

## Convergence of complex martingales in the branching random walk: the boundary

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### Abstract

Biggins [Uniform convergence of martingales in the branching random walk. *Ann. Probab.*, 20(1):137–151, 1992] proved local uniform convergence of additive martingales in  $d$ -dimensional supercritical branching random walks at complex parameters  $\lambda$  from an open set  $\Lambda \subseteq \mathbb{C}^d$ . We investigate the martingales corresponding to parameters from the boundary  $\partial\Lambda$  of  $\Lambda$ . The boundary can be decomposed into several parts. We demonstrate by means of an example that there may be a part of the boundary, on which the martingales do not exist. Where the martingales exist, they may diverge, vanish in the limit or converge to a non-degenerate limit. We provide mild sufficient conditions for each of these three types of limiting behaviors to occur. The arguments that give convergence to a non-degenerate limit also apply in  $\Lambda$  and require weaker moment assumptions than the ones used by Biggins.

**Keywords:** branching random walk; complex martingales.

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## 1 Introduction

Biggins [8] proved local uniform convergence of additive martingales in a supercritical branching random walk on  $\mathbb{R}^d$  at complex parameters within a certain open set  $\Lambda \subseteq \mathbb{C}^d$ . He used the results obtained to derive a local large deviation result for the point process of the positions in the  $n$ th generation as  $n \rightarrow \infty$ .

In some situations, the arguments from [8] cover parts of the boundary  $\partial\Lambda$  of  $\Lambda$ , but typically only a proper, possibly empty, subset of  $\partial\Lambda$ . However, the ideas and results required to deal with the boundary are available in the literature nowadays, but spread over different papers [1, 11, 14] and not directly applicable. In this paper, we gather these techniques and results and provide a complete treatment (up to mild moment assumptions) of the convergence of additive martingales on the boundary  $\partial\Lambda$ .

Besides its value in the study of large deviation results for the branching random walk and its intrinsic interest, there is further motivation to study the convergence of additive martingales at complex parameters, particularly on the boundary  $\partial\Lambda$ .

First, in the recent applied probability literature, there are several examples of limit theorems, in which the limiting behavior of a quantity of interest is described by the

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solution to a complex smoothing equation, see [20] for a discussion and a collection of examples including fragmentation processes and Pólya urns. In each known example and in the whole setup of [20], the main result of which is the description of the set of all solutions to non-critical smoothing equations, this solution can always be chosen as the limit of an additive martingale in a suitable branching random walk at a complex parameter from  $\Lambda$ . If one aims at extending the results from [20] from non-critical to critical smoothing equations, then limits of additive martingales in suitable branching random walks at complex parameters from the boundary of  $\Lambda$  figure. As the application to critical smoothing equations is the main motivation for us for writing the note at hand, we will describe the link between smoothing equations and the convergence of additive martingales at complex parameters in greater detail in Section 5.

Second, the additive martingales are intimately connected with cascade measures, processes that have initially been introduced by Mandelbrot as statistical models for turbulence [18, 19]. The parameters on the boundary  $\partial\Lambda$  correspond to boundaries between different phases of the cascade model, see e.g. [6, 17] and the references therein. Our main result, Theorem 2.1, suggests that one might expect that the general, not necessarily Gaussian, complex cascade measures converge also on the phase boundary between the diffuse phase and the glassy phase (on  $\partial\Lambda^{(1,2)}$  in our notation given below).

The cascade measures are also toy models for multiplicative chaos measures. In the particular Gaussian case studied in [15] these are measures of the form

$$e^{\gamma X(x) + i\beta Y(x)} dx, \tag{1.1}$$

where  $X$  and  $Y$  are two independent log-correlated Gaussian fields. The processes  $X$  and  $Y$  cannot be defined as functions and therefore (1.1) is not a proper definition. To overcome this issue one needs to work with approximations  $X_\epsilon$ ,  $Y_\epsilon$  and proper normalizations (see [15] for more details). Our main result, Theorem 2.1, might be viewed as a discrete, non-Gaussian counterpart of the convergence of the suitably normalized approximations to complex Gaussian multiplicative chaos measures in the diffuse phase and the boundary between the diffuse and the glassy phase.

## 2 Main results

**Model description.** We consider a branching random walk in  $\mathbb{R}^d$  where  $d \in \mathbb{N} = \{1, 2, \dots\}$ . The process starts with an initial ancestor at the origin. The ancestor forms generation 0 of the process and produces offspring placed on  $\mathbb{R}^d$  at the points of a point process  $\mathcal{Z} = \sum_{j=1}^N \delta_{X_j}$  with intensity measure  $\mu$ . The children of the ancestor form the first generation of the process. Each member of the first generation has children with positions relative to their parent's position given by an independent copy of  $\mathcal{Z}$ , and so on. We suppose that the branching random walk is supercritical, that is,  $\mu(\mathbb{R}^d) = \mathbb{E}[N] > 1$ .

More formally, let  $\mathcal{I} := \bigcup_{n \geq 0} \mathbb{N}^n$  be the set of finite tuples of positive integers. If  $u = (u_1, \dots, u_n) \in \mathbb{N}^n$  and  $v = (v_1, \dots, v_m) \in \mathbb{N}^m$ , we write  $u_1 \dots u_n$  for  $u$  and  $uv$  for  $(u_1, \dots, u_n, v_1, \dots, v_m)$ . Further, we write  $u|_k$  for  $(u_1, \dots, u_{k \wedge n})$ ,  $k \in \mathbb{N}_0$ .

The ancestor is identified with the empty tuple  $\emptyset$  and its position is  $S(\emptyset) = 0$ . On some probability space  $(\Omega, \mathcal{A}, \mathbb{P})$ , let  $(\mathcal{Z}(u))_{u \in \mathcal{I}}$  be a family of i.i.d. copies of  $\mathcal{Z}$ . For ease of notation, we assume  $\mathcal{Z}(\emptyset) = \mathcal{Z}$ . We write  $\mathcal{Z}(u) = \sum_{i=1}^{N(u)} \delta_{X_i(u)}$  where  $(N(u), X_1(u), X_2(u), \dots)$  is an independent copy of  $(N, X_1, X_2, \dots)$ . In particular,  $N(u) = \mathcal{Z}(u)(\mathbb{R}^d)$ ,  $u \in \mathcal{I}$ . Then  $\mathcal{G}_0 := \{\emptyset\}$  is generation 0 of the process and, recursively,

$$\mathcal{G}_{n+1} := \{uj \in \mathbb{N}^{n+1} : u \in \mathcal{G}_n \text{ and } 1 \leq j \leq N(u)\}$$

is generation  $n + 1$  of the process,  $n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$ . Define the set of all individuals by  $\mathcal{G} := \bigcup_{n \in \mathbb{N}_0} \mathcal{G}_n$ . We write  $|u| = n$  for  $u \in \mathcal{G}_n$  and  $|u| < n$  if  $u \in \mathcal{G}_k$  for some  $k < n$ . The

position of an individual  $u = u_1 \dots u_n \in \mathcal{G}_n$  is given by

$$S(u) := X_{u_1}(\emptyset) + \dots + X_{u_n}(u_1 \dots u_{n-1}).$$

The point process of the  $n$ th generation people will be denoted by  $\mathcal{Z}_n$ , that is,

$$\mathcal{Z}_n = \sum_{|u|=n} \delta_{S(u)}.$$

The sequence of point processes  $(\mathcal{Z}_n)_{n \in \mathbb{N}_0}$  is then called a *branching random walk*.

The multivariate Laplace transform  $m$  of  $\mu$  is denoted by

$$m(\lambda) = \int e^{-\lambda x} \mu(dx),$$

where  $\lambda \in \mathbb{C}^d$  and  $\lambda = \theta + i\eta$  with  $\theta, \eta \in \mathbb{R}^d$ . (We adopt the convention from [8] and always write  $\theta$  for  $\text{Re}(\lambda)$  and  $\eta$  for  $\text{Im}(\lambda)$ .) We are only interested in those  $\lambda$  for which  $m(\lambda)$  is well-defined, i. e.,  $\lambda$  from the set

$$\mathcal{D} = \{\lambda \in \mathbb{C}^d : m(\lambda) \text{ converges absolutely}\} = \{\theta \in \mathbb{R}^d : m(\theta) < \infty\} + i\mathbb{R}^d.$$

Throughout, we assume  $\text{int } \mathcal{D} \neq \emptyset$ . Let  $\mathcal{F}_0$  be the trivial  $\sigma$ -field and, for  $n \in \mathbb{N}$ ,

$$\mathcal{F}_n := \sigma(\mathcal{Z}(u) : u \in \mathbb{N}^k \text{ for some } k < n).$$

Then, for  $\lambda \in \mathcal{D}$  with  $m(\lambda) \neq 0$ , the family

$$Z_n(\lambda) = m(\lambda)^{-n} \sum_{|u|=n} e^{-\lambda S(u)}, \quad n \in \mathbb{N}_0$$

forms a complex martingale with respect to  $(\mathcal{F}_n)_{n \in \mathbb{N}_0}$ .

**Point of departure.** Biggins [8, Theorem 1] proved that if

$$\mathbb{E}[Z_1(\theta)^\gamma] < \infty \quad \text{for some } \gamma \in (1, 2] \tag{2.1}$$

and

$$\frac{m(p\theta)}{|m(\lambda)|^p} < 1 \quad \text{for some } p \in (1, \gamma], \tag{2.2}$$

then  $(Z_n(\lambda))_{n \geq 0}$  converges almost surely and in  $p$ th mean to a limit variable  $Z(\lambda)$ . What is more, Biggins [8, Theorem 2] proved that this convergence is locally uniform (almost surely and in mean) on the set  $\Lambda = \bigcup_{\gamma \in (1, 2]} \Lambda_\gamma$  where  $\Lambda_\gamma = \Lambda_\gamma^1 \cap \Lambda_\gamma^3$  and, for  $\gamma \in (1, 2]$ ,

$$\Lambda_\gamma^1 = \text{int}\{\lambda \in \mathcal{D} : \mathbb{E}[Z_1(\theta)^\gamma] < \infty\} \quad \text{and} \quad \Lambda_\gamma^3 = \text{int}\{\lambda \in \mathcal{D} : \inf_{1 \leq p \leq \gamma} \frac{m(p\theta)}{|m(\lambda)|^p} < 1\}.$$

**The boundary of  $\Lambda$ .** We decompose  $\partial\Lambda$  into several parts. The first part is  $\partial\Lambda^0 := \partial\Lambda \cap \mathcal{D}^c$ . Notice that  $\partial\Lambda^0$  may be non-empty, see Example 3.2 in Section 3. The martingale  $(Z_n(\lambda))_{n \geq 0}$  is not defined on  $\partial\Lambda^0$ , so we will exclude this set from the further discussion. We introduce a weaker form of (2.2), namely,

$$\frac{m(\alpha\theta)}{|m(\lambda)|^\alpha} = 1 \quad \text{and} \quad \mathbb{E}\left[\sum_{|u|=1} \theta S(u) \frac{e^{-\alpha\theta S(u)}}{|m(\lambda)|^\alpha}\right] \geq -\log(|m(\lambda)|) \quad \text{for some } \alpha \in [1, 2]. \tag{C1}$$

Notice that the second condition in (C1) can be rewritten as  $\frac{d}{d\alpha} m(\theta\alpha)/|m(\lambda)|^\alpha \leq 0$ . Additionally, we use the following moment condition:

$$\mathbb{E}[|Z_1(\lambda)|^\alpha \log_+^{2+\epsilon}(|Z_1(\lambda)|)] < \infty \quad \text{for some } \epsilon > 0 \tag{C2}$$

with the same  $\alpha$  as in (C1). Subject to the moment condition (C2), there is convergence almost surely and in mean of the martingales at  $\lambda$  from

$$\partial\Lambda^{(1,2)} := \{\lambda \in \partial\Lambda \cap \mathcal{D} : (C1) \text{ holds with } \alpha \in (1, 2)\},$$

see Theorem 2.1 below. On the set

$$\partial\Lambda^1 := \{\lambda \in \mathcal{D} \cap \partial\Lambda : (C1) \text{ holds with } \alpha = 1\},$$

we have from the first condition in (C1) with  $\alpha = 1$  that  $m(\theta) = |m(\lambda)|$ . Notice that this does not imply that  $\lambda$  is real as (higher-dimensional) lattice-type effects may occur, see e.g. Example 3.2. Hence,  $Z_1(\lambda) = Z_1(\theta)$  almost surely. Consequently,  $(Z_n(\lambda))_{n \geq 0}$  is a nonnegative martingale for  $\lambda \in \partial\Lambda^1$ . Whether or not the additive martingale in the branching random walk converges in the real case is known from Biggins' martingale convergence theorem [3, 7, 16]. We therefore omit the treatment of  $\partial\Lambda^1$  in what follows. Further, typically (see Proposition 2.2 for the details), there is no convergence on

$$\begin{aligned} \partial\Lambda^2 &:= \{\lambda \in \mathcal{D} \cap \partial\Lambda : (C1) \text{ holds with } \alpha = 2\} \\ \text{and } \partial\Lambda^3 &:= \{\lambda \in \mathcal{D} \cap \partial\Lambda : \mathbb{E}[|Z_1(\lambda)|^\gamma] = \infty \text{ for every } \gamma > 1\}. \end{aligned}$$

In most situations, it will hold that

$$\partial\Lambda = \partial\Lambda^0 \cup \partial\Lambda^1 \cup \partial\Lambda^{(1,2)} \cup \partial\Lambda^2 \cup \partial\Lambda^3, \tag{2.3}$$

i.e., the sets defined above exhaust  $\partial\Lambda$ . There is a discussion including a set of (mild) conditions that ensure (2.3) to hold in Section 3 below.

**Main theorems.** To unburden the notation, we fix  $\lambda \in \mathcal{D}$  and set  $L(u) := m(\lambda)^{-n} e^{-\lambda S(u)}$  if  $u \in \mathcal{G}_n$  for some  $n \in \mathbb{N}_0$ , and  $L(u) := 0$ , otherwise. We write  $Z_n$  for  $Z_n(\lambda)$ ,  $n \in \mathbb{N}$  and  $Z$  for  $Z(\lambda)$  if the latter exists. By construction,  $(Z_n)_{n \geq 0}$  is a complex martingale with

$$\mathbb{E}[Z_1] = 1. \tag{2.4}$$

To avoid trivialities, we assume that  $\mathbb{P}(Z_1 = 1) < 1$ . Condition (C1) in the simplified notation becomes

$$\mathbb{E}\left[\sum_{|u|=1} |L(u)|^\alpha\right] = 1 \text{ and } \mathbb{E}\left[\sum_{|u|=1} |L(u)|^\alpha \log(|L(u)|)\right] \leq 0 \text{ for some } \alpha \in [1, 2]. \tag{C1}$$

Condition (C2) in the simplified notation reads

$$\mathbb{E}[|Z_1|^\alpha \log_+^{2+\epsilon}(|Z_1|)] < \infty \text{ for some } \epsilon > 0. \tag{C2}$$

Sometimes, we will refer to the following condition:

$$\mathbb{E}\left[\sum_{|u|=1} |L(u)|^\vartheta\right] < \infty \text{ for some } \vartheta \in [0, \alpha]. \tag{C3}$$

We further define  $W_n := \sum_{|u|=n} |L(u)|^\alpha$ ,  $n \in \mathbb{N}_0$ . Then, by (C1),  $(W_n)_{n \geq 0}$  is a nonnegative martingale. The martingale convergence theorem and Fatou's lemma give  $W_n \rightarrow W$  almost surely for a nonnegative random variable  $W$  with  $\mathbb{E}[W] \in \{0, 1\}$ . Whether  $\mathbb{E}[W] = 0$  or  $\mathbb{E}[W] = 1$  is known from Biggins' martingale convergence theorem [3, 7, 16].

The following theorem is the main result of the paper. It gives convergence of the additive martingales to non-degenerate limits on  $\partial\Lambda^{(1,2)}$ .

**Theorem 2.1.** *Suppose that (C1) and (C2) hold with  $\alpha \in (1, 2)$ . Then  $(Z_n)_{n \geq 0}$  converges almost surely and in  $L^p$  for every  $p < \alpha$  to a non-degenerate limit  $Z$ .*

It is natural to wonder when  $(Z_n)_{n \geq 0}$  converges in  $L^p$  for some  $p \geq \alpha$ . It follows from classical martingale theory that  $Z_n \rightarrow Z$  in  $L^p$  holds iff  $Z_n \rightarrow Z$  in  $L^1$  and  $Z \in L^p$ . Hence, in order to answer the question, one has to determine the tail behavior of the limit  $Z$ , which solves a smoothing equation. The tail behavior of such solutions has been studied extensively in the literature, we confine ourselves to referring to [10, Theorem 2.11]. Under a certain set of assumptions, the cited theorem guarantees that  $\mathbb{P}(|Z| > t)$  is of the order  $t^{-\beta}$  where  $\beta$  is defined by the two requirements  $\beta > \alpha$  and  $\mathbb{E}[\sum_{|u|=1} |L(u)|^\beta] = 1$ .

The following propositions are essentially contained in [14] and provide sufficient conditions for the divergence of the additive martingales on  $\partial\Lambda^2$  and  $\partial\Lambda^3$ , respectively.

**Proposition 2.2.** *Suppose that  $\mathbb{P}(N < \infty) = 1$  and that (C1) holds with  $\alpha = 2$ . Then each of the following two conditions is sufficient for  $(Z_n)_{n \geq 0}$  not to converge in probability.*

- (i)  $\mathbb{E}[\sum_{|u|=1} |L(u)|^2 \log(|L(u)|)] \in (-\infty, 0)$  and  $\mathbb{E}[W_1 \log_+ W_1] < \infty$ ,
- (ii)  $\mathbb{E}[\sum_{|u|=1} |L(u)|^2 \log(|L(u)|)] = 0$ , (C3) holds and

$$\mathbb{E}[\sum_{|u|=1} |L(u)|^2 \log^2(|L(u)|)] < \infty, \mathbb{E}[W_1 \log_+^2(W_1)] < \infty \text{ and } \mathbb{E}[\widetilde{W}_1 \log_+(\widetilde{W}_1)] < \infty$$

where  $\widetilde{W}_1 := \sum_{|u|=1} |L(u)|^2 \log_-(|L(u)|)$ .

**Proposition 2.3.** *Suppose that  $\mathbb{P}(N < \infty) = 1$  and that (C1) with  $\alpha > 1$  and (C3) hold. If  $\mathbb{E}[|Z_1|^p] = \infty$  for some  $p \in (1, \alpha)$ , then  $(Z_n)_{n \geq 0}$  does not converge in probability.*

**Remark 2.4.** It is natural to ask whether on parts of the boundary where the martingales do not converge there are constants  $a_n(\lambda)$  such that  $a_n(\lambda)Z_n(\lambda)$  converges as  $n \rightarrow \infty$ . For instance, for  $\lambda \in \partial\Lambda^1$ ,  $Z_n(\lambda)$  is nonnegative and hence converges almost surely to a limit  $Z(\lambda)$ . There is a criterion for  $\mathbb{P}(Z(\lambda) = 0) \neq 1$ , see [3]. If  $\mathbb{P}(Z(\lambda) = 0) = 1$ , there are results that give sufficient conditions for the existence of scaling constants  $a_n(\lambda)$  such that  $a_n(\lambda)Z_n(\lambda)$  converges in probability to a nondegenerate limit. Biggins and Kyprianou [9] proved the existence of such norming constants in the case where an “ $X \log X$ ” condition fails. Aïdékon and Shi [1] showed that in the “boundary case” the scaling constants can be chosen as  $a_n(\lambda) = \sqrt{n}$ ,  $n \in \mathbb{N}$ . In both papers,  $a_n(\lambda)/a_{n+1}(\lambda) \rightarrow 1$  so that the limit distribution again is a solution (with infinite mean) of a smoothing equation.

The situation is different on  $\partial\Lambda^2$  and  $\partial\Lambda^3$ . It follows from the theory developed in [20] that when studying a given complex smoothing equation, there is an associated additive martingale that is of major importance for understanding the smoothing equation. If the martingale corresponds to a parameter from  $\partial\Lambda^2$  or  $\partial\Lambda^3$ , then there are no scaling constants such that the scaled martingale converges in probability to a solution of the original smoothing equation. This is the reason why we do not address the problem of finding norming constants here. More details can be found in Section 5.

Nevertheless, there is interest in renormalizing the martingales corresponding to parameters from  $\Lambda^c$  that vanish in the limit or do not converge. The case of Gaussian multiplicative chaos studied in [15] suggests that one can indeed find  $a_n(\lambda)$  such that  $a_n(\lambda)Z_n(\lambda)$  converges in distribution. On  $\partial\Lambda$ , from [15], we expect that  $a_n(\lambda) = n^{-1/2}$  if condition (i) of Proposition 2.2 is fulfilled and  $a_n(\lambda) = n^{-1/4}$  for  $\lambda \in \text{cl}(\partial\Lambda^{(1,2)}) \cap \partial\Lambda^{(2)}$  (corresponding to the red curves and the red dots, respectively, in Figure 1).

**Remark 2.5.** In both propositions, we require  $\mathbb{P}(N < \infty) = 1$ . This is because their proofs are based on arguments from [4, 14] involving complex multiplicative martingales and convergence of triangular arrays. It may be possible, but certainly tedious, to extend those arguments to the case  $\mathbb{P}(N = \infty) > 0$ . As we want to keep the presentation short and accessible, we refrain from trying to remove the assumption.

The rest of the paper is organized as follows. In Section 3, we give a brief discussion of the shape of  $\Lambda$ , its boundary and the parts in which the boundary can be divided. We further give an example to illustrate our results. Section 4 contains the proofs of

our results, while Section 5 contains extensions of the main results to a more general, multidimensional situation. Finally, there is an appendix comprising an auxiliary result required in the proof of Theorem 2.1.

### 3 Discussion and examples

It is illustrative to first consider examples.

**Examples.** We begin with an example which is strongly reminiscent of the situation studied in [17]. We also refer to [15], where the problem of convergence on  $\Lambda_\gamma^{(1,2)}$  is studied in the different context of Gaussian multiplicative chaos.

**Example 3.1** (The Gaussian case with binary splitting). Consider a branching random walk with independent standard Gaussian increments and binary splitting, i. e.,  $\mathcal{Z} = \delta_{X_1} + \delta_{X_2}$  where  $X_1, X_2$  are i.i.d. random variables with standard normal laws. Then  $m(\lambda) = 2 \exp(\lambda^2/2)$  for all  $\lambda \in \mathbb{C}$ . For every  $\theta \in \mathbb{R}$  and every  $\gamma > 1$ , we have

$$\mathbb{E}[Z_1(\theta)^\gamma] = \frac{1}{m(\theta)^\gamma} \mathbb{E}[(e^{-\theta X_1} + e^{-\theta X_2})^\gamma] \leq \frac{2^\gamma}{m(\theta)^\gamma} \mathbb{E}[e^{-\theta \gamma X_1}] = \frac{2^\gamma m(\theta \gamma)}{m(\theta)^\gamma} < \infty.$$

Hence  $\Lambda = \{\lambda \in \mathbb{C} : m(p\theta)/|m(\lambda)|^p < 1 \text{ for some } p \in (1, 2]\}$ . Thus,  $\lambda \in \Lambda$  if and only if there exists some  $p \in (1, 2]$  with  $m(p\theta)/|m(\lambda)|^p < 1$ . The latter inequality is equivalent to

$$(1 - p)2 \log 2 + p^2 \theta^2 - p(\theta^2 - \eta^2) < 0. \tag{3.1}$$

By symmetry, it suffices to consider  $\theta, \eta \geq 0$  only. Next notice that  $\sup\{\theta : \lambda \in \Lambda\} = \sqrt{2 \log 2}$ . For fixed  $\theta \in [0, \sqrt{2 \log 2}]$ , making (3.1) explicit in  $\eta^2$  gives:

$$\eta^2 < \frac{p-1}{p} 2 \log 2 - (p-1)\theta^2.$$

The right-hand side assumes its maximum (as a function of  $p \in (1, 2]$ ) at  $p = (\sqrt{2 \log 2}/\theta) \wedge 2$  giving  $\eta < \sqrt{2 \log 2} - \theta$  for all  $\sqrt{\log 2}/\sqrt{2} \leq \theta < \sqrt{2 \log 2}$ . For  $0 \leq \theta \leq \sqrt{(\log 2)/2}$ , we get  $\theta^2 + \eta^2 < \log 2$ . In conclusion, we get the shape depicted in Figure 1 for  $\Lambda$ .

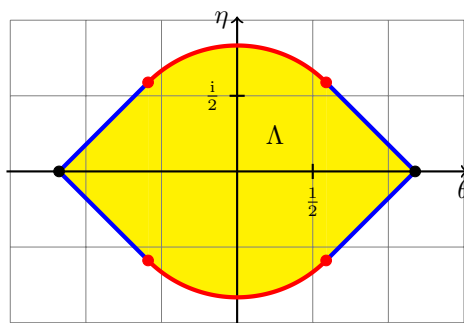


Figure 1: The figure shows  $\Lambda$  (in yellow) and  $\partial\Lambda$  (in red, blue and with two black dots) for the branching random walk with binary splitting and independent standard Gaussian increments. Convergence of the additive martingales for  $\lambda$  from the yellow phase follows from [8, Theorem 1], but also from our Theorem 2.1. The black dots form  $\partial\Lambda^1$  and correspond to the real martingale in what is called the boundary case in the literature. There is no convergence at the black dots. The blue lines form  $\partial\Lambda^{(1,2)}$  and thus correspond to the case  $1 < \alpha < 2$ . Theorem 2.1 yields that there is convergence to a nontrivial limit on the blue lines. The red lines including the endpoints form  $\partial\Lambda^2$ , which is dealt with in Proposition 2.2. The proposition yields that there is no convergence on the red arcs without the endpoints and in the endpoints, respectively.

We continue with a somewhat pathological example in which  $\partial\Lambda \cap \mathcal{D}^c \neq \emptyset$ .

**Example 3.2.** Let  $\mathcal{Z} = \sum_{k=1}^N \delta_{X_k}$  with  $\mathbb{P}(N = n(n+1)) = 1/(n(n+1))$  for all  $n \in \mathbb{N}$  and  $\mathbb{P}(X_k = n \mid N = n(n+1)) = 1$  for  $k = 1, \dots, n(n+1)$ . Then, for  $\theta > 0$ ,  $m(\theta) = e^{-\theta}/(1-e^{-\theta})$ . It is easily checked that  $\mathbb{E}[Z_1(\theta)^2] < \infty$  for all  $\theta > 0$ . We now explicitly determine  $\Lambda$ . To this end, notice that any  $\lambda$  with  $\theta > 0$  is in  $\Lambda$  iff for some  $p \in (1, 2]$ , we have  $m(p\theta)/|m(\lambda)|^p < 1$ , equivalently,

$$|1 - e^{-(\theta+i\eta)}|^p = (|1 - e^{-(\theta+i\eta)}|^2)^{p/2} = (1 - 2e^{-\theta} \cos \eta + e^{-2\theta})^{p/2} < 1 - e^{-p\theta}.$$

Making this inequality explicit in  $\cos \eta$  results in

$$\frac{1}{2}(e^\theta - (1 - e^{-p\theta})^{2/p}e^\theta + e^{-\theta}) < \cos \eta.$$

Since  $p = 2$  is the minimizer for the left-hand side as a function of  $p \in [1, 2]$ , we have  $\lambda \in \Lambda$  iff  $e^{-\theta} < \cos \eta$ . Thus, since  $m(0) = \infty$ , it holds that

$$\Lambda = \{\theta + i\eta : \theta > 0, e^{-\theta} < \cos \eta\} = \bigcup_{n \in \mathbb{Z}} (2\pi i n + \{\theta + i\eta : \theta > 0, |\eta| < \frac{\pi}{2}, e^{-\theta} < \cos \eta\}).$$

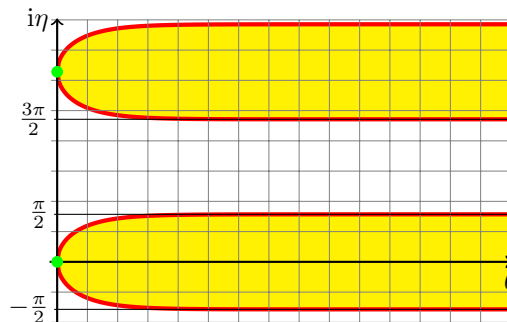


Figure 2: The figure shows  $\Lambda$  (in yellow) and  $\partial\Lambda$  (the red curves and green dots). As  $\mathcal{Z}$  is concentrated on  $\mathbb{Z}$ , the Laplace transform  $m$  is  $2\pi i$ -periodic, hence  $\Lambda$  consists of a countable family of shifted copies of the connected part of  $\Lambda$  intersecting the halfline  $\{\lambda : \theta > 0, \eta = 0\}$ . Convergence of the additive martingales for  $\lambda$  from the yellow phase follows from [8, Theorem 1] and Theorem 2.1. The green dots correspond to the domain  $\partial\Lambda^0 = \partial\Lambda \cap \mathcal{D}^c$ . The martingale is not defined on this set. The red curves form  $\partial\Lambda^2$ , i.e., they correspond to the case  $\alpha = 2$ . There is no convergence on the red curves by Proposition 2.2(a) (there is some checking required to see that the proposition applies).

**Discussion of the assumptions.** There is a discussion of the shape of  $\Lambda$  on p. 141 of [8]. Here, we want to confine ourselves to explaining why one can expect that (2.3) holds, i.e., that the boundary is typically exhausted by  $\partial\Lambda^0 \cup \partial\Lambda^1 \cup \partial\Lambda^{(1,2)} \cup \partial\Lambda^2 \cup \partial\Lambda^3$ .

**Lemma 3.3.** Let  $\lambda \in \partial\Lambda_\gamma$  for some  $\gamma \in (1, 2]$ . If  $\mathbb{P}(Z_1(\lambda) \in [0, \infty)) < 1$ , then (C1) holds with  $\alpha \in (1, \gamma]$ .

*Proof.* We conclude  $m(\theta) < \infty$  from  $\mathbb{E}[Z_1(\theta)^\gamma] < \infty$ . We further have

$$m(\gamma\theta) = \mathbb{E}[\sum_{|u|=1} e^{-\gamma\theta S(u)}] \leq \mathbb{E}[(\sum_{|u|=1} e^{-\theta S(u)})^\gamma] = \mathbb{E}[Z_1(\theta)^\gamma] < \infty.$$

Define the functions,  $p \mapsto f(p) := m(p\theta)/|m(\lambda)|^p$  and  $p \mapsto f_n(p) := m(p\theta_n)/|m(\lambda_n)|^p$ , where  $\lambda_n \in \Lambda_\gamma$  are such that  $\lambda_n \rightarrow \lambda$ . Then  $f, f_1, f_2, \dots$  are finite and continuous on  $[1, \gamma]$ .

Further,  $\lim_{n \rightarrow \infty} \lambda_n = \lambda$  implies  $f_n \rightarrow f$  pointwise on  $[1, \gamma]$  and hence  $\inf_{1 \leq p \leq \gamma} f(p) \leq 1$ . Let  $\alpha \in [1, \gamma]$  be minimal with  $f(\alpha) = m(\alpha\theta)/|m(\lambda)|^\alpha = 1$ . This is the first condition of (C1). Clearly,  $\alpha > 1$  since  $\mathbb{P}(Z_1(\lambda) \in [0, \infty)) < 1$ . Thus,  $f$  is differentiable at  $\alpha$  (from the left if  $\alpha = \gamma$ ) with  $f'(\alpha) \leq 0$ , which translates into the second condition of (C1).  $\square$

The lemma explains the choice of  $\partial\Lambda^{(1,2)}$ . In the situation of the lemma, (C2) is automatically fulfilled if  $\gamma > \alpha$ . If  $\alpha = \gamma$ , we have  $\mathbb{E}[|Z_1(\lambda)|^\alpha] \leq \mathbb{E}[Z_1(\theta)^\alpha] < \infty$  and (C2) thus constitutes only a very mild additional moment assumption.

### 4 Proofs of the main results

**Many-to-one lemma and auxiliary results for random walks.** There is a well-known simple formula with far-reaching implications that connects the branching random walk  $(Z_n)_{n \in \mathbb{N}_0}$  with an associated standard random walk  $(S_n)_{n \in \mathbb{N}_0}$  on  $\mathbb{R}$ . This formula is sometimes called the many-to-one lemma and takes the following form here:

$$\mathbb{E}[f(S_0, \dots, S_n)] = \mathbb{E} \left[ \sum_{|u|=n} |L(u)|^\alpha f(0, -\log(|L(u_1)|), \dots, -\log(|L(u)|)) \right] \tag{4.1}$$

for all nonnegative Borel-measurable functions  $f : \mathbb{R}^{n+1} \rightarrow \mathbb{R}$ . The formula is used in many (possibly all) papers on branching random walks, a proof can be found, for instance, in [9, Lemma 4.1]. We just mention an important consequence of (4.1), namely, choosing  $n = 1$  and  $f(x, y) = y$ , whenever the positive or negative part of  $S_1$  or  $\sum_{|u|=1} |L(u)|^\alpha (-\log(|L(u)|))$  is integrable, we get

$$\mathbb{E}[S_1] = \mathbb{E} \left[ \sum_{|u|=1} |L(u)|^\alpha (-\log(|L(u)|)) \right]. \tag{4.2}$$

**Proofs of Theorem 2.1, Proposition 2.2 and Proposition 2.3.** For the remainder of this section, we denote by  $[\cdot]_u$ ,  $u \in \mathcal{I}$  the canonical shift-operators, i.e., if  $\Psi$  is a function of  $(Z(v))_{v \in \mathcal{I}}$ , then  $[\Psi]_u$  is the same function of  $(Z(uv))_{v \in \mathcal{I}}$ . For  $n \in \mathbb{N}$ , introduce the  $n$ th martingale difference  $D_n := Z_n - Z_{n-1} = \sum_{|u|=n-1} L(u)([Z_1]_u - 1)$ .

*Proof of Theorem 2.1.* Let  $\epsilon > 0$  be as in (C2) and choose  $\phi$  as in Lemma A.1 with  $\delta := 1 + \epsilon/2$ . We extend  $\phi$  to a function on  $\mathbb{C}$  by letting  $\phi(x + iy) := \phi(x) + \phi(y)$ ,  $x, y \in \mathbb{R}$ . Set  $\ell(x) := \phi(x)x^{-\alpha}$  for  $x > 0$  and notice that condition (C2) implies  $C_{\phi\ell} := \mathbb{E}[\phi(|Z_1 - 1| \vee 1)\ell(|Z_1 - 1| \vee 1)] < \infty$ .

For  $t > 0$ , we write  $D_n^{(t)}$  for the truncated martingale differences

$$D_n^{(t)} = \sum_{|u|=n-1} L(u) \mathbb{1}_{\{|L(u_j)| \leq t \text{ for } j=0, \dots, n-1\}} ([Z_1]_u - 1)$$

and set  $Z_0^{(t)} = 0$  and  $Z_n^{(t)} := D_1^{(t)} + \dots + D_n^{(t)}$ ,  $n \in \mathbb{N}$ . It is easy to check that  $(Z_n^{(t)})_{n \geq 0}$  is a martingale with respect to  $(\mathcal{F}_n)_{n \geq 0}$ . Clearly,  $Z_n^{(t)} = Z_n$  for all  $n \geq 0$  on the set  $\{\sup_{u \in \mathcal{G}} |L(u)| \leq t\}$ . As in the proof of Proposition 2.1 in [11], we infer from (4.1) that

$$\begin{aligned} \mathbb{P}(\sup_{u \in \mathcal{G}} |L(u)| > t) &\leq \mathbb{E}[\#\{u : |L(u)| > t \text{ and } |L(u_k)| \leq t \text{ for all } k < |u|\}] \\ &= \mathbb{E} \left[ \sum_{n \geq 0} e^{\alpha S_n} \mathbb{1}_{\{S_n < -\log t \text{ and } S_k \geq -\log t \text{ for } k=0, \dots, n-1\}} \right] < t^{-\alpha}. \end{aligned} \tag{4.3}$$

In particular,  $\lim_{t \rightarrow \infty} \mathbb{P}(\sup_{u \in \mathcal{G}} |L(u)| > t) = 0$ . Therefore, if we show that  $(Z_n^{(t)})_{n \geq 0}$  converges almost surely for every  $t > 0$ , then we infer that  $(Z_n)_{n \geq 0}$  converges almost surely to some finite limit  $Z$ .



To prove convergence of  $(Z_n^{(t)})_{n \geq 0}$ , we apply the Topchiĭ-Vatutin inequality for martingales [5, Theorem 1] twice (for the second application note that  $D_k^{(t)}$  conditional on  $\mathcal{F}_{k-1}$  is a weighted sum of independent, centered and  $\phi$ -integrable random variables)

$$\begin{aligned} \mathbb{E}[\phi(Z_n^{(t)} - 1)] &\leq 2 \sum_{k=1}^n \mathbb{E}[\phi(D_k^{(t)})] \\ &\leq 4 \sum_{k=1}^n \mathbb{E} \left[ \sum_{|u|=k-1} \phi(L(u)([Z_1]_u - 1)) \mathbf{1}_{\{|L(u)_j| \leq t \text{ for } j=1, \dots, k-1\}} \right] \\ &\leq 8 \sum_{k=1}^n \mathbb{E} \left[ \sum_{|u|=k-1} \phi(|L(u)([Z_1]_u - 1)|) \mathbf{1}_{\{|L(u)_j| \leq t \text{ for } j=1, \dots, k-1\}} \right], \end{aligned}$$

where we have used that  $\phi(z) \leq 2\phi(|z|)$  for all  $z \in \mathbb{C}$ . Using that

$$\ell(|zw|) \leq \ell(|z|)\ell(|w|)^2 \tag{4.4}$$

for all  $z, w \in \mathbb{C}$  with  $|w| \geq 1$  and the change of measure (4.1), we get

$$\begin{aligned} \mathbb{E}[\phi(Z_n^{(t)} - 1)] &\leq 8C_{\phi\ell} \sum_{k=0}^{n-1} \mathbb{E}[\phi(e^{-S_k})e^{\alpha S_k} \mathbf{1}_{\{S_i \geq -\log t \text{ for } i=1, \dots, k\}}] \\ &\leq 8C_{\phi\ell} \sum_{k=0}^{\infty} \mathbb{E}[\ell(e^{-S_k}) \mathbf{1}_{\{S_i \geq -\log t \text{ for } i=1, \dots, k\}}]. \end{aligned} \tag{4.5}$$

To see that the latter series is finite, let  $\tau_0 := 0$  and let  $\tau_n$  denote the  $n$ th strictly descending ladder epoch for the walk  $(S_k)_{k \geq 0}$ ,  $n \in \mathbb{N}$ , we refer to [13, Chapter XII] for the definition of and background on ladder epochs.

Notice that  $\mathbb{E}[S_1] \geq 0$  by (4.2), hence  $\tau_n$  may be infinite with positive probability. Then, for any  $k \geq 0$ , there exist unique (random) numbers  $n \in \mathbb{N}$  and  $j \in \mathbb{N}_0$  such that  $\tau_{n-1} \leq k = \tau_{n-1} + j < \tau_n$ . In this case,  $S_i \geq -\log t$  for all  $i = 0, \dots, k$  if and only if  $S_{\tau_{n-1}} \geq -\log t$ , and we infer from (4.4)

$$\begin{aligned} &\ell(e^{-S_k}) \mathbf{1}_{\{S_i \geq -\log t \text{ for } i=1, \dots, k\}} \\ &= \ell(e^{-(S_{\tau_{n-1}+j-S_{\tau_{n-1}})})e^{-S_{\tau_{n-1}}}) \mathbf{1}_{\{S_{\tau_{n-1}} + \log t \geq 0\}} \\ &\leq \ell(e^{-(S_{\tau_{n-1}+j-S_{\tau_{n-1}})})\ell(e^{-S_{\tau_{n-1}}})^2 \mathbf{1}_{\{S_{\tau_{n-1}} + \log t \geq 0\}} \\ &\leq \ell(t)^4 \ell(e^{-(S_{\tau_{n-1}+j-S_{\tau_{n-1}})})\ell(e^{-(S_{\tau_{n-1}} + \log t)})^2 \mathbf{1}_{\{S_{\tau_{n-1}} + \log t \geq 0\}}. \end{aligned}$$

We thus infer for the infinite series in (4.5):

$$\begin{aligned} &\sum_{k=0}^{\infty} \mathbb{E}[\ell(e^{-S_k}) \mathbf{1}_{\{S_i \geq -\log t \text{ for } i=1, \dots, k\}}] \\ &\leq \ell(t)^4 \mathbb{E} \left[ \sum_{n=1}^{\infty} \mathbf{1}_{\{\tau_{n-1} < \infty\}} \ell(e^{-(S_{\tau_{n-1}} + \log t)})^2 \mathbf{1}_{\{S_{\tau_{n-1}} + \log t \geq 0\}} \sum_{j=0}^{\tau_n - \tau_{n-1} - 1} \ell(e^{-(S_{\tau_{n-1}+j-S_{\tau_{n-1}})}) \right] \\ &= \ell(t)^4 \mathbb{E} \left[ \sum_{n=1}^{\infty} \mathbf{1}_{\{\tau_{n-1} < \infty\}} \ell(e^{-(S_{\tau_{n-1}} + \log t)})^2 \mathbf{1}_{\{S_{\tau_{n-1}} + \log t \geq 0\}} \right] \mathbb{E} \left[ \sum_{j=0}^{\tau_1 - 1} \ell(e^{-S_j}) \right], \end{aligned}$$

where we have used the strong Markov property for the random walk  $(S_k)_{k \geq 0}$ . Let  $\sigma_0 := 0$  and  $\sigma_n$  the  $n$ th weakly ascending ladder epoch of the walk  $(S_k)_{k \geq 0}$ , i.e.,  $\sigma_n := \inf\{k > \sigma_{n-1} : S_k \geq S_{\sigma_{n-1}}\}$ ,  $n \in \mathbb{N}$ . Then the duality lemma [13, p. 395] gives

$$\mathbb{E} \left[ \sum_{j=0}^{\tau_1 - 1} \ell(e^{-S_j}) \right] = \mathbb{E} \left[ \sum_{n=0}^{\infty} \ell(e^{-S_{\sigma_n}}) \right]. \tag{4.6}$$

By the choice of  $\phi$  (see Lemma A.1),  $\ell(e^{-x})$  is decreasing and  $\ell(e^{-x}) \sim c^{-1}x^{-1-\epsilon/2}$  as  $x \rightarrow \infty$ . Thus,  $x \mapsto \ell(e^{-x})\mathbb{1}_{[0,\infty)}$  is directly Riemann integrable, see [13, p. 362] for the definition. Now  $\mathbb{E}[S_1] \geq 0$  implies  $\mathbb{P}(\sigma_n < \infty)$  for all  $n \in \mathbb{N}$ . Hence  $(S_{\sigma_n})_{n \geq 0}$  is a random walk drifting to  $+\infty$ . Taken together, we infer that the expectation in (4.6) is finite. Again from the direct Riemann integrability of  $x \mapsto \ell(e^{-x})\mathbb{1}_{[0,\infty)}$ , we conclude that

$$\sup_{t>0} \mathbb{E} \left[ \sum_{n=1}^{\infty} \mathbb{1}_{\{\tau_{n-1} < \infty\}} \ell(e^{-(S_{\tau_{n-1}} + \log t)})^2 \mathbb{1}_{\{S_{\tau_{n-1}} + \log t \geq 0\}} \right] < \infty.$$

So far we have shown that there is a constant  $C > 0$ , not depending on  $t$ , such that

$$\sup_{n \geq 1} \mathbb{E}[\phi(Z_n^{(t)} - 1)] \leq C\ell(t)^4 \tag{4.7}$$

for all  $t > 0$ . This implies that  $Z_n^{(t)} \rightarrow Z^{(t)}$  almost surely for some random variable  $Z^{(t)}$  and, upon letting  $t \rightarrow \infty$ , also  $Z_n \rightarrow Z$  almost surely for  $Z := \lim_{t \rightarrow \infty} Z^{(t)}$ . What is more,

$$\begin{aligned} \mathbb{P}(|Z_n - 1| > t) &\leq \mathbb{P}(\phi(|Z_n - 1|) > \phi(t), \sup_{u \in \mathcal{G}} |L(u)| \leq t) + \mathbb{P}(\sup_{u \in \mathcal{G}} |L(u)| > t) \\ &\leq \mathbb{P}(\phi(|Z_n^{(t)} - 1|) > \phi(t)) + t^{-\alpha} \leq \phi(t)^{-1} \sup_{n \geq 1} \mathbb{E}[\phi(Z_n^{(t)} - 1)] + t^{-\alpha} \\ &\leq t^{-\alpha}(C\ell(t)^3 + 1) \end{aligned}$$

for all sufficiently large  $t$ . As  $\ell(t)$  is of the order  $\log^{1+\epsilon/2} t$  as  $t \rightarrow \infty$ , the bound above implies that  $(|Z_n - 1|)^p_{n \geq 0}$  is uniformly integrable for all  $p < \alpha$ . Consequently,  $Z_n \rightarrow Z$  in  $L^p$  for all  $p < \alpha$ . In particular,  $\mathbb{E}[Z] = 1$ .  $\square$

Proposition 2.2 and Proposition 2.3 can be proved using minor modifications of the corresponding results in [14]. For the reader's convenience, we sketch the corresponding arguments in the given context.

Both propositions are based on the following lemma.

**Lemma 4.1.** *Suppose that  $\mathbb{P}(N < \infty) = 1$  and that (C1) holds. Further, assume that  $\mathbb{E}[\sum_{|u|=1} |L(u)|^\alpha \log(|L(u)|)] \in (-\infty, 0)$  and  $\mathbb{E}[W_1 \log_+ W_1] < \infty$ , or that (C3) holds. Then  $Z_n \rightarrow Z$  in probability as  $n \rightarrow \infty$  implies  $\mathbb{P}(|Z| \geq t) = o(t^{-p})$  as  $t \rightarrow \infty$  and, in particular,  $\mathbb{E}[|Z_1|^p] < \infty$  for every  $p \in (1, \alpha)$ .*

The proof of the lemma is lengthy and follows along the lines of the proofs of [4, Lemma 4.9] and [14, Lemma 4.7]. We will therefore only give a sketch of the proof.

*Sketch of the proof.* First notice that if  $Z_n \rightarrow Z$  in probability as  $n \rightarrow \infty$ , then  $Z$  satisfies

$$Z = \sum_{|u|=n} L(u)[Z]_u \quad \text{almost surely} \tag{4.8}$$

for every  $n \in \mathbb{N}$ . This means that  $Z$  is a fixed point of a smoothing transformation. The proof of Lemma 4.1 is based on a comparison of the survival probability  $\mathbb{P}(|Z| > t)$  with the Laplace transform  $\varphi$  at 0 solving the functional equation of a suitable smoothing transform. To be more precise, there exists a probability measure on  $[0, \infty)$ , non-degenerate at 0, such that its Laplace transform  $\varphi$  satisfies

$$\varphi(t) = \mathbb{E}[\prod_{|u|=1} \varphi(|L(u)|^\alpha t)], \quad t \geq 0. \tag{4.9}$$

Indeed,  $\varphi$  is the Laplace transform of a fixed point of a smoothing transformation on the nonnegative halfline with tilted weights  $|L(u)|^\alpha, |u| = 1$ . Further,  $\varphi$  is such that  $1 - \varphi(t)$  is regularly varying of index 1 at 0. These facts are summarized in [2], see in particular Proposition 2.1 and Theorem 3.1 there. As in [14, Section 3.5], using multiplicative martingales and the theory of independent, infinitesimal triangular arrays, one can deduce that  $\mathbb{P}(|Z| > t) = o(1 - \varphi(t^{-\alpha}))$  as  $t \rightarrow \infty$ . Thus,  $\mathbb{P}(|Z| > t) = o(t^{-p})$  as  $t \rightarrow \infty$  for every  $p \in (1, \alpha)$ . In particular, for any  $p \in (1, \alpha)$ , we have  $\mathbb{E}[|Z|^p] < \infty$  and thus, by standard martingale theory,  $\mathbb{E}[|Z_1|^p] = \mathbb{E}[\mathbb{E}[|Z|\mathcal{F}_1]^p] \leq \mathbb{E}[\mathbb{E}[|Z|^p|\mathcal{F}_1]] = \mathbb{E}[|Z|^p] < \infty$ .  $\square$

Proposition 2.2 can be proved as Theorem 2.3 in [14]. We therefore keep the presentation short here.

*Proof of Proposition 2.2.* Suppose that  $\mathbb{P}(N < \infty) = 1$  and that (C1) holds with  $\alpha = 2$  and that one of the additional conditions holds. Further, assume for a contradiction that  $Z_n \rightarrow Z$  in probability as  $n \rightarrow \infty$ . Then we can apply Lemma 4.1 and deduce that  $\mathbb{E}[|Z|^p] < \infty$  for every  $p \in (1, \alpha)$ . Standard martingale theory gives  $\mathbb{E}[|Z_n - Z|^p] \rightarrow 0$  as  $n \rightarrow \infty$  for each such  $p$ . On the other hand, from the Burkholder-Davis-Gundy inequality [12, Theorem 11.3.1] and Jensen's inequality for the concave function  $x \mapsto x^{p/2}$  for  $x \geq 0$ , we get as in the proof of Theorem 2.3 in [14] that there exists a constant  $c_p > 0$  such that

$$\begin{aligned} \mathbb{E}[|\operatorname{Re}(Z_n) - 1|^p + |\operatorname{Im}(Z_n)|^p] &\geq c_p \mathbb{E}[(\sum_{k=1}^n |\operatorname{Re}(D_k)|^2)^{p/2} + (\sum_{k=1}^n |\operatorname{Im}(D_k)|^2)^{p/2}] \\ &\geq c_p n^{p/2-1} \mathbb{E}[(\sum_{k=1}^n |\operatorname{Re}(D_k)|^p) + (\sum_{k=1}^n |\operatorname{Im}(D_k)|^p)]. \end{aligned}$$

Here, using that given  $\mathcal{F}_{k-1}$ ,  $D_k$  is a weighted sum of centered i.i.d. random variables, we can again apply the Burkholder-Davis-Gundy inequality and then Jensen's inequality on  $\{W_{k-1} > 0\}$  to infer

$$\begin{aligned} &\mathbb{E}[|\operatorname{Re}(D_k)|^p + |\operatorname{Im}(D_k)|^p] \\ &\geq c_p \mathbb{E}[(\sum_{|u|=k-1} \operatorname{Re}(L(u)([Z_1]_u - 1))^2)^{p/2} + (\sum_{|u|=k-1} \operatorname{Im}(L(u)([Z_1]_u - 1))^2)^{p/2}] \\ &\geq c_p 2^{p/2-1} \mathbb{E}[(\sum_{|u|=k-1} (\operatorname{Re}(L(u)([Z_1]_u - 1))^2 + \operatorname{Im}(L(u)([Z_1]_u - 1))^2)^{p/2}] \\ &= c_p 2^{p/2-1} \mathbb{E}[(\sum_{|u|=k-1} |L(u)([Z_1]_u - 1)|^2)^{p/2}] \\ &\geq c_p 2^{p/2-1} \mathbb{E}[|Z_1 - 1|^p] \mathbb{E}[W_{k-1}^{p/2}]. \end{aligned}$$

Consequently,

$$\begin{aligned} \mathbb{E}[|Z_n - 1|^p] &\geq 2^{p/2-1} \mathbb{E}[|\operatorname{Re}(Z_n) - 1|^p + |\operatorname{Im}(Z_n)|^p] \\ &\geq c_p^2 2^{p-2} \mathbb{E}[|Z_1 - 1|^p] n^{p/2-1} \sum_{k=0}^{n-1} \mathbb{E}[W_k^{p/2}]. \end{aligned} \tag{4.10}$$

Condition (i) of Proposition 2.2 implies that  $W_n \rightarrow W$  in  $L^1$ , see [16]. Hence the lower bound in (4.10) is of the order  $n^{p/2}$  which tends to  $+\infty$  as  $n \rightarrow \infty$ . Condition (ii) of Proposition 2.2 implies that  $n^{p/4} W_n^{p/2}$ ,  $n \in \mathbb{N}$  converges in distribution as  $n \rightarrow \infty$  to a non-degenerate limit and is also uniformly integrable, see [1, Theorem 1.1] and [14, Remark 4.8]. Thus the lower bound in (4.10) is of the order  $n^{p/4}$  and again diverges as  $n \rightarrow \infty$ .  $\square$

*Proof of Proposition 2.3.* The proposition follows from Lemma 4.1 via contraposition.  $\square$

## 5 Results for higher dimensions and the connection to smoothing transforms

As already pointed out in the introduction, to a large extent, our interest in the problem of complex martingale convergence in the branching random walk comes from its significance in the fixed-point theory for smoothing transformations. We will therefore give a more detailed description of the link between smoothing equations and martingale convergence here.

Suppose that  $\mathcal{Z} = \sum_{u=1}^N \delta_{L(u)}$  is a point process on  $\mathbb{C}$ , i.e., the  $L(u)$ ,  $u = 1, \dots, N$  are complex random variables and  $N$  is a nonnegative integer-valued random variable. A fixed point of the smoothing transform associated with  $\mathcal{Z}$  is a complex random variable  $X$  satisfying

$$X \stackrel{\text{law}}{=} \sum_{u=1}^N L(u) X_u \tag{5.1}$$

where  $X_1, X_2, \dots$  are i.i.d. copies of  $X$  and independent of  $\mathcal{Z}$ . Eq. (5.1) is also called a smoothing equation and the law of  $X$  is a solution to this equation. Fixed points of smoothing transforms arise frequently as limit laws of quantities of interest in models that have some kind of recursive structure.

It has been shown in [20, Theorem 1.2] that the solutions of a given complex smoothing equation are the laws of random variables of the form

$$aZ + X_W \tag{5.2}$$

where  $X_W$  is the value of a suitable independent Lévy process  $(X_t)_{t \geq 0}$  evaluated at an independent random time  $W$ . The Lévy process  $(X_t)_{t \geq 0}$  further has a certain invariance property related to  $\alpha$ -stability. Moreover,  $Z$  is either 0 or the limit of a martingale  $(Z_n)_{n \in \mathbb{N}_0}$  defined as follows. Take independent copies  $\mathcal{Z}(u)$ ,  $u \in \mathcal{I}$  of  $\mathcal{Z}$  on a suitable probability space  $(\Omega, \mathcal{A}, \mathbb{P})$  and define  $\mathcal{G}_n, \mathcal{G}, \mathcal{F}_n$  in obvious analogy to the corresponding objects defined in Section 2. Let  $L(\emptyset) := 1$  and, for  $uj \in \mathcal{G}$ , define recursively  $L(uj) := L(u)[L(j)]_u$ . Then  $Z_n := \sum_{|u|=n} L(u)$ ,  $n \in \mathbb{N}_0$ . If  $\mathbb{E}[Z_1] = 1$ , then  $(Z_n)_{n \in \mathbb{N}_0}$  is a martingale, in fact, it is the additive martingale in a suitable branching random walk. Here, it is important to stress that  $Z$  in (5.1) must be (up to a scaling constant) the limit of  $(Z_n)_{n \in \mathbb{N}_0}$  without additional renormalization. This explains why in the paper at hand, we focus on martingale convergence and do not consider Seneta-Heyde norming constants.

As the theory of smoothing equations has applications to problems that go (with regard to the dimension) beyond the complex case, we now switch to a more general multivariate setup and also explain how our main results can be extended to this setup.

To be precise, fix a dimension  $d \in \mathbb{N}$  and let  $\mathcal{S}(d)$  denote the set of real  $d \times d$  similarity matrices. A similarity matrix is the product of a positive scaling factor and an orthogonal  $d \times d$  matrix. Now suppose that  $\mathcal{Z} = \sum_{u=1}^N \delta_{L(u)}$  is a point process on  $\mathcal{S}(d)$ . A fixed point of the smoothing transform associated with  $\mathcal{Z}$  is a  $d$ -dimensional random vector  $X$  satisfying (5.1) where  $X_1, X_2, \dots$  are i.i.d. copies of  $X$  and independent of  $\mathcal{Z}$ . A similar representation theorem as in the complex case holds for the set of solutions to (5.1). In particular, an important problem arising when solving (5.1) is the following. Construct a probability space  $(\Omega, \mathcal{A}, \mathbb{P})$  which carries i.i.d. copies  $\mathcal{Z}(u)$ ,  $u \in \mathcal{I}$  of  $\mathcal{Z}$  and  $\mathcal{G}_n, \mathcal{G}, \mathcal{F}_n$  as in Section 2. Let  $L(\emptyset)$  be the  $d \times d$  identity matrix, and, for  $uj \in \mathcal{G}$ , define recursively  $L(uj) := L(u)[L(j)]_u$ . Now suppose that the matrix  $\mathbb{E}[\sum_{|u|=1} L(u)]$  has finite entries only and that it has a right eigenvector  $w \neq 0$  to the eigenvalue 1. Then the sequence  $(Z_n w)_{n \in \mathbb{N}_0}$  defined via

$$Z_n w := \sum_{|u|=n} L(u)w, \quad n \in \mathbb{N}_0$$

defines a  $d$ -dimensional martingale with respect to  $(\mathcal{F}_n)_{n \geq 0}$ . In slight abuse of notation, we write  $|\cdot|$  not only for the standard Euclidean norm in  $\mathbb{R}^d$  but also for the usual matrix norm. Since we only work with similarity matrices, this should cause no confusion.

Condition (C1) makes perfect sense in the given situation, and the following result can be proved along the lines of the proof of Theorem 2.1:

**Theorem 5.1.** *Suppose that (C1) holds with  $\alpha \in (1, 2)$  and that (C2) holds with  $Z_1$  replaced by  $Z_1 w$ . Then  $(Z_n w)_{n \geq 0}$  converges almost surely and in  $L^p$  for every  $p < \alpha$  to a non-degenerate limit  $Z^w$ .*

This improves Proposition 1.1(c) in [20] in two ways. First of all, the assumptions on finite absolute moments of  $Z_1 w$  are relaxed. Second, the theorem above includes the boundary case  $m'(\alpha) = 0$ , which is not covered in [20].

Also, with  $W_n := \sum_{|u|=n} |L(u)|^\alpha$ ,  $n \in \mathbb{N}_0$ , the analog of Lemma 4.1 holds in the given context and thus allows to conclude the analogs of Propositions 2.2 and 2.3 with  $Z_n$  replaced by  $Z_n^w$ ,  $n \in \mathbb{N}_0$ . We refrain from reformulating the corresponding results in the more general context.

## A Auxiliary results

**Lemma A.1.** *Let  $\alpha \in (1, 2)$ ,  $\delta > 0$ . Then there is an even convex function  $\phi : \mathbb{R} \rightarrow [0, \infty)$  with  $\phi(0) = 0$  having a concave derivative on  $(0, \infty)$  such that  $\ell(x) := \phi(x)x^{-\alpha}$ ,  $x > 0$  is increasing and satisfies the following assertions:*

- (i) *For all  $x > 0$ , we have  $\ell(x^{-1}) = \ell(x)^{-1} > 0$ .*
- (ii) *There exists a constant  $c > 0$  such that  $\ell(x) \sim c \log^\delta(x)$  as  $x \rightarrow \infty$ .*
- (iii) *For all  $x \geq 1$  and  $y > 0$ , we have  $\ell(xy) \leq \ell(x)^2 \ell(y)$ .*

*Proof.* We set  $\varepsilon(u) := \delta u_0^{-1} \mathbb{1}_{[0, u_0]}(|u|) + \delta |u|^{-1} \mathbb{1}_{(u_0, \infty)}(|u|)$  for some  $u_0 > 0$  to be specified below, and

$$\ell(x) := \exp\left(\int_0^{\log x} \varepsilon(u) du\right), \quad x > 0$$

where the integral has to be understood as an (oriented) Riemann integral. We then define  $\phi(0) := 0$  and  $\phi(x) := |x|^\alpha \ell(|x|)$  for  $x \neq 0$ . Then  $\ell$  satisfies (i) since  $\varepsilon$  is symmetric around 0. From  $\varepsilon(u) = \delta u^{-1}$  for all  $u \geq u_0$  we conclude that (ii) holds. For the proof of (iii), first notice that since  $\varepsilon$  is decreasing on  $[0, \infty)$ , the integral  $\int_0^x \varepsilon(u) du$  is subadditive as a function of  $x \geq 0$ . Consequently,  $\ell(xy) \leq \ell(x)\ell(y) \leq \ell(x)^2 \ell(y)$  for all  $x, y \geq 1$ . Now suppose  $x \geq 1$  and  $y < 1$ . We distinguish two cases. If  $xy < 1$ , then

$$\int_0^{\log(xy)} \varepsilon(u) du = \int_{\log y}^{\log x + \log y} \varepsilon(u) du - \int_{\log y}^0 \varepsilon(u) du \leq \int_0^{\log x} \varepsilon(u) du + \int_0^{\log y} \varepsilon(u) du,$$

where we have used that  $\varepsilon$  is symmetric and decreasing on  $[0, \infty)$ . Again, we conclude that  $\ell(xy) \leq \ell(x)\ell(y) \leq \ell(x)^2 \ell(y)$ . Next, suppose  $xy \geq 1$ . Then

$$\int_0^{\log(xy)} \varepsilon(u) du \leq \int_0^{\log x} \varepsilon(u) du \leq 2 \int_0^{\log x} \varepsilon(u) du + \int_0^{\log y} \varepsilon(u) du,$$

hence  $\ell(xy) \leq \ell(x)^2 \ell(y)$ .

Finally, we have to show that we can choose  $u_0 > 0$  such that  $\phi$  is convex on  $\mathbb{R}$  with concave derivative on  $(0, \infty)$ . Clearly,  $\phi$  is continuously differentiable with derivative

$$\phi'(x) = x^{\alpha-1} \ell(x) (\alpha + \varepsilon(\log x)), \quad t > 0.$$

As  $\varepsilon$  is smooth on  $(0, \infty) \setminus \{u_0\}$ , so is  $\phi$ , and we get for the higher order derivatives:

$$\begin{aligned} \phi''(x) &= x^{\alpha-2} \ell(x) (\alpha(\alpha-1) + (2\alpha-1)\varepsilon(\log x) + \varepsilon^2(\log x) + \varepsilon'(\log x)), \\ \phi'''(x) &= x^{\alpha-3} \ell(x) (\alpha(\alpha-1)(\alpha-2) + p_0(\varepsilon(\log x)) + p_1(\varepsilon'(\log x)) + p_2(\varepsilon''(\log x))) \end{aligned}$$

for  $x > 0$ ,  $x \neq u_0$ , where  $p_0, p_1, p_2$  are polynomials with  $p_j(0) = 0$  for  $j = 0, 1, 2$  and coefficients depending only on  $\alpha$ . Consequently, there exists a constant  $\eta > 0$  such that  $\phi''(x) > 0$  and  $\phi'''(x) < 0$  for all  $x > 0$ ,  $x \neq u_0$  such that  $\max\{|\varepsilon(x)|, |\varepsilon'(x)|, |\varepsilon''(x)|\} \leq \eta$ . Now fix  $u_0 > 0$  so large that  $\max\{|\varepsilon(u)|, |\varepsilon'(u)|, |\varepsilon''(u)|\} \leq \eta$  for all  $u \geq u_0$ . Then  $\phi'''(x) < 0$  for all  $x > 0$ ,  $x \neq e^{u_0}$ , hence  $\phi''$  is strictly decreasing on  $(0, e^{u_0})$  and  $(e^{u_0}, \infty)$ . From the explicit expression for  $\phi''$  above, we conclude that  $\phi''(u_0-) > \phi''(u_0^+)$  (the difference between these expressions is given exactly by the difference of the limits of  $\varepsilon'(\log x)$  as  $x \uparrow e^{u_0}$ , which is 0, and as  $x \downarrow e^{u_0}$ , which is  $-\delta u_0^{-2} < 0$ ). Thus  $\phi'$  is (strictly) concave. Analogously, we infer that  $\phi$  is (strictly) convex on  $[0, \infty)$ .  $\square$

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