in V \ : \ \rho(v,x) = n\},$$

$$A(v,m,n) = B(v,n) \setminus B(v,m-1).$$

Consider Bernoulli bond percolation on G with parameter $p \in [0,1]$ and denote the corresponding probability measure by $\mathbb{P}_p$. The open cluster of $v \in V$ is denoted by $C(v)$. Let p_c be the critical threshold for percolation, i.e., for $v \in V$,

$$p_c = \inf \{p : \mathbb{P}_p[|C(v)| = \infty] > 0\}.$$

For $x, y \in V$ and $X, Y, Z \subset V$, we write $x \leftrightarrow y$ in Z if there is a nearest neighbor path of open edges such that all its vertices are in Z, $X \leftrightarrow Y$ in Z if there exist $x \in X$ and $y \in Y$ such that $x \leftrightarrow y$ in Z, and $x \leftrightarrow Y$ in Z, if there exist $y \in Y$ such that $x \leftrightarrow y$ in Z. If $Z = V$, we omit "in Z" from the notation. We use $\not\leftrightarrow$ instead of $\leftrightarrow$ to denote complements of the respective events.

In this note we are interested in the existence and equality of the limits

$$\lim_{n\to\infty} \mathbb{P}_{p_c}[E \mid w \longleftrightarrow S(w,n)] \quad \text{and} \quad \lim_{p \searrow p_c} \mathbb{P}_p[E \mid |C(w)| = \infty], \tag{1.1}$$

*Max-Planck Institute for Mathematics in the Sciences, Inselstrasse 22, 04103 Leipzig, Germany.
E-mail: deepan.basu@mis.mpg.de
†University of Leipzig, Department of Mathematics, Augustusplatz 10, 04109 Leipzig, Germany.
E-mail: artem.sapozhnikov@math.uni-leipzig.de

where E is a cylinder event. The question is highly non-trivial if $\mathbb{P}_{p_c}[|C(w)| = \infty] = 0$. The seminal result of Kesten [18, Theorem (3)] states that if G is from a class of planar graphs, such as $\mathbb{Z}^2$, then the above two limits exist and have the same value $\nu_{G,w}(E)$. By Kolmogorov's extension theorem, $\nu_{G,w}$ extends uniquely to a probability measure on configurations of edges, which is often called *Kesten's incipient infinite cluster* measure. It is immediate that $\nu_{G,w}[|C(w)| = \infty] = 1$. Kesten's argument is based on the existence of an infinite collection of open circuits around w in disjoint annuli and the properties that (a) each path from w to infinity intersects every such circuit and (b) by conditioning on the innermost open circuit in an annulus, the occupancy configuration in the region not surrounded by the circuit is still an independent Bernoulli percolation. These properties are no longer valid when one considers higher dimensional lattices. In fact, the existence of Kesten's IIC on $\mathbb{Z}^d$ for $d \geq 3$ is still an open problem. Partial progress has been recently made in sufficiently high dimensions by Heydenreich, van der Hofstad and Hulshof [14, Theorem 1.2], who showed using lace expansions the existence of the first limit in (1.1) under the assumption that $n^{-2}\,\mathbb{P}_{p_c}[0 \longleftrightarrow S(0,n)]$ converges. Concerning low dimensional lattices, almost nothing is known there about critical and near critical percolation, and the existence of Kesten's IIC seems particularly hard to show. Several other constructions of incipient infinite clusters are obtained by Járai [17] for planar lattices and van der Hofstad and Járai [15] for high dimensional lattices.

The main result of this note is the existence and the equality of the two limits in (1.1) for graphs satisfying two assumptions: (A1) uniqueness of the infinite open cluster and (A2) quasi-multiplicativity of crossing probabilities. While (A1) is satisfied by many amenable graphs, most notably $\mathbb{Z}^d$, (A2) can be expected only in low dimensional graphs. For instance, we argue below that (A2) should hold for $\mathbb{Z}^d$ when $d < 6$, but not when $d > 6$. In our second result, we prove that (A2) is satisfied by slabs $\mathbb{Z}^2 \times \{0, \ldots, k\}^{d-2}$ ($d \geq 2$, $k \geq 0$), thus showing for these graphs the existence and equality of the limits in (1.1). We now state the assumptions and the main result, and then comment more on the assumptions.

(A1) (Uniqueness of the infinite open cluster) For any $p \in [0,1]$ there exists almost surely at most one infinite open cluster.

(A2) (Quasi-multiplicativity of crossing probabilities) Let $v \in V$ and $\delta > 0$. There exists $c_* > 0$ such that for any $p \in [p_c, p_c + \delta]$, integer $m > 0$, a finite connected set $Z \subset V$ such that $Z \supseteq A(v, m, 4m)$, and sets $X \subset Z \cap B(v, m)$ and $Y \subset Z \setminus B(v, 4m)$,

$$\mathbb{P}_p[X \leftrightarrow Y \text{ in } Z] \geq c_* \cdot \mathbb{P}_p[X \leftrightarrow S(v, 2m) \text{ in } Z] \cdot \mathbb{P}_p[Y \leftrightarrow S(v, 2m) \text{ in } Z]. \tag{1.2}$$

Theorem 1.1. *Assume that the graph G satisfies the assumptions (A1) and (A2) for some choice of $v \in V$ and $\delta > 0$. Then, for any cylinder event E, the two limits in* (1.1) *exist and have the same value.*

If the assumptions (A1) and (A2) are satisfied at $p = p_c$, then the first limit in (1.1) *exists.*

Before we discuss the strategy of the proof, let us comment on the assumptions.

Comments on (A1):

1. (A1) is satisfied by many sufficiently regular (e.g., vertex transitive) amenable graphs, most notably lattices $\mathbb{Z}^d$ and slabs $\mathbb{Z}^2 \times \{0, \ldots, k\}^{d-2}$ ($d \geq 2$, $k \geq 0$), see, e.g., [4].
2. (A1) is equivalent to the assumption that for some $\delta > 0$ there exists at most one infinite open cluster for any fixed $p \in [p_c, p_c + \delta]$. Indeed, if for a given p the infinite

open cluster is unique almost surely, then the same holds for any $p' > p$, see, e.g., [11, 22].

3. For $v \in V$ and $m \le n$, let $E_1(v,m,n) = \{S(v,m) \leftrightarrow S(v,n)\}$ and $E_2(v,m,n)$ the event that in the annulus $A(v,m,n)$ there are at least two disjoint open crossing clusters.

   Assumption (A1) is equivalent to the following one, which will be used in the proof of Theorem 1.1: For any $v \in V$, $\varepsilon > 0$ and $m \in \mathbb{N}$, there exists $n > 4m$ such that

$$\sup_{p\in[0,1]} \mathbb{P}_p\left[E_2(v,m,n)\right] < \varepsilon \tag{1.3}$$

   or, equivalently,

$$\sup_{p\in[0,1]} \mathbb{P}_p\left[E_2(v,m,n) \mid E_1(v,m,n)\right] < \varepsilon. \tag{1.4}$$

   The equivalence of the claims (1.3) and (1.4) follows from the inequalities

$$\mathbb{P}_p\left[E_2(v,m,n)\right] \le \mathbb{P}_p\left[E_2(v,m,n) \mid E_1(v,m,n)\right] \le \mathbb{P}_p\left[E_2(v,m,n)\right]^{\frac{1}{2}},$$

   where the second one is a consequence of the BK inequality.

   It is elementary to see that (1.3) implies (A1). On the other hand, if (1.3) does not hold, then there exist $v_0 \in V$, $\varepsilon_0 > 0$ and $m_0 \in \mathbb{N}$ such that for all $n > 4m_0$, $\sup_{p\in[0,1]} \mathbb{P}_p\left[E_2(v_0,m_0,n)\right] \ge \varepsilon_0$. The function $\mathbb{P}_p\left[E_2(v_0,m_0,n)\right]$ is continuous in $p \in [0,1]$ and monotone decreasing in n. Thus, there exists $p_0 \in [0,1]$ such that $\mathbb{P}_{p_0}\left[E_2(v_0,m_0,n)\right] \ge \varepsilon_0$ for all $n > 4m_0$. By passing to the limit as $n \to \infty$, we conclude that for $p = p_0$, with positive probability there exist at least two infinite open clusters and (A1) does not hold.

Comments on (A2):

4. If there exists a unique infinite cluster at some $p' > p_c$, i.e., (A1) holds at $p = p'$, then (1.2) automatically holds at $p = p'$ with $c_* = c_*(p') > 0$. However, (A1) does not imply that there exists $c_* > 0$ such that (1.2) holds *for all* p' near p_c, see, e.g., the discussion about $G = \mathbb{Z}^d$ with $d > 6$ below.

5. It follows from the Russo-Seymour-Welsh theorem [21, 23] and planar arguments that (A2) holds for planar graphs, such as $\mathbb{Z}^2$, considered by Kesten in [18]. RSW ideas have been recently extended to slabs in [20, 3], after the absence of percolation at criticality in slabs was proved by Duminil-Copin, Sidoravicius and Tassion [9]. In Section 3 we show that (A2) is fulfilled by slabs $\mathbb{Z}^2 \times \{0,\ldots,k\}^{d-2}$ ($d \ge 2$, $k \ge 0$). Our proof is based on an adaptation of planar arguments to slabs and uses recent results and ideas of Newman, Tassion and Wu [20]. A classical observation is that in the plane the existence of two open paths from $X \subset B(v,m)$ to $S(v,2m)$ and from $Y \subset \mathbb{Z}^d \setminus B(v,2m)$ to $S(v,m)$ and an open circuit in $A(v,m,2m)$ around $B(v,m)$ implies the existence of an open path from X to Y, since the two “short” paths are always linked through the circuit. This is no longer the case on slabs, nevertheless it is still true that with positive probability, if such two short paths and a circuit exist then they are connected, and this along with RSW results from [20] suffices for (A2). We prove this fact by adapting a local modification argument (“glueing” an open path to an open circuit) from [20].

6. We believe that assumption (A2) holds for lattices $\mathbb{Z}^d$ if $d < 6$, but does not hold if $d > 6$. Dimension $d_c = 6$ is called the *upper critical dimension* above which the percolation phase transition should be described by mean-field theory, see, e.g., [7].

This was rigorously confirmed in sufficiently high dimensions by Hara and Slade [13, 12] and recently in $d > 10$ by Fitzner and van der Hofstad [10].

It is easy to see that the mean-field behavior excludes (A2). Indeed, it is believed that above d_c, the two point function decays as

$$\mathbb{P}_{p_c}[x \leftrightarrow y] \asymp (1+\rho(x,y))^{2-d}.$$

(Here $f(z) \asymp g(z)$ if for some c, $cf(z) \leq g(z) \leq c^{-1} f(z)$ for all z.) Hara [12] proved it rigorously in sufficiently high dimensions. Given this asymptotics, Aizenman showed in [1, Theorem 4(2)] that for all $m(n) \leq n$ such that $\frac{m(n)}{n^{2/(d-4)}} \to \infty$ (which exists only if $d > 6$),

$$\mathbb{P}_{p_c}\left[S(0,m(n)) \leftrightarrow S(0,n)\right] \to 1, \quad \text{as } n \to \infty,$$

and Kozma and Nachmias [19] that $\mathbb{P}_{p_c}\left[0 \leftrightarrow S(0,n)\right] \asymp n^{-2}$. Thus, the inequality

$$\mathbb{P}_{p_c}[0 \leftrightarrow S(0,n)] \geq c\, \mathbb{P}_{p_c}[0 \leftrightarrow S(0,m(n))]\, \mathbb{P}_{p_c}[S(0,m(n)) \leftrightarrow S(0,n)]$$

cannot hold for large n if $n^{2/(d-4)} \ll m(n) \ll n$.

The situation below d_c is much more subtle. With the exception of $d = 2$, where planarity helps enormously, the (near-)critical behavior below d_c is widely unknown. Let us nevertheless give a few words about why we think (A2) should hold below d_c. It is believed that the number of clusters crossing any annulus $A(0,m,2m)$ is bounded uniformly in m if $d < d_c$ and grows at $p = p_c$ like m^{d-6} above d_c, with log-correction for $d = d_c$, and this dichotomy is intimately linked to the transition at d_c from the hyperscaling to the mean-field; see [6, 5]. Thus, it would be not unreasonable to expect that below d_c, for some $c > 0$,

$$\mathbb{P}_p[\exists!\, \text{crossing cluster of } A(0,m,2m) \mid X \leftrightarrow S(0,2m) \text{ in } Z,\, Y \leftrightarrow S(0,m) \text{ in } Z] \geq c,$$

which is enough to establish (A2). We are not able to prove it yet or give a simpler sufficient condition for it. It would already be very nice if, for instance, (A2) was derived from the assumption that $\mathbb{P}_p[\exists!\, \text{crossing cluster of } A(0,m,2m)] \geq c$ or from the assumptions of [5].

We finish the introduction with a brief description of the proof of Theorem 1.1. Our proof follows the general scheme proposed by Kesten in [18] by attempting to decouple the configuration near w from infinity on multiple scales. The implementations are however rather different. Using (1.4) we identify a sufficiently fast growing sequence N_i such that given $w \leftrightarrow S(w,n)$, the probability that the annulus $A(v, N_i, N_{i+1}) \subset B(w,n)$ contains a unique crossing cluster is asymptotically close to 1; see (2.2). Next, let an annulus $A(v, N_i, N_{i+1})$ contain a unique crossing cluster. We explore all the open clusters in this annulus that intersect the interior boundary $S(v, N_i)$, call their union $\mathcal{C}_i$, and let $\mathcal{D}_i$ be the subset of $S(v, N_{i+1}+1)$ of vertices connected by an open edge to $\mathcal{C}_i$; see (2.3). Then, the state of the edges not incident to any vertex of $\mathcal{C}_i$ is distributed as the original independent percolation and every vertex from $\mathcal{D}_i$ is connected by an edge to the same (crossing) cluster from $\mathcal{C}_i$. Thus, $w \leftrightarrow S(w,n)$ if and only if (a) w is connected to $\mathcal{D}_i$ (this event only depends on the edges intersecting $S(v,N_i) \cup \mathcal{C}_i$) and (b) $\mathcal{D}_i$ is connected to $S(w,n)$ outside $\mathcal{C}_i$ (this only depends on the edges outside $\mathcal{C}_i$). This allows to factorize $\mathbb{P}_p[E,\, w \leftrightarrow S(w,n)]$; see (2.4). The rest of the proof is essentially the same as that of Kesten [18]. We repeat the described factorization on several scales, obtaining in (2.6) an approximation of $\mathbb{P}_p[E | w \leftrightarrow S(w,n)]$ in terms of products of positive matrices. Finally, we use (A2) to prove that the matrix operators are uniformly contracting, which is enough to conclude the proof; see (2.7) and the text below.

2 Proof of Theorem 1.1

We will prove the first claim of the theorem. The proof of the second one follows from the proof below by replacing everywhere p by p_c. The general outline of the proof is the same as the original one of Kesten [18, Theorem (3)], but the choice of scales and the decoupling are done differently.

First of all, it suffices to prove that for any $w \in V$ and a cylinder event E,

$$\mathbb{P}_p[E|w \leftrightarrow S(w,n)] \text{ converges to some } \nu_p(E) \textit{ uniformly} \text{ on } [p_c, p_c+\delta] \text{ for some } \delta > 0. \tag{2.1}$$

Indeed, (2.1) implies the existence of the first limit in (1.1) and that $\nu_p(E)$ is continuous. Since for any $p > p_c$, $\nu_p(E) = \mathbb{P}_p[E \mid |C(v)| = \infty]$, the existence of the second limit in (1.1) and its equality to the first one follows from the continuity of $\nu_p(E)$.

Actually, by the inclusion-exclusion formula, it suffices to prove (2.1) for all events E of the form $\{\text{edges } e_1, \ldots, e_k \text{ are open}\}$. Although our proof could be implemented for any cylinder event E, calculations are neater for increasing events.

Fix $w \in V$ and an increasing event E. Also fix $v \in V$ and $\delta > 0$ for which the assumption (A2) is satisfied. Consider a sequence of scales N_i such that $N_{i+1} > 4N_i$ for all i, $B(v, N_0)$ contains w and the states of its edges determine E. We will write $B_i = B(v, N_i)$, $S_i = S(v, N_i)$ and $A_i = A(v, N_i, N_{i+1})$. Let F_i be the event that there exists a unique open crossing cluster in A_i. Define

$$\varepsilon_i = \sup_{p\in[p_c,p_c+\delta]} \mathbb{P}_p\left[F_i^c \mid S_i \leftrightarrow S_{i+1}\right].$$

By (1.4), we can choose the scales N_i so that $\varepsilon_i \to 0$ as $i \to \infty$.

We first note that for $n > N_{i+1} + N_0$,

$$\mathbb{P}_p[w \leftrightarrow S(w,n), F_i^c] \le c_*^{-2}\varepsilon_i \cdot \mathbb{P}_p[w \leftrightarrow S(w,n)], \tag{2.2}$$

where c_* is the constant in the assumption (A2). Indeed, by independence,

$$\begin{aligned}
\mathbb{P}_p[w \leftrightarrow S(w,n), F_i^c] &\le \mathbb{P}_p[w \leftrightarrow S_i] \cdot \mathbb{P}_p[S_i \leftrightarrow S_{i+1}, F_i^c] \cdot \mathbb{P}_p[S_{i+1} \leftrightarrow S(w,n)] \\
&\le \varepsilon_i \cdot \mathbb{P}_p[w \leftrightarrow S_i] \cdot \mathbb{P}_p[S_i \leftrightarrow S_{i+1}] \cdot \mathbb{P}_p[S_{i+1} \leftrightarrow S(w,n)] \\
&\le c_*^{-2}\varepsilon_i \cdot \mathbb{P}_p\left[w \leftrightarrow S(w,n)\right],
\end{aligned}$$

where the last inequality follows from the assumption (A2).

We begin to describe the main decomposition step. Consider the random sets

$$\begin{aligned}
\mathcal{C}_i &= \{x \in B(v, N_{i+1}) \;:\; x \leftrightarrow B(v, N_i) \text{ in } B(v, N_{i+1})\}, \\
\mathcal{D}_i &= \{x \in S(v, N_{i+1}+1) \;:\; \exists\, y \in \mathcal{C}_i, \text{ a neighbor of } x, \text{ such that edge } \langle x, y\rangle \text{ is open}\}.
\end{aligned} \tag{2.3}$$

Note that $\mathcal{C}_i$ contains $B(v, N_i)$, the event $\{\mathcal{C}_i = U\}$ depends only on the states of edges in $B(v, N_{i+1})$ with at least one end-vertex in U, and either $\{\mathcal{C}_i = U\} \subset F_i$ or $\{\mathcal{C}_i = U\} \cap F_i = \emptyset$. Also note that the event $\{\mathcal{C}_i = U, \mathcal{D}_i = R\}$ depends only on the states of edges in $B(v, N_{i+1}+1)$ with at least one end-vertex in U.

For any $U \subset B(v, N_{i+1})$ and $R \subset S(v, N_{i+1}+1)$, consider the event

$$F_i(U,R) = \{\mathcal{C}_i = U,\, \mathcal{D}_i = R\},$$

and let Π_i be the collection of all such pairs (U,R) that $\{\mathcal{C}_i = U\} \subset F_i$ and $F_i(U,R) \neq \emptyset$.

Then $F_i = \cup_{(U,R)\in\Pi_i} F_i(U,R)$, and for all $n > N_{i+1} + N_0$,

$$\begin{aligned}\mathbb{P}_p\left[E, w \leftrightarrow S(w,n), F_i\right] &= \sum_{(U,R)\in\Pi_i} \mathbb{P}_p\left[E, w \leftrightarrow S(w,n), F_i(U,R)\right] \\ &= \sum_{(U,R)\in\Pi_i} \mathbb{P}_p\left[E, w \leftrightarrow S_{i+1}, F_i(U,R)\right] \cdot \mathbb{P}_p\left[R \leftrightarrow S(w,n) \text{ in } B(w,n)\setminus U\right].\end{aligned}$$

Together with (2.2), this gives the inequality

$$\begin{aligned}\Big|\mathbb{P}_p\left[E, w \leftrightarrow S(w,n)\right] \\ - \sum_{(U,R)\in\Pi_i} \mathbb{P}_p\left[E, w \leftrightarrow S_{i+1}, F_i(U,R)\right] \cdot \mathbb{P}_p\left[R \leftrightarrow S(w,n) \text{ in } B(w,n)\setminus U\right]\Big| \\ \le c_*^{-2}\varepsilon_i \cdot \mathbb{P}_p[w \leftrightarrow S(w,n)] \le \frac{c_*^{-2}\varepsilon_i}{\mathbb{P}_{p_c}[E]} \cdot \mathbb{P}_p[E, w \leftrightarrow S(w,n)], \quad (2.4)\end{aligned}$$

where the last step follows from the FKG inequality, since E is increasing. Define the constant $C_* = (c_*^2\,\mathbb{P}_{p_c}[E])^{-1}$ and for $(U,R)\in\Pi_i$, let

$$\begin{aligned}u'_p(U,R) &= \mathbb{P}_p\left[E, w \leftrightarrow S_{i+1}, F_i(U,R)\right], \\ u''_p(U,R) &= \mathbb{P}_p\left[w \leftrightarrow S_{i+1}, F_i(U,R)\right], \\ \gamma_p(U,R,n) &= \mathbb{P}_p\left[R \leftrightarrow S(w,n) \text{ in } B(w,n)\setminus U\right].\end{aligned}$$

In this notation, (2.4) becomes

$$\begin{aligned}(1 - C_*\varepsilon_i)\ \mathbb{P}_p\left[E, w \leftrightarrow S(w,n)\right] &\le \sum_{(U,R)\in\Pi_i} u'_p(U,R)\,\gamma_p(U,R,n) \\ &\le (1 + C_*\varepsilon_i)\ \mathbb{P}_p\left[E, w \leftrightarrow S(w,n)\right]\end{aligned}$$

and by replacing E above with the sure event, we also get

$$\begin{aligned}(1 - C_*\varepsilon_i)\ \mathbb{P}_p\left[w \leftrightarrow S(w,n)\right] &\le \sum_{(U,R)\in\Pi_i} u''_p(U,R)\,\gamma_p(U,R,n) \\ &\le (1 + C_*\varepsilon_i)\ \mathbb{P}_p\left[w \leftrightarrow S(w,n)\right].\end{aligned}$$

Now we iterate. Let $(U,R)\in\Pi_i$. We can apply a similar reasoning as in (2.2) and (2.4) to $\gamma_p(U,R,n)$ and obtain that for any $j > i+2$ and $n > N_{j+1} + N_0$,

$$\begin{aligned}\Big|\gamma_p(U,R,n) - \sum_{(U',R')\in\Pi_j} \mathbb{P}_p\left[R \leftrightarrow S_{j+1} \text{ in } B_{j+1}\setminus U, F_{j-1}, F_j(U',R')\right] \cdot \gamma_p(U',R',n)\Big| \\ \le c_*^{-2}(\varepsilon_{j-1} + \varepsilon_j) \cdot \gamma_p(U,R,n). \quad (2.5)\end{aligned}$$

For $j > i+2$, $(U,R)\in\Pi_i$ and $(U',R')\in\Pi_j$, define

$$M_p(U,R;\,U',R') = \mathbb{P}_p\left[R \leftrightarrow S_{j+1} \text{ in } B_{j+1}\setminus U, F_{j-1}, F_j(U',R')\right].$$

Then (2.5) becomes

$$\begin{aligned}(1 - c_*^{-2}\,(\varepsilon_{j-1} + \varepsilon_j))\,\gamma_p(U,R,n) &\le \sum_{(U',R')\in\Pi_j} M_p(U,R;\,U',R')\,\gamma_p(U',R',n) \\ &\le (1 + c_*^{-2}\,(\varepsilon_{j-1} + \varepsilon_j))\,\gamma_p(U,R,n).\end{aligned}$$

Iterating further gives that for any $\varepsilon > 0$ and $s \in \mathbb{N}$, there exist indices $i_1, \dots, i_s$ such that $i_{k+1} > i_k + 2$ and for all $n > N_{i_s+1} + N_0$,

$$\begin{aligned} e^{-\varepsilon}\, \mathbb{P}_p\left[E \,|\, w \leftrightarrow S(w,n)\right] \le & \\ \frac{\sum u_p'(U_1,R_1)\, M_p(U_1,R_1;\, U_2,R_2) \dots M_p(U_{s-1},R_{s-1};, U_s,R_s)\, \gamma_p(U_s,R_s,n)}{\sum u_p''(U_1,R_1)\, M_p(U_1,R_1;\, U_2,R_2) \dots M_p(U_{s-1},R_{s-1};, U_s,R_s)\, \gamma_p(U_s,R_s,n)} & \\ \le e^{\varepsilon}\, \mathbb{P}_p\left[E \,|\, w \leftrightarrow S(w,n)\right], & \end{aligned} \tag{2.6}$$

where the two sums are over $(U_1,R_1) \in \Pi_{i_1}, \dots, (U_s,R_s) \in \Pi_{i_s}$.

We will prove that (A2) implies that there exists $\kappa > 1$ such that for all i, $j > i+2$, all pairs $(U_1,R_1),(U_2,R_2) \in \Pi_i$, $(U_1',R_1'),(U_2',R_2') \in \Pi_j$, and all $p \in [p_c, p_c+\delta]$,

$$\frac{M_p(U_1,R_1;\, U_1',R_1')\, M_p(U_2,R_2;\, U_2',R_2')}{M_p(U_1,R_1;\, U_2',R_2')\, M_p(U_2,R_2;\, U_1',R_1')} \le \kappa^2. \tag{2.7}$$

(This is an analogue of [18, Lemma (23)].) Before giving the proof of (2.7), we show how to use it to finish the proof of the theorem. Although the argument is essentially the same as in [18, pages 377-378], we provide the details for completeness.

For a pair of vectors v' and v'' of the same dimension and strictly positive components, define $\mathrm{osc}(v',v'') = \max_{i,j} \left|\frac{v_i'}{v_i''} - \frac{v_j'}{v_j''}\right|$. Hopf's contraction theorem [16, Theorem 1] (see also [18, (24) and (25)]) states that if u' and u'' are vectors in $\mathbb{R}^m$ with strictly positive components and M is an $m \times n$–matrix with strictly positive real entries such that for some $\kappa > 1$ and all indices i,j,k,l, $\max_{i,j,k,l} \frac{M_{ij}\, M_{kl}}{M_{il}\, M_{kj}} \le \kappa^2$, then $\mathrm{osc}(u'M, u''M) \le \frac{\kappa-1}{\kappa+1}\, \mathrm{osc}(u',u'')$. Applying this theorem to the vectors u_p' and u_p'' and the matrices M_p, we obtain that

$$\begin{aligned} \max_{(U_s,R_s),(U_s',R_s') \in \Pi_{i_s}} \Bigg| & \frac{\sum u_p'(U_1,R_1)\, M_p(U_1,R_1;\, U_2,R_2) \dots M_p(U_{s-1},R_{s-1};, U_s,R_s)}{\sum u_p''(U_1,R_1)\, M_p(U_1,R_1;\, U_2,R_2) \dots M_p(U_{s-1},R_{s-1};, U_s,R_s)} \\ & - \frac{\sum u_p'(U_1,R_1)\, M_p(U_1,R_1;\, U_2,R_2) \dots M_p(U_{s-1},R_{s-1};, U_s',R_s')}{\sum u_p''(U_1,R_1)\, M_p(U_1,R_1;\, U_2,R_2) \dots M_p(U_{s-1},R_{s-1};, U_s',R_s')}\Bigg| \\ & \le \left(\frac{\kappa-1}{\kappa+1}\right)^{s-1} \mathrm{osc}(u_p',u_p'') \le \left(\frac{\kappa-1}{\kappa+1}\right)^{s-1}, \end{aligned}$$

where each sum is over all pairs $(U_1,R_1) \in \Pi_{i_1}, \dots, (U_{s-1},R_{s-1}) \in \Pi_{i_{s-1}}$.[1] In particular, there exists $\xi \le 1$, which depends on E, p, and the scales $i_1, \dots, i_s$, such that

$$\begin{aligned} \max_{(U_s,R_s) \in \Pi_{i_s}} \left| \frac{\sum u_p'(U_1,R_1)\, M_p(U_1,R_1;\, U_2,R_2) \dots M_p(U_{s-1},R_{s-1};, U_s,R_s)}{\sum u_p''(U_1,R_1)\, M_p(U_1,R_1;\, U_2,R_2) \dots M_p(U_{s-1},R_{s-1};, U_s,R_s)} - \xi \right| & \\ \le \left(\frac{\kappa-1}{\kappa+1}\right)^{s-1}. & \end{aligned}$$

In combination with (2.6), this gives that for all $n > N_{i_s+1} + N_0$,

$$e^{-\varepsilon}\left(\xi - \left(\frac{\kappa-1}{\kappa+1}\right)^{s-1}\right) \le \mathbb{P}_p\left[E \,|\, w \leftrightarrow S(w,n)\right] \le e^{\varepsilon}\left(\xi + \left(\frac{\kappa-1}{\kappa+1}\right)^{s-1}\right). \tag{2.8}$$

It follows from (2.8) and the fact that $\xi \le 1$ that for any $m,n > N_{i_s+1} + N_0$ and $p \in [p_c, p_c+\delta]$,

$$\Big|\mathbb{P}_p\left[E \,|\, w \leftrightarrow S(w,m)\right] - \mathbb{P}_p\left[E \,|\, w \leftrightarrow S(w,n)\right]\Big| \le \left(e^{\varepsilon} - e^{-\varepsilon}\right) + \left(e^{\varepsilon} + e^{-\varepsilon}\right)\, \left(\frac{\kappa-1}{\kappa+1}\right)^{s-1},$$

[1] The corresponding inequality in [18]—the first inequality on [18, page 378]—contains a mathematical typo, $\mathrm{osc}(u',u'')$ is missing. However, one can show using RSW techniques that the missing term there is bounded from above by a constant independent of j_1, and the remaining argument goes through. In our case, the situation is simpler, since for our choice of u' and u'', $\mathrm{osc}(u',u'') \le 1$.

which implies (2.1).

It remains to prove (2.7). Let $j > i+2$. Consider the random sets

$$\mathcal{X}_j = \{x \in A_{j-1} \,:\, x \leftrightarrow S_j \text{ in } A_{j-1}\},$$
$$\mathcal{Y}_j = \{y \in S(v, N_{j-1}-1) \,:\, \exists\, x \in \mathcal{X}_j, \text{ a neighbor of } y, \text{ such that the edge } \langle x,y\rangle \text{ is open}\}.$$

Note that $\mathcal{X}_j$ contains S_j, the event $\{\mathcal{X}_j = X\}$ depends only on the states of edges in A_{j-1} with at least one end-vertex in X, and either $\{\mathcal{X}_j = X\} \subset F_{j-1}$ or $\{\mathcal{X}_j = X\} \cap F_{j-1} = \emptyset$. Also note that the event $\{\mathcal{X}_j = X, \, \mathcal{Y}_j = Y\}$ depends only on the states of edges in B_j with at least one end-vertex in X. For any $X \subset A_{j-1}$ and $Y \subset S(v, N_{j-1}-1)$, consider the event

$$G_j(X,Y) = \{\mathcal{X}_j = X, \, \mathcal{Y}_j = Y\},$$

and let Γ_j be the collection of all such pairs (X,Y) that $\{\mathcal{X}_j = X\} \subset F_{j-1}$ and $G_j(X,Y) \neq \emptyset$. Then $F_{j-1} = \cup_{(X,Y)\in\Gamma_j} G_j(X,Y)$ and for any $(U,R) \in \Pi_i$, $(U',R') \in \Pi_j$,

$$\begin{aligned} &M_p(U,R;\, U',R') \\ &\qquad = \sum_{(X,Y)\in\Gamma_j} \mathbb{P}_p\left[R \leftrightarrow Y \text{ in } B_j \setminus (X \cup U)\right] \cdot \mathbb{P}_p\left[G_j(X,Y), F_j(U',R'), Y \leftrightarrow R'\right]. \end{aligned}$$

By the assumption (A2),

$$\begin{aligned} &\mathbb{P}_p\left[R \leftrightarrow Y \text{ in } B_j \setminus (X \cup U)\right] \\ &\quad \geq c_* \cdot \mathbb{P}_p\left[R \leftrightarrow S(v, 2N_{i+1}) \text{ in } B(v, 2N_{i+1}) \setminus U\right] \cdot \mathbb{P}_p\left[S(v, 2N_{i+1}) \leftrightarrow Y \text{ in } B_j \setminus X\right] \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad \geq c_* \cdot \mathbb{P}_p\left[R \leftrightarrow Y \text{ in } B_j \setminus (X \cup U)\right]. \end{aligned}$$

This easily implies (2.7) with $\kappa = c_*^{-1}$. The proof of Theorem 1.1 is complete. ☐

Remark 2.1. Instead of conditioning on the events $\{w \leftrightarrow S(w,n)\}$, one could condition on $\{w \leftrightarrow Y_n \text{ in } Z_n\}$, where $Z_n \supset B(w,n)$ and $Y_n \subseteq Z_n \setminus B(w,n)$, and obtain the same limits as in (1.1). Indeed, by going through the same proof one observes that $\mathbb{P}_p[E | w \leftrightarrow Y_n \text{ in } Z_n]$ satisfies inequalities (2.8) with the same ξ.

3 Quasi-multiplicativity for slabs

In this section we prove that the assumption (A2) is fulfilled by slabs $\mathbb{Z}^2 \times \{0,\dots,k\}^{d-2}$ for any $d \geq 2$ and $k \geq 0$ and for any $\delta > 0$ such that $p_c + \delta < 1$, thus proving

Theorem 3.1. *The two limits* (1.1) *exist and coincide on* $\mathbb{Z}^2 \times \{0,\dots,k\}^{d-2}$ ($d \geq 2$, $k \geq 0$).

Fix $d \geq 2$ and $k \geq 0$ and define $\mathbb{S} = \mathbb{Z}^2 \times \{0,\dots,k\}^{d-2}$. For positive integers $m \leq n$, let $Q(n) = [-n,n]^2 \times \{0,\dots,k\}^{d-2}$ be the box of side length $2n$ in $\mathbb{S}$ centered at 0, $\partial Q(n) = Q(n) \setminus Q(n-1)$ the inner boundary of $Q(n)$, and $\mathrm{An}(m,n) = Q(n) \setminus Q(m-1)$ the annulus of side lengths $2m$ and $2n$. We will prove the following lemma.

Lemma 3.2. *Let* $d \geq 2$ *and* $k \geq 0$. *Let* $\delta > 0$ *such that* $p_c + \delta < 1$. *There exists* $c > 0$ *such that for any* $p \in [p_c, p_c+\delta]$, *integer* $m > 0$, *any finite connected* $Z \subset \mathbb{S}$ *such that* $Z \supseteq \mathrm{An}(m,3m)$, *and any* $X \subset Z \cap Q(m)$ *and* $Y \subset Z \setminus Q(3m)$,

$$\mathbb{P}_p[X \leftrightarrow Y \text{ in } Z] \geq c \cdot \mathbb{P}_p[X \leftrightarrow \partial Q(2m) \text{ in } Z] \cdot \mathbb{P}_p[Y \leftrightarrow \partial Q(2m) \text{ in } Z]. \tag{3.1}$$

To see that Lemma 3.2 implies (A2), note that it suffices to prove (1.2) for $m \geq m_0$ and sufficiently large m_0. One can choose $m_0 = m_0(d,k)$ large enough so that $A(0,m,4m) \supset \mathrm{An}(m,3m)$. Thus, Lemma 3.2 implies (A2).

Proof of Lemma 3.2. Instead of (3.1), it suffices to prove that there exists $c > 0$ such that for any $m > 0$, any finite connected $Z \subset \mathbb{S}$ such that $Z \supseteq \mathrm{An}(2m, 3m)$, and any $X \subset Z \cap Q(2m)$ and $Y \subset Z \setminus Q(3m)$,

$$\mathbb{P}_p[X \leftrightarrow Y \text{ in } Z] \geq c \cdot \mathbb{P}_p[X \leftrightarrow \partial Q(3m) \text{ in } Z] \cdot \mathbb{P}_p[Y \leftrightarrow \partial Q(2m) \text{ in } Z]. \tag{3.2}$$

Indeed, for Z as in the statement of the lemma, by (3.2),

$$\mathbb{P}_p[X \leftrightarrow \partial Q(3m) \text{ in } Z] \geq c \cdot \mathbb{P}_p[X \leftrightarrow \partial Q(2m) \text{ in } Z] \cdot \mathbb{P}_p[\partial Q(\frac{4}{3}m) \leftrightarrow \partial Q(3m) \text{ in } Z],$$

and $\mathbb{P}_p[\partial Q(\frac{4}{3}m) \leftrightarrow \partial Q(3m) \text{ in } Z] \geq \mathbb{P}_{p_c}[\partial Q(\frac{4}{3}m) \leftrightarrow \partial Q(3m)] \geq c > 0$, as proved in [3, 20].

We proceed to prove (3.2). Let E be the event that there exists an open circuit (nearest neighbor path with the same start and end points) around $Q(2m)$ contained in $\mathrm{An}(2m, 3m)$. It is shown in [20] that $\mathbb{P}_p[E] \geq \mathbb{P}_{p_c}[E] > c > 0$ for some $c > 0$ independent of m. Thus, by the FKG inequality,

$$\begin{aligned}\mathbb{P}_p[X \leftrightarrow \partial Q(3m) \text{ in } Z,\ Y \leftrightarrow \partial Q(2m) \text{ in } Z,\ E]\\ \geq c \cdot \mathbb{P}_p[X \leftrightarrow \partial Q(3m) \text{ in } Z] \cdot \mathbb{P}_p[Y \leftrightarrow \partial Q(2m) \text{ in } Z].\end{aligned}$$

Consider an arbitrary deterministic ordering of all circuits in $\mathbb{S}$, and for a configuration in E, let Γ be the minimal (with respect to this ordering) open circuit around $Q(2m)$ contained in $\mathrm{An}(2m, 3m)$. For $W \subset \mathbb{S}$, let

$$\overline{W} = \{z = (z_1, \ldots, z_d) \in \mathbb{S}\ :\ (z_1, z_2, x_3, \ldots, x_d) \in W \text{ for some } x_3, \ldots, x_d\ \}.$$

Note that

$$\mathbb{P}_p[X \leftrightarrow \partial Q(3m) \text{ in } Z,\ Y \leftrightarrow \partial Q(2m) \text{ in } Z,\ E] \leq \mathbb{P}_p[X \leftrightarrow \overline{\Gamma} \text{ in } Z,\ Y \leftrightarrow \overline{\Gamma} \text{ in } Z,\ E].$$

Thus, to prove (3.2), it suffices to show that for some $C < \infty$,

$$\mathbb{P}_p[X \leftrightarrow \overline{\Gamma} \text{ in } Z,\ Y \leftrightarrow \overline{\Gamma} \text{ in } Z,\ E] \leq C \cdot \mathbb{P}_p[X \leftrightarrow Y \text{ in } Z].$$

This will be achieved using local modification arguments similar to those in [20]. In fact, for the above inequality to hold, it suffices to show that for some $C < \infty$,

$$\mathbb{P}_p[X \leftrightarrow \overline{\Gamma} \text{ in } Z,\ Y \leftrightarrow \overline{\Gamma} \text{ in } Z,\ E, X \not\leftrightarrow Y \text{ in } Z] \leq C \cdot \mathbb{P}_p[X \leftrightarrow Y \text{ in } Z]. \tag{3.3}$$

We write the event in the left hand side of (3.3) as the union of three subevents satisfying additionally

(a) $X \not\leftrightarrow \Gamma$ in Z, $Y \not\leftrightarrow \Gamma$ in Z,

(b) $X \not\leftrightarrow \Gamma$ in Z, $Y \leftrightarrow \Gamma$ in Z,

(c) $X \leftrightarrow \Gamma$ in Z, $Y \not\leftrightarrow \Gamma$ in Z.

It suffices to prove that the probability of each of the three subevents can be bounded from above by $C \cdot \mathbb{P}_p[X \leftrightarrow Y \text{ in } Z]$. The cases (b) and (c) can be handled similarly, thus we only consider (a) and (b).

Case (a): We prove that for some $C < \infty$,

$$\mathbb{P}_p\left[\begin{array}{c} X \leftrightarrow \overline{\Gamma} \text{ in } Z,\ Y \leftrightarrow \overline{\Gamma} \text{ in } Z,\ E, X \not\leftrightarrow Y \text{ in } Z \\ X \not\leftrightarrow \Gamma \text{ in } Z, Y \not\leftrightarrow \Gamma \text{ in } Z \end{array}\right] \leq C \cdot \mathbb{P}_p[X \leftrightarrow Y \text{ in } Z]. \tag{3.4}$$

Denote by E_a the event on the left hand side. It suffices to construct a map $f : E_a \to \{X \leftrightarrow Y \text{ in } Z\}$ such that for some constant $D < \infty$, (1) for each $\omega \in E_a$, ω and $f(\omega)$ differ in at most D edges, (2) at most D ω's can be mapped to the same configuration, i.e., for each $\omega \in E_a$, $|\{\omega' \in E_a : f(\omega') = f(\omega)\}| \le D$. If so, the desired inequality is satisfied with $C = \frac{D}{\min(p_c, 1-p_c-\delta))^D}$.

Take a configuration $\omega \in E_a$. Let U be the set of all points $u \in \overline{\Gamma}$ such that u is connected to X in Z by an open self-avoiding path that from the first step on does not visit $\overline{\{u\}}$. For each $u \in U$, choose one such open self-avoiding path and denote it by π_u. Similarly, let V be the set of all points $v \in \overline{\Gamma}$ such that v is connected to Y in Z by an open self-avoiding path that from the first step on does not visit $\overline{\{v\}}$. For each $v \in V$, choose one such open self-avoiding path and denote it by π_v.

Assume first that we can choose $u \in U$ and $v \in V$ such that $\overline{\{u\}} = \overline{\{v\}}$. For such ω's, the configuration $f(\omega)$ is defined as follows. We

(a) close all the edges with an end-vertex in $\overline{\{u\}}$ except for the (unique) edge of π_u, the (unique) edge of π_v, and the edges belonging to Γ,

(b) open all the edges in $\overline{\{u\}}$ that belong to a shortest path ρ (line segment if $d = 3$) between u and Γ in $\overline{\{u\}}$,

(c) open all the edges in $\overline{\{u\}}$ that belong to a shortest path between v and $\Gamma \cup \rho$ in $\overline{\{u\}}$.

Notice that ω and $f(\omega)$ differ in at most $2d\,(k+1)^{d-2}$ edges. Also, since u, v and Γ are all in different open clusters in ω, after connecting them by simple open paths as in (b) and (c), no new open circuits are created. Thus, the set $\overline{\{u\}}$ can be uniquely reconstructed in $f(\omega)$ as the unique set of the form $\overline{\{z\}}$ where X (and Y) is connected to Γ.

Assume next that $\overline{U} \cap \overline{V} = \emptyset$. Choose $u \in U$ and $v \in V$. Note that $\overline{\{u\}}$ is not connected to Y in Z and $\overline{\{v\}}$ is not connected to X in Z. The configuration $f(\omega)$ is defined as follows. We

(a) close all the edges with an end-vertex in $\overline{\{u\}} \cup \overline{\{v\}}$ except for those of π_u, π_v and Γ,

(b) open all the edges in $\overline{\{u\}}$ that belong to a shortest path between u and Γ in $\overline{\{u\}}$,

(c) open all the edges in $\overline{\{v\}}$ that belong to a shortest path between v and Γ in $\overline{\{v\}}$.

Notice that ω and $f(\omega)$ differ in at most $4d\,(k+1)^{d-2}$ edges. Step (a) of the construction does not alter the paths π_u and π_v. Finally, since u, v, and Γ are all in different open clusters in ω, after connecting u, v, and Γ by simple open paths as in (b) and (c), no new open circuits are created. Thus, the set $\overline{\{u\}} \cup \overline{\{v\}}$ can be uniquely reconstructed in $f(\omega)$ as the unique such set where X and Y are connected to Γ.

The constructed function f satisfies the requirement (1) with $D = 4d\,(k+1)k^{d-2}$ and the requirement (2) with $D = 2^{4d\,(k+1)^{d-2}}$. The proof of (3.4) is complete.

Case (b): We prove that for some $C < \infty$,

$$\mathbb{P}_p \left[\begin{array}{c} X \leftrightarrow \overline{\Gamma} \text{ in } Z,\ Y \leftrightarrow \overline{\Gamma} \text{ in } Z,\ E, X \not\leftrightarrow Y \text{ in } Z \\ X \not\leftrightarrow \Gamma \text{ in } Z, Y \leftrightarrow \Gamma \text{ in } Z \end{array} \right] \le C \cdot \mathbb{P}_p[X \leftrightarrow Y \text{ in } Z]. \qquad (3.5)$$

Denote by E_b the event on the left hand side. As in Case (a), (3.5) will follow if we construct a map $f : E_b \to \{X \leftrightarrow Y \text{ in } Z\}$ such that for some constant $D < \infty$, (1) for each $\omega \in E_b$, ω and $f(\omega)$ differ in at most D edges, (2) at most D ω's are mapped to the same configuration.

Take a configuration $\omega \in E_b$. Let U be the set of all points $u \in \overline{\Gamma}$ such that u is connected to X in Z by an open self-avoiding path that from the first step on does not visit $\overline{\{u\}}$. For each $u \in U$, choose one such open self-avoiding path and denote it by π_u.

We first assume that there exists $u \in U$ such that Y is connected to Γ in $Z \setminus \overline{\{u\}}$. For such ω's, we define $f(\omega)$ as follows. We

(a) close all the edges with an end-vertex in $\overline{\{u\}}$ except for the edges of π_u and Γ,

(b) open all the edges in $\overline{\{u\}}$ that belong to a shortest path between u and Γ in $\overline{\{u\}}$.

Notice that ω and $f(\omega)$ differ in at most $2d\,(k+1)^{d-2}$ edges. Y is connected to Γ in $Z\setminus\overline{\{u\}}$ in the configuration $f(\omega)$. Finally, since u and Γ are in different open clusters in ω, after connecting u and Γ by a simple open path as in (b), no new open circuits are created. Thus, the set $\overline{\{u\}}$ can be uniquely reconstructed in $f(\omega)$ as the unique such set where X is connected to Γ.

Assume next that for any $u \in U$, Y is not connected to Γ in $Z \setminus \overline{\{u\}}$. Take $u \in U$. There exists $v \in \overline{\{u\}}$ such that v is connected to Y in Z by an open self-avoiding path that from the first step on does not visit $\overline{\{v\}}$. Choose one such open self-avoiding path and denote it by π_v. For such ω's, we define $f(\omega)$ exactly as in the first part of Case (a). We

(a) close all the edges with an end-vertex in $\overline{\{u\}}$ except for the edges of π_u, π_v, and Γ,

(b) open all the edges in $\overline{\{u\}}$ that belong to a shortest path ρ between u and Γ in $\overline{\{u\}}$,

(c) open all the edges in $\overline{\{u\}}$ that belong to a shortest path between v and $\Gamma \cup \rho$ in $\overline{\{u\}}$.

Notice that unlike in Case (a), it is allowed here that $v \in \Gamma$, but this makes no difference for the construction. Indeed, after closing edges as in (a), Y remains connected to Γ only if $v \in \Gamma$. Thus, after modifying ω according to (a), either u, v, and Γ are all in different open clusters or $v \in \Gamma$ and the clusters of u and Γ are different. In both cases, after connecting u, v, and Γ by simple open paths as in (b) and (c), no new open circuits are created. Thus, the set $\overline{\{u\}}$ can be uniquely reconstructed in $f(\omega)$ as the unique set of the form $\overline{\{z\}}$ where X (and Y) is connected to Γ.

The function f satisfies requirements (1) and (2), and the proof of (3.5) is complete.

Since the proof of Case (c) is essentially the same as the proof of Case (b), we omit it. Cases (a)-(c) imply (3.3). The proof of Lemma 3.2 is complete. □

Remark 3.3. (1) Theorem 3.1 and Remark 2.1 are used in [2] to extend various results of Járai [17] to slabs. For instance, to prove that the local limit of the occupancy configurations around vertices in the bulk of a crossing cluster of large box is given by the IIC measure from Theorem 3.1.

(2) Using Lemma 3.2, one can show that the expected number of vertices of the IIC in $Q(n)$ is comparable to $n^2\mathbb{P}[0 \leftrightarrow \partial Q(n)]$.

(3) In [8], the so-called multiple-armed IIC measures were introduced for planar lattices, which are supported on configurations with several disjoint infinite open clusters meeting in a neighborhood of the origin. These measures describe the local occupancy configurations around outlets of the invasion percolation [8] and pivotals for open crossings of large boxes [2]. It would be interesting to construct multiple-armed IIC measures on slabs, but at the moment it seems quite difficult.

References

[1] Aizenman, M.: On the number of incipient spanning clusters. *Nucl. Phys. B* **485**, (1997), 551–582. MR-1431856

[2] Basu, D.: Generalizations and interpretations of Incipient Infinite Cluster measure on planar lattices and slabs. PhD thesis, (2016).

[3] Basu, D. and Sapozhnikov, A.: Crossing probabilities for critical Bernoulli percolation on slabs. *To appear in Ann. Inst. Henri Poincaré Probab. Stat.*, arXiv:1512.05178

[4] Benjamini, I. and Schramm, O.: Percolation beyond $\mathbb{Z}^d$, many questions and a few answers. *Electron. Comm. Probab.* **1**, (1996), 71–82. MR-1423907

[5] Borgs, C., Chayes, J. T., Kesten, H. and Spencer, J.: Uniform boundedness of critical crossing probabilities implies hyperscaling. *Random Structures Algorithms* **15**, (1999), 368–413. MR-1716769

[6] Coniglio, A.: Shapes, surfaces and interfaces in percolation clusters. *Proc Les Houches Conf on Physics of Finely Divided Matter*, M. Daoud and N. Boccara (Editors), Springer-Verlag, Berlin, (1985), 84–109.

[7] Chayes, J. T. and Chayes, L.: On the upper critical dimension of Bernoulli percolation. *Comm. Math. Phys.* **113(1)**, (1987), 27–48. MR-0918403

[8] Damron, M. and Sapozhnikov, A.: Outlets of $2D$ invasion percolation and multiple-armed incipient infinite clusters. *Probab. Th. Rel. Fields* **150**, (2011), 257–294. MR-2800910

[9] Duminil-Copin, H., Sidoravicius, V. and Tassion, V.: Absence of infinite cluster for critical Bernoulli percolation on slabs. *Comm. Pure Appl. Math.* **69(7)**, (2016), 1397–1411. MR-3503025

[10] Fitzner, R. and van der Hofstad, R.: Mean-field behavior for nearest-neighbor percolation in $d > 10$ arXiv:1506.07977

[11] Häggström, O. and Peres, Y.: Monotonicity of uniqueness for percolation on Cayley graphs: all infinite clusters are born simultaneously. *Probab. Th. Rel. Fields* **113**, (1999), 273–285. MR-1676835

[12] Hara, T.: Decay of correlations in nearest-neighbor self-avoiding walk, percolation, lattice trees and animals. *Ann. Probab.* **36(2)**, (2008), 530–593. MR-2393990

[13] Hara, T. and Slade, G.: Mean-field critical behaviour for percolation in high dimensions. *Comm. Math. Phys.* **128(2)**, (1990), 333–391. MR-1043524

[14] Heydenreich, M., van der Hofstad, R. and Hulshof, T.: High-dimensional incipient infinite clusters revisited. *J. Stat. Phys.* **155**, (2014), 966–1025. MR-3198669

[15] van der Hofstad, R. and Jarai, A.: The incipient infinite cluster for high-dimensional unoriented percolation. *J. Stat. Phys.* **114(3)**, (2004), 625–663. MR-2035627

[16] Hopf, E.: An inequality for positive linear integral operators. *J. Math. Mech.* **12**, (1963), 683–692. MR-0165325

[17] Jarai, A.: Incipient infinite percolation clusters in $2D$. *Ann. Probab.* **31(1)**, (2003), 444–485. MR-1959799

[18] Kesten, H.: The incipient infinite cluster in two-dimensional percolation. *Probab. Th. Rel. Fields* **73**, (1986), 369–394. MR-0859839

[19] Kozma, G. and Nachmias, A.: Arm exponents in high dimensional percolation. *J. Amer. Math. Soc.* **24(2)**, (2011), 375–409. MR-2748397

[20] Newman, Ch., Tassion, V. and Wu, W.: Critical percolation and the minimal spanning tree in slabs, arXiv:1512.09107

[21] Russo, L.: A note on percolation. *Z. Wahrsch. Verw. Gebiete* **43(1)**, (1978), 39–48. MR-0488383

[22] Schonmann, R.: Stability of infinite clusters in supercritical percolation. *Probab. Th. Rel. Fields* **113(2)**, (1999), 287–300. MR-1676831

[23] Seymour, P. D. and Welsh, D. J. A.: Percolation probabilities on the square lattice. *Annals of Discrete Mathematics* **3**, (1978), 227–245. MR-0494572

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