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Parameter estimation for discretely observed non-ergodic fractional Ornstein–Uhlenbeck processes of the second kind

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Abstract. We use the least squares type estimation to estimate the drift parameter $\theta > 0$ of a non-ergodic fractional Ornstein–Uhlenbeck process of the second kind defined as $dX_t = \theta X_t\,dt + dY_t^{(1)}$, $X_0 = 0$, $t \geq 0$, where $Y_t^{(1)} = \int_0^t e^{-s}\,dB_{a_s}$ with $a_t = He^{\frac{t}{H}}$, and $\{B_t, t \geq 0\}$ is a fractional Brownian motion of Hurst parameter $H \in (\frac{1}{2}, 1)$. We assume that the process $\{X_t, t \geq 0\}$ is observed at discrete time instants $t_1 = \Delta_n, \dots, t_n = n\Delta_n$. We construct two estimators $\hat{\theta}_n$ and $\check{\theta}_n$ of θ which are strongly consistent and we prove that these estimators are $\sqrt{n\Delta_n}$-consistent, in the sense that the sequences $\sqrt{n\Delta_n}(\hat{\theta}_n - \theta)$ and $\sqrt{n\Delta_n}(\check{\theta}_n - \theta)$ are tight.

1 Introduction

Parameter estimation for non-ergodic type diffusion processes has been developed in several papers. For motivation and further references, we refer the reader to Basawa and Scott (1983), Dietz and Kutoyants (2003), Jacod (2006), Shimizu (2009).

Let $B = \{B_t, t \geq 0\}$ be a fractional Brownian motion (fBm) with Hurst parameter $H \in (0, 1)$. In recent years, the study of various statistical estimation problems related to the (so-called) fractional Ornstein–Uhlenbeck (fOU) of the first kind, that is, to the solution X of

$$X_0 = 0, \qquad dX_t = \theta X_t\,dt + dB_t, \qquad t \geq 0 \tag{1.1}$$

has attracted interest. In the case of fOU (1.1), the parameter estimation for θ has been extensively studied by using several approaches. For a comprehensive review on maximum likelihood method, we refer to Kleptsyna and Le Breton (2002), Bercu, Coutin and Savy (2011), Tanaka (2015). A least squares approach has been proposed in the papers Hu and Nualart (2010), Belfadli, Es-Sebaiy and Ouknine (2011), Es-Sebaiy and Ndiaye (2014), El Machkouri, Es-Sebaiy and Ouknine (2016). For a more recent comprehensive discussion via method of moments, we refer to El Onsy, Es-Sebaiy and Viens (2017), Es-Sebaiy and Viens (2016).

Key words and phrases. Drift parameter estimation, non-ergodic fractional Ornstein–Uhlenbeck process of the second kind.

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In the present work, we consider the non-ergodic fractional Ornstein–Uhlenbeck process of the second kind (FOUSK) $\{X_t, t \geq 0\}$ given by the following linear stochastic differential equation

$$X_0 = 0, \qquad dX_t = \theta X_t\, dt + dY_t^{(1)}, \qquad t \geq 0, \tag{1.2}$$

where $Y_t^{(1)} := \int_0^t e^{-s}\, dB_{a_s}$ with $a_t = He^{\frac{t}{H}}$, and B is a fBm of Hurst index $H \in (\frac{1}{2}, 1)$, whereas $\theta > 0$ is considered as an unknown parameter.

The drift parameter estimation for (1.2) based on continuous-time observations has been studied in El Onsy, Es-Sebaiy and Tudor (2014) by using the least squares estimator (LSE) defined by

$$\widetilde{\theta}_t = \frac{\int_0^t X_s\, dX_s}{\int_0^t X_s^2\, ds}, \qquad t \geq 0 \tag{1.3}$$

as estimator of θ, where the integral with respect to X is a Young integral. Let us describe what is known about this problem: $\widetilde{\theta}_t$ is a strongly consistent estimator of θ and it is asymptotically Cauchy. More precisely, as $t \to \infty$

$$e^{\theta t}(\widetilde{\theta}_t - \theta) \xrightarrow{\text{Law}} 2\theta H^{2(\theta-1)H} \mathcal{C}(1),$$

with $\mathcal{C}(1)$ the standard Cauchy distribution.

From a practical point of view, in parametric inference, it is more realistic and interesting to consider asymptotic estimation for FOUSK based on discrete observations. Then, we will assume that the process X given in (1.2) is observed equidistantly in time with the step size Δ_n: $t_i = i\Delta_n, i = 0, \dots, n$, and $T_n = n\Delta_n$ denotes the length of the "observation window". Let us consider the following discrete version of $\widetilde{\theta}_t$ defined in (1.3),

$$\hat{\theta}_n = \frac{\sum_{i=1}^n X_{t_{i-1}}(X_{t_i} - X_{t_{i-1}})}{\Delta_n \sum_{i=1}^n X_{t_{i-1}}^2}. \tag{1.4}$$

Since we can rewrite $\widetilde{\theta}_t$ as follows,

$$\widetilde{\theta}_t = \frac{X_t^2}{2\int_0^t X_s^2\, ds},$$

we can also consider this second discrete version of $\widetilde{\theta}_t$,

$$\check{\theta}_n = \frac{X_{T_n}^2}{2\Delta_n \sum_{i=1}^n X_{t_{i-1}}^2}. \tag{1.5}$$

Our purpose is to study the asymptotic behavior and the rate consistency of the estimators $\hat{\theta}_n$ and $\check{\theta}_n$ based on the sampling data $X_{t_i}, i = 0, \dots, n$.

Recall that in the case of ergodic-type FOUSK, corresponding to $\theta < 0$, the drift estimation based on continuous and discrete observations of X has been studied for example, in Es-Sebaiy and Viens (2016), Azmoodeh and Morlanes (2013), Azmoodeh and Viitasaari (2015).

The paper is organized as follows. In Section 2, we give some properties of the FOUSK process. Section 3 is devoted to the study of the strong consistency of the above estimators $\hat{\theta}_n$ and $\check{\theta}_n$. In Section 4, we study the rate consistency of those estimators. Finally, in Section 5 we give simulation examples to show the performance of these estimators and the standard error is also proposed as a criterion of validation.

2 Preliminaries

Throughout this paper, we assume that B is a fractional Brownian motion with Hurst parameter $H \in (\frac{1}{2}, 1)$, defined on a complete probability space $(\Omega, \mathcal{F}, P)$, that is, B is a centered Gaussian process $B = \{B_t, t \geq 0\}$ with the covariance function

$$E(B_t B_s) = \frac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H}).$$

In this section, we recall some properties of the FOUSK process (see Kaarakka and Salminen (2011), El Onsy, Es-Sebaiy and Tudor (2014)). These properties will be needed in the next sections in order to analyze the behavior of the estimators $\hat{\theta}_n$ and $\check{\theta}_n$ of θ. Let us first note that the unique solution to (1.2) can be written as

$$X_t = e^{\theta t} \int_0^t e^{-\theta s} \, dY_s^{(1)}, \qquad t \geq 0. \tag{2.1}$$

In order to make the analysis of this process easier, we will express the Wiener integral with respect to the process $Y^{(1)}$ as a Wiener integral with respect to the fractional Brownian motion B. Define the process

$$\zeta_t = \int_0^t e^{-\theta s} \, dY_s^{(1)}, \qquad t \geq 0. \tag{2.2}$$

From El Onsy, Es-Sebaiy and Tudor (2014), we can write

$$\zeta_t = H^{(\theta+1)H} \int_{a_0}^{a_t} s^{-(\theta+1)H} \, dB_s, \qquad t \geq 0. \tag{2.3}$$

Moreover, since $H > \frac{1}{2}$, we have

$$E\big[(\zeta_t - \zeta_s)^2\big] = H(2H-1)H^{2(\theta+1)H} \int_{a_s}^{a_t} \int_{a_s}^{a_t} (uv)^{-(\theta+1)H} |u-v|^{2H-2} \, du \, dv, \quad 0 \leq s < t. \tag{2.4}$$

We will also need the following result, which is proved in El Onsy, Es-Sebaiy and Tudor (2014).

Lemma 2.1. *Let ζ be the process defined in (2.2). Then*

(i) *For all $\varepsilon \in (0, H)$, the process ζ admits a modification with $(H-\varepsilon)$-Hölder continuous paths, still denoted ζ in the sequel.*

(ii) *As $t \to \infty$*

$$\zeta_t \to \zeta_\infty := H^{(\theta+1)H} \int_{a_0}^{\infty} t^{-(\theta+1)H} \, dB_t \qquad \text{almost surely and in } L^2(\Omega).$$

3 Strong consistency of the estimators

Let X be the FOUSK process given by (1.2), and let us introduce the following sequences

$$S_n := \Delta_n \sum_{i=1}^{n} X_{t_{i-1}}^2$$

and

$$\Lambda_n := \sum_{i=1}^{n} e^{\theta t_i} (\zeta_{t_i} - \zeta_{t_{i-1}}) X_{t_{i-1}}.$$

So, we can write $\widehat{\theta}_n$ and $\check{\theta}_n$ as follows

$$\widehat{\theta}_n = \frac{e^{\theta \Delta_n} - 1}{\Delta_n} + \frac{\Lambda_n}{S_n} \tag{3.1}$$

and

$$\check{\theta}_n = \frac{X_{t_n}^2}{2S_n}. \tag{3.2}$$

In order to study the strong consistency, let us state the following direct consequence of the Borel–Cantelli Lemma (see Kloeden and Neuenkirch (2007)), which allows us to turn convergence rates in the p-th mean into pathwise convergence rates.

Lemma 3.1. *Let $\gamma > 0$ and $p_0 \in \mathbb{N}$. Moreover let $(Z_n)_{n \in \mathbb{N}}$ be a sequence of random variables. If for every $p \geq p_0$ there exists a constant $c_p > 0$ such that for all $n \in \mathbb{N}$,*

$$\left(E|Z_n|^p\right)^{1/p} \leq c_p \cdot n^{-\gamma},$$

then for all $\varepsilon > 0$ there exists a random variable η_ε such that

$$|Z_n| \leq \eta_\varepsilon \cdot n^{-\gamma+\varepsilon} \qquad \text{almost surely}$$

for all $n \in \mathbb{N}$. Moreover, $E|\eta_\varepsilon|^p < \infty$ for all $p \geq 1$.

From now on, the generic constant is always denoted by $C(\cdot)$ which depends on certain parameters in the parentheses. The following lemma plays an important role in this paper.

Lemma 3.2. *Let $\{R_n, n \geq 1\}$ be a sequence of random variables defined by*

$$R_n := \sum_{i=1}^{n-1} (\zeta_{t_i}^2 - \zeta_{t_{i-1}}^2) e^{-2\theta(n-i+1)\Delta_n}.$$

Then,

$$e^{-2\theta T_n} S_n = \frac{\Delta_n}{e^{2\theta\Delta_n} - 1} (\zeta_{t_{n-1}}^2 - R_n). \tag{3.3}$$

Moreover, if we assume that $\Delta_n \to 0$ and $n\Delta_n^{1+\alpha} \to \infty$ for some $\alpha > 0$,

$$R_n \longrightarrow 0 \qquad \text{almost surely.} \tag{3.4}$$

In particular,

$$e^{-2\theta T_n} S_n \underset{n\to\infty}{\longrightarrow} \frac{\zeta_\infty^2}{2\theta} \qquad \text{almost surely.} \tag{3.5}$$

Proof. Since for every $t \geq 0$, $X_t = e^{\theta t}\zeta_t$, we have

$$\begin{aligned}
e^{-2\theta T_n} S_n &= \frac{\Delta_n}{e^{2\theta\Delta_n} - 1} \sum_{i=1}^{n} e^{-2\theta(n-i)\Delta_n} \left(\frac{e^{2\theta\Delta_n} - 1}{e^{2\theta\Delta_n}} \right) \zeta_{t_{i-1}}^2 \\
&= \frac{\Delta_n}{e^{2\theta\Delta_n} - 1} \sum_{i=1}^{n} e^{-2\theta(n-i)\Delta_n} \left(1 - \frac{1}{e^{2\theta\Delta_n}} \right) \zeta_{t_{i-1}}^2 \\
&= \frac{\Delta_n}{e^{2\theta\Delta_n} - 1} \sum_{i=1}^{n} (e^{-2\theta(n-i)\Delta_n} - e^{-2\theta(n-i+1)\Delta_n}) \zeta_{t_{i-1}}^2 \\
&= \frac{\Delta_n}{e^{2\theta\Delta_n} - 1} \left[\zeta_{t_{n-1}}^2 - \sum_{i=2}^{n} (\zeta_{t_{i-1}}^2 - \zeta_{t_{i-2}}^2) e^{-2\theta(n-i+1)\Delta_n} \right] \\
&= \frac{\Delta_n}{e^{2\theta\Delta_n} - 1} (\zeta_{t_{n-1}}^2 - R_n),
\end{aligned}$$

which proves (3.3).

To prove (3.4), let us first calculate $E[(\zeta_{t_i} - \zeta_{t_{i-1}})^2]$ for every $i = 1, \ldots, n$. Using (2.4) and making the change of variables $x = u/a_{t_{i-1}}$ and $y = v/a_{t_{i-1}}$, we obtain

$$\begin{aligned}
&E\big[(\zeta_{t_i} - \zeta_{t_{i-1}})^2\big] \\
&\quad = H(2H-1)H^{2(\theta+1)H} \int_{a_{t_{i-1}}}^{a_{t_i}} \int_{a_{t_{i-1}}}^{a_{t_i}} (uv)^{-(\theta+1)H} |u-v|^{2H-2}\, du\, dv
\end{aligned}$$

$$
\begin{aligned}
&= H^{2H+1}(2H-1)e^{-2\theta(i-1)\Delta_n} \\
&\quad \times \int_1^{e^{\frac{\Delta_n}{H}}} \int_1^{e^{\frac{\Delta_n}{H}}} (xy)^{-(\theta+1)H}|x-y|^{2H-2}\,dx\,dy \\
&\le H^{2H+1}(2H-1)e^{-2\theta(i-1)\Delta_n} \int_1^{e^{\frac{\Delta_n}{H}}} \int_1^{e^{\frac{\Delta_n}{H}}} |x-y|^{2H-2}\,dx\,dy \\
&= H^{2H}e^{-2\theta(i-1)\Delta_n}\big(e^{\frac{\Delta_n}{H}}-1\big)^{2H}.
\end{aligned} \tag{3.6}
$$

On the other hand, by using the point (ii) of Lemma 2.1 and the fact that ζ is Gaussian, we have for every $p \ge 1$

$$
\big(E\big[|\zeta_{t_i}^2-\zeta_{t_{i-1}}^2|^p\big]\big)^{1/p} \le C(p)\big(E\big[(\zeta_{t_i}-\zeta_{t_{i-1}})^2\big]\big)^{1/2}. \tag{3.7}
$$

Combining (3.7) and (3.6), we can deduce that

$$
\begin{aligned}
\big(E\big[|R_n|^p\big]\big)^{1/p} &\le \sum_{i=1}^{n-1} e^{-2\theta(n-i+1)\Delta_n}\big(E\big[|\zeta_{t_i}^2-\zeta_{t_{i-1}}^2|^p\big]\big)^{1/p} \\
&\le C(p)H^H e^{-\theta(n+1)\Delta_n}\big(e^{\frac{\Delta_n}{H}}-1\big)^H \sum_{i=1}^{n-1} e^{-\theta(n-i)\Delta_n} \\
&\le C(p)H^H e^{-\theta n\Delta_n}\big(e^{\frac{\Delta_n}{H}}-1\big)^H \sum_{i=1}^{n-1} e^{-\theta(n-i+1)\Delta_n} \\
&= C(p)H^H e^{-\theta n\Delta_n}\big(e^{\frac{\Delta_n}{H}}-1\big)^H e^{-2\theta\Delta_n}\frac{1-e^{-\theta(n-1)\Delta_n}}{1-e^{-\theta\Delta_n}} \\
&\le C(p,\theta,H)\Delta_n^{H-1}e^{-\theta n\Delta_n}.
\end{aligned} \tag{3.8}
$$

The last equality comes from $\Delta_n \to 0$, $n\Delta_n \to \infty$ and the fact that, as $x \to 0$,

$$
\frac{e^x-1}{x} \longrightarrow 1.
$$

Now, let $\gamma > 0$ be a constant verifying $\frac{1-H}{\gamma} < \alpha < \gamma$, then there exists $\varepsilon_0 > 0$ such that

$$
\alpha = \frac{\varepsilon_0+1-H}{\gamma-\varepsilon_0}.
$$

This allows us to write

$$
(n\Delta_n)^\gamma \Delta_n^{1-H} = n^{\varepsilon_0}\big(n\Delta_n^{1+\alpha}\big)^{\gamma-\varepsilon_0}. \tag{3.9}
$$

Thus, by combining (3.8), (3.9) and Lemma 3.1, the convergence (3.4) is proved.

Finally, the convergence (3.5) is a direct consequence of (3.3), (3.4) and the point (ii) of Lemma 2.1. □

Thus we arrive at our main result of this section.

Theorem 3.1. *Suppose that* $\Delta_n \to 0$ *and* $n\Delta_n^{1+\alpha} \to \infty$ *for some* $\alpha > 0$. *Then, as* $n \to \infty$,

$$\widehat{\theta}_n \to \theta \qquad \textit{almost surely,} \tag{3.10}$$

and also

$$\check{\theta}_n \to \theta \qquad \textit{almost surely.} \tag{3.11}$$

Proof. We first prove (3.10). From (3.1), we can write

$$\widehat{\theta}_n = \frac{e^{\theta\Delta_n} - 1}{\Delta_n} + \frac{e^{-2\theta T_n}\Lambda_n}{e^{-2\theta T_n} S_n}.$$

Then, by (3.5) and the fact that $(e^{\theta\Delta_n} - 1)/\Delta_n \longrightarrow \theta$, it suffices to show that $e^{-2\theta T_n}\Lambda_n$ converges to 0 almost surely. We have

$$\begin{aligned}(E[|\Lambda_n|^2])^{1/2} &\leq \sum_{i=1}^{n} e^{\theta t_i}\big(E[(\zeta_{t_i} - \zeta_{t_{i-1}})^2 X_{t_{i-1}}^2]\big)^{1/2} \\ &\leq \sum_{i=1}^{n} e^{\theta(t_i + t_{i-1})}\big(E[(\zeta_{t_i} - \zeta_{t_{i-1}})^4]\big)^{1/4}\big(E[\zeta_{t_{i-1}}^4]\big)^{1/4}.\end{aligned}$$

Thanks to (3.6) and the point (ii) of Lemma 2.1, we obtain

$$\begin{aligned}(E[|\Lambda_n|^2])^{1/2} &\leq C(\theta, H)\big(e^{\frac{\Delta_n}{H}} - 1\big)^H \sum_{i=1}^{n} e^{\theta i\Delta_n} \\ &= C(\theta, H) e^{\theta\Delta_n}\big(e^{\frac{\Delta_n}{H}} - 1\big)^H \left(\frac{e^{\theta n\Delta_n} - 1}{e^{\theta\Delta_n} - 1}\right).\end{aligned}$$

Furthermore, since $\Delta_n \to 0$ and $n\Delta_n \to \infty$, we get

$$\big(E\big[|e^{-2\theta T_n}\Lambda_n|^2\big]\big)^{1/2} \leq C(\theta, H)\Delta_n^{H-1} e^{-\theta T_n}. \tag{3.12}$$

Using similar arguments as in the proof of the convergence (3.4), we deduce that

$$e^{-2\theta T_n}\Lambda_n \longrightarrow 0 \qquad \text{almost surely,}$$

which proves (3.10).

Since

$$\check{\theta}_n = \frac{\zeta_{t_n}^2}{2e^{-2\theta T_n} S_n},$$

the convergence (3.11) is a direct consequence of (3.5) and (ii) of Lemma 2.1. □

4 Rate consistency of the estimators

In this section, we will establish that the sequences of random variables $\sqrt{n\Delta_n}(\widehat{\theta}_n - \theta)$ and $\sqrt{n\Delta_n}(\check{\theta}_n - \theta)$ are tight.

Definition 4.1. Let $\{Z_n\}$ be a sequence of random variables defined on $(\Omega, \mathcal{F}, P)$. We say $\{Z_n\}$ is tight (or bounded in probability), if for every $\varepsilon > 0$, there exists $M_\varepsilon > 0$ such that,

$$P\big(|Z_n| > M_\varepsilon\big) < \varepsilon \qquad \text{for all } n.$$

Theorem 4.1. *Suppose that $\Delta_n \to 0$ and $n\Delta_n^{1+\alpha} \to \infty$ for some $\alpha > 0$. Then, for any $q \geq 0$,*

$$\Delta_n^q e^{\theta T_n}(\widehat{\theta}_n - \theta) \qquad \textit{is not tight}. \tag{4.1}$$

In addition, if we assume that $n\Delta_n^3 \to 0$ as $n \to \infty$, the estimator $\widehat{\theta}_n$ is $\sqrt{T_n}$-consistent in the sense that the sequence

$$\sqrt{T_n}(\widehat{\theta}_n - \theta) \qquad \textit{is tight}. \tag{4.2}$$

Proof. We first start with the case $q \geq \frac{1}{2}$. By (3.1), we have

$$\Delta_n^q e^{\theta T_n}(\widehat{\theta}_n - \theta) = \Delta_n^{q+1} e^{\theta T_n}\left(\frac{e^{\theta\Delta_n} - 1 - \theta\Delta_n}{\Delta_n^2}\right) + \frac{\Delta_n^q e^{-\theta T_n}\Lambda_n}{e^{-2\theta T_n} S_n}. \tag{4.3}$$

We have $\frac{e^{\theta\Delta_n} - 1 - \theta\Delta_n}{\Delta_n^2} \to \theta^2/2$. Moreover,

$$\begin{aligned} \Delta_n^{q+1} e^{\theta T_n} &= (n\Delta_n)^{\frac{q+1}{\alpha}} \Delta_n^{q+1} \frac{e^{\theta T_n}}{T_n^{\frac{q+1}{\alpha}}} \\ &= \big(n\Delta_n^{1+\alpha}\big)^{\frac{q+1}{\alpha}} \frac{e^{\theta T_n}}{T_n^{\frac{q+1}{\alpha}}} \\ &\longrightarrow \infty \end{aligned}$$

because $T_n \to \infty$ and $n\Delta_n^{1+\alpha} \to \infty$.

Thus,

$$\Delta_n^{q+1} e^{\theta T_n}\left(\frac{e^{\theta\Delta_n} - 1 - \theta\Delta_n}{\Delta_n^2}\right) \longrightarrow \infty. \tag{4.4}$$

On the other hand, it follows from (3.12) that

$$\big(E\big[\big|\Delta_n^q e^{-\theta T_n}\Lambda_n\big|^2\big]\big)^{1/2} \leq C(\theta, H)\Delta_n^{q+H-1} \longrightarrow 0 \tag{4.5}$$

since $H > \frac{1}{2}$.

Consequently, by combining (4.3), (4.4), (4.5) and (3.5), we conclude that for every $q \geq \frac{1}{2}$, $\Delta_n^q e^{\theta T_n}(\widehat{\theta}_n - \theta)$ is not tight.

The case $0 \leq q < \frac{1}{2}$ is a direct consequence of

$$\Delta_n^q e^{\theta T_n}(\widehat{\theta}_n - \theta) = \Delta_n^{q-\frac{1}{2}}\left(\Delta_n^{\frac{1}{2}} e^{\theta T_n}(\widehat{\theta}_n - \theta)\right),$$

$\Delta_n^{q-\frac{1}{2}} \longrightarrow \infty$ and the previous case. Thus the proof (4.1) is finished.

Let us now prove (4.2). From (3.1), we can write

$$\sqrt{T_n}(\widehat{\theta}_n - \theta) = \sqrt{n\Delta_n^3}\left(\frac{e^{\theta\Delta_n} - 1 - \theta\Delta_n}{\Delta_n^2}\right) + \frac{\sqrt{T_n}e^{-2\theta T_n}\Lambda_n}{e^{-2\theta T_n}S_n}. \tag{4.6}$$

Since $n\Delta_n^3 \to 0$ and $\frac{e^{\theta\Delta_n}-1-\theta\Delta_n}{\Delta_n^2} \to \theta^2/2$, we have

$$\sqrt{n\Delta_n^3}\left(\frac{e^{\theta\Delta_n} - 1 - \theta\Delta_n}{\Delta_n^2}\right) \to 0. \tag{4.7}$$

Furthermore, (3.12) leads to

$$\begin{aligned} \left(E\left[\left|\sqrt{T_n}e^{-2\theta T_n}\Lambda_n\right|^2\right]\right)^{1/2} &\leq C(\theta, H)\Delta_n^{H-1}\sqrt{T_n}e^{-\theta T_n} \\ &= C(\theta, H)\frac{T_n^{\frac{1}{2}+\frac{1-H}{\alpha}}e^{-\theta T_n}}{(n\Delta_n^{1+\alpha})^{\frac{1-H}{\alpha}}} \\ &\longrightarrow 0 \end{aligned} \tag{4.8}$$

by using $T_n \to \infty$ and $n\Delta_n^{1+\alpha} \to \infty$.

Consequently, by (4.6), (4.7), (4.8) and (3.5) we deduce (4.2). □

Theorem 4.2. *Suppose that $\Delta_n \to 0$ and $n\Delta_n^{1+\alpha} \to \infty$ for some $\alpha > 0$. Then, for any $q \geq 0$,*

$$\Delta_n^q e^{\theta T_n}(\check{\theta}_n - \theta) \qquad \text{is not tight.} \tag{4.9}$$

In addition, we assume that $n\Delta_n^3 \to 0$ as $n \to \infty$. Then the estimator $\check{\theta}_n$ is $\sqrt{T_n}-$consistent in the sense that the sequence

$$\sqrt{T_n}(\check{\theta}_n - \theta) \qquad \text{is tight.} \tag{4.10}$$

Proof. Fix $q \geq 1/2$. We have

$$\begin{aligned} &\Delta_n^q e^{\theta T_n}(\check{\theta}_n - \theta) \\ &\quad = \Delta_n^q e^{\theta T_n}\left(\frac{e^{2\theta T_n}\zeta_{t_n}^2}{2S_n} - \theta\right) \\ &\quad = \frac{\Delta_n^q e^{\theta T_n}}{2e^{-2\theta T_n}S_n}\left[(\zeta_{t_n}^2 - \zeta_{t_{n-1}}^2) + \left(1 - \frac{2\theta\Delta_n}{e^{2\theta\Delta_n} - 1}\right)\zeta_{t_{n-1}}^2\right. \end{aligned} \tag{4.11}$$

$$
\begin{aligned}
&- 2\theta\left(e^{-2\theta T_n} S_n - \frac{\Delta_n}{e^{2\theta\Delta_n}-1}\zeta_{t_{n-1}}^2\right)\Bigg]\\
&= \frac{\Delta_n^q e^{\theta T_n}}{2e^{-2\theta T_n} S_n}\left[(\zeta_{t_n}^2 - \zeta_{t_{n-1}}^2) + \left(1-\frac{2\theta\Delta_n}{e^{2\theta\Delta_n}-1}\right)\zeta_{t_{n-1}}^2 + \left(\frac{2\theta\Delta_n}{e^{2\theta\Delta_n}-1}\right)R_n\right].
\end{aligned}
$$

Using (3.6),

$$
\begin{aligned}
\left(E\left[\left(\Delta_n^q e^{\theta T_n}(\zeta_{t_n}^2 - \zeta_{t_{n-1}}^2)\right)^2\right]\right)^{1/2} &\le C(\theta, H)\Delta_n^{q+H}\left(\frac{e^{\frac{\Delta_n}{H}}-1}{\Delta_n}\right)^H\\
&\longrightarrow 0.
\end{aligned}
\tag{4.12}
$$

We also have,

$$
\begin{aligned}
&\Delta_n^q e^{\theta T_n}\left(1-\frac{2\theta\Delta_n}{e^{2\theta\Delta_n}-1}\right)\\
&\quad = \Delta_n^{q+1} e^{\theta T_n}\left(\frac{e^{2\theta\Delta_n}-1-2\theta\Delta_n}{\Delta_n^2}\frac{\Delta_n}{e^{2\theta\Delta_n}-1}\right)\\
&\quad \longrightarrow \infty.
\end{aligned}
\tag{4.13}
$$

Moreover,

$$
\begin{aligned}
\left(E\left[\left(\Delta_n^q e^{\theta T_n} R_n\right)^2\right]\right)^{1/2} &\le C(\theta, H)\Delta_n^{q+H-1}\\
&\longrightarrow 0.
\end{aligned}
\tag{4.14}
$$

Combining (4.11), (4.12), (4.13), (4.14) and (3.5), we conclude that for every $q \ge \frac{1}{2}$, $\Delta_n^q e^{\theta T_n}(\check{\theta}_n - \theta)$ is not tight.

It is obvious that (4.9) is satisfied for $0 \le q < \frac{1}{2}$ by using a similar argument as in the proof of (4.1). Thus, the proof of (4.9) is finished.

We prove now (4.10). Thanks to (4.11), we can write

$$
\begin{aligned}
&\sqrt{T_n}(\check{\theta}_n - \theta)\\
&= \frac{\sqrt{T_n}}{2e^{-2\theta T_n} S_n}\left[(\zeta_{t_n}^2 - \zeta_{t_{n-1}}^2) + \left(1-\frac{2\theta\Delta_n}{e^{2\theta\Delta_n}-1}\right)\zeta_{t_{n-1}}^2 + \left(\frac{2\theta\Delta_n}{e^{2\theta\Delta_n}-1}\right)R_n\right].
\end{aligned}
$$

This implies that (4.10) is satisfied as a result of the convergence (3.5),

$$
\begin{aligned}
\left(E\left[\left(\sqrt{T_n}(\zeta_{t_n}^2 - \zeta_{t_{n-1}}^2)\right)^2\right]\right)^{1/2} &\le C(\theta, H)\Delta_n^H\sqrt{T_n}e^{-\theta T_n}\left(\frac{e^{\frac{\Delta_n}{H}}-1}{\Delta_n}\right)^H\\
&\longrightarrow 0,\\
\sqrt{T_n}\left(1-\frac{2\theta\Delta_n}{e^{2\theta\Delta_n}-1}\right) &= \sqrt{n\Delta_n^3}\left(\frac{e^{2\theta\Delta_n}-1-2\theta\Delta_n}{\Delta_n^2}\frac{\Delta_n}{e^{2\theta\Delta_n}-1}\right)\\
&\longrightarrow 0,
\end{aligned}
$$

and

$$\begin{aligned}\left(E\left[(\sqrt{T_n}R_n)^2\right]\right)^{1/2} &\leq C(\theta,H)\Delta_n^{H-1}\sqrt{T_n}e^{-\theta T_n}\\ &= C(\theta,H)\frac{T_n^{\frac{1}{2}+\frac{1-H}{\alpha}}e^{-\theta T_n}}{(n\Delta_n^{1+\alpha})^{\frac{1-H}{\alpha}}}\\ &\longrightarrow 0.\end{aligned}$$

□

Remark 4.1. Let $\widetilde{\theta}_t$ be the LSE, defined in (1.3), based on continuous-time observations of (1.2). The authors of El Onsy, Es-Sebaiy and Tudor (2014) proved that $e^{\theta t}(\widetilde{\theta}_t-\theta)$ is asymptotically Cauchy. But, for the discrete versions $\widehat{\theta}_n$ and $\check{\theta}_n$ of $\widetilde{\theta}_t$, Theorems 4.1 and 4.2 which have been proved above, state that the sequences $\Delta_n^q e^{\theta T_n}(\widehat{\theta}_n-\theta)$ and $\Delta_n^q e^{\theta T_n}(\check{\theta}_n-\theta)$ are not tight. Moreover, $\sqrt{T_n}(\widehat{\theta}_n-\theta)$ and $\sqrt{T_n}(\check{\theta}_n-\theta)$ are tight and converge in probability to 0, which means that the rate are actually 'larger' than $\sqrt{T_n}$.

5 Numerical illustrations

Let us start with the following simulated path of the fractional Ornstein Uhlenbeck with second kind process $dX_t=\theta X_t\,dt+dY_t^{(1)}$ with $X_0=0$, $\theta=0.78$, $H=0.70$.

- First, we generate the fractional Brownian motion using the wavelet method Abry and Sellan (1996).
- Then, we simulate $Y^{(1)}$ by the discretization of the stochastic integral $\int_0^t e^{-s}\,dB_{a_s}$.
- After that we simulate the process X using the Euler–Maruyama method for different values of H and θ (see Figure 1).

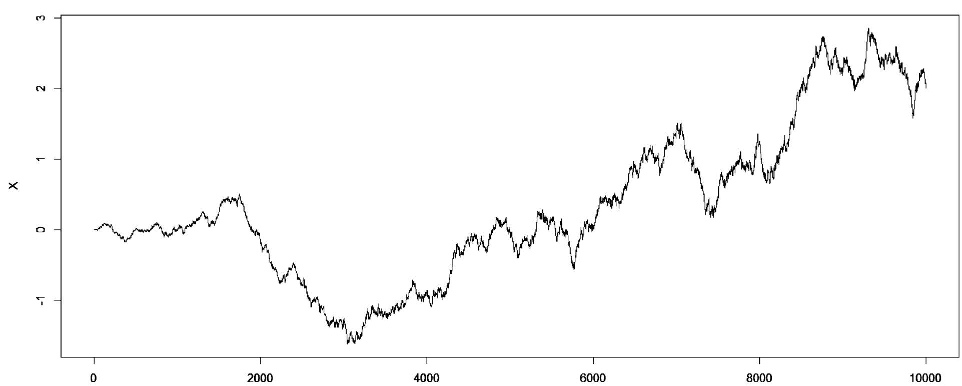

Figure 1 *FOUSK.*

Table 1 *The means and standard deviations of estimators*

	$H = 0.55$		$H = 0.60$		$H = 0.65$		$H = 0.70$	
	$\widehat{\theta}$	$\check{\theta}$	$\widehat{\theta}$	$\check{\theta}$	$\widehat{\theta}$	$\check{\theta}$	$\widehat{\theta}$	$\check{\theta}$
Panel A. Low parameter value $\theta = 0.7880$								
Mean	0.4140	0.7642	0.5989	0.7621	0.7170	0.7847	0.7424	0.7801
Median	0.7540	0.8153	0.7909	0.8287	0.8065	0.8254	0.8127	0.8219
Std. dev.	0.8525	0.2888	0.5440	0.2797	0.3754	0.2448	0.3250	0.2941
Panel B. Medium parameter value $\theta = 1.6811$								
Mean	1.5286	1.6374	1.5774	1.6310	1.6155	1.6403	1.6142	1.6304
Median	1.6768	1.6837	1.6745	1.6811	1.6816	1.6836	1.6799	1.6820
Std. dev.	0.6684	0.2429	0.4206	0.2409	0.3474	0.2576	0.3122	0.2619
Panel C. High parameter value $\theta = 3.6977$								
Mean	3.6964	3.6982	3.6884	3.6927	3.6967	3.6983	3.6973	3.6987
Median	3.6977	3.6990	3.6976	3.6990	3.6977	3.6990	3.6977	3.6991
Std. dev.	0.0186	0.0160	0.1847	0.1275	0.0115	0.0109	0.0062	0.0062

Now, we present numerical examples for different values of H and θ to investigate the efficiency of our estimators $\widehat{\theta}$ and $\check{\theta}$. For a fixed length $h = 0.0002$, we simulate 500 sample paths on the interval [0, 2] using a regular partition of 10,000 intervals. Finally, we implement these generated data sets to obtain the estimators by (1.4) and (1.5). The simulated of these estimators $\widehat{\theta}$ and $\check{\theta}$ are given in Table 1 (true value is the parameter value used in the Monte Carlo simulation; Mean, Median and Std.dev. are the sample statistics computed with the 500 estimated parameter values).

As shown in Table 1, we can see that the standard deviations of $\widehat{\theta}$ and $\check{\theta}$ are small. These results also demonstrate that the mean and the median values of all considered parameters are close to the true values, which indicates a pretty good finite sample behavior of our method. As consequence, this simulation study confirms the theoretical results.

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