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The set of connective constants of Cayley graphs contains a Cantor space

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Abstract

The connective constant of a transitive graph is the exponential growth rate of its number of self-avoiding walks. We prove that the set of connective constants of the so-called Cayley graphs contains a Cantor set. In particular, this set has the cardinality of the continuum.

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In this paper, graphs are implicitly taken to be simple, unoriented, non-empty, connected and locally finite. Besides, we denote by $\mathbb{N}$ the set consisting of the non-negative integers and by $\mathbb{N}^\star$ that of all positive integers.

A graph is said to be **transitive** if it admits an action by graph automorphisms that is transitive on its set of vertices. Given a transitive graph $\mathcal{G}$ and a vertex o of $\mathcal{G}$, denote by c_n the number of paths starting at o, going through n edges and not visiting any vertex more than once. By Fekete's Subadditive Lemma, the sequence $c_n^{1/n}$ converges to some real number $\mu(\mathcal{G})$. This number does not depend on the choice of o and is called the **connective constant** of $\mathcal{G}$.

Let us now define Cayley graphs. Given a group G and a finite generating subset S of G, the **Cayley graph** associated with (G, S) is the graph with vertex-set G and such that two *distinct* elements g and h of G are connected by an edge if and only if $g^{-1}h \in S \cup S^{-1}$. This defines a transitive graph $\mathsf{Cay}(G, S)$ which satisfies the implicit assumptions of this paper.

Our purpose is to prove the following theorem.

Theorem. *The set* $\{x \in \mathbb{R} : \exists (G, S),\ x = \mu(\mathsf{Cay}(G, S))\}$ *contains a Cantor space. In particular, this subset of* $\mathbb{R}$ *has cardinality* $2^{\aleph_0}$.

This theorem implies the following result of Leader and Markström: the set of isomorphism classes of Cayley graphs has cardinality $2^{\aleph_0}$. See [8].

An unpublished argument of Kozma [7] shows that the set of p_c's of Cayley graphs contains a Cantor space, where p_c denotes the critical parameter for bond Bernoulli percolation. The strategy of proof used in the present paper is inspired by [7].

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Proof. Let Conn denote $\{x : \exists(G,S),\ x = \mu(\mathsf{Cay}(G,S))\}$. Let Ω_∞ stand for $\{0,1\}^{\mathbb{N}}$, which is endowed with the product topology. It is enough to show that there is a continuous injection f from Ω_∞ to Conn. Indeed, as Ω_∞ is compact and $\mathbb{R}$ is Hausdorff, if there is such an f, then f induces a homeomorphism from the Cantor space Ω_∞ to $f(\Omega_\infty)$. Besides, as Conn is a subset of $\mathbb{R}$ and both $\mathbb{R}$ and Ω_∞ have cardinality $2^{\aleph_0}$, the Cantor-Schröder-Bernstein Theorem implies that if there is an injection from Ω_∞ to Conn, then Conn has cardinality $2^{\aleph_0}$.

To prove the existence of a function f as above, we will rely on several facts, which are listed below. Fact G is about "Groups". For it, the reader is referred to the *proof* of Lemma III.40 in [3]. Facts L and SI, respectively on "Locality" and "Strict Inequalities", are due to Grimmett and Li: see respectively [5] and [6]. Facts I and C are easy and classical. They provide an "Inequality" and a "Convergence". The image of a set/element X by a quotient map which is clear from the context is denoted by $\overline{X}$.

- G There are a finitely generated group H and a subgroup C_H of H such that C_H is isomorphic to $\bigoplus_{n\in\mathbb{N}}\mathbb{Z}$ and central in H.
- L Let $(\mathcal{G}_n)_{n\le\infty}$ be a sequence of Cayley graphs such that $\mathcal{G}_n$ converges locally [(1)] to $\mathcal{G}_\infty$. Denote by $\mathcal{Z}$ the graph $\mathsf{Cay}(\mathbb{Z},\{1\})$ and by $\mathcal{G}_n \times \mathcal{Z}$ the Cartesian product of the graphs $\mathcal{G}_n$ and $\mathcal{Z}$.

  Then, $\mu(\mathcal{G}_n \times \mathcal{Z})$ converges to $\mu(\mathcal{G}_\infty \times \mathcal{Z})$. [(2)]
- SI Let G be a group generated by a finite subset S, and let N be a normal subgroup of G. Assume that $N \neq \{1\}$ and that the ball of centre 1 and radius 2 of $\mathsf{Cay}(G,S)$ intersects N only at 1.

  Then, $\mu(\mathsf{Cay}(G/N,\overline{S})) < \mu(\mathsf{Cay}(G,S))$.
- I Let G be a group generated by a finite subset S, and let N be a normal subgroup of G.

  Then, $\mu(\mathsf{Cay}(G/N,\overline{S})) \le \mu(\mathsf{Cay}(G,S))$.
- C Let G be a group generated by a finite subset S. Let $(N_n)_{n\le\infty}$ be a sequence of normal subgroups of G such that for every finite subset F of G, for n large enough, $N_n \cap F = N_\infty \cap F$.

  Then, $\mathsf{Cay}(G/N_n,\overline{S})$ converges locally to $\mathsf{Cay}(G/N_\infty,\overline{S})$.

The proof may now begin. Let us fix (H, C_H) satisfying G. Let S_H be a finite generating subset of H. Let $\langle a\rangle$ denote the free group with one generator, with multiplicative notation (1 denotes the identity element). Let $G := H \times \langle a\rangle$ and $S := (S_H \times \{1\}) \cup \{(1,a)\}$. The finite subset S of G generates the group G. The subgroup $C := C_H \times \{1\}$ of G is central and isomorphic to $\bigoplus_{n\in\mathbb{N}}\langle a\rangle$. Fix a basis (g_n) of the free abelian group C.

Let Ω denote the set of the (finite and infinite) words on the alphabet $\{0,1\}$. If $\mathcal{P}$ denotes a property which may be satisfied or not by elements of $\mathbb{N}\cup\{\infty\}$, denote by $\Omega_{\mathcal{P}}$ the set of the elements of Ω whose length satisfies $\mathcal{P}$. In this context, we may use the subscript "k" as an abbreviation for "$= k$". The Ω_∞ introduced at the beginning of the proof agrees with this notation.

For every $\omega \in \Omega_{<\infty}$, we will define a group G_ω, which will be a quotient of G. Before stating our conditions, let us point out that we set $G_{\text{empty word}}$ to be G. As a result, G with no subscript or with an empty subscript are both defined, and refer to the same object.

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(1) This means the following. Denote by ρ_n the vertex corresponding to the identity element of G_n, where $\mathcal{G}_n = \mathsf{Cay}(G_n, S_n)$. For $n \le \infty$ and $r \in \mathbb{N}$, let $B_n(r)$ be the ball of centre ρ_n and radius r in $\mathcal{G}_n$, considered as a *rooted graph*, rooted at ρ_n. We say that $\mathcal{G}_n$ **converges locally** to $\mathcal{G}_\infty$ if $\forall r,\ \exists n_0,\ \forall n \ge n_0,\ B_n(r) \simeq B_\infty(r)$. See [1, 2, 4].

(2) This results from [5] as, for every $\mathcal{G}_n$, the projection on the $\mathcal{Z}$-factor induces a "height function" with $d = 1$ and $r = 0$.

We will proceed by induction on $n \in \mathbb{N}$, with the following Induction Hypothesis. See the figure below. Given a subset/element X of a group, $\langle X \rangle$ stands for the *subgroup* it generates. Notice that as C is central in G, every subgroup of C is central hence normal in G.

IH For every $\omega \in \Omega_{\leq n}$, we have built a group G_ω which is G or a quotient of G by a subgroup of $\langle \{g_i : i < n\} \rangle$. We denote by $\mathcal{G}_\omega$ the Cayley graph of G_ω with respect to $\overline{S}$.

For every $\omega \in \Omega_{<n}$, we have constructed a real number denoted by $b_{\omega\star}$, and we have $G_{\omega 0} = G_\omega$.

Setting "no letter" $< 0 < \star < 1$ and ordering lexicographically the words on the alphabet $\{0, \star, 1\}$, the set

$$\mathcal{S}_n := \{(\mu(\mathcal{G}_\omega), \omega) : \omega \in \Omega_n\} \cup \{(b_{\omega\star}, \omega\star) : \omega \in \Omega_{<n}\}$$

satisfies $\forall (x, \eta), (x', \eta') \in \mathcal{S}_n,\ \eta < \eta' \iff x > x'$.

Notice that IH holds for $n = 0$.

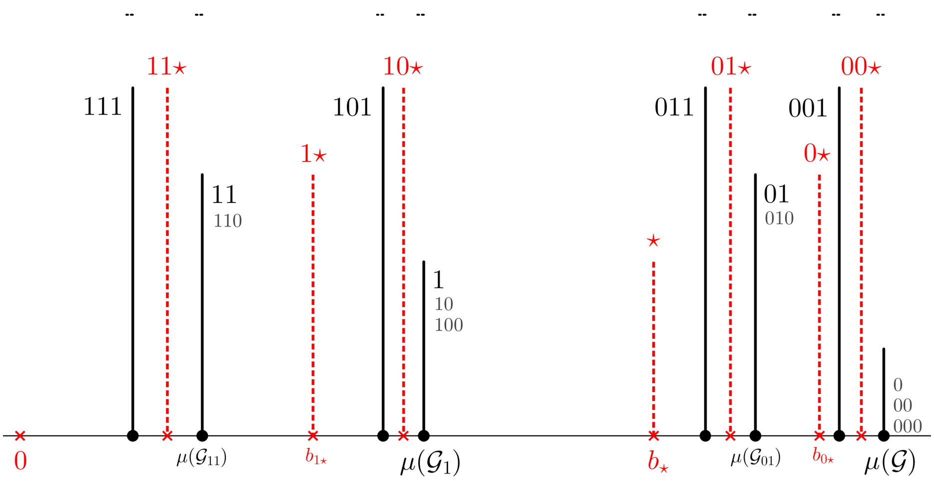

Illustration of IH at rank 3. Above the vertical lines is "represented" the Cantor subset of Conn that we will build.

Let $n \in \mathbb{N}$ be such that IH holds at rank n, and let us prove that it holds at rank $n+1$. For $\omega \in \Omega_n$ and $k \in \mathbb{N}^\star$, let N_k^ω denote the subgroup of G_ω generated by $\overline{g_n}^k$, which is central hence normal in G_ω. Let F be a finite subset of G_ω. By IH at rank n, the map $k \mapsto \overline{g_n}^k$ is injective from $\mathbb{Z}$ to G_ω. The set $Z_F^\omega := \{j \in \mathbb{Z} : \overline{g_n}^j \in F\}$ is thus finite. For every $k > \max_{j \in Z_F^\omega} |j|$, we have $k\mathbb{Z} \cap Z_F^\omega \subset \{0\}$. As a result, for k large enough $N_k^\omega \cap F = \{1\} \cap F$. It follows from C that $\mathsf{Cay}(G_\omega / \langle k\overline{g_n} \rangle, \overline{S})$ converges locally to $\mathcal{G}_\omega$ when k goes to infinity.

Thus, by taking $m_n \in \mathbb{N}^\star$ large enough, L and SI guarantee that for every $\omega \in \Omega_n$, the connective constant $x := \mu(\mathsf{Cay}(G_\omega / \langle m_n \overline{g_n} \rangle, \overline{S}))$ satisfies $x < \mu(\mathcal{G}_\omega)$ and, for every strict prefix α of ω, $b_{\alpha\star} < x$. Taking m_n to be minimal such that the above holds and letting $G_{\omega 0} := G_\omega$, $G_{\omega 1} := G_\omega / \langle m_n \overline{g_n} \rangle$ and $b_{\omega\star} := (\mu(\mathcal{G}_{\omega 0}) + \mu(\mathcal{G}_{\omega 1}))/2$, we get IH at rank $n+1$. By induction, the G_ω's are constructed with the desired properties, together with the "byproduct" sequence (m_n).

Now, for $\omega \in \Omega_\infty$, let $G_\omega := G / \langle \{m_i g_i : i \text{ such that } \omega(i) = 1\} \rangle$ and let $\mathcal{G}_\omega := \mathsf{Cay}(G_\omega, \overline{S})$. To conclude the proof, it is enough to show that $f : \omega \mapsto \mu(\mathcal{G}_\omega)$ is injective and continuous as a function from Ω_∞ to $\mathbb{R}$.

Let (ω_n) be a converging sequence of elements of Ω_∞, and let ω_∞ denote its limit. For $n \in \mathbb{N} \cup \{\infty\}$, define N_n to be $\langle\{m_i g_i : i \text{ such that } \omega_n(i) = 1\}\rangle$. For every element $g = \prod_{i \leq i_0} g_i^{a_i}$ of the free abelian group C, we have

$$g \in N_n \iff \text{“}\{i : a_i \neq 0\} \subset \{i : \omega_n(i) = 1\} \text{ and } \forall i \leq i_0,\ m_i | a_i\text{”}.$$

Consequently, $(N_n)_{n \leq \infty}$ satisfies the hypotheses of C. By C and L, $f(\omega_n)$ converges to $f(\omega_\infty)$, so that f is continuous.

It remains to establish the injectivity of f. Let ω and ω' be two distinct elements of Ω_∞. Let $i \in \mathbb{N}$ be minimal such that $\omega(i) \neq \omega'(i)$. Without loss of generality, we may assume that $\omega(i) = 0$ and $\omega'(i) = 1$. For $n \in \mathbb{N}$, denote by ω_n the prefix of ω of length n. Note that $\omega_i = \omega'_i$. By I and the construction, we have

$$\forall n > i,\ \mu(\mathcal{G}_{\omega'}) \leq \mu(\mathcal{G}_{\omega_i}) < b_{\omega_i\star} < \mu(\mathcal{G}_{\omega_n}).$$

By C and L, $\mu(\mathcal{G}_{\omega_n})$ converges to $\mu(\mathcal{G}_\omega)$. Therefore,

$$\mu(\mathcal{G}_{\omega'}) \leq \mu(\mathcal{G}_{\omega_i}) < b_{\omega_i\star} \leq \mu(\mathcal{G}_\omega).$$

In particular, $\mu(\mathcal{G}_{\omega'}) \neq \mu(\mathcal{G}_\omega)$. The function f is thus injective, and the theorem is proved. □

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