

A large class of new bivariate copulas and their properties

Zahra Sharifonnasabi, Mohammad Hossein Alamatsaz¹ and Iraj Kazemi

University of Isfahan

Abstract. In this paper, we shall construct a large class of new bivariate copulas. This class happens to contain several known classes of copulas, such as Farlie–Gumbel–Morgenstern, Ali–Mikhail–Haq and Barnett–Gumbel, as its especial members. It is shown that the proposed copulas improve the range of values of correlation coefficient and thus they are more applicable in data modeling. We shall also reveal that the dependent properties of the base copula are preserved by the generated copula under certain conditions. Several members of the new class are introduced as instances and their range of correlation coefficients are computed.

1 Introduction

Copulas are useful tools in constructing joint distributions. In fact, a copula is a multidimensional distribution function (d.f.) whose marginals are uniformly distributed on $[0, 1]$. We shall only focus on bivariate copulas and their properties in this manuscript.

A bivariate copula is a function $C : I^2 \rightarrow I$ ($I = [0, 1]$) which satisfies the boundary conditions

$$C(t, 0) = C(0, t) = 0 \quad \text{and} \quad C(t, 1) = C(1, t) = t, \quad t \in I \quad (1.1)$$

and the 2-increasing condition

$$C(u_2, v_2) - C(u_1, v_2) - C(u_2, v_1) + C(u_1, v_1) \geq 0 \quad (1.2)$$

for all u_1, u_2, v_1, v_2 in $[0, 1]$ such that $u_1 \leq u_2$ and $v_1 \leq v_2$.

According to Sklar's theorem (see Sklar (1959)), using a two-dimensional copula $C(\cdot, \cdot)$, any bivariate d.f. $H_{X,Y}$ can be represented as a function of its marginal distributions F_X and F_Y . In fact, if F_X and F_Y are continuous d.f.'s, then there exists a unique copula such that

$$H_{X,Y}(x, y) = C(F_X(x), F_Y(y)), \quad (1.3)$$

where

$$C(u, v) = P(U \leq u, V \leq v), \quad U \equiv F_X(x), \quad V \equiv F_Y(y). \quad (1.4)$$

¹Corresponding author.

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Assuming that C is differentiable with respect to its both arguments, equation (1.2) is satisfied if

$$c(u, v) \equiv \frac{\partial^2 C(u, v)}{\partial u \partial v} \geq 0. \tag{1.5}$$

Furthermore, it is known that

$$\begin{aligned} \frac{\partial C(u, v)}{\partial u} &= P(V \leq v | U = u) \geq 0, \\ \frac{\partial C(u, v)}{\partial v} &= P(U \leq u | V = v) \geq 0. \end{aligned} \tag{1.6}$$

For further details about copulas, see [Nelsen \(2006\)](#).

A known and useful family of copulas is *FGM* family. This family is used, e.g., for competing risk problems [Tolley and Norman \(1979\)](#), the joint occurrence of certain trace elements in water [Cook and Johnson \(1986\)](#) and a robustness issue [Delahorra and Fernandez \(1995\)](#). However, a shortcoming of *FGM* copulas is that they do not allow modeling of high dependencies because their spearman's ρ is limited to $[-\frac{1}{3}, \frac{1}{3}]$. To overcome this problem, many authors such as [Huang and Kotz \(1999\)](#), [Bairamov and Kotz \(2002\)](#), [Amblard and Girard \(2002, 2009\)](#), [Amini, Jabbari and Mohtashami Borzadaran \(2011\)](#) have tried to extend this family. Recently [Klein and Christa \(2011\)](#) proposed a class of copulas which contains *FGM* copulas as follows: A bivariate function

$$C(u, v; \theta, \gamma) = uv[1 + \theta\phi(u)\phi(v)]^{\frac{1}{\gamma}} \quad \forall u, v \in I \tag{1.7}$$

is a copula if the function $\phi : [0, 1] \rightarrow \Re$ is a differentiable and monotone function satisfying

1. $\phi(1) = 0$,
2. $|\phi(u)| \leq 1, u \in [0, 1]$,
3. $D = \{\gamma \neq 0; |\phi(u) + \frac{u\phi'(u)}{\gamma}| \leq 1, u \in [0, 1]\}$,
4. either $\gamma \in D$ and $\theta \in [-1, 1]$ such that $\theta\gamma > 0$, or ϕ is nonnegative and monotonely decreasing or nonpositive and monotonely increasing on $u \in [0, 1]$ and

$$\gamma \in D \cap (0, 1] \quad \text{and} \quad \theta \in [-1, 0)$$

or

$$\gamma \in D \cap (-\infty, -1] \quad \text{and} \quad \theta \in (0, 1].$$

Also, the bivariate function

$$C(u, v; \theta) = uve^{\theta\phi(u)\phi(v)}, \quad u, v \in I \tag{1.8}$$

is a copula for $\theta \in [0, 1]$ if

1. $\phi(1) = 0$,

2. $|\phi(u)| \leq 1, u \in [0, 1],$

where $\phi : [0, 1] \rightarrow \Re$ is a differentiable and monotone function, and it is a copula for $\theta \in [-1, 0]$ if

1. $\phi(1) = 0,$
2. $|u\phi'(u)| \leq 1, u \in [0, 1],$

where $\phi : [0, 1] \rightarrow \Re$ is differentiable, nonnegative and monotonely decreasing or nonpositive and monotonely increasing on $[0, 1].$

The remainder of the paper is organized as follows. In Section 2, replacing the independent copula uv in (1.7) and (1.8) by a general copula $K(u, v),$ we shall introduce a new class of copulas which extends several known classes of copulas, such as FGM, AMH and BG and several other families. We shall discuss conditions under which our generalization is justified. Several illustrating examples are provided in Section 3. Dependent properties of the new class are studied in Section 4. In Section 5, we shall discuss tail dependencies of members of this new family. Measures of association are discussed in Section 6, where it is observed that the new family can be used to model data with a wider range of correlation than the FGM family.

2 A new class of bivariate copulas

Consider the bivariate copula (1.7) and (1.8) of Klein and Christa (2011). We replace the coefficient $uv,$ the independent copula, by an arbitrary copula $K(u, v).$ That is, we consider the bivariate function

$$C(u, v; \theta, \gamma) = K(u, v)[1 + \theta\phi(u)\phi(v)]^{\frac{1}{\gamma}} \quad \forall u, v \in I, \tag{2.1}$$

where $\theta \in [-1, 1]$ and $\gamma \in \Re/\{0\}$ and

$$C(u, v; \theta) = K(u, v)e^{\theta\phi(u)\phi(v)} \quad \forall u, v \in I, \tag{2.2}$$

where $\theta \in [-1, 1].$

The next theorems provide certain conditions under which $C(u, v; \gamma, \theta)$ and $C(u, v; \theta)$ in (2.1) and (2.2), respectively, are indeed well defined copulas. First, in order to simplify the statements of our theorems, we set up the following assumptions and definitions:

Assumptions:

1. $\frac{\partial^2}{\partial u \partial v} \ln K(u, v) \geq 0, \forall (u, v) \in I^2.$
2. $G = \{\gamma \neq 0; |\phi(u) + \frac{\phi'(u)}{\gamma \frac{\partial}{\partial u} \ln K(u, v)}| \leq 1, |\phi(v) + \frac{\phi'(v)}{\gamma \frac{\partial}{\partial v} \ln K(u, v)}| \leq 1, (u, v) \in I^2\}.$
3. $0 \leq \phi(u) \leq 1$ and be decreasing or $-1 \leq \phi(u) \leq 0$ and be increasing.
4. $\phi(u)$ is a negative and nondecreasing function and

$$\phi'(u) \leq \frac{\partial}{\partial u} \ln K(u, v), \quad \forall v \in I.$$

5. $|\phi'(u)| \leq 1$ and $\frac{k(u,v)}{K(u,v)} \geq 1$ or $\frac{k(u,v)}{K(u,v)} \geq \phi'(u)\phi'(v)$ for $u, v \in [0, 1]$, where

$$k(u, v) = \frac{\partial^2}{\partial u \partial v} K(u, v).$$

Definitions:

$$Q_0 = \left\{ \gamma \neq 0; \frac{k(u, v)}{K(u, v)} \geq \frac{\phi'(u)\phi'(v)}{\gamma^2[1 - \phi(u)\phi(v)]}, (u, v) \in I^2 \right\},$$

$$Q_1 = \{ \gamma \neq 0; \phi(u) + A(u, v, \gamma) \geq 0, \phi(v) + B(u, v, \gamma) \geq 0, (u, v) \in I^2 \},$$

$$Q_2 = \left\{ \gamma \neq 0; \left| \frac{\phi'(u)}{\sqrt{|\gamma|} \frac{\partial}{\partial u} \ln K(u, v)} \right| \leq 1, \left| \frac{\phi'(v)}{\sqrt{|\gamma|} \frac{\partial}{\partial v} \ln K(u, v)} \right| \leq 1, (u, v) \in I^2 \right\},$$

where

$$A(u, v, \gamma) = \frac{\phi'(u)}{\gamma \frac{\partial}{\partial u} \ln K(u, v)}, \quad B(u, v, \gamma) = \frac{\phi'(v)}{\gamma \frac{\partial}{\partial v} \ln K(u, v)}.$$

Theorem 2.1. Let $\phi : [0, 1] \rightarrow [-1, 1]$ be a monotone and differentiable function with $\phi(1) = 0$ and $K(u, v)$ be differentiable with respect to its both arguments u and v . Then, $C(u, v; \theta, \gamma) = K(u, v)[1 + \theta\phi(u)\phi(v)]^{\frac{1}{\gamma}}$ is a copula

- (i) under Assumptions 1 and 2 for $\theta \in [-1, 1]$ provided that $\theta\gamma > 0$;
- (ii) under Assumption 3 for $\gamma \in (0, 1] \cap Q_0$ or $\gamma \in (0, 1] \cap Q_1$ and $-1 \leq \theta \leq 0$;
- (iii) under Assumption 1 and 3 and $K(u, v) = K(v, u)$ for all $u, v \in I^2$, for $\gamma \in (-\infty, -1] \cap Q_2$, $0 \leq \theta \leq 1$.

Proof. Obviously boundary conditions are satisfied. Thus, it is sufficient to prove that under the mentioned conditions $C(u, v; \theta, \gamma)$ is 2-increasing. Let $c(u, v; \theta, \gamma) = \frac{\partial^2}{\partial u \partial v} C(u, v; \theta, \gamma)$. Hence, we have

$$\begin{aligned} & \frac{c(u, v; \theta, \gamma)}{C(u, v; \theta, \gamma)} \\ &= \frac{\partial^2}{\partial u \partial v} \ln C(u, v; \theta, \gamma) + \frac{\partial}{\partial u} \ln C(u, v; \theta, \gamma) \frac{\partial}{\partial v} \ln C(u, v; \theta, \gamma). \end{aligned} \tag{2.3}$$

(i) Clearly, since $K(u, v)$ is differentiable, under Assumption 2 we obtain

$$\frac{\partial}{\partial u} \ln C(u, v; \theta, \gamma) = \frac{\frac{\partial}{\partial u} \ln K(u, v)}{1 + \theta\phi(u)\phi(v)} \left\{ 1 + \theta\phi(v) \left[\phi(u) + \frac{\phi'(u)}{\gamma \frac{\partial}{\partial u} \ln K(u, v)} \right] \right\}$$

and

$$\frac{\partial}{\partial v} \ln C(u, v; \theta, \gamma) = \frac{\frac{\partial}{\partial v} \ln K(u, v)}{1 + \theta\phi(u)\phi(v)} \left\{ 1 + \theta\phi(v) \left[\phi(u) + \frac{\phi'(v)}{\gamma \frac{\partial}{\partial v} \ln K(u, v)} \right] \right\}$$

are nonnegative. Moreover, by Assumption 1,

$$\frac{\partial^2}{\partial u \partial v} \ln C(u, v; \theta, \gamma) = \frac{\partial^2}{\partial u \partial v} \ln K(u, v) + \frac{\frac{\theta}{\gamma} \phi'(u) \phi'(v)}{[1 + \theta \phi(u) \phi(v)]^2}$$

is nonnegative. Therefore, by (2.3), $c(u, v) \geq 0$ and, thus, by (1.5) the 2-increasing property holds.

(ii) We prove the case when $\phi(u)$ is nonnegative and decreasing, the other case follows by replacing $\phi(u)$ with $-\phi(u)$ in the copula function. According to (2.3)

$$\begin{aligned} \frac{c(u, v; \theta, \gamma)}{C(u, v; \theta, \gamma)} &= \frac{\partial^2}{\partial u \partial v} \ln K(u, v) + \frac{\frac{\theta}{\gamma} \phi'(u) \phi'(v)}{[1 + \theta \phi(u) \phi(v)]^2} \\ &+ \left[\frac{\partial}{\partial u} \ln K(u, v) + \frac{\frac{\theta}{\gamma} \phi'(u) \phi(v)}{1 + \theta \phi(u) \phi(v)} \right] \\ &\times \left[\frac{\partial}{\partial v} \ln K(u, v) + \frac{\frac{\theta}{\gamma} \phi(u) \phi'(v)}{1 + \theta \phi(u) \phi(v)} \right] \tag{2.4} \\ &= \frac{k(u, v)}{K(u, v)} + \frac{\theta}{\gamma} \frac{\phi(u) \phi'(v)}{1 + \theta \phi(u) \phi(v)} \frac{\partial}{\partial u} \ln K(u, v) + \frac{\partial}{\partial v} \ln K(u, v) \\ &\times \frac{\theta \phi'(u)}{\gamma [1 + \theta \phi(u) \phi(v)]} \left\{ \phi(v) + \frac{\phi'(v) [1 + \frac{\theta}{\gamma} \phi(u) \phi(v)]}{[1 + \theta \phi(u) \phi(v)] \frac{\partial}{\partial v} \ln K(u, v)} \right\}, \end{aligned}$$

where $k(u, v) = \frac{\partial^2}{\partial u \partial v} K(u, v)$. If $0 < \gamma \leq 1$, then

$$\frac{1 + \frac{\theta}{\gamma} \phi(u) \phi(v)}{1 + \theta \phi(u) \phi(v)} = \frac{1}{\gamma} \frac{\gamma + \theta \phi(u) \phi(v)}{1 + \theta \phi(u) \phi(v)} \leq \frac{1}{\gamma}$$

and since $\phi'(u) \leq 0$ and $\theta \leq 0$, we have

$$\theta \phi'(u) \left\{ \phi(v) + \frac{\phi'(v) [1 + \frac{\theta}{\gamma} \phi(u) \phi(v)]}{[1 + \theta \phi(u) \phi(v)] \frac{\partial}{\partial v} \ln K(u, v)} \right\} \geq \theta \phi'(u) \{ \phi(v) + B(u, v, \gamma) \}.$$

Consequently, if $\gamma \in (0, 1] \cap Q_1$

$$\begin{aligned} \frac{c(u, v; \theta, \gamma)}{C(u, v; \theta, \gamma)} &\geq \frac{k(u, v)}{K(u, v)} + \frac{\theta}{\gamma} \frac{\phi(u) \phi'(v)}{1 + \theta \phi(u) \phi(v)} \frac{\partial}{\partial u} \ln K(u, v) \\ &+ \frac{\partial}{\partial v} \ln K(u, v) \frac{\theta \phi'(u)}{\gamma [1 + \theta \phi(u) \phi(v)]} \{ \phi(v) + B(u, v, \gamma) \} \geq 0, \end{aligned}$$

and if $\gamma \in (0, 1] \cap Q_0$ we can rewrite

$$\begin{aligned} \frac{c(u, v; \theta, \gamma)}{C(u, v; \theta, \gamma)} &\geq \frac{k(u, v)}{K(u, v)} + \frac{\theta}{\gamma} \frac{\phi(u) \phi'(v)}{1 + \theta \phi(u) \phi(v)} \frac{\partial}{\partial u} \ln K(u, v) \\ &+ \frac{\theta}{\gamma} \frac{\phi'(u) \phi(v)}{1 + \theta \phi(u) \phi(v)} \frac{\partial}{\partial v} \ln K(u, v) + \frac{\theta \phi'(u) \phi'(v)}{\gamma^2 [1 + \theta \phi(u) \phi(v)]} \geq 0, \end{aligned}$$

because

$$\frac{\theta\phi'(u)\phi'(v)}{\gamma^2[1 + \theta\phi(u)\phi(v)]} \geq \frac{-\phi'(u)\phi'(v)}{\gamma^2[1 - \phi(u)\phi(v)]}.$$

(iii) Similar to part (b), we just prove the case when $\phi(u)$ is nonnegative and decreasing. Since $\gamma \in (-\infty, -1) \cap Q_2$, it follows that $\phi(u) + \frac{\phi'(u)}{\gamma \frac{\partial}{\partial u} \ln K(u, v)} \geq 0$. Thus, we have

$$\begin{aligned} \frac{c(u, v; \theta, \gamma)}{C(u, v; \theta, \gamma)} &= \frac{\partial^2}{\partial u \partial v} \ln K(u, v) + \frac{\frac{\theta}{\gamma}\phi'(u)\phi'(v)}{[1 + \theta\phi(u)\phi(v)]^2} \\ &\quad + \{1 + \theta\phi(v)[\phi(u) + A(u, v, \gamma)]\} \\ &\quad \times \{1 + \theta\phi(u)[\phi(v) + B(u, v, \gamma)]\} \\ &\quad \times \frac{\frac{\partial}{\partial u} \ln K(u, v) \frac{\partial}{\partial v} \ln K(u, v)}{[1 + \theta\phi(u)\phi(v)]^2} \\ &\geq \frac{\partial^2}{\partial u \partial v} \ln K(u, v) + \frac{\frac{\theta}{\gamma}\phi'(u)\phi'(v)}{[1 + \theta\phi(u)\phi(v)]^2} \\ &\quad + \frac{\frac{\partial}{\partial u} \ln K(u, v) \frac{\partial}{\partial v} \ln K(u, v)}{[1 + \theta\phi(u)\phi(v)]^2} \\ &= \frac{\partial^2}{\partial u \partial v} \ln K(u, v) + \frac{\frac{\partial}{\partial u} \ln K(u, v) \frac{\partial}{\partial v} \ln K(u, v)}{[1 + \phi(u)\phi(v)]^2} \\ &\quad \times \left\{ 1 - \theta \frac{\phi'(u)}{\sqrt{|\gamma|} \frac{\partial}{\partial u} \ln K(u, v)} \frac{\phi'(v)}{\sqrt{|\gamma|} \frac{\partial}{\partial v} \ln K(u, v)} \right\} \geq 0. \end{aligned}$$

This completes the proof. □

Theorem 2.2. *Let ϕ be a differentiable function and $\phi(1) = 0$. Then, $C(u, v; \theta) = K(u, v)e^{\theta\phi(u)\phi(v)}$ is a copula*

- (i) for $\theta \in [0, 1]$ under Assumptions 1 and 4;
- (ii) for $\theta \in [-1, 0]$ under Assumptions 3 and 5.

Proof. (i) Obviously boundary conditions are satisfied. As before, it is sufficient to show that $c(u, v) \geq 0$. According to Assumption 4, we have

$$\frac{\partial^2}{\partial u \partial v} \ln C(u, v; \theta) = \frac{\partial^2}{\partial u \partial v} \ln K(u, v) + \theta\phi'(u)\phi'(v) \geq 0 \tag{2.5}$$

and since $\theta\phi(v) \leq 0$ we obtain

$$\theta\phi'(u)\phi(v) \geq \theta\phi(v) \frac{\partial}{\partial u} \ln K(u, v).$$

Thus,

$$\frac{\partial}{\partial u} \ln C(u, v; \theta) \geq \frac{\partial}{\partial u} \ln K(u, v)[1 + \theta\phi(v)] \geq 0,$$

and thus, (2.3) is nonnegative. Hence, the result follows.

(ii) Again, we have

$$\begin{aligned} \frac{c(u, v; \theta)}{C(u, v; \theta)} &= \frac{k(u, v)}{K(u, v)} - \frac{\partial}{\partial u} \ln K(u, v) \frac{\partial}{\partial v} \ln K(u, v) + \theta\phi'(u)\phi'(v) \\ &\quad + \frac{\partial}{\partial u} \ln K(u, v) \frac{\partial}{\partial v} \ln K(u, v) \left[1 + \theta \frac{\phi'(u)\phi(v)}{\frac{\partial}{\partial u} \ln K(u, v)} \right] \\ &\quad \times \left[1 + \theta \frac{\phi(u)\phi'(v)}{\frac{\partial}{\partial v} \ln K(u, v)} \right]. \end{aligned}$$

Since $\theta \in [-1, 0]$, by the properties of $\phi(u)$, we obtain

$$\frac{c(u, v; \theta)}{C(u, v; \theta)} \geq \frac{k(u, v)}{K(u, v)} + \theta\phi'(u)\phi'(v) \geq 0$$

that is, $c(u, v) \geq 0$. □

3 Some examples

In this section, we introduce several members of our class of copulas. First, we need to recall some copulas $K(u, v)$ in Table 1 which are used in the sequel.

Table 1 Some bivariate copulas recalled

$K(u, v)$	Parameter spaces
$uv[1 + \delta(1 - u)^p(1 - v)^p]$	$p > 0, -1 \leq \delta \leq \left(\frac{p+1}{p-1}\right)^{p-1}$ (see Huang and Kotz (1999))
$uv[1 + \delta(u^n - 1)(v^n - 1)]$	$n \in \mathbb{N}, -\frac{1}{n^2} \leq \delta \leq 1$ (see Grane (2009))
$uv[1 + \delta(1 - u^p)(1 - v^p)]$	$p > 0, -(\max\{1, p\})^{-2} \leq \delta \leq \frac{1}{p}$ (see Huang and Kotz (1999))
$uv[1 + \alpha(1 - u)(1 - v) + \beta(1 - u)^2(1 - v)^2]$	$ \alpha \leq 1, \alpha + \beta \geq -1, \beta \leq \frac{3 - \alpha + \sqrt{9 - 6\alpha - 3\alpha^2}}{2}$ (see Lin (1987))
$uv[1 + \alpha(1 - u)(1 - v) + \beta uv(1 - u)(1 - v)]$	$ \alpha \leq 1, \alpha + \beta \geq -1, \beta \leq \frac{3 - \alpha + \sqrt{9 - 6\alpha - 3\alpha^2}}{2}$ (see Lin (1987))

Example 3.1. Based on Theorem 2.1(i) the bivariate function

$$\begin{aligned}
 C(u, v; \theta, \gamma, \delta, k) &= uv[1 + \delta(1 - u)^2(1 - v)^2][1 + \theta(1 - u^k)(1 - v^k)]^{\frac{1}{\gamma}}, \tag{3.1}
 \end{aligned}$$

is a copula for $u, v \in I$ when

$$(a) \quad 0 \leq \theta \leq 1, \quad 0 \leq \delta \leq 1, \quad 0 \leq k \leq 1, \quad \gamma \geq 1$$

or

$$(b) \quad 0 \leq \theta \leq 1, \quad 0 \leq \delta \leq 2, \quad 1 \leq k \leq 3, \quad \gamma \geq 3$$

and for $-1 \leq \theta \leq 0, 0 \leq \delta \leq 1$ when

$$(c) \quad 0 \leq k \leq 1, \quad \gamma \leq -2$$

or

$$(d) \quad 1 \leq k \leq 2, \quad \gamma \leq -4,$$

or

$$(e) \quad 2 \leq k \leq 3, \quad \gamma \leq -6.$$

Here, $\phi(u) = 1 - u^k$, for which boundary condition of Theorem 2.1 holds. Moreover, by using Table 1

$$K(u, v; \delta) = uv[1 + \delta(1 - u)^2(1 - v)^2], \quad \delta \in [-1, 3],$$

is a copula and

$$\frac{\partial^2}{\partial v \partial u} \ln K(u, v; \delta) = \frac{4\delta(1 - u)(1 - v)}{[1 + \delta(1 - u)^2(1 - v)^2]^2} \geq 0, \quad \delta \in [0, 3]$$

for $\forall u, v \in I$. Now, we define

$$\begin{aligned}
 h(u, v; \delta, \gamma, k) &= \phi(u) + \frac{\phi'(u)}{\gamma \frac{\partial}{\partial u} \ln K(u, v; \delta)} \\
 &= 1 - u^k - \frac{ku^k[1 + \delta(1 - u)^2(1 - v)^2]}{\gamma[1 + \delta(1 - u)^2(1 - v)^2 - 2\delta u(1 - u)(1 - v)^2]} \\
 &= 1 - u^k f(u, v; \delta, \gamma, k),
 \end{aligned}$$

where

$$f(u, v; \delta, \gamma, k) = 1 + \frac{k(1 + \delta(1 - u)^2(1 - v)^2)}{\gamma[1 + \delta(1 - u)^2(1 - v)^2 - 2\delta u(1 - u)(1 - v)^2]}.$$

(a) In this case, since $f(u, v; \delta, \gamma, k) \geq 1$, it is sufficient to show that

$$u^k f(u, v; \delta, \gamma, k) \leq 2.$$

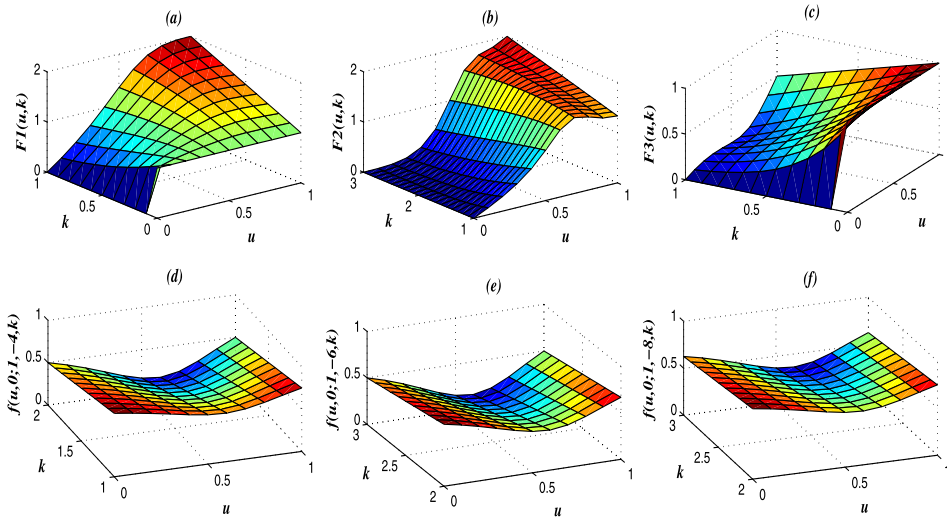


Figure 1 Illustrations of $F_i(u, k)$ for $i = 1, 2, 3$ and $f(u, 0; 1, \gamma, k)$ for $\gamma = -4, -6, -8$.

On the other hand, $f(u, v; \delta, \gamma, k)$ is decreasing in γ and v , so $f(u, v; \delta, \gamma, k) \leq f(u, 0; \delta, 1, k)$ for $u \in [0, 1]$, $\delta \in [0, 1]$ and $k \in [0, 1]$ and $f(u, 0; \delta, 1, k)$ is increasing function in δ , because

$$\frac{\partial}{\partial \delta} f(u, 0; \delta, 1, k) = \frac{2k(1-u)u}{(1+\delta-4\delta u+3\delta u^2)^2} \geq 0.$$

So, the maximum of $f(u, v; \delta, \gamma, k)$ happens at $v = 0$, $\gamma = 1$ and $\delta = 1$. Moreover, As we see in Figure 1(a), $0 \leq F_1(u, k) = u^k f(u, 0; 1, 1, k) \leq 2$ for $u \in [0, 1]$, $k \in [0, 1]$. Thus, we can conclude that $0 \leq u^k f(u, v; \delta, \gamma, k) \leq 2$ for $u, v \in [0, 1]$, $0 \leq \delta \leq 1$, $0 \leq k \leq 1$ and $\gamma \geq 1$.

(b) By a similar argument, we can show that $0 \leq F_2(u, k) = u^k f(u, 0; 2, 3, k) \leq 2$. Figure 1(b) confirms this claim for $0 \leq u \leq 1$ and $1 \leq k \leq 3$.

(c) In this case, since $f(u, v; \delta, \gamma, k) \leq 1$, it is sufficient to show that $f(u, v; \delta, \gamma, k) \geq 0$. By a similar argument, we just investigate $f(u, v; \delta, \gamma, k)$ at $v = 0$, $\gamma = -2$ and $\delta = 1$. Figure 1(c) illustrates $F_3(u, k) = f(u, 0; 1, -2, k) \geq 0$ for $0 \leq u \leq 1$ and $0 \leq k \leq 1$.

Other cases are similarly proved. Figure 1(d)–(f) illustrates that $0 \leq f(u, 0; 1, \gamma, k) \leq 1$ for certain values of γ .

Example 3.2. Based on Theorem 2.1(i) the bivariate function

$$C(u, v; \theta, \gamma, \alpha, k) = uv[1 + \alpha(u^2 - 1)(v^2 - 1)][1 + \theta(1 - u^k)(1 - v^k)]^{1/\gamma}, \tag{3.2}$$

is a copula for

$$-1 \leq \theta \leq 0, \quad 0 \leq \alpha \leq 0.4, \quad \gamma \leq -5k, \quad k > 0,$$

or

$$0 \leq \theta \leq 1, \quad 0 \leq \alpha \leq 0.4, \quad \gamma \geq 5k, \quad k > 0.$$

First, we note that

$$K(u, v; \alpha) = uv[1 + \alpha(u^2 - 1)(v^2 - 1)],$$

is a copula for $\alpha \in [-0.25, 1]$ and

$$\frac{\partial^2}{\partial v \partial u} \ln K(u, v; \alpha) = \frac{4\alpha uv}{[1 + \alpha(u^2 - 1)(v^2 - 1)]^2} \geq 0, \quad \alpha \in [0, 1].$$

Now, we define

$$r(u, v; \alpha, \gamma, k) = \phi(u) + \frac{\phi'(u)}{\gamma \frac{\partial}{\partial u} \ln K(u, v; \alpha)} = 1 - u^k \left\{ 1 + \frac{k}{\gamma} R(u, v; \alpha) \right\},$$

where

$$R(u, v; \alpha) = \frac{1 + \alpha(u^2 - 1)(v^2 - 1)}{1 + \alpha(u^2 - 1)(v^2 - 1) + 2\alpha u^2(v^2 - 1)}.$$

We note that it is sufficient to investigate the range of $r(u, v; \alpha, \gamma, k)$ at $\gamma = -5k$ and $\gamma = 5k$. Moreover, we can easily show that the function $R(u, v; \alpha)$ is a decreasing function in v and increasing function in α and u , so $1 \leq R(u, v; \alpha) \leq 5$. Hence,

$$\begin{aligned} 0 \leq u^k \left\{ 1 - \frac{1}{5} R(u, v; \alpha) \right\} &\leq 1, \\ u, v \in [0, 1] &\Rightarrow 0 \leq r(u, v; \alpha, \gamma, k) \leq 1, \\ 0 \leq u^k \left\{ 1 + \frac{1}{5} R(u, v; \alpha) \right\} &\leq 2, \\ u, v \in [0, 1] &\Rightarrow -1 \leq r(u, v; \alpha, \gamma, k) \leq 1. \end{aligned}$$

for $\alpha \in [0, 0.4]$. Figure 2 illustrates our claim for certain values of the parameters.

It is also worth noting that, similarly, the bivariate function

$$\begin{aligned} C(u, v; \delta, \theta, \gamma, p) & \\ &= uv[1 + \delta(1 - u^p)(1 - v^p)][1 + \theta(1 - u)(1 - v)]^{\frac{1}{\gamma}}, \end{aligned} \tag{3.3}$$

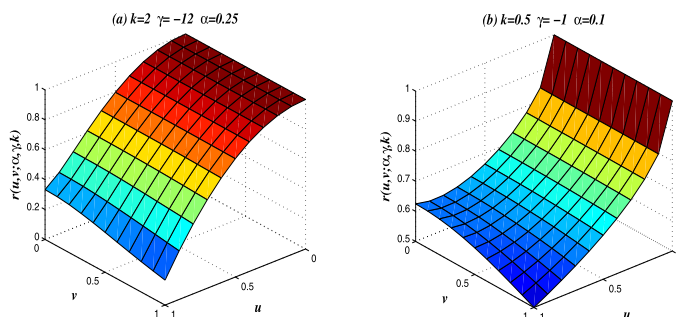


Figure 2 Illustrations of $r(u, v; \alpha, \gamma, k)$.

is a copula for $p > 0, 0 \leq \delta \leq \frac{1}{2p}, \gamma \geq 2$ and $0 \leq \theta \leq 1$ or $\gamma \leq -2$ and $-1 \leq \theta \leq 0$. Also, the bivariate function

$$C(u, v; \alpha, \beta, \theta, \gamma) = uv[1 + \alpha(1 - u)(1 - v) + \beta(1 - u)^2(1 - v)^2] \times [1 + \theta(e^{-u} - e^{-1})(e^{-v} - e^{-1})]^{\frac{1}{\gamma}}, \tag{3.4}$$

is also a copula for $0 \leq \beta \leq 1, 0 \leq \alpha < 1, -1 \leq \theta \leq 0$ and $\gamma \leq -\frac{2}{e(1-\alpha)}$ or $0 \leq \theta \leq 1$ and $\gamma \geq \frac{2}{e(1-\alpha)}$ and the bivariate function

$$C(u, v; \alpha, \beta, \theta, \gamma, k) = uv[1 + \alpha(1 - u)(1 - v) + \beta(1 - u)^2(1 - v)^2] \times [1 + \theta(1 - u)(1 - v)]^{\frac{1}{\gamma}}, \tag{3.5}$$

is a copula for $0 \leq \alpha < 1, 0 \leq \beta \leq 1, -1 \leq \theta \leq 0, \gamma \leq -\frac{2}{1-\alpha}$ or $0 \leq \theta \leq 1, \gamma \geq \frac{2}{1-\alpha}$.

Example 3.3. The bivariate function

$$C(u, v; \theta, \delta, \gamma) = uv[1 + \delta(1 - u)^2(1 - v)^2][1 + \theta(1 - u)(1 - v)]^{\frac{1}{\gamma}}, \tag{3.6}$$

is a copula for $u, v \in I, 0 \leq \theta \leq 1, 0 \leq \delta \leq 1.5$ and $\gamma \leq -3$ because it satisfies the conditions of Theorem 2.1(ii).

Clearly, since $\phi(u) = 1 - u$, boundary condition and Assumption 3 hold. Furthermore, similar to Example 3.1, we have

$$\frac{\partial^2}{\partial v \partial u} \ln K(u, v; \theta, \delta, \gamma) \geq 0, \quad \delta \in [0, 1.5].$$

Now, we define

$$\begin{aligned} t(u, v; \delta, \gamma) &= \frac{-\phi'(u)}{\sqrt{|\gamma|} \frac{\partial}{\partial u} \ln K(u, v)} \\ &= \frac{u[1 + \delta(1 - u)^2(1 - v)^2]}{\sqrt{|\gamma|[1 + \delta(1 - u)^2(1 - v)^2 - 2\delta u(1 - u)(1 - v)^2]}, \end{aligned}$$

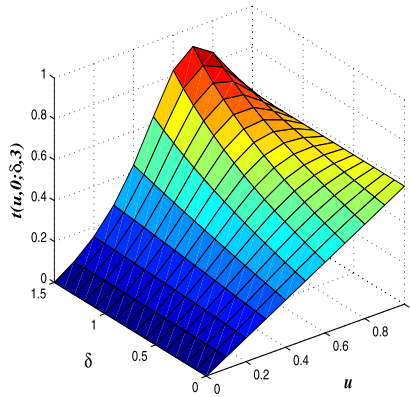


Figure 3 Illustrations of $t(u, 0; \delta, -3)$.

which is obviously nonnegative. Thus, it is sufficient to show that $t(u, v; \delta, \gamma) \leq 1$. Obviously $t(u, v; \delta, \gamma)$ is a decreasing function in v and γ . So, we need to investigate the values of $t(u, 0; \delta, -3)$. As Figure 3 illustrates we can show that $t(u, 0; \delta, -3) \leq 1$ for $0 \leq u \leq 1$ and $0 \leq \delta \leq 1.5$.

Example 3.4. The bivariate function

$$C(u, v; \theta, \alpha) = uv[1 + \alpha(1 - u)(1 - v)(1 + uv)]e^{\theta(e^{-1} - e^{-u})(e^{-1} - e^{-v})}, \quad (3.7)$$

is a copula for $0 \leq \theta \leq 1$ and $0 \leq \alpha \leq 1 - e^{-1}$, because it satisfies the conditions of Theorem 2.2(i). Clearly, since $\phi(u) = e^{-1} - e^{-u}$, the boundary condition holds. Furthermore, by using Table 1, we have that

$$K(u, v; \alpha) = uv[1 + \alpha(1 - u)(1 - v)(1 + uv)],$$

is a copula for $-0.5 \leq \alpha \leq 1$ and

$$\frac{\partial^2}{\partial v \partial u} \ln K(u, v; \alpha) = \frac{\alpha[1 + \alpha(1 - u)^2(1 - v)^2 + (1 - 2u)(1 - 2v)]}{[1 + \alpha(1 - u)(1 - v)(1 + uv)]^2} \geq 0,$$

for $\alpha \in [0, 1]$, because $(1 - 2u)(1 - 2v) \geq -1$. Now, we define

$$\begin{aligned} q(u, v; \alpha) &= \frac{\partial}{\partial u} \ln K(u, v; \alpha) - \phi'(u) \\ &= \frac{1}{u} + \frac{\alpha(1 - v)(v - 2uv - 1)}{1 + \alpha(1 - u)(1 - v)(1 + uv)} - e^{-u} \\ &= g(u) + f_1(u, v; \alpha), \end{aligned}$$

where, obviously

$$g(u) = \frac{1}{u} - e^{-u} \geq 0, \quad f_1(u, v; \alpha) = \frac{\alpha(1 - v)(v - 2uv - 1)}{1 + \alpha(1 - u)(1 - v)(1 + uv)} \leq 0.$$

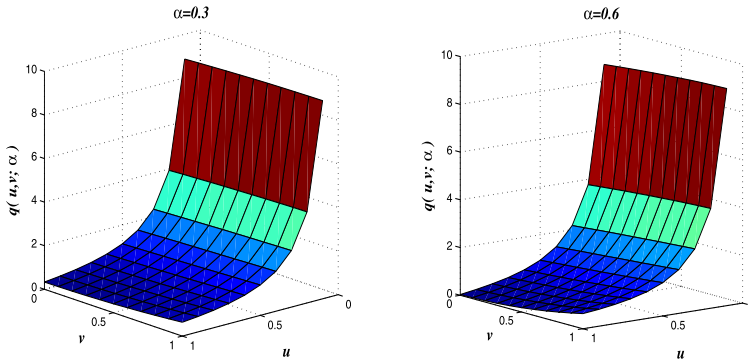


Figure 4 Illustration of $q(u, v, \alpha)$.

Both functions g and f_1 are decreasing in u , because

$$g'(u) = e^{-u} - \frac{1}{u^2} \leq e^{-u} - 1 \leq 0,$$

$$\frac{\partial}{\partial u} f_1(u, v; \alpha) = \frac{1}{[1 + \alpha(1 - u)(1 - v)(1 + uv)]^2} [-\alpha^2(1 - v)^2(v - 2uv - 1)^2 - 2\alpha v(1 - v)(1 + \alpha(1 - u)(1 - v)(1 + uv))] \leq 0.$$

g decreases very faster than f_1 , so to find values α such that $q(u, v; \alpha) \geq 0$ for $u, v \in [0, 1]$, it is sufficient to investigate the sign of $q(1, v; \alpha)$ for $v \in [0, 1]$. But, clearly, we have

$$q(1, v; \alpha) = 1 - e^{-1} - \alpha(1 - v^2) \geq 0, \quad 0 \leq \alpha \leq 1 - e^{-1}.$$

Figure 4 also confirms the validity of Assumption 4 for $\alpha = 0.3$ and 0.6 .

Example 3.5. By Theorem 2.2(i), the bivariate function

$$C(u, v; \alpha, \beta, \theta) = uv[1 + \alpha(1 - u)(1 - v) + \beta(1 - u)^2(1 - v)^2] \times e^{\theta(e^{-1} - e^{-u})(e^{-1} - e^{-v})} \tag{3.8}$$

is a copula for $0 \leq \theta \leq 1$, $0 \leq \beta \leq 0.5$ and $0 \leq \alpha \leq 1 - e^{-1}$. Similar to Example 3.3, $\phi(u) = e^{-1} - e^{-u}$ and the boundary condition holds and

$$K(u, v; \alpha, \beta) = uv[1 + \alpha(1 - u)(1 - v) + \beta(1 - u)^2(1 - v)^2]$$

is a copula for $|\alpha| \leq 1$, $\alpha + \beta \geq -1$ and $\beta \leq \frac{3 - \alpha + \sqrt{9 - 6\alpha - 3\alpha^2}}{2}$ with

$$\frac{\partial^2}{\partial v \partial u} \ln K(u, v; \alpha, \beta) = \frac{\alpha + 4\beta(1 - u)(1 - v) + \alpha\beta(1 - u)^2(1 - v)^2}{[1 + \alpha(1 - u)(1 - v) + \beta(1 - u)^2(1 - v)^2]^2} \geq 0,$$

for $0 \leq \alpha \leq 1$ and $0 \leq \beta \leq \frac{3-\alpha+\sqrt{9-6\alpha-3\alpha^2}}{2}$. Now, we define

$$\begin{aligned} s(u, v; \alpha, \beta) &= \frac{\partial}{\partial u} \ln K(u, v; \alpha, \beta) - e^{-u} \\ &= \frac{1}{u} - \frac{\alpha(1-v) + 2\beta(1-u)(1-v)^2}{1 + \alpha(1-u)(1-v) + \beta(1-u)^2(1-v)^2} - e^{-u} \\ &= g(u) - f_2(u, v; \alpha, \beta), \end{aligned}$$

where $g(u) = \frac{1}{u} - e^{-u} > 0$ and

$$f_2(u, v; \alpha, \beta) = \frac{\alpha(1-v) + 2\beta(1-u)(1-v)^2}{1 + \alpha(1-u)(1-v) + \beta(1-u)^2(1-v)^2} \geq 0,$$

for $0 \leq \alpha \leq 1$ and $0 \leq \beta \leq \frac{3-\alpha+\sqrt{9-6\alpha-3\alpha^2}}{2}$. f_2 is a decreasing function in v for $0 \leq \alpha \leq 1$ and $0 \leq \beta \leq 1$, because

$$\begin{aligned} &\frac{\partial}{\partial v} f_2(u, v; \alpha, \beta) \\ &= -\frac{\alpha + 4\beta(1-u)(1-v)}{A} + \frac{(1-u)(1-v)(\alpha + 2\beta(1-u)(1-v))}{A^2} \\ &\leq -\frac{\alpha + 2\beta(1-u)(1-v)}{A} + \frac{(1-u)(1-v)(\alpha + 2\beta(1-u)(1-v))}{A^2} \\ &= \frac{\alpha + 2\beta(1-u)(1-v)}{A} \left[-1 + \frac{\beta(1-u)^2(1-v)^2}{A} \right] \leq 0, \end{aligned}$$

where $A = 1 + \alpha(1-u)(1-v) + \beta(1-u)^2(1-v)^2 > 0$. So, it is sufficient to investigate the behavior of $f_2(u, 0; \alpha, \beta)$. This function will be increasing, decreasing or upsidedown U-shaped in u , depending on different values of α and β . Thus, we consider two cases: (i) $f_2(u, 0; \alpha, \beta)$ is an increasing or decreasing function and (ii) $f_2(u, 0; \alpha, \beta)$ is upsidedown U-shaped function in u .

For case (i), since g is decreasing very fast, we just investigate the sign of $s(u, 0; \alpha, \beta)$ for $u = 0$ and $u = 1$ and conclude that $s(u, 0; \alpha, \beta) \geq 0$ for $u, v \in [0, 1]$, $0 \leq \alpha \leq 1 - e^{-1}$.

In case (ii), we note that f_2 is an increasing function in α and β . So, we define

$$h(u) = f_2(u, 0; 1 - e^{-1}, 0.5) = \frac{2 - u - e^{-1}}{1 + (1 - e^{-1})(1 - u) + 0.5(1 - u)^2}.$$

$h(u)$ has a maximum at $u_0 = 0.3670$ and

$$h(u) \geq \frac{\alpha + 2\beta(1-u)}{1 + \alpha(1-u) + \beta(1-u)^2}, \quad \alpha \in [0, 1 - e^{-1}], \beta \in [0, 0.5].$$

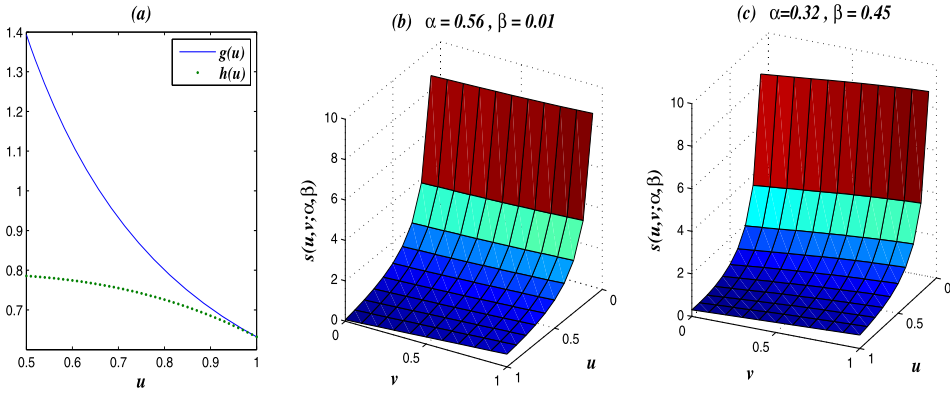


Figure 5 Illustrations of $s(u, v; \alpha, \beta)$.

Thus, since $g(u)$ is decreasing very fast compared to $h(u)$, we just plot $g(u)$ and $h(u)$ for $u \in [0.5, 1]$, in Figure 5(a). Then, we have

$$g(u) - h(u) \geq 0, \quad \forall u \in [0, 1].$$

Hence, we conclude that $s(u, v; \alpha, \beta) \geq 0$ for $\alpha \in [0, 1 - e^{-1}]$ and $\beta \in [0, 0.5]$. Moreover, Figure 5(b)–(c) confirms the validity of Assumption 4 of Theorem 2.2(a) for certain values of the parameters.

Example 3.6. The bivariate function

$$C(u, v; \theta, \alpha, \beta) = uv[1 + \alpha(1 - u)(1 - v) + \beta(1 - u)^2(1 - v)^2] \times e^{\theta(1-u)(1-v)}, \tag{3.9}$$

is a copula for $\theta \in [-1, 0]$, $\alpha \in [0, 1]$ and $\beta \in [-1, 1]$. In this example, $\phi(u) = 1 - u$ and clearly the boundary condition and Assumption 3 of Theorem 2.2(ii) hold and

$$K(u, v; \alpha, \beta) = uv[1 + \alpha(1 - u)(1 - v) + \beta(1 - u)^2(1 - v)^2]$$

is a copula for $|\alpha| \leq 1$, $\alpha + \beta \geq -1$ and $\beta \leq \frac{3-\alpha+\sqrt{9-6\alpha-3\alpha^2}}{2}$. Now, to check the validity of Assumption 5, we define

$$\begin{aligned} g(u, v; \alpha, \beta) &= \frac{k(u, v; \alpha, \beta)}{K(u, v; \alpha, \beta)} - 1 \\ &= \frac{k(u, v; \alpha, \beta) - K(u, v; \alpha, \beta)}{K(u, v; \alpha, \beta)} \\ &= \frac{1}{K(u, v; \alpha, \beta)} \{1 + \alpha(1 - u)(1 - v) + \beta(1 - u)^2(1 - v)^2\} \end{aligned}$$

$$\begin{aligned}
& - (u + v - 2uv)[\alpha + 2\beta(1 - u)(1 - v)] \\
& + uv[\alpha + 4\beta(1 - u)(1 - v)] \\
& - uv[1 + \alpha(1 - u)(1 - v) + \beta(1 - u)^2(1 - v)^2]}.
\end{aligned}$$

First, let $0 \leq \beta \leq 1$, we have

$$g(u, v; \alpha, \beta) = \frac{f_1(u, v; \alpha, \beta) - f_2(u, v; \alpha, \beta)}{K(u, v; \alpha, \beta)} = \frac{F(u, v; \alpha, \beta)}{K(u, v; \alpha, \beta)},$$

where

$$\begin{aligned}
f_1(u, v; \alpha, \beta) &= 1 + \alpha(1 - u)(1 - v) + \beta(1 - u)^2(1 - v)^2 \\
&\quad + uv[\alpha + 4\beta(1 - u)(1 - v)] \geq 0, \\
f_2(u, v; \alpha, \beta) &= uv[1 + \alpha(1 - u)(1 - v) + \beta(1 - u)^2(1 - v)^2] \\
&\quad + (u + v - 2uv)[\alpha + 2\beta(1 - u)(1 - v)] \geq 0
\end{aligned}$$

and

$$F(u, v; \alpha, \beta) = f_1(u, v; \alpha, \beta) - f_2(u, v; \alpha, \beta).$$

f_1 and f_2 are increasing functions in α and β . Moreover, f_1 and f_2 are linear functions of α and β . So, to find the signs of g , it is sufficient to investigate the signs of $F(u, v; 0, 0)$ and $F(u, v; 1, 1)$. Hence, we have

$$\begin{aligned}
F(u, v; 0, 0) &= 1 - uv \geq 0, \quad u, v \in [0, 1] \\
F(u, v; 1, 1) &= [1 + (1 - u)(1 - v) + (1 - u)^2(1 - v)^2](1 - uv) + 4uv \\
&\quad \times (1 - u)(1 - v) - [1 + 2(1 - u)(1 - v)](u + v - 2uv) + uv \\
&\geq 0, \quad u, v \in [0, 1].
\end{aligned}$$

Figure 6(a) illustrates $F(u, v; 1, 1) \geq 0$ for $u, v \in [0, 1]$. Thus, $g \geq 0$ and hence Assumption 5 holds for $0 \leq \beta \leq 1$.

Now, we let $-1 \leq \beta \leq 0$ and

$$g(u, v; \alpha, \beta) = \frac{f_3(u, v; \alpha, \beta) - f_4(u, v; \alpha, \beta)}{K(u, v; \alpha, \beta)} = \frac{G(u, v; \alpha, \beta)}{K(u, v; \alpha, \beta)},$$

where

$$\begin{aligned}
f_3(u, v; \alpha, \beta) &= 1 + \alpha(1 - u)(1 - v) + \alpha uv - 2\beta(1 - u)(1 - v) \\
&\quad \times (u + v - 2uv) - \beta uv(1 - u)^2(1 - v)^2 \geq 0, \\
f_4(u, v; \alpha, \beta) &= \alpha(u + v - 2uv) + uv[1 + \alpha(1 - u)(1 - v)] \\
&\quad - \beta(1 - u)^2(1 - v)^2 - 4\beta uv(1 - u)(1 - v) \geq 0
\end{aligned}$$

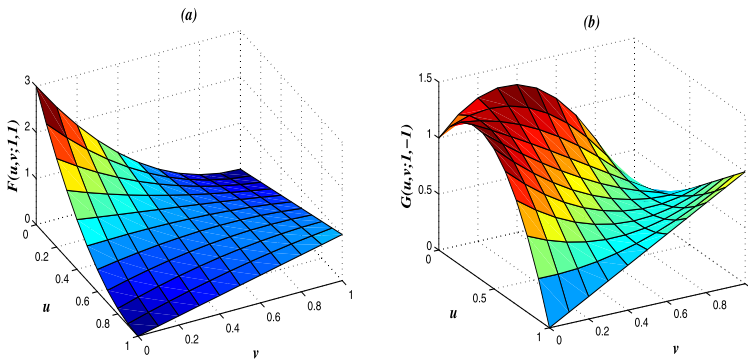


Figure 6 Illustrations of $F(u, v; 1, 1)$ and $G(u, v; 1, -1)$.

and

$$G(u, v; \alpha, \beta) = f_3(u, v; \alpha, \beta) - f_4(u, v; \alpha, \beta).$$

f_3 and f_4 are increasing functions in α and decreasing functions in β . Moreover, these functions are linear with respect to α and β . So, similar to the previous case, it is sufficient to investigate the signs of $G(u, v; 0, 0)$ and $G(u, v; 1, -1)$. Here, we have

$$G(u, v; 0, 0) = 1 - uv \geq 0, \quad u, v \in [0, 1],$$

$$G(u, v; 1, -1)$$

$$\begin{aligned} &= 1 + (1 - u)(1 - v) - (1 - u)^2(1 - v)^2 + uv[1 - 4(1 - u) \\ &\quad \times (1 - v)] - (u + v - 2uv)[1 - 2(1 - u)(1 - v)] - uv \\ &\quad - uv(1 - u)(1 - v) + uv(1 - u)^2(1 - v)^2 \geq 0, \quad u, v \in [0, 1]. \end{aligned}$$

This is illustrated by Figure 6(b). Then, we conclude that $g(u, v; \alpha, \beta) \geq 0$, for $\alpha \in [0, 1]$ and $\beta \in [-1, 1]$. Moreover, Figure 7 also illustrates that $G(u, v; \alpha, \beta) \geq 0$ for certain values of α and β .

It is also noted that, based on Theorem 2.2(ii), the bivariate functions

$$\begin{aligned} C(u, v; \alpha, \theta) &= uv[1 + \alpha(u^2 - 1)(v^2 - 1)]e^{\theta(1-u)(1-v)}, \\ 0 \leq \alpha \leq 0.45, \quad -1 \leq \theta \leq 0 \end{aligned} \tag{3.10}$$

and

$$\begin{aligned} C(u, v; \delta, \theta) &= uv[1 + \delta(1 - u^{0.5})(1 - v^{0.5})]e^{\theta(1-u)(1-v)}, \\ 0 \leq \delta \leq 2, \quad -1 \leq \theta \leq 0 \end{aligned} \tag{3.11}$$

are copulas.

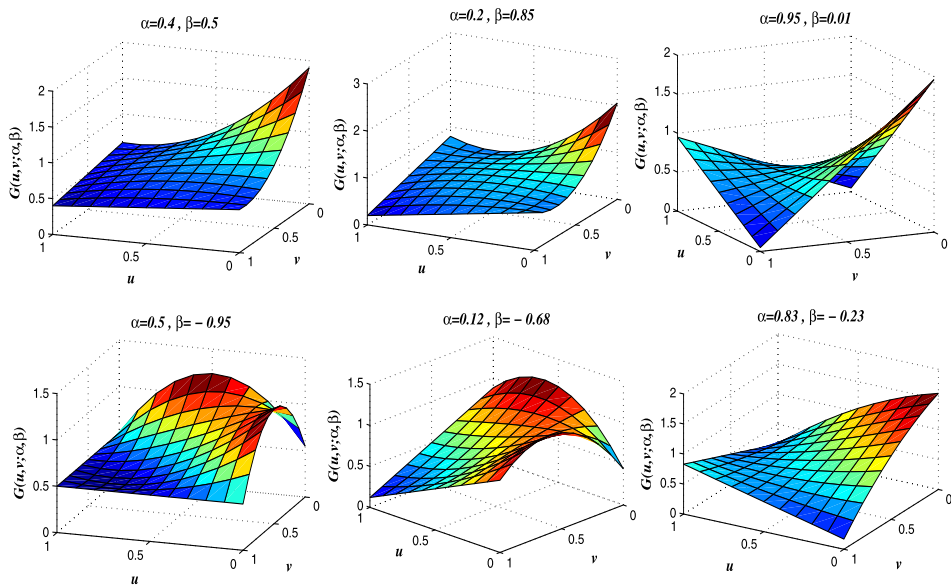


Figure 7 Illustrations of $G(u, v; \alpha, \beta)$.

Remark 3.1. An important feature of the family introduced here is that it is very large so that it contains many copulas, several but not all Archimedean copulas included. For instance, by our Theorems 2.1 and 2.2 it follows that

$$\begin{aligned}
 C(u, v; \delta, \theta, \gamma) &= uv[1 + \delta(1-u)^2(1-v)^2][1 + \theta(1-u)(1-v)]^{\frac{1}{\gamma}} \\
 0 \leq \delta \leq 1, \quad 0 \leq \theta \leq 1, \quad \gamma \geq 1, \quad \gamma \leq -1, \\
 0 \leq \delta \leq 1, \quad -1 \leq \theta \leq 1, \quad \gamma = 1, \\
 \delta = 0, \quad -1 \leq \theta \leq 1, \quad \gamma \leq -1,
 \end{aligned} \tag{3.12}$$

is a bivariate copula which is not Archimedean. However, in the special case when $\delta = 0$ and $\gamma = 1$ it reduces to *FGM* (the first order approximation to the Frank copula, which is Archimedean) and when $\delta = 0$ and $\gamma = -1$ it reduces to the Archimedean copula *AMH*. Also, by Theorem 2.2, the bivariate function

$$\begin{aligned}
 C(u, v; \delta, \theta) &= uv e^{[(-\ln(u))^{-\delta} + (-\ln(v))^{-\delta}]^{\frac{1}{\delta}} - \theta \ln(u) \ln(v)}, \\
 0 < \delta \leq 1, \quad 0 \leq \theta \leq 1
 \end{aligned} \tag{3.13}$$

is a bivariate copula but it is not Archimedean in general. In the special case when $\theta = 0$, it reduces to *Galambos* copula and when $\delta \rightarrow 0$ reduces to *BG* copula which is an Archimedean copula.

Remark 3.2. Note that when ϕ is unbounded, such as $\ln(u)$, for any values of γ

$$R(u, v; \gamma) = \phi(u) + \frac{\phi'(u)}{\gamma \frac{\partial}{\partial u} \ln K(u, v)} \quad \forall u, v \in I$$

does not lie between $[-1, 1]$ and thus G is empty. Even if ϕ is bounded, ϕ' may be unbounded such as

$$\phi(u) = (1 - u)^k, \quad 0 < k < 1$$

in which case again G is empty. As another instance, let ϕ and ϕ' be bounded, e.g., $\phi = 1 - u$, but K be the Clayton copula

$$K(u, v; \theta) = (u^{-\theta} + v^{-\theta})^{-\frac{1}{\theta}}.$$

Here, we have

$$R(u, v; \theta, \gamma) = 1 - u - \frac{u^{\theta+1}(u^{-\theta} + v^{-\theta})}{\gamma},$$

which can be unbounded for $v = 0$ and thus G is empty.

Similarly, for Q_0, Q_1 and Q_2 to be empty, one can consider $\phi(u) = 1 - u$ and K as the Galambos copula.

4 Dependence properties

In this section, we shall investigate dependence properties of our new class of copulas. We shall show that, under certain conditions, dependent properties of $C(u, v)$ are the same as those of $K(u, v)$. First, we need to recall some known concepts of dependencies.

Definition 4.1. Let r.v.'s X and Y be related by a copula $C(u, v)$. Then,

1. X and Y are Positively Quadrant Dependent (PQD) (Negatively Quadrant Dependent (NQD)), if

$$\forall (x, y) \in \mathfrak{R}^2, \quad P(X \leq x, Y \leq y) \geq (\leq) P(X \leq x)P(Y \leq y) \quad (4.1)$$

or, equivalently,

$$\forall (u, v) \in I^2, \quad C(u, v) \geq (\leq) uv. \quad (4.2)$$

2. Y is Left Tail Decreasing (LTD) in X if $P(Y \leq y|X \leq x)$ is nonincreasing in x for all y .
3. X and Y are Left Corner Set Decreasing ($LCSD$) if $P(X \leq x, Y \leq y|X \leq x', Y \leq y')$ is nonincreasing in x' and y' for all x and y .
4. Y is Stochastically Increasing (SI) in X if $P(Y > y|X = x)$ is nondecreasing in x for all y .

Theorem 4.1. Let either $\forall u \in I, \phi(u) \geq 0$ or $\forall u \in I, \phi(u) \leq 0$.

(i) If $\theta\gamma \geq 0$ ($\theta\gamma \leq 0$), then *PQD* (*NQD*) property of $K(u, v)$ is preserved by $C(u, v; \theta, \gamma) = K(u, v)[1 + \theta\phi(u)\phi(v)]^{\frac{1}{\gamma}}$.

(ii) If $\theta \in [0, 1]$ ($\theta \in [-1, 0]$), then *PQD* (*NQD*) property of $K(u, v)$ is preserved by $C(u, v; \theta) = K(u, v)e^{\theta\phi(u)\phi(v)}$.

Proof. (i) Suppose that $\gamma > 0$ and $0 \leq \theta \leq 1$, then

$$1 + \theta\phi(u)\phi(v) \geq 1 \quad \forall u, v \in I,$$

and, thus,

$$[1 + \theta\phi(u)\phi(v)]^{\frac{1}{\gamma}} \geq 1 \quad \forall u, v \in I.$$

Hence, if $K(u, v)$ is *PQD*, it follows that

$$C(u, v; \theta, \gamma) = K(u, v)[1 + \theta\phi(u)\phi(v)]^{\frac{1}{\gamma}} \geq uv \quad \forall u, v \in I,$$

as required. For $\gamma < 0$ and $-1 \leq \theta \leq 0$, we have

$$0 \leq 1 + \theta\phi(u)\phi(v) \leq 1,$$

or, equivalently,

$$[1 + \theta\phi(u)\phi(v)]^{\frac{1}{\gamma}} \geq 1.$$

Thus, if $K(u, v)$ is *PQD*, it follows that

$$K(u, v)[1 + \theta\phi(u)\phi(v)]^{\frac{1}{\gamma}} \geq uv \quad \forall u, v \in I,$$

as required. For $\gamma < 0$ and $0 \leq \theta \leq 1$, we have

$$1 + \theta\phi(u)\phi(v) \geq 1,$$

or, equivalently,

$$[1 + \theta\phi(u)\phi(v)]^{\frac{1}{\gamma}} \leq 1.$$

If $K(u, v)$ is *NQD*, it follows that

$$K(u, v)[1 + \theta\phi(u)\phi(v)]^{\frac{1}{\gamma}} \leq uv \quad \forall u, v \in I,$$

as required. For $\gamma > 0$ and $-1 \leq \theta \leq 0$, we have

$$0 \leq 1 + \theta\phi(u)\phi(v) \leq 1.$$

Thus, if $K(u, v)$ is *NQD*, we have

$$K(u, v)[1 + \theta\phi(u)\phi(v)]^{\frac{1}{\gamma}} \leq uv \quad \forall u, v \in I.$$

i.e. $C(u, v; \theta, \gamma)$ is *NQD*.

(ii) For $\theta \in [0, 1]$ ($\theta \in [-1, 0]$), $e^{\theta\phi(u)\phi(v)} \geq 1$ (≤ 1). Thus, if $K(u, v)$ is PQD (NQD), $C(u, v; \theta)$ is also PQD (NQD). \square

First, in what follow assume that X and Y are related according to copula $C(u, v; \theta, \gamma) = K(u, v)[1 + \theta\phi(u)\phi(v)]^{\frac{1}{\gamma}}$ and X' and Y' are related according to the corresponding base copula K .

Theorem 4.2. *Let $\phi : [0, 1] \rightarrow [-1, 1]$ be a differentiable function and $\theta\gamma > 0$. Then*

(i) *Y is LTD X if Y' is LTD in X' and ϕ is positive and nonincreasing or negative and nondecreasing.*

(ii) *X and Y are LCSD if X' and Y' are LCSD and ϕ is positive and nonincreasing or negative and nondecreasing.*

(iii) *Y is SI in X if Y' is SI in X' and ϕ is positive, nonincreasing and concave or negative, nondecreasing and convex for $0 \leq \theta \leq 1$ and $\gamma \geq 1$.*

Proof. (i) By Theorem 5.2.5 in Nelsen (2006), it is sufficient to prove that $\frac{C(u,v;\theta,\gamma)}{u}$ is nonincreasing in u for all $v \in I$. Since

$$\frac{C(u, v; \theta, \gamma)}{u} = \frac{K(u, v)}{u} [1 + \theta\phi(u)\phi(v)]^{\frac{1}{\gamma}},$$

according to the condition of the theorem, $\frac{K(u,v)}{u}$ is nonincreasing in u . Obviously $[1 + \theta\phi(u)\phi(v)]^{\frac{1}{\gamma}}$ is also nonincreasing in u , because

$$\frac{d}{du} [1 + \theta\phi(u)\phi(v)]^{\frac{1}{\gamma}} = \frac{\theta}{\gamma} \phi'(u)\phi(v) [1 + \theta\phi(u)\phi(v)]^{\frac{1}{\gamma}-1} \leq 0.$$

(ii) By Corollary 5.2.17 in Nelsen (2006), we should show that for all $0 \leq u_1 \leq u_2 \leq 1$ and $0 \leq v_1 \leq v_2 \leq 1$,

$$C(u_1, v_1; \theta, \gamma)C(u_2, v_2; \theta, \gamma) - C(u_1, v_2; \theta, \gamma)C(u_2, v_1; \theta, \gamma) \geq 0, \tag{4.3}$$

or equivalently,

$$K(u_1, v_1)K(u_2, v_2) \{ [1 + \theta\phi(u_1)\phi(v_1)][1 + \theta\phi(u_2)\phi(v_2)] \}^{\frac{1}{\gamma}} - K(u_1, v_2)K(u_2, v_1) \{ [1 + \theta\phi(u_1)\phi(v_2)][1 + \theta\phi(u_2)\phi(v_1)] \}^{\frac{1}{\gamma}} \geq 0. \tag{4.4}$$

First, suppose that $0 \leq \theta \leq 1$ and $\gamma \geq 0$. Since X' and Y' are LCSD, it is sufficient to show that

$$\{ [1 + \theta\phi(u_1)\phi(v_1)][1 + \theta\phi(u_2)\phi(v_2)] \}^{\frac{1}{\gamma}} \geq \{ [1 + \theta\phi(u_1)\phi(v_2)] \times [1 + \theta\phi(u_2)\phi(v_1)] \}^{\frac{1}{\gamma}},$$

or, equivalently,

$$[\phi(v_2) - \phi(v_1)][\phi(u_2) - \phi(u_1)] \geq 0. \tag{4.5}$$

But, by monotonicity of ϕ , (4.5) obviously holds and, thus, we have the result. By the same argument, (4.4) holds for $-1 \leq \theta \leq 0$ and $\gamma \leq 0$.

(iii) By Corollary 5.2.11 in Nelsen (2006) we should show that $\frac{\partial C(u,v)}{\partial u}$ is non-increasing in u . But, we have

$$\begin{aligned} \frac{\partial C(u, v; \theta, \gamma)}{\partial u} &= \frac{\partial}{\partial u} \{K(u, v)[1 + \theta\phi(u)\phi(v)]^{\frac{1}{\gamma}}\} \\ &= \frac{\partial K(u, v)}{\partial u} \underbrace{[1 + \theta\phi(u)\phi(v)]^{\frac{1}{\gamma}}}_I \end{aligned} \tag{4.6}$$

$$+ \frac{K(u, v)}{u} \underbrace{\left\{ \frac{\theta}{\gamma} u\phi(v)\phi'(u)[1 + \theta\phi(u)\phi(v)]^{\frac{1}{\gamma}-1} \right\}}_{II}. \tag{4.7}$$

Since Y' is *SI* in X' and *SI* implies *LTD* (Corollary 5.2.17 in Nelsen, 2006), $\frac{\partial K(u,v)}{\partial u}$ and $\frac{K(u,v)}{u}$ are nonincreasing. Expressions I and II are also nonincreasing because

$$\frac{\partial}{\partial u} [1 + \theta\phi(u)\phi(v)]^{\frac{1}{\gamma}} = \frac{\theta}{\gamma} \phi'(u)\phi(v)[1 + \theta\phi(u)\phi(v)]^{\frac{1}{\gamma}-1} \leq 0 \tag{4.8}$$

and

$$\begin{aligned} &\frac{\partial}{\partial u} \left\{ \frac{\theta}{\gamma} u\phi'(u)\phi(v)[1 + \theta\phi(u)\phi(v)]^{\frac{1}{\gamma}-1} \right\} \\ &= \frac{\theta}{\gamma} \phi'(u)\phi(v)[1 + \theta\phi(u)\phi(v)]^{\frac{1}{\gamma}-1} \\ &\quad + \frac{\theta}{\gamma} \phi''(u)u\phi(v)[1 + \theta\phi(u)\phi(v)]^{\frac{1}{\gamma}-1} + \frac{\theta^2}{\gamma^2} (1 - \gamma)(\phi'(u))^2(\phi(v))^2u \\ &\quad \times [1 + \theta\phi(u)\phi(v)]^{\frac{1}{\gamma}-2} \leq 0. \end{aligned} \tag{4.9}$$

Consequently, by (4.8) and (4.9), the Expressions (4.6) and (4.7) are nonincreasing in u . This completes the proof. \square

Now, in what follows, we assume that X and Y are related according to copula $C(u, v; \theta) = K(u, v)e^{\theta\phi(u)\phi(v)}$ and X' and Y' are related according to the corresponding base copula K .

Theorem 4.3. *Let ϕ be a differentiable function and $0 \leq \theta \leq 1$. Then*

(i) Y is LTD in X if Y' is LTD in X' and ϕ is positive and nonincreasing or negative and nondecreasing.

(ii) Y is LCSD in X if Y' is LCSD in X' and ϕ is positive and nonincreasing or negative and nondecreasing.

Proof. (i) It is sufficient to prove that $\frac{C(u,v;\theta)}{u}$ is nonincreasing in u . According to the condition of the theorem, $\frac{K(u,v)}{u}$ is nonincreasing in u and obviously $e^{\theta\phi(u)\phi(v)}$ is nonincreasing in u too, because we have

$$\frac{\partial}{\partial u} e^{\theta\phi(u)\phi(v)} = \theta\phi'(u)\phi(v)e^{\theta\phi(u)\phi(v)} \leq 0.$$

Hence,

$$\frac{C(u, v; \theta)}{u} = \frac{K(u, v)}{u} e^{\theta\phi(u)\phi(v)}$$

is nonincreasing.

(ii) Similar to the proof of Theorem 4.2(ii). □

5 Tail dependence

In this section, we consider two concepts of dependencies that measure the amount of dependency in the upper-right quadrant and lower-right quadrant on I^2 and show that upper tail dependence coefficient of (X, Y) equals to that of (X', Y') .

Definition 5.1. Let X and Y be continuous random variables with d.f.'s F and G , respectively. The upper tail dependence λ_U is defined by

$$\lambda_U = \lim_{u \rightarrow 1^-} P[Y > G^{-1}(u) | X > F^{-1}(u)] = 2 - \lim_{u \rightarrow 1^-} \frac{1 - C(u, u)}{1 - u}.$$

Similarly, lower tail dependence λ_L is defined by

$$\lambda_L = \lim_{u \rightarrow 0^+} P[Y \leq G^{-1}(u) | X \leq F^{-1}(u)] = \lim_{u \rightarrow 0^+} \frac{C(u, u)}{u}.$$

In the following theorem, we assume that $\lambda_{U,K}$ and $\lambda_{L,K}$ are upper and lower tail dependencies of copula $K(u, v)$, and $\lambda_{U,C}$ and $\lambda_{L,C}$ are those of copula $C(u, v; \theta, \gamma)$ (or $C(u, v; \theta)$).

Theorem 5.1.

(i) If $C(u, v; \theta, \gamma) = K(u, v)[1 + \theta\phi(u)\phi(v)]^{\frac{1}{\gamma}}$ for $\gamma \neq 0$, then

$$\lambda_{U,C} = \lambda_{U,K}, \quad \lambda_{L,C} = \lambda_{L,K}[1 + \theta\phi(0)]^{\frac{1}{\gamma}}.$$

(ii) if $C(u, v; \theta) = K(u, v)e^{\theta\phi(u)\phi(v)}$, then

$$\lambda_{U,C} = \lambda_{U,K}, \quad \lambda_{L,C} = \lambda_{L,K}e^{\theta\phi^2(0)}.$$

Proof. (i) We have

$$\begin{aligned} \lambda_{U,C} &= 2 - \lim_{u \rightarrow 1^-} \frac{1 - C(u, u)}{1 - u} \\ &= 2 - \lim_{u \rightarrow 1^-} \frac{1 - K(u, u)[1 + \theta\phi^2(u)]^{\frac{1}{\gamma}}}{1 - u} \\ &= 2 - \lim_{u \rightarrow 1^-} \left\{ \frac{\partial}{\partial u} K(u, u)[1 + \theta\phi^2(u)]^{\frac{1}{\gamma}} + \frac{2\theta}{\gamma} K(u, u)\phi(u)\phi'(u) \right. \\ &\quad \left. \times [1 + \theta\phi^2(u)]^{\frac{1}{\gamma}-1} \right\} \\ &= 2 - \lim_{u \rightarrow 1^-} \frac{\partial}{\partial u} K(u, u) = \lambda_{U,K} \end{aligned}$$

and

$$\begin{aligned} \lambda_{L,C} &= \lim_{u \rightarrow 0^+} \frac{C(u, u)}{u} \\ &= \lim_{u \rightarrow 0^+} \frac{\partial}{\partial u} C(u, u) \\ &= \lim_{u \rightarrow 0^+} \left\{ \frac{\partial}{\partial u} K(u, u)[1 + \theta\phi^2(u)]^{\frac{1}{\gamma}} + \frac{2\theta}{\gamma} K(u, u)\phi(u)\phi'(u) \right. \\ &\quad \left. \times [1 + \theta\phi^2(u)]^{\frac{1}{\gamma}-1} \right\} \\ &= \left(\lim_{u \rightarrow 0^+} \frac{\partial}{\partial u} K(u, u) \right) [1 + \theta\phi^2(0)]^{\frac{1}{\gamma}} \\ &= \lambda_{L,K} [1 + \theta\phi^2(0)]^{\frac{1}{\gamma}}. \end{aligned}$$

As required. The prove of part (ii) is similar. □

6 Measures of association

In this section, we obtain Spearman's ρ_s for our new copulas for certain values of their parameters. As theoretical calculations are not feasible here, we shall use a Maple software to compute ρ_s numerically. Our new copulas prove to improve the range of correlation coefficient of the FGM family. Recall that Spearman's ρ_s is

Table 2 Spearman's ρ_s for various values of k, δ, θ and γ for copula (3.1)

k	δ	θ	γ	ρ_s	k	δ	θ	γ	ρ_s
1	1	1	1	0.4467	1.5	1	-1	-4	0.2972
3	2	1	3	0.5116	2	1	-1	-6	0.2945
3	0.5	0.75	4	0.2241	0	1	-1	-2	0.0833
2.4	2	1	3	0.4588	1	1	-1	-2	0.3230
1	2	0	3	0.1667	1	1	-1	-4	0.1969
0.5	1	1	1	0.2160	2	1	-1	-4	0.4143
3	1	-1	-6	0.4597	2.5	1	-1	-6	0.4847

Table 3 The Spearman's ρ_s for various values of k, α, θ and γ for copula (3.2)

k	δ	θ	γ	ρ_s	k	δ	θ	γ	ρ_s
0.1	0.4	1	0.5	0.3167	0.2	0.001	1	1	0.0256
0.5	0.4	1	2.5	0.3561	1	0.4	-1	-5	0.3981
1	0.4	1	5	0.3728	2	0.4	-1	-10	0.4339
1	0.4	1	6	0.3604	3	0.4	-1	-15	0.4516
3	0.4	1	15	0.3677	3	0.4	0	-10	0.3000
1	0.01	1	5	0.0689	5	0.4	-1	-25	0.4681
1	0.25	1	5	0.2559	1	0.4	-1	-8	0.3604
0.75	0.4	1	3.75	0.3671	1	0.4	-1	-7	0.3693
6	0.4	-1	-30	0.4723	4	0.4	-1	-20	0.4618

defined by

$$\rho_s = 12 \int_0^1 \int_0^1 C(u, v) du dv - 3.$$

Example 6.1.

- 6.1.1. Consider the copula in (3.1). The Spearman's ρ_s calculated for various values of k, δ, θ and γ appear in Table 2. Here, ρ_s is an increasing function in δ and k , increasing (decreasing) function in θ when $\gamma \geq 0$ ($\gamma \leq 0$) and increasing (decreasing) function in γ when $\theta \leq 0$ ($\theta \geq 0$). Furthermore, by Theorem 4.1, $C(u, v; k, \delta, \theta, \gamma)$ is *PQD*. So, $\rho_s \in [0, 0.5116]$.
- 6.1.2. Consider the copula in (3.2). ρ_s is increasing in α and k , increasing (decreasing) function in θ if $\gamma \geq 0$ ($\gamma \leq 0$), increasing (decreasing) function in γ if $\theta \leq 0$ ($\theta \geq 0$). The Spearman's ρ_s calculated for various values of its parameters appear in Table 3. The value of ρ_s equals 0.4723 for $k = 6, \alpha = 0.4, \theta = -1$ and $\gamma = -30$ which is greater than the upper bound of the correlation coefficient of an FGM copula.
- 6.1.3. Consider the copula in (3.3). The Spearman's ρ_s calculated for various values of δ, θ, γ and p appear in Table 4. Here, ρ_s is increasing function in p

Table 4 The Spearman's ρ_s for various values of δ, θ, γ and p for copula (3.3)

δ	θ	γ	p	ρ_s	δ	θ	γ	p	ρ_s
5	1	2	0.1	0.1964	0.1	-1	-2	5	0.3857
0.5	1	2	1	0.3437	0.5	-1	-2	1	0.4122
0.25	1	2	2	0.3643	0.25	-1	-2	2	0.4311
0.1	1	2	5	0.3233	5	-1	-2	0.1	0.2571
1.67	1	2	0.3	0.2542	0.5	-1	-3	1	0.3248
0.25	1	2	2	0.3643	0.05	-0.5	-3	6	0.1480
0.17	0.5	2	3	0.2733	5	0	-2	0.1	0.0340
0	1	3	1	0.1033	0.5	0	3	1	0.1667

Table 5 The Spearman's ρ_s for various values of α, θ, β and γ for copula (3.4)

α	θ	β	γ	ρ_s	α	θ	β	γ	ρ_s
0.1	-0.35	0.5	-3	0.0849	0	1	1	1.5	0.1397
0.2	-0.75	0.5	-1	0.1769	0.2	0.5	0.9	1	0.1862
0.2	-1	1	-1	0.2484	0.5	1	1	1.5	0.3134
0.3	-0.95	0.8	-2	0.2124	0.5	1	1	2	0.2973
0.45	-0.5	1	-1.5	0.2663	0.6	0.85	0.8	1.23	0.3332
0.65	-1	1	-2.5	0.3425	0.8	1	1	4	0.3750
0.8	-1	0	-4	0.2916	0.95	1	0.7	14.72	0.3818
0.99	-1	1	-75	0.4148	0.99	0	1	-75	0.4133
0.99	-1	1	-73.58	0.4149	0.99	1	1	73.58	0.4147

and δ , increasing (decreasing) function in θ when $\gamma \geq 0$ ($\gamma \leq 0$) and increasing (decreasing) function in γ when $\theta \leq 0$ ($\theta \geq 0$). The value of ρ_s equals 0.4311 for $\delta = 0.25, \theta = -1, \gamma = -2$ and $p = 2$ which is greater than the upper bound of the correlation coefficient of an FGM copula.

- 6.1.4. Consider the copula in (3.4). ρ_s is an increasing function in α and β , increasing (decreasing) function in θ when $\gamma \geq 0$ ($\gamma \leq 0$) and increasing (decreasing) function in γ if $\theta \geq 0$ ($\theta \leq 0$). The Spearman's ρ_s calculated for various values of its parameters appear in Table 5. The value of ρ_s equals 0.4149 for $\alpha = 0.99, \theta = -1, \beta = 1$ and $\gamma = -73.58$ which is greater than the upper bound of the correlation coefficient of an FGM copula.
- 6.1.5. Consider the copula in (3.8). Table 6 shows Spearman's ρ_s values calculated for various values of α, β and θ . ρ_s is an increasing function in α, β and θ . Furthermore, $C(u, v; \alpha, \beta, \theta)$ is PQD, so $\rho_s \in [0, 0.3515]$.
- 6.1.6. Consider the copula in (3.9). ρ_s is increasing in α, β and θ . As seen in Table 7, the maximum of ρ_s equals 0.4167 which is greater than the upper bound of an FGM copula and the minimum of ρ_s equals -0.3552 which is less than the lower bound of an FGM copula.

Table 6 The Spearman's ρ_s for various values of parameters of copula (3.8)

α	β	θ	ρ_s	α	β	θ	ρ_s
0	0	0.1	0.0078	0.45	0.3	0.6	0.2300
0.1	0.1	0.2	0.0579	0.5	0.35	0.7	0.2614
0.2	0.15	0.3	0.1043	0.55	0.4	0.8	0.2931
0.3	0.2	0.4	0.1514	0.6	0.45	0.9	0.3253
0.4	0.25	0.5	0.1991	$1 - e^{-1}$	0.5	1	0.3515

Table 7 The Spearman's ρ_s for various values of parameters of copulas (3.9) and (3.10)

α	(3.9)			(3.10)		
	β	θ	ρ_s	δ	θ	ρ_s
0.1	-0.45	-0.5	-0.1588	0	-1	-0.2962
0.4	-.65	-0.8	-0.1733	0.1	-0.9	-0.2073
0.8	-0.95	-1	-0.1418	0.15	-0.8	-0.1471
0.8	-0.95	0	0.1875	0.2	-0.7	-0.0850
1	-1	-1	-0.0920	0.25	-0.6	-0.0208
0	-1	-1	-0.3552	0.3	-0.5	0.0455
0.6	0.2	-0.35	0.0865	0.35	-0.4	0.1141
0.9	0.8	-0.7	0.0911	0.4	-0.3	0.1849
1	1	-1	0.0258	0.45	-0.2	0.2582
1	1	0	0.4167	0.45	0	0.3375

- 6.1.7. Consider the copula in (3.10). ρ_s is an increasing in δ and θ . Hence, as we see in Table 7, $\rho_s \in [-0.2962, 0.3375]$.

Remark 6.1. For copula (3.12) which contains *FGM* and *AMH*, $\rho_s \in [-0.28, 0.4784]$.

Conclusion

In this study, we started with a rather arbitrary base bivariate copula and found sufficient and necessary conditions under which two new classes of bivariate functions would be copulas. In particular, they include various already known copulas previously studied in the literature such as *FGM*, *AMH* and *BG*. Dependence properties of variables involved were investigated and revealed that the dependent properties of the base copula are preserved by the generated copula under certain conditions. Furthermore, we have provided several examples and computed the values of Spearman's ρ_s as measures of association. It is shown that the proposed copulas improve this range of ρ_s comparing to *FGM* copula.

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Department of Statistics
University of Isfahan
Isfahan 81744
Iran
E-mail: zahra.sharif@sci.ui.ac.ir
alamatho@sci.ui.ac.ir
i.kazemi@stat.ui.ac.ir