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Product space for two processes with independent increments under nonlinear expectations *

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Abstract

In this paper, we consider the product space for two processes with independent increments under nonlinear expectations. By introducing a discretization method, we construct a nonlinear expectation under which the given two processes can be seen as a new process with independent increments.

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1 Introduction

Peng [4, 5] introduced the notions of distribution and independence under nonlinear expectation spaces. Under sublinear case, Peng [8] obtained the corresponding central limit theorem for a sequence of i.i.d. random vectors. The limit distribution is called G-normal distribution. Based on this distribution, Peng [6, 7] gave the definition of G-Brownian motion, which is a kind of processes with stationary and independent increments, and then discussed the Itô stochastic analysis with respect to G-Brownian motion.

It is well-known that the existence for a sequence of i.i.d. random vectors is important for central limit theorem. In the nonlinear case, Peng [9] introduced the product space technique to construct a sequence of i.i.d. random vectors. But this product space technique does not hold in the continuous time case. More precisely, let $(M_t)_{t\geq 0}$ and $(N_t)_{t\geq 0}$ be two d-dimensional processes with independent increments defined respectively on nonlinear expectation spaces $(\Omega_1, \mathcal{H}_1, \hat{\mathbb{E}}_1)$ and $(\Omega_2, \mathcal{H}_2, \hat{\mathbb{E}}_2)$, we want to construct a $2d$-dimensional process $(\tilde{M}_t, \tilde{N}_t)_{t\geq 0}$ with independent increments defined on a nonlinear expectation space $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$ such that $(\tilde{M}_t)_{t\geq 0} \overset{d}{=} (M_t)_{t\geq 0}$ and $(\tilde{N}_t)_{t\geq 0} \overset{d}{=} (N_t)_{t\geq 0}$. Usually,

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set $\Omega = \Omega_1 \times \Omega_2$, $\tilde{M}_t(\omega) = M_t(\omega_1)$, $\tilde{N}_t(\omega) = N_t(\omega_2)$ for each $\omega = (\omega_1, \omega_2) \in \Omega$, $t \geq 0$. If we use Peng's product space technique, then we can only get a $2d$-dimensional process $(\tilde{M}_t, \tilde{N}_t)_{t\geq 0}$ such that $(\tilde{M}_t)_{t\geq 0}$ is independent from $(\tilde{N}_t)_{t\geq 0}$ or $(\tilde{N}_t)_{t\geq 0}$ is independent from $(\tilde{M}_t)_{t\geq 0}$. Different from linear expectation case, the independence is not mutual under nonlinear case (see [2]). So this $(\tilde{M}_t, \tilde{N}_t)_{t\geq 0}$ is not a process with independent increments.

In this paper, we introduce a discretization method, which can overcome the problem of independence. More precisely, for each given $\mathcal{D}_n = \{i2^{-n} : i \geq 0\}$, we can construct a nonlinear expectation $\hat{\mathbb{E}}^n$ under which $(\tilde{M}_t, \tilde{N}_t)_{t\in\mathcal{D}_n}$ possesses independent increments. But $\hat{\mathbb{E}}^n$, $n \geq 1$, are not consistent, i.e., the values of the same random variable under $\hat{\mathbb{E}}^n$ are not equal. Fortunately, we can prove that the limit of $\hat{\mathbb{E}}^n$ exisits by using the notion of tightness, which was introduced by Peng in [10] to prove central limit theorem under sublinear case. Denote the limit of $\hat{\mathbb{E}}^n$ by $\hat{\mathbb{E}}$, we show that $(\tilde{M}_t, \tilde{N}_t)_{t\geq 0}$ is the process with independent increments under $\hat{\mathbb{E}}$.

This paper is organized as follows: In Section 2, we recall some basic notions and results of nonlinear expectations. The main theorem is stated and proved in Section 3.

2 Preliminaries

We present some basic notions and results of nonlinear and sublinear expectations in this section. More details can be found in [1, 3, 9–11].

Let Ω be a given nonempty set and $\mathcal{H}$ be a linear space of real-valued functions on Ω such that if $X_1,\ldots,X_d \in \mathcal{H}$, then $\varphi(X_1, X_2, \ldots, X_d) \in \mathcal{H}$ for each $\varphi \in C_{b.Lip}(\mathbb{R}^d)$, where $C_{b.Lip}(\mathbb{R}^d)$ denotes the set of all bounded and Lipschitz functions on $\mathbb{R}^d$. $\mathcal{H}$ is considered as the space of random variables. Similarly, $\{X = (X_1, \ldots, X_d) : X_i \in \mathcal{H}, i \leq d\}$ denotes the space of d-dimensional random vectors.

Definition 2.1. *A sublinear expectation $\hat{\mathbb{E}}$ on $\mathcal{H}$ is a functional $\hat{\mathbb{E}} : \mathcal{H} \to \mathbb{R}$ satisfying the following properties: for each $X, Y \in \mathcal{H}$,*

(i) ***Monotonicity:*** $\hat{\mathbb{E}}[X] \geq \hat{\mathbb{E}}[Y]$ *if $X \geq Y$;*

(ii) ***Constant preserving:*** $\hat{\mathbb{E}}[c] = c$ *for $c \in \mathbb{R}$;*

(iii) ***Sub-additivity:*** $\hat{\mathbb{E}}[X + Y] \leq \hat{\mathbb{E}}[X] + \hat{\mathbb{E}}[Y]$*;*

(iv) ***Positive homogeneity:*** $\hat{\mathbb{E}}[\lambda X] = \lambda\hat{\mathbb{E}}[X]$ *for $\lambda \geq 0$.*

The triple $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$ is called a sublinear expectation space. If (i) and (ii) are satisfied, $\hat{\mathbb{E}}$ is called a nonlinear expectation and the triple $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$ is called a nonlinear expectation space.

Let $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$ be a nonlinear (resp. sublinear) expectation space. For each given d-dimensional random vector X, we define a functional on $C_{b.Lip}(\mathbb{R}^d)$ by

$$\hat{\mathbb{F}}_X[\varphi] := \hat{\mathbb{E}}[\varphi(X)] \text{ for each } \varphi \in C_{b.Lip}(\mathbb{R}^d).$$

It is easy to verify that $(\mathbb{R}^d, C_{b.Lip}(\mathbb{R}^d), \hat{\mathbb{F}}_X)$ forms a nonlinear (resp. sublinear) expectation space. $\hat{\mathbb{F}}_X$ is called the distribution of X. Two d-dimensional random vectors X_1 and X_2 defined respectively on nonlinear expectation spaces $(\Omega_1, \mathcal{H}_1, \hat{\mathbb{E}}_1)$ and $(\Omega_2, \mathcal{H}_2, \hat{\mathbb{E}}_2)$ are called identically distributed, denoted by $X_1 \overset{d}{=} X_2$, if $\hat{\mathbb{F}}_{X_1} = \hat{\mathbb{F}}_{X_2}$, i.e.,

$$\hat{\mathbb{E}}_1[\varphi(X_1)] = \hat{\mathbb{E}}_2[\varphi(X_2)] \text{ for each } \varphi \in C_{b.Lip}(\mathbb{R}^d).$$

Similar to the classical case, Peng [10] gave the following definition of convergence in distribution.

Definition 2.2. *Let X_n, $n \geq 1$, be a sequence of d-dimensional random vectors defined respectively on nonlinear (resp. sublinear) expectation spaces $(\Omega_n, \mathcal{H}_n, \hat{\mathbb{E}}_n)$. $\{X_n : n \geq 1\}$ is said to converge in distribution if, for each fixed $\varphi \in C_{b.Lip}(\mathbb{R}^d)$, $\{\hat{\mathbb{F}}_{X_n}[\varphi] : n \geq 1\}$ is a Cauchy sequence. Define*

$$\hat{\mathbb{F}}[\varphi] = \lim_{n\to\infty} \hat{\mathbb{F}}_{X_n}[\varphi],$$

then the triple $(\mathbb{R}^d, C_{b.Lip}(\mathbb{R}^d), \hat{\mathbb{F}})$ forms a nonlinear (resp. sublinear) expectation space.

If X_n, $n \geq 1$, is a sequence of d-dimensional random vectors defined on the same sublinear expectation space $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$ satisfying

$$\lim_{n,m\to\infty} \hat{\mathbb{E}}[|X_n - X_m|] = 0,$$

then we can deduce that $\{X_n : n \geq 1\}$ converges in distribution by

$$|\hat{\mathbb{F}}_{X_n}[\varphi] - \hat{\mathbb{F}}_{X_m}[\varphi]| = |\hat{\mathbb{E}}[\varphi(X_n)] - \hat{\mathbb{E}}[\varphi(X_m)]| \leq C_\varphi \hat{\mathbb{E}}[|X_n - X_m|],$$

where C_φ is the Lipschitz constant of φ.

The following definition of tightness is important for obtaining a subsequence which converges in distribution.

Definition 2.3. *Let X be a d-dimensional random vector defined on a sublinear expectation space $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$. The distribution of X is called tight if, for each $\varepsilon > 0$, there exist an $N > 0$ and a $\varphi \in C_{b.Lip}(\mathbb{R}^d)$ with $I_{\{|x|\geq N\}} \leq \varphi$ such that $\hat{\mathbb{F}}_X[\varphi] = \hat{\mathbb{E}}[\varphi(X)] < \varepsilon$.*

Definition 2.4. *Let $\{\hat{\mathbb{E}}_\lambda : \lambda \in I\}$ be a family of nonlinear expectations and $\hat{\mathbb{E}}$ be a sublinear expectation defined on $(\Omega, \mathcal{H})$. $\{\hat{\mathbb{E}}_\lambda : \lambda \in I\}$ is said to be dominated by $\hat{\mathbb{E}}$ if, for each $\lambda \in I$,*

$$\hat{\mathbb{E}}_\lambda[X] - \hat{\mathbb{E}}_\lambda[Y] \leq \hat{\mathbb{E}}[X - Y] \text{ for each } X, Y \in \mathcal{H}.$$

Definition 2.5. *Let X_λ, $\lambda \in I$, be a family of d-dimensional random vectors defined respectively on nonlinear expectation spaces $(\Omega_\lambda, \mathcal{H}_\lambda, \hat{\mathbb{E}}_\lambda)$. $\{\hat{\mathbb{F}}_{X_\lambda} : \lambda \in I\}$ is called tight if there exists a tight sublinear expectation $\hat{\mathbb{F}}$ on $(\mathbb{R}^d, C_{b.Lip}(\mathbb{R}^d))$ which dominates $\{\hat{\mathbb{F}}_{X_\lambda} : \lambda \in I\}$.*

Remark 2.6. A family of sublinear expectations $\{\hat{\mathbb{F}}_{X_\lambda} : \lambda \in I\}$ on $(\mathbb{R}^d, C_{b.Lip}(\mathbb{R}^d))$ is tight if and only if $\hat{\mathbb{F}}[\varphi] = \sup_{\lambda\in I} \hat{\mathbb{F}}_{X_\lambda}[\varphi]$ for each $\varphi \in C_{b.Lip}(\mathbb{R}^d)$ is a tight sublinear expectation.

Theorem 2.7. *([10]) Let X_n, $n \geq 1$, be a sequence of d-dimensional random vectors defined respectively on nonlinear expectation spaces $(\Omega_n, \mathcal{H}_n, \hat{\mathbb{E}}_n)$. If $\{\hat{\mathbb{F}}_{X_n} : n \geq 1\}$ is tight, then there exists a subsequence $\{X_{n_i} : i \geq 1\}$ which converges in distribution.*

The following definition of independence is fundamental in nonlinear expectation theory.

Definition 2.8. *Let $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$ be a nonlinear expectation space. A d-dimensional random vector Y is said to be independent from another m-dimensional random vector X under $\hat{\mathbb{E}}[\cdot]$ if, for each test function $\varphi \in C_{b.Lip}(\mathbb{R}^{m+d})$, we have*

$$\hat{\mathbb{E}}[\varphi(X, Y)] = \hat{\mathbb{E}}[\hat{\mathbb{E}}[\varphi(x, Y)]_{x=X}].$$

Remark 2.9. It is important to note that "Y is independent from X" does not imply that "X is independent from Y" (see [2]).

Remark 2.10. In the above definitions, the condition "X is a d-dimensional random vector" can be weakened to "X is a mapping from Ω into $\mathbb{R}^d$ such that $\varphi(X) \in \mathcal{H}$ for each $\varphi \in C_{b.Lip}(\mathbb{R}^d)$". In the latter case, the nonlinear expectation of X may not exist.

A d-dimensional stochastic process in a nonlinear expectation space $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$ is a family of mappings $(X_t)_{t\geq 0}$ from Ω into $\mathbb{R}^d$ such that $\varphi(X_{t_1}, \ldots, X_{t_n}) \in \mathcal{H}$ for each $0 \leq t_1 < \cdots < t_n$ and $\varphi \in C_{b.Lip}(\mathbb{R}^{n\times d})$.

Definition 2.11. *Two d-dimensional processes $(X_t)_{t\geq 0}$ and $(Y_t)_{t\geq 0}$ defined respectively on nonlinear expectation spaces $(\Omega_1, \mathcal{H}_1, \hat{\mathbb{E}}_1)$ and $(\Omega_2, \mathcal{H}_2, \hat{\mathbb{E}}_2)$ are called identically distributed, denoted by $(X_t)_{t\geq 0} \overset{d}{=} (Y_t)_{t\geq 0}$, if for each $n \in \mathbb{N}$, $0 \leq t_1 < \cdots < t_n$, $(X_{t_1}, \ldots, X_{t_n}) \overset{d}{=} (Y_{t_1}, \ldots, Y_{t_n})$, i.e.,*

$$\hat{\mathbb{E}}_1[\varphi(X_{t_1}, \ldots, X_{t_n})] = \hat{\mathbb{E}}_2[\varphi(Y_{t_1}, \ldots, Y_{t_n})] \text{ for each } \varphi \in C_{b.Lip}(\mathbb{R}^{n\times d}).$$

Definition 2.12. *A d-dimensional process $(X_t)_{t\geq 0}$ with $X_0 = 0$ on a nonlinear expectation space $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$ is said to have independent increments if, for each $0 \leq t_1 < \cdots < t_n$, $X_{t_n} - X_{t_{n-1}}$ is independent from $(X_{t_1}, \ldots, X_{t_{n-1}})$. A d-dimensional process $(X_t)_{t\geq 0}$ with $X_0 = 0$ is said to have stationary increments if, for each t, $s \geq 0$, $X_{t+s} - X_s \overset{d}{=} X_t$.*

We give a typical example of processes with stationary and independent increments.

Example 2.13. Let Γ be a given bounded subset in $\mathbb{R}^{d\times d}$, where $\mathbb{R}^{d\times d}$ denotes the set of all $d \times d$ matrices. Define $G : \mathbb{S}(d) \to \mathbb{R}$ by

$$G(A) = \frac{1}{2} \sup_{Q\in\Gamma} \mathrm{tr}[AQQ^T] \text{ for each } A \in \mathbb{S}(d),$$

where $\mathbb{S}(d)$ denotes the set of all $d \times d$ symmetric matrices. A d-dimensional process $(B_t)_{t\geq 0}$ on a sublinear expectation space $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$ is called a G-Brownian motion if the following properties are satisfied:

(1) $B_0 = 0$;

(2) It is a process with independent increments;

(3) For each t, $s \geq 0$, $\hat{\mathbb{E}}[\varphi(B_{t+s} - B_s)] = u^\varphi(t, 0)$ for each $\varphi \in C_{b.Lip}(\mathbb{R}^d)$, where u^φ is the viscosity solution of the following G-heat equation:

$$\begin{cases} \partial_t u(t,x) - G(D_x^2 u(t,x)) = 0, \\ u(0,x) = \varphi(x). \end{cases}$$

Obviously, (3) implies that the process $(B_t)_{t\geq 0}$ has stationary increments.

3 Main result

Let $(M_t)_{t\geq 0}$ and $(N_t)_{t\geq 0}$ be two d-dimensional processes with independent increments defined respectively on nonlinear (resp. sublinear) expectation spaces $(\Omega_1, \mathcal{H}_1, \hat{\mathbb{E}}_1)$ and $(\Omega_2, \mathcal{H}_2, \hat{\mathbb{E}}_2)$. We need the following assumption:

(A) There exist two sublinear expectations $\tilde{\mathbb{E}}_1 : \mathcal{H}_1 \to \mathbb{R}$ and $\tilde{\mathbb{E}}_2 : \mathcal{H}_2 \to \mathbb{R}$ satisfying:

(1) $\tilde{\mathbb{E}}_1$ and $\tilde{\mathbb{E}}_2$ dominate $\hat{\mathbb{E}}_1$ and $\hat{\mathbb{E}}_2$ respectively;

(2) For each $t \geq 0$, the distributions of M_t and N_t are tight under $\tilde{\mathbb{E}}_1$ and $\tilde{\mathbb{E}}_2$ respectively;

(3) For each $t \geq 0$,

$$\lim_{s\to t} (\tilde{\mathbb{E}}_1[|M_s - M_t| \wedge 1] + \tilde{\mathbb{E}}_2[|N_s - N_t| \wedge 1]) = 0.$$

Remark 3.1. Noting that for any $K > 0$,

$$|M_s - M_t| \wedge K \leq (K \vee 1)(|M_s - M_t| \wedge 1),$$

the assumption (3) implies

$$\lim_{s\to t} (\tilde{\mathbb{E}}_1[|M_s - M_t| \wedge K] + \tilde{\mathbb{E}}_2[|N_s - N_t| \wedge K]) = 0.$$

Remark 3.2. If $\hat{\mathbb{E}}_1$ and $\hat{\mathbb{E}}_2$ are sublinear expectations, then we can get

$$\lim_{s\to t}(\hat{\mathbb{E}}_1[|M_s - M_t| \wedge 1] + \hat{\mathbb{E}}_2[|N_s - N_t| \wedge 1]) = 0$$

by $\hat{\mathbb{E}}_1[\cdot] \le \tilde{\mathbb{E}}_1[\cdot]$ and $\hat{\mathbb{E}}_2[\cdot] \le \tilde{\mathbb{E}}_2[\cdot]$. So we can replace $\tilde{\mathbb{E}}_1$ and $\tilde{\mathbb{E}}_2$ by $\hat{\mathbb{E}}_1$ and $\hat{\mathbb{E}}_2$ in the assumption (A) respectively.

Now we give our main theorem.

Theorem 3.3. *Let $(M_t)_{t\ge0}$ and $(N_t)_{t\ge0}$ be two d-dimensional processes with independent increments defined respectively on nonlinear (resp. sublinear) expectation spaces $(\Omega_1, \mathcal{H}_1, \hat{\mathbb{E}}_1)$ and $(\Omega_2, \mathcal{H}_2, \hat{\mathbb{E}}_2)$ satisfying the assumption (A). Then there exists a $2d$-dimensional process $(\tilde{M}_t, \tilde{N}_t)_{t\ge0}$ with independent increments defined on a nonlinear (resp. sublinear) expectation space $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$ such that $(\tilde{M}_t)_{t\ge0} \overset{d}{=} (M_t)_{t\ge0}$ and $(\tilde{N}_t)_{t\ge0} \overset{d}{=} (N_t)_{t\ge0}$. Furthermore, $(\tilde{M}_t, \tilde{N}_t)_{t\ge0}$ is a process with stationary and independent increments if $(M_t)_{t\ge0}$ and $(N_t)_{t\ge0}$ are two processes with stationary and independent increments.*

In the following, we only prove the sublinear expectation case. The nonlinear expectation case can be proved by the same method. Moreover, the following lemma shows that we only need to prove the theorem for $t \in [0, 1]$.

Lemma 3.4. *Let $(X_t^i, Y_t^i)_{t\in[0,1]}$, $i \ge 0$, be a sequence of $2d$-dimensional processes with independent increments defined respectively on sublinear expectation spaces $(\bar{\Omega}_i, \bar{\mathcal{H}}_i, \bar{\mathbb{E}}_i)$ such that $(X_t^i)_{t\in[0,1]} \overset{d}{=} (M_{i+t} - M_i)_{t\in[0,1]}$ and $(Y_t^i)_{t\in[0,1]} \overset{d}{=} (N_{i+t} - N_i)_{t\in[0,1]}$. Then there exists a $2d$-dimensional process $(\tilde{M}_t, \tilde{N}_t)_{t\ge0}$ with independent increments defined on a sublinear expectation space $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$ such that $(\tilde{M}_t)_{t\ge0} \overset{d}{=} (M_t)_{t\ge0}$ and $(\tilde{N}_t)_{t\ge0} \overset{d}{=} (N_t)_{t\ge0}$.*

Proof. Set $(\Omega, \mathcal{H}, \hat{\mathbb{E}}) = (\Pi_{i=0}^{\infty}\bar{\Omega}_i, \otimes_{i=0}^{\infty}\bar{\mathcal{H}}_i, \otimes_{i=0}^{\infty}\bar{\mathbb{E}}_i)$ which is the product space of $\{(\bar{\Omega}_i, \bar{\mathcal{H}}_i, \bar{\mathbb{E}}_i) : i \ge 0\}$ (see [9]). For each $\omega = (\omega_i)_{i=0}^{\infty}$, define

$$\tilde{M}_t(\omega) = \sum_{i=0}^{[t]-1} X_1^i(\omega_i) + X_{t-[t]}^{[t]}(\omega_{[t]}),\ \tilde{N}_t(\omega) = \sum_{i=0}^{[t]-1} Y_1^i(\omega_i) + Y_{t-[t]}^{[t]}(\omega_{[t]}).$$

By Proposition 3.15 in Chapter I in [9], we can easily obtain that $(\tilde{M}_t, \tilde{N}_t)_{t\ge0}$ has independent increments property, $(\tilde{M}_t)_{t\ge0} \overset{d}{=} (M_t)_{t\ge0}$ and $(\tilde{N}_t)_{t\ge0} \overset{d}{=} (N_t)_{t\ge0}$. □

Set $\Omega = \Omega_1 \times \Omega_2 = \{\omega = (\omega_1, \omega_2) : \omega_1 \in \Omega_1, \omega_2 \in \Omega_2\}$. For each $\omega = (\omega_1, \omega_2) \in \Omega$, define

$$\tilde{M}_t(\omega) = M_t(\omega_1),\ \tilde{N}_t(\omega) = N_t(\omega_2) \text{ for } t \in [0, 1].$$

For notation simplicity, we denote $X_t = (\tilde{M}_t, \tilde{N}_t)$. Define the space of random variables as follows:

$$\mathcal{H} = \{\varphi(X_{t_1}, X_{t_2} - X_{t_1}, \ldots, X_{t_n} - X_{t_{n-1}}) : \forall n \ge 1, \forall 0 \le t_1 < t_2 < \cdots < t_n \le 1, \\ \forall \varphi \in C_{b.Lip}(\mathbb{R}^{n\times 2d})\}.$$

In the following, we will construct a sublinear expectation $\hat{\mathbb{E}} : \mathcal{H} \to \mathbb{R}$ such that $(\tilde{M}_t)_{t\in[0,1]} \overset{d}{=} (M_t)_{t\in[0,1]}$, $(\tilde{N}_t)_{t\in[0,1]} \overset{d}{=} (N_t)_{t\in[0,1]}$ and $(\tilde{M}_t, \tilde{N}_t)_{t\in[0,1]}$ possessing independent increments. In order to construct $\hat{\mathbb{E}}$, we set, for each fixed $n \ge 1$,

$$\mathcal{H}^n = \{\varphi(X_{\delta_n}, X_{2\delta_n} - X_{\delta_n}, \ldots, X_{2^n\delta_n} - X_{(2^n-1)\delta_n}) : \forall \varphi \in C_{b.Lip}(\mathbb{R}^{2^n\times 2d})\},$$

where $\delta_n = 2^{-n}$. Define $\hat{\mathbb{E}}^n : \mathcal{H}^n \to \mathbb{R}$ as follows:

Step 1. For each given $\phi(X_{i\delta_n}-X_{(i-1)\delta_n})=\phi(\tilde{M}_{i\delta_n}-\tilde{M}_{(i-1)\delta_n},\tilde{N}_{i\delta_n}-\tilde{N}_{(i-1)\delta_n})\in\mathcal{H}^n$ with $i\leq 2^n$ and $\phi\in C_{b.Lip}(\mathbb{R}^{2d})$, define

$$\hat{\mathbb{E}}^n[\phi(X_{i\delta_n}-X_{(i-1)\delta_n})]=\hat{\mathbb{E}}_1[\psi(M_{i\delta_n}-M_{(i-1)\delta_n})],$$

where

$$\psi(x)=\hat{\mathbb{E}}_2[\phi(x,N_{i\delta_n}-N_{(i-1)\delta_n})] \text{ for each } x\in\mathbb{R}^d.$$

Step 2. For each given $\varphi(X_{\delta_n},X_{2\delta_n}-X_{\delta_n},\ldots,X_{2^n\delta_n}-X_{(2^n-1)\delta_n})\in\mathcal{H}^n$ with $\varphi\in C_{b.Lip}(\mathbb{R}^{2^n\times 2d})$, define

$$\hat{\mathbb{E}}^n[\varphi(X_{\delta_n},X_{2\delta_n}-X_{\delta_n},\ldots,X_{2^n\delta_n}-X_{(2^n-1)\delta_n})]=\varphi_0,$$

where φ_0 is obtained backwardly by Step 1 in the following sense:

$$\varphi_{2^n-1}(x_1,x_2,\ldots,x_{2^n-1})=\hat{\mathbb{E}}^n[\varphi(x_1,x_2,\ldots,x_{2^n-1},X_{2^n\delta_n}-X_{(2^n-1)\delta_n})],$$

$$\varphi_{2^n-2}(x_1,x_2,\ldots,x_{2^n-2})=\hat{\mathbb{E}}^n[\varphi_{2^n-1}(x_1,x_2,\ldots,x_{2^n-2},X_{(2^n-1)\delta_n}-X_{(2^n-2)\delta_n})],$$

$$\vdots$$

$$\varphi_1(x_1)=\hat{\mathbb{E}}^n[\varphi_2(x_1,X_{2\delta_n}-X_{\delta_n})],$$

$$\varphi_0=\hat{\mathbb{E}}^n[\varphi_1(X_{\delta_n})].$$

Lemma 3.5. *Let $(\Omega,\mathcal{H}^n,\hat{\mathbb{E}}^n)$ be defined as above. Then*

(1) *$(\Omega,\mathcal{H}^n,\hat{\mathbb{E}}^n)$ forms a sublinear expectation space;*

(2) *For each $2\leq i\leq 2^n$, $X_{i\delta_n}-X_{(i-1)\delta_n}$ is independent from $(X_{\delta_n},\ldots,X_{(i-1)\delta_n}-X_{(i-2)\delta_n})$;*

(3) *$(\tilde{M}_{\delta_n},\tilde{M}_{2\delta_n}-\tilde{M}_{\delta_n},\ldots,\tilde{M}_{2^n\delta_n}-\tilde{M}_{(2^n-1)\delta_n})\overset{d}{=}(M_{\delta_n},M_{2\delta_n}-M_{\delta_n},\ldots,M_{2^n\delta_n}-M_{(2^n-1)\delta_n})$, $(\tilde{N}_{\delta_n},\tilde{N}_{2\delta_n}-\tilde{N}_{\delta_n},\ldots,\tilde{N}_{2^n\delta_n}-\tilde{N}_{(2^n-1)\delta_n})\overset{d}{=}(N_{\delta_n},N_{2\delta_n}-N_{\delta_n},\ldots,N_{2^n\delta_n}-N_{(2^n-1)\delta_n})$.*

Proof. (1) It is easy to check that $\hat{\mathbb{E}}^n:\mathcal{H}^n\to\mathbb{R}$ is well-defined. We only prove that $\hat{\mathbb{E}}^n$ satisfies monotonicity, the other properties can be similarly obtained. For each given $Y=\varphi_1(X_{\delta_n},X_{2\delta_n}-X_{\delta_n},\ldots,X_{2^n\delta_n}-X_{(2^n-1)\delta_n})$, $Z=\varphi_2(X_{\delta_n},X_{2\delta_n}-X_{\delta_n},\ldots,X_{2^n\delta_n}-X_{(2^n-1)\delta_n})\in\mathcal{H}^n$ with $Y\geq Z$, it is easy to verify that $Y=(\varphi_1\vee\varphi_2)(X_{\delta_n},X_{2\delta_n}-X_{\delta_n},\ldots,X_{2^n\delta_n}-X_{(2^n-1)\delta_n})$. Then by the definition of $\hat{\mathbb{E}}^n$ and the monotonicity of $\hat{\mathbb{E}}_1$ and $\hat{\mathbb{E}}_2$, we can get $\hat{\mathbb{E}}^n[Y]\geq\hat{\mathbb{E}}^n[Z]$. (2) and (3) can be easily obtained by the definition of $\hat{\mathbb{E}}^n$. □

Corollary 3.6. *Set $\mathcal{D}_n=\{i2^{-n}:0\leq i\leq 2^n\}$. Then*

(1) *For each $0\leq t_1<\cdots<t_m$ with $t_i\in\mathcal{D}_n$, $i\leq m$, $X_{t_m}-X_{t_{m-1}}$ is independent from $(X_{t_1},\ldots,X_{t_{m-1}})$ under $\hat{\mathbb{E}}^k$ for any $k\geq n$;*

(2) *For each $0\leq t_1<\cdots<t_m$ with $t_i\in\mathcal{D}_n$, $i\leq m$, $(\tilde{M}_{t_1},\ldots,\tilde{M}_{t_m})\overset{d}{=}(M_{t_1},\ldots,M_{t_m})$ and $(\tilde{N}_{t_1},\ldots,\tilde{N}_{t_m})\overset{d}{=}(N_{t_1},\ldots,N_{t_m})$ under $\hat{\mathbb{E}}^k$ for any $k\geq n$.*

Proof. Noting that $X_{t_m}-X_{t_{m-1}}$ is the sum of finite $X_{i\delta_k}-X_{(i-1)\delta_k}$, then by Lemma 3.5 and the definition of independence and distribution, it is easy to obtain (1) and (2). □

Obviously, $\mathcal{H}^n \subset \mathcal{H}^{n+1}$ for each $n \geq 1$. We set

$$\mathcal{L} = \bigcup_{n\geq 1} \mathcal{H}^n.$$

It is easily seen that $\mathcal{L}$ is a subspace of $\mathcal{H}$ such that if $Y_1,\ldots,Y_m \in \mathcal{L}$, then $\varphi(Y_1,\ldots,Y_m) \in \mathcal{L}$ for each $\varphi \in C_{b.Lip}(\mathbb{R}^m)$.

In the following, we want to define a sublinear expectation $\hat{\mathbb{E}} : \mathcal{L} \to \mathbb{R}$. Unfortunately, $\hat{\mathbb{E}}^{n+1}[\cdot] \neq \hat{\mathbb{E}}^n[\cdot]$ on $\mathcal{H}^n$, because the order of independence under sublinear expectation space is unchangeable. For example, let $(M_t)_{t\geq 0}$ and $(N_t)_{t\geq 0}$ be two 1-dimensional G-Brownian motions with the same $G(a) = \frac{1}{2}(\bar{\sigma}^2 a^+ - \underline{\sigma}^2 a^-)$, where $0 \leq \underline{\sigma}^2 < \bar{\sigma}^2 < \infty$. Since $\hat{\mathbb{E}}_1[|M_t|^k] + \hat{\mathbb{E}}_2[|N_t|^k] < \infty$ for any $k \geq 1$ and $t \geq 0$, we can use local Lipschitz functions to define $\mathcal{H}^n$ (see [9]). Take $\varphi(X_{2^{-1}}) = (\tilde{M}_{2^{-1}})^2 \tilde{N}_{2^{-1}} \in \mathcal{H}^1$, by simple calculation, we can get $\hat{\mathbb{E}}^1[\varphi(X_{2^{-1}})] = 0$ and $\hat{\mathbb{E}}^2[\varphi(X_{2^{-1}})] = 2^{-2}(\bar{\sigma}^2 - \underline{\sigma}^2)\hat{\mathbb{E}}_2[(N_{2^{-2}})^+] > 0$. Thus $\hat{\mathbb{E}}^1[\cdot] \neq \hat{\mathbb{E}}^2[\cdot]$ on $\mathcal{H}^1$ for this case. But the following lemma will allow us to construct $\hat{\mathbb{E}}$.

Lemma 3.7. *For each fixed $n \geq 1$, let $\hat{\mathbb{F}}^n_k$, $k \geq n$, be the distribution of $(X_{\delta_n}, X_{2\delta_n} - X_{\delta_n}, \ldots, X_{2^n\delta_n} - X_{(2^n-1)\delta_n})$ under $\hat{\mathbb{E}}^k$. Then $\{\hat{\mathbb{F}}^n_k : k \geq n\}$ is tight.*

Proof. For each given $N > 1$, let $\varphi_N \in C_{b.Lip}(\mathbb{R}^{2^n\times 2d})$ satisfy $I_{\{|x|\geq N\}} \leq \varphi_N \leq I_{\{|x|\geq N-1\}}$. Taking $x = (x_1 - x_0, y_1 - y_0, \cdots, x_{2^n} - x_{2^n-1}, y_{2^n} - y_{2^n-1})$, x_i, $y_i \in \mathbb{R}^d$, $i \leq 2^n$, it is easy to verify that

$$\begin{aligned}
I_{\{|x|\geq N-1\}} &\leq \sum_{i=1}^{2^n} (I_{\{|x_i - x_{i-1}|\geq (N-1)\sqrt{2^{-(n+1)}}\}} + I_{\{|y_i - y_{i-1}|\geq (N-1)\sqrt{2^{-(n+1)}}\}}) \\
&\leq 2\sum_{i=0}^{2^n} (I_{\{|x_i|\geq (N-1)\sqrt{2^{-(n+3)}}\}} + I_{\{|y_i|\geq (N-1)\sqrt{2^{-(n+3)}}\}}) \\
&\leq 2\sum_{i=0}^{2^n} (\psi_N(x_i) + \psi_N(y_i)),
\end{aligned}$$

where $\psi_N \in C_{b.Lip}(\mathbb{R}^d)$ such that $I_{\{|z|\geq (N-1)\sqrt{2^{-(n+3)}}\}} \leq \psi_N \leq I_{\{|z|\geq (N-2)\sqrt{2^{-(n+3)}}\}}$. Thus we have

$$\begin{aligned}
\hat{\mathbb{F}}^n_k[\varphi_N] &= \hat{\mathbb{E}}^k[\varphi_N(X_{\delta_n}, X_{2\delta_n} - X_{\delta_n}, \ldots, X_{2^n\delta_n} - X_{(2^n-1)\delta_n})] \\
&\leq 2\sum_{i=1}^{2^n} (\hat{\mathbb{E}}^k[\psi_N(\tilde{M}_{i\delta_n})] + \hat{\mathbb{E}}^k[\psi_N(\tilde{N}_{i\delta_n})]) \\
&= 2\sum_{i=1}^{2^n} (\hat{\mathbb{E}}_1[\psi_N(M_{i\delta_n})] + \hat{\mathbb{E}}_2[\psi_N(N_{i\delta_n})]),
\end{aligned}$$

where the last equality is due to (2) in Corollary 3.6. By (2) in the assumption (A) and the definition of tightness, for each $\varepsilon > 0$, we can take N large enough such that

$$\sup_{k\geq n} \hat{\mathbb{F}}^n_k[\varphi_N] \leq 2\sum_{i=1}^{2^n} (\hat{\mathbb{E}}_1[\psi_N(M_{i\delta_n})] + \hat{\mathbb{E}}_2[\psi_N(N_{i\delta_n})]) < \varepsilon.$$

Thus $\{\hat{\mathbb{F}}^n_k : k \geq n\}$ is tight. □

Now we will use this lemma to construct a sublinear expectation $\hat{\mathbb{E}} : \mathcal{L} \to \mathbb{R}$.

Lemma 3.8. *Set $\mathcal{D} = \{i2^{-n} : n \geq 1, 0 \leq i \leq 2^n\}$. Then there exists a sublinear expectation $\hat{\mathbb{E}} : \mathcal{L} \to \mathbb{R}$ satisfying the following properties:*

(1) *For each* $0 \le t_1 < \cdots < t_n$ *with* $t_i \in \mathcal{D}$*,* $i \le n$*,* $X_{t_n} - X_{t_{n-1}}$ *is independent from* $(X_{t_1}, \ldots, X_{t_{n-1}})$*;*

(2) *For each* $0 \le t_1 < \cdots < t_n$ *with* $t_i \in \mathcal{D}$*,* $i \le n$*,* $(\tilde{M}_{t_1}, \ldots, \tilde{M}_{t_n}) \overset{d}{=} (M_{t_1}, \ldots, M_{t_n})$ *and* $(\tilde{N}_{t_1}, \ldots, \tilde{N}_{t_n}) \overset{d}{=} (N_{t_1}, \ldots, N_{t_n})$*.*

Proof. For $n = 1$, by Lemma 3.7, we know $\{\hat{\mathbb{F}}^1_k : k \ge 1\}$ is tight. Then, by Theorem 2.7, there exists a subsequence $\{\hat{\mathbb{F}}^1_{k^1_j} : j \ge 1\}$ which converges in distribution, i.e., for each $\varphi \in C_{b.Lip}(\mathbb{R}^{2\times 2d})$, $\{\hat{\mathbb{F}}^1_{k^1_j}[\varphi] : j \ge 1\}$ is a Cauchy sequence. Note that $\hat{\mathbb{F}}^1_{k^1_j}[\varphi] = \hat{\mathbb{E}}^{k^1_j}[\varphi(X_{2^{-1}}, X_1 - X_{2^{-1}})]$, then for each $Y \in \mathcal{H}^1$, $\{\hat{\mathbb{E}}^{k^1_j}[Y] : j \ge 1\}$ is a Cauchy sequence.

For $n = 2$, by Lemma 3.7 and Theorem 2.7, we can find a subsequence $\{k^2_j : j \ge 1\} \subset \{k^1_j : j \ge 1\}$ such that for each $Y \in \mathcal{H}^2$, $\{\hat{\mathbb{E}}^{k^2_j}[Y] : j \ge 1\}$ is a Cauchy sequence.

Repeat this process, for each $n \ge 2$, we can find a subsequence $\{k^n_j : j \ge 1\} \subset \{k^{n-1}_j : j \ge 1\}$ such that for each $Y \in \mathcal{H}^n$, $\{\hat{\mathbb{E}}^{k^n_j}[Y] : j \ge 1\}$ is a Cauchy sequence. Taking the diagonal sequence $\{k^j_j : j \ge 1\}$, then for each $Y \in \mathcal{L}$, $\{\hat{\mathbb{E}}^{k^j_j}[Y] : j \ge 1\}$ is a Cauchy sequence, where $\hat{\mathbb{E}}^{k^j_j}[Y] = \infty$ if $Y \notin \mathcal{H}^{k^j_j}$. Define

$$\hat{\mathbb{E}}[Y] = \lim_{j\to\infty} \hat{\mathbb{E}}^{k^j_j}[Y] \text{ for each } Y \in \mathcal{L}.$$

For each Y, $Z \in \mathcal{L}$, there exists a j_0 such that Y, $Z \in \mathcal{H}^{k^j_j}$ for $j \ge j_0$. From this we can easily deduce that $\hat{\mathbb{E}}$ is a sublinear expectation.

Now we prove that this $\hat{\mathbb{E}}$ satisfies (1) and (2). For each $0 \le t_1 < \cdots < t_n$ with $t_i \in \mathcal{D}$, $i \le n$, there exists a j_0 such that $\varphi(X_{t_1}, \ldots, X_{t_{n-1}}, X_{t_n} - X_{t_{n-1}}) \in \mathcal{H}^{k^j_j}$ for each $j \ge j_0$ and $\varphi \in C_{b.Lip}(\mathbb{R}^{n\times 2d})$. Thus, from (2) in Lemma 3.5, we can get

$$\begin{aligned}\hat{\mathbb{E}}[\varphi(X_{t_1}, \ldots, X_{t_{n-1}}, X_{t_n} - X_{t_{n-1}})] &= \lim_{j\to\infty} \hat{\mathbb{E}}^{k^j_j}[\varphi(X_{t_1}, \ldots, X_{t_{n-1}}, X_{t_n} - X_{t_{n-1}})] \\ &= \lim_{j\to\infty} \hat{\mathbb{E}}^{k^j_j}[\psi_j(X_{t_1}, \ldots, X_{t_{n-1}})],\end{aligned}$$

where $\psi_j(x_1, \ldots, x_{n-1}) = \hat{\mathbb{E}}^{k^j_j}[\varphi(x_1, \ldots, x_{n-1}, X_{t_n} - X_{t_{n-1}})]$ for each $x_i \in \mathbb{R}^{2d}$, $i \le n-1$. Define $\psi(x_1, \ldots, x_{n-1}) = \hat{\mathbb{E}}[\varphi(x_1, \ldots, x_{n-1}, X_{t_n} - X_{t_{n-1}})]$ for each $x_i \in \mathbb{R}^{2d}$, $i \le n-1$. In order to prove (1), we only need to show that

$$\lim_{j\to\infty} \hat{\mathbb{E}}^{k^j_j}[\psi_j(X_{t_1}, \ldots, X_{t_{n-1}})] = \hat{\mathbb{E}}[\psi(X_{t_1}, \ldots, X_{t_{n-1}})]. \tag{3.1}$$

It is clear that ψ_j, $j \ge j_0$, and ψ are bounded Lipschitz functions with the common bound K_φ and the common Lipschitz constant L_φ, where K_φ and L_φ are respective the bound and Lipschitz constant of φ. On the other hand, for each $x = (x^1_1, x^2_1, \ldots, x^1_{n-1}, x^2_{n-1}) \in \mathbb{R}^{(n-1)\times 2d}$, by the definition of $\hat{\mathbb{E}}$, we can get $\psi_j(x) \to \psi(x)$. Thus, from the common Lipschitz constant L_φ and pointwise convergence, we can easily obtain that $\{\psi_j : j \ge j_0\}$ converges uniformly to ψ on any compact set in $\mathbb{R}^{(n-1)\times 2d}$. For each given $N > 0$, we have

$$|\psi_j(x) - \psi(x)| \le a_j + 2K_\varphi I_{\{|x|>N\}} \le a_j + 2K_\varphi \sum_{i=1}^{n-1} (\varphi_N(x^1_i) + \varphi_N(x^2_i)),$$

where $a_j = \sup_{|x|\le N} |\psi_j(x) - \psi(x)|$ and $\varphi_N \in C_{b.Lip}(\mathbb{R}^d)$ such that $I_{\{|z|\ge \frac{N}{\sqrt{2(n-1)}}\}} \le \varphi_N \le I_{\{|z|\ge \frac{N-1}{\sqrt{2(n-1)}}\}}$. From the uniform convergence on any compact set, we know $a_j \to 0$ as

$j \to \infty$. Thus

$$\begin{aligned}
&|\hat{\mathbb{E}}^{k_j^j}[\psi_j(X_{t_1},\ldots,X_{t_{n-1}})] - \hat{\mathbb{E}}^{k_j^j}[\psi(X_{t_1},\ldots,X_{t_{n-1}})]| \\
&\le \hat{\mathbb{E}}^{k_j^j}[|\psi_j(X_{t_1},\ldots,X_{t_{n-1}}) - \psi(X_{t_1},\ldots,X_{t_{n-1}})|] \\
&\le \hat{\mathbb{E}}^{k_j^j}[a_j + 2K_\varphi \sum_{i=1}^{n-1}(\varphi_N(\tilde{M}_{t_i}) + \varphi_N(\tilde{N}_{t_i}))] \\
&\le a_j + 2K_\varphi \sum_{i=1}^{n-1}(\hat{\mathbb{E}}^{k_j^j}[\varphi_N(\tilde{M}_{t_i})] + \hat{\mathbb{E}}^{k_j^j}[\varphi_N(\tilde{N}_{t_i})]) \\
&= a_j + 2K_\varphi \sum_{i=1}^{n-1}(\hat{\mathbb{E}}_1[\varphi_N(M_{t_i})] + \hat{\mathbb{E}}_2[\varphi_N(N_{t_i})]).
\end{aligned}$$

Noting that $\hat{\mathbb{E}}^{k_j^j}[\psi(X_{t_1},\ldots,X_{t_{n-1}})] \to \hat{\mathbb{E}}[\psi(X_{t_1},\ldots,X_{t_{n-1}})]$ as $j \to \infty$, we have

$$\begin{aligned}
&\limsup_{j\to\infty} |\hat{\mathbb{E}}^{k_j^j}[\psi_j(X_{t_1},\ldots,X_{t_{n-1}})] - \hat{\mathbb{E}}[\psi(X_{t_1},\ldots,X_{t_{n-1}})]| \\
&\le 2K_\varphi \sum_{i=1}^{n-1}(\hat{\mathbb{E}}_1[\varphi_N(M_{t_i})] + \hat{\mathbb{E}}_2[\varphi_N(N_{t_i})]).
\end{aligned}$$

Due to the tightness of M and N, we get relation (3.1). Thus (1) is obtained. (2) is obvious by the definition of $\hat{\mathbb{E}}$ and (2) in Corollary 3.6. □

Proof of Theorem 3.3. We first extend the sublinear expectation $\hat{\mathbb{E}} : \mathcal{L} \to \mathbb{R}$ to $\hat{\mathbb{E}} : \mathcal{H} \to \mathbb{R}$. Here we still use $\hat{\mathbb{E}}$ for notation simplicity. For each $\varphi(X_{t_1}, X_{t_2}-X_{t_1},\ldots,X_{t_n}-X_{t_{n-1}}) \in \mathcal{H}$ with $\varphi \in C_{b.Lip}(\mathbb{R}^{n\times 2d})$, we can choose $t_k^i \in \mathcal{D}$, $k \le n$, $i \ge 1$, such that $t_k^i < t_{k+1}^i$ and $t_k^i \downarrow t_k$ as $i \to \infty$. By (3) in the assumption (A), we have

$$\begin{aligned}
&|\hat{\mathbb{E}}[\varphi(X_{t_1^i}, X_{t_2^i} - X_{t_1^i},\ldots,X_{t_n^i} - X_{t_{n-1}^i})] - \hat{\mathbb{E}}[\varphi(X_{t_1^j}, X_{t_2^j} - X_{t_1^j},\ldots,X_{t_n^j} - X_{t_{n-1}^j})]| \\
&\le L_\varphi \hat{\mathbb{E}}[(\textstyle\sum_{k=1}^n |X_{t_k^i} - X_{t_k^j} - X_{t_{k-1}^i} + X_{t_{k-1}^j}|) \wedge \frac{2K_\varphi}{L_\varphi}] \\
&\le L_\varphi \hat{\mathbb{E}}[\textstyle\sum_{k=1}^n (|X_{t_k^i} - X_{t_k^j} - X_{t_{k-1}^i} + X_{t_{k-1}^j}| \wedge \frac{2K_\varphi}{L_\varphi})] \\
&\le 2L_\varphi \textstyle\sum_{k=1}^n \{\hat{\mathbb{E}}[|\tilde{M}_{t_k^i} - \tilde{M}_{t_k^j}| \wedge \frac{2K_\varphi}{L_\varphi}] + \hat{\mathbb{E}}[|\tilde{N}_{t_k^i} - \tilde{N}_{t_k^j}| \wedge \frac{2K_\varphi}{L_\varphi}]\} \\
&= 2L_\varphi \textstyle\sum_{k=1}^n \{\hat{\mathbb{E}}_1[|M_{t_k^i} - M_{t_k^j}| \wedge \frac{2K_\varphi}{L_\varphi}] + \hat{\mathbb{E}}_2[|N_{t_k^i} - N_{t_k^j}| \wedge \frac{2K_\varphi}{L_\varphi}]\} \\
&\le 2L_\varphi \textstyle\sum_{k=1}^n \{\hat{\mathbb{E}}_1[|M_{t_k^i} - M_{t_k}| \wedge \frac{2K_\varphi}{L_\varphi} + |M_{t_k^j} - M_{t_k}| \wedge \frac{2K_\varphi}{L_\varphi}] \\
&\quad + \hat{\mathbb{E}}_2[|N_{t_k^i} - N_{t_k}| \wedge \frac{2K_\varphi}{L_\varphi} + |N_{t_k^j} - N_{t_k}| \wedge \frac{2K_\varphi}{L_\varphi}]\} \\
&\to 0 \text{ as } i, j \to \infty,
\end{aligned} \tag{3.2}$$

where $L_\varphi > 0$ is the Lipschitz constant of φ, $K_\varphi = \sup|\varphi|$ and $t_0^i = 0$ for $i \ge 1$. So we can define

$$\hat{\mathbb{E}}[\varphi(X_{t_1}, X_{t_2} - X_{t_1},\ldots,X_{t_n} - X_{t_{n-1}})] = \lim_{i\to\infty} \hat{\mathbb{E}}[\varphi(X_{t_1^i}, X_{t_2^i} - X_{t_1^i},\ldots,X_{t_n^i} - X_{t_{n-1}^i})].$$

It is easy to check that the limit does not depend on the choice of t_k^i by using the same estimate as above.

Our next task is to show that $\hat{\mathbb{E}} : \mathcal{H} \to \mathbb{R}$ is well-defined, that is, if $\varphi(X_{t_1}, X_{t_2} - X_{t_1},\ldots,X_{t_n} - X_{t_{n-1}}) = \tilde{\varphi}(X_{t_1}, X_{t_2} - X_{t_1},\ldots,X_{t_n} - X_{t_{n-1}})$ with φ, $\tilde{\varphi} \in C_{b.Lip}(\mathbb{R}^{n\times 2d})$, then

$$\hat{\mathbb{E}}[\varphi(X_{t_1}, X_{t_2} - X_{t_1},\ldots,X_{t_n} - X_{t_{n-1}})] = \hat{\mathbb{E}}[\tilde{\varphi}(X_{t_1}, X_{t_2} - X_{t_1},\ldots,X_{t_n} - X_{t_{n-1}})]. \tag{3.3}$$

Set

$$U_1 = \{(\tilde{M}_{t_1}(\omega), \tilde{M}_{t_2}(\omega) - \tilde{M}_{t_1}(\omega), \ldots, \tilde{M}_{t_n}(\omega) - \tilde{M}_{t_{n-1}}(\omega)) : \omega \in \Omega\},$$
$$U_2 = \{(\tilde{N}_{t_1}(\omega), \tilde{N}_{t_2}(\omega) - \tilde{N}_{t_1}(\omega), \ldots, \tilde{N}_{t_n}(\omega) - \tilde{N}_{t_{n-1}}(\omega)) : \omega \in \Omega\}.$$

It is clear that $\varphi = \tilde{\varphi}$ on $\overline{U_1} \times \overline{U_2}$, where $\overline{U_i}$, $i = 1, 2$, is the closure of U_i. For each $\varepsilon > 0$, denote

$$U_i^\varepsilon = \{x \in \mathbb{R}^{n\times d} : d(x, U_i) < \varepsilon\},\ i = 1, 2.$$

By Tietze's extension theorem, we can choose a Lipschitz function ψ_i^ε, $i = 1, 2$, such that $I_{(U_i^\varepsilon)^c} \leq \psi_i^\varepsilon \leq I_{\overline{U_i}^c}$. Then

$$\begin{aligned}
&|(\varphi - \tilde{\varphi})(X_{t_1^i}, X_{t_2^i} - X_{t_1^i}, \ldots, X_{t_n^i} - X_{t_{n-1}^i})| \\
&\leq (|\varphi - \tilde{\varphi}| I_{U_1^\varepsilon \times U_2^\varepsilon} + |\varphi - \tilde{\varphi}| I_{(U_1^\varepsilon \times U_2^\varepsilon)^c})(X_{t_1^i}, X_{t_2^i} - X_{t_1^i}, \ldots, X_{t_n^i} - X_{t_{n-1}^i}) \\
&\leq (L_\varphi + L_{\tilde{\varphi}})\sqrt{2}\varepsilon + (K_\varphi + K_{\tilde{\varphi}})(\psi_1^\varepsilon(\tilde{M}_{t_1^i}, \tilde{M}_{t_2^i} - \tilde{M}_{t_1^i}, \ldots, \tilde{M}_{t_n^i} - \tilde{M}_{t_{n-1}^i}) \\
&\quad + \psi_2^\varepsilon(\tilde{N}_{t_1^i}, \tilde{N}_{t_2^i} - \tilde{N}_{t_1^i}, \ldots, \tilde{N}_{t_n^i} - \tilde{N}_{t_{n-1}^i})),
\end{aligned}$$

where $L_{\tilde{\varphi}}$, $K_{\tilde{\varphi}}$ are defined as above. Taking $\hat{\mathbb{E}}[\cdot]$ on both sides, we have

$$\begin{aligned}
&|\hat{\mathbb{E}}[\varphi(X_{t_1^i}, X_{t_2^i} - X_{t_1^i}, \ldots, X_{t_n^i} - X_{t_{n-1}^i})] - \hat{\mathbb{E}}[\tilde{\varphi}(X_{t_1^i}, X_{t_2^i} - X_{t_1^i}, \ldots, X_{t_n^i} - X_{t_{n-1}^i})]| \\
&\leq (L_\varphi + L_{\tilde{\varphi}})\sqrt{2}\varepsilon + (K_\varphi + K_{\tilde{\varphi}})(\hat{\mathbb{E}}[\psi_1^\varepsilon(\tilde{M}_{t_1^i}, \tilde{M}_{t_2^i} - \tilde{M}_{t_1^i}, \ldots, \tilde{M}_{t_n^i} - \tilde{M}_{t_{n-1}^i})] \\
&\quad + \hat{\mathbb{E}}[\psi_2^\varepsilon(\tilde{N}_{t_1^i}, \tilde{N}_{t_2^i} - \tilde{N}_{t_1^i}, \ldots, \tilde{N}_{t_n^i} - \tilde{N}_{t_{n-1}^i})]) \\
&= (L_\varphi + L_{\tilde{\varphi}})\sqrt{2}\varepsilon + (K_\varphi + K_{\tilde{\varphi}})(\hat{\mathbb{E}}_1[\psi_1^\varepsilon(M_{t_1^i}, M_{t_2^i} - M_{t_1^i}, \ldots, M_{t_n^i} - M_{t_{n-1}^i})] \\
&\quad + \hat{\mathbb{E}}_2[\psi_2^\varepsilon(N_{t_1^i}, N_{t_2^i} - N_{t_1^i}, \ldots, N_{t_n^i} - N_{t_{n-1}^i})]).
\end{aligned}$$

By the definition of $\tilde{M}$, it is easy to see $U_1 = \{(M_{t_1}(\omega_1), M_{t_2}(\omega_1) - M_{t_1}(\omega_1), \ldots, M_{t_n}(\omega_1) - M_{t_{n-1}}(\omega_1)) : \omega_1 \in \Omega_1\}$, which implies $\psi_1^\varepsilon(M_{t_1}, M_{t_2} - M_{t_1}, \ldots, M_{t_n} - M_{t_{n-1}}) = 0$. Thus, by the same method as (3.2), we have

$$\begin{aligned}
&|\hat{\mathbb{E}}_1[\psi_1^\varepsilon(M_{t_1^i}, M_{t_2^i} - M_{t_1^i}, \ldots, M_{t_n^i} - M_{t_{n-1}^i})]| \\
&= |\hat{\mathbb{E}}_1[\psi_1^\varepsilon(M_{t_1^i}, M_{t_2^i} - M_{t_1^i}, \ldots, M_{t_n^i} - M_{t_{n-1}^i}) - \psi_1^\varepsilon(M_{t_1}, M_{t_2} - M_{t_1}, \ldots, M_{t_n} - M_{t_{n-1}})]| \\
&\to 0 \text{ as } i \to \infty.
\end{aligned}$$

Similarly, $\hat{\mathbb{E}}_2[\psi_2^\varepsilon(N_{t_1^i}, N_{t_2^i} - N_{t_1^i}, \ldots, N_{t_n^i} - N_{t_{n-1}^i})] \to 0$ as $i \to \infty$. It follows that

$$\begin{aligned}
&\limsup_{i\to\infty} |\hat{\mathbb{E}}[\varphi(X_{t_1^i}, X_{t_2^i} - X_{t_1^i}, \ldots, X_{t_n^i} - X_{t_{n-1}^i})] - \hat{\mathbb{E}}[\tilde{\varphi}(X_{t_1^i}, X_{t_2^i} - X_{t_1^i}, \ldots, X_{t_n^i} - X_{t_{n-1}^i})]| \\
&\leq (L_\varphi + L_{\tilde{\varphi}})\sqrt{2}\varepsilon.
\end{aligned}$$

Letting $\varepsilon \to 0$, we obtain (3.3).

Moreover, it is easy to verify that $\hat{\mathbb{E}}$ is a sublinear expectation. By (2) in Lemma 3.8 and the assumption (A), we can deduce that $(\tilde{M}_t)_{t\in[0,1]} \overset{d}{=} (M_t)_{t\in[0,1]}$ and $(\tilde{N}_t)_{t\in[0,1]} \overset{d}{=} (N_t)_{t\in[0,1]}$. Then we verify that $(X_t)_{t\in[0,1]}$ has independent increments. For each $0 \leq t_1 < \cdots < t_n \leq 1$, we can choose $t_k^i \in \mathcal{D}$ as above. By the definition of $\hat{\mathbb{E}}$ and (1) in Lemma 3.8, we can get that for each $\varphi \in C_{b.Lip}(\mathbb{R}^{n\times 2d})$,

$$\begin{aligned}
\hat{\mathbb{E}}[\varphi(X_{t_1}, \ldots, X_{t_{n-1}}, X_{t_n} - X_{t_{n-1}})] &= \lim_{i\to\infty} \hat{\mathbb{E}}[\varphi(X_{t_1^i}, \ldots, X_{t_{n-1}^i}, X_{t_n^i} - X_{t_{n-1}^i})] \\
&= \lim_{i\to\infty} \hat{\mathbb{E}}[\psi_i(X_{t_1^i}, \ldots, X_{t_{n-1}^i})],
\end{aligned}$$

where $\psi_i(x_1,\ldots,x_{n-1}) = \hat{\mathbb{E}}[\varphi(x_1,\ldots,x_{n-1},X_{t_n^i} - X_{t_{n-1}^i})]$. Define

$$\psi(x_1,\ldots,x_{n-1}) = \hat{\mathbb{E}}[\varphi(x_1,\ldots,x_{n-1},X_{t_n} - X_{t_{n-1}})].$$

Then

$$\begin{aligned}&|\psi_i(x_1,\ldots,x_{n-1}) - \psi(x_1,\ldots,x_{n-1})|\\ &\leq L_\varphi \hat{\mathbb{E}}[|X_{t_n^i} - X_{t_{n-1}^i} - X_{t_n} + X_{t_{n-1}}| \wedge \frac{2K_\varphi}{L_\varphi}]\\ &\leq L_\varphi\{\hat{\mathbb{E}}_1[|M_{t_n^i} - M_{t_n}| \wedge \frac{2K_\varphi}{L_\varphi} + |M_{t_{n-1}^i} - M_{t_{n-1}}| \wedge \frac{2K_\varphi}{L_\varphi}]\\ &\quad + \hat{\mathbb{E}}_2[|N_{t_n^i} - N_{t_n}| \wedge \frac{2K_\varphi}{L_\varphi} + |N_{t_{n-1}^i} - N_{t_{n-1}}| \wedge \frac{2K_\varphi}{L_\varphi}]\},\end{aligned}$$

which implies

$$\begin{aligned}&|\hat{\mathbb{E}}[\psi_i(X_{t_1^i},\ldots,X_{t_{n-1}^i})] - \hat{\mathbb{E}}[\psi(X_{t_1^i},\ldots,X_{t_{n-1}^i})]|\\ &\leq L_\varphi\{\hat{\mathbb{E}}_1[|M_{t_n^i} - M_{t_n}| \wedge \frac{2K_\varphi}{L_\varphi} + |M_{t_{n-1}^i} - M_{t_{n-1}}| \wedge \frac{2K_\varphi}{L_\varphi}]\\ &\quad + \hat{\mathbb{E}}_2[|N_{t_n^i} - N_{t_n}| \wedge \frac{2K_\varphi}{L_\varphi} + |N_{t_{n-1}^i} - N_{t_{n-1}}| \wedge \frac{2K_\varphi}{L_\varphi}]\}.\end{aligned}$$

From this we deduce

$$\begin{aligned}\hat{\mathbb{E}}[\varphi(X_{t_1},\ldots,X_{t_{n-1}},X_{t_n} - X_{t_{n-1}})] &= \lim_{i\to\infty} \hat{\mathbb{E}}[\psi(X_{t_1^i},\ldots,X_{t_{n-1}^i})]\\ &= \hat{\mathbb{E}}[\psi(X_{t_1},\ldots,X_{t_{n-1}})],\end{aligned}$$

which proves that $(X_t)_{t\in[0,1]}$ has independent increments.

If $(M_t)_{t\in[0,1]}$ and $(N_t)_{t\in[0,1]}$ are two processes with stationary and independent increments, then from the construction of $\hat{\mathbb{E}}$, $(X_t)_{t\in[0,1]}$ has stationary increments. The proof is complete. □

In the following, we give an example to calculate $\hat{\mathbb{E}}$.

Example 3.9. Let Γ_i, $i = 1, 2$, be two given bounded subset in $\mathbb{R}^{d\times d}$. Define $G_i : \mathbb{S}(d) \to \mathbb{R}$ by

$$G_i(A) = \frac{1}{2}\sup_{Q\in\Gamma_i} \mathrm{tr}[AQQ^T] \text{ for each } A \in \mathbb{S}(d).$$

Let $(B_t^i)_{t\geq 0}$ be a d-dimensional G_i-Brownian motion defined on sublinear expectation space $(\Omega_i, \mathcal{H}_i, \hat{\mathbb{E}}_i)$, $i = 1,\ 2$. In the above, we construct a sublinear expectation space $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$ and a $2d$-dimensional process $(\tilde{B}_t)_{t\geq 0} = (\tilde{B}_t^1, \tilde{B}_t^2)_{t\geq 0}$ with stationary and independent increments satisfying $(\tilde{B}_t^1)_{t\geq 0} \overset{d}{=} (B_t^1)_{t\geq 0}$ and $(\tilde{B}_t^2)_{t\geq 0} \overset{d}{=} (B_t^2)_{t\geq 0}$. Since

$$\hat{\mathbb{E}}[|\tilde{B}_t|^3] \leq 4(\hat{\mathbb{E}}[|\tilde{B}_t^1|^3] + \hat{\mathbb{E}}[|\tilde{B}_t^2|^3]) = 4(\hat{\mathbb{E}}_1[|B_t^1|^3] + \hat{\mathbb{E}}_2[|B_t^2|^3]) = 4Ct^{\frac{3}{2}},$$

where $C = \hat{\mathbb{E}}_1[|B_1^1|^3] + \hat{\mathbb{E}}_2[|B_1^2|^3]$, by Theorem 1.6 in Chapter III in [9], we can obtain that $(\tilde{B}_t)_{t\geq 0}$ is a G-Brownian motion with

$$G(A) = \frac{1}{2}\hat{\mathbb{E}}[\langle A\tilde{B}_1, \tilde{B}_1\rangle] \text{ for each } A = \begin{bmatrix} A_1 & D \\ D^T & A_2 \end{bmatrix} \in \mathbb{S}(2d).$$

By our construction, it is easy to check that $\hat{\mathbb{E}}^n[\langle D\tilde{B}_1^2, \tilde{B}_1^1\rangle] = \hat{\mathbb{E}}^n[-\langle D\tilde{B}_1^2, \tilde{B}_1^1\rangle] = 0$ for each $n \geq 1$. Thus $\hat{\mathbb{E}}[\langle D\tilde{B}_1^2, \tilde{B}_1^1\rangle] = \hat{\mathbb{E}}[-\langle D\tilde{B}_1^2, \tilde{B}_1^1\rangle] = 0$. By subadditivity, we can get $\hat{\mathbb{E}}[\langle A\tilde{B}_1, \tilde{B}_1\rangle] = \hat{\mathbb{E}}[\langle A_1\tilde{B}_1^1, \tilde{B}_1^1\rangle + \langle A_2\tilde{B}_1^2, \tilde{B}_1^2\rangle]$. Furthermore,

$$\hat{\mathbb{E}}^n[\langle A_1\tilde{B}_1^1, \tilde{B}_1^1\rangle + \langle A_2\tilde{B}_1^2, \tilde{B}_1^2\rangle] = \hat{\mathbb{E}}_1[\langle A_1B_1^1, B_1^1\rangle] + \hat{\mathbb{E}}_2[\langle A_2B_1^2, B_1^2\rangle]$$

for each $n \geq 1$ by our construction. Thus $G(A) = G_1(A_1) + G_2(A_2)$.

Finally, we must observe that $\hat{\mathbb{E}}$ in our main Theorem 3.3 may not be unique. This point will be illustrated by an example.

Example 3.10. Let $(B_t^i)_{t\geq 0}$ be a 1-dimensional G-Brownian motion defined on sublinear expectation space $(\Omega_i, \mathcal{H}_i, \hat{\mathbb{E}}_i)$, $i = 1$, 2, where

$$G(a) = \frac{1}{2}(\overline{\sigma}^2 a^+ - \underline{\sigma}^2 a^-),\ a \in \mathbb{R},$$

here $0 \leq \underline{\sigma}^2 < \overline{\sigma}^2 < \infty$. For each fixed $\lambda \in [0,1]$, define

$$\tilde{G}_\lambda(A) = \lambda G(a_{11} + a_{22}) + (1-\lambda)[G(a_{11}) + G(a_{22})] \text{ for each } A = (a_{ij}) \in \mathbb{S}(2).$$

Following Chapter III in [9], we can construct a sublinear expectation space $(\Omega, \mathcal{H}, \hat{\mathbb{E}}_\lambda)$ and a $\tilde{G}_\lambda$-Brownian motion $(\tilde{B}_t)_{t\geq 0} = (\tilde{B}_t^1, \tilde{B}_t^2)_{t\geq 0}$. By Proposition 1.4 in Chapter III in [9], it is easy to verify that $(\tilde{B}_t^1)_{t\geq 0} \overset{d}{=} (B_t^1)_{t\geq 0}$ and $(\tilde{B}_t^2)_{t\geq 0} \overset{d}{=} (B_t^2)_{t\geq 0}$. Thus in Theorem 3.3 we can take $\hat{\mathbb{E}} = \hat{\mathbb{E}}_\lambda$, $\lambda \in [0,1]$, where $\hat{\mathbb{E}}_{\lambda_1} \neq \hat{\mathbb{E}}_{\lambda_2}$ for $\lambda_1 \neq \lambda_2$.

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