Bayesian Analysis (2017) **12**, Number 4, pp. 1217–1219

Correction to: "Posterior Consistency of Bayesian Quantile Regression Based on the Misspecified Asymmetric Laplace Density"*

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Abstract. In this note, we highlight and provide corrections to two errors in the paper: *Karthik Sriram, R.V. Ramamoorthi, Pulak Ghosh (2013) "Posterior Consistency of Bayesian Quantile Regression Based on the Misspecified Asymmetric Laplace Density", Bayesian Analysis, Vol 8, Num 2, pg 479–504.*

Keywords: asymmetric Laplace, Bayesian, correction, posterior consistency, quantile regression.

MSC 2010 subject classifications: Primary 62C10.

1 Introduction

In this note, we highlight and provide corrections to two errors that inadvertently occurred in the paper: *Karthik Sriram, R.V. Ramamoorthi, Pulak Ghosh (2013) "Posterior Consistency of Bayesian Quantile Regression Based on the Misspecified Asymmetric Laplace Density", Bayesian Analysis, Vol 8, Num 2, pg 479–504.*

1. First error is in the proof of Lemma 4 (pg 492), where we stated and used an inequality viz, $\forall t < 1$, $e^t < 1/(1-t)$. We note that this is not true for $t < 0$. We acknowledge and thank Michael Guggisberg, a PhD candidate at University of California, Irvine for bringing this to our attention. In this note, we restate and provide an alternative argument for Lemma 4. With this, the rest of the arguments in the paper continue to hold with only minor modifications.

2. We also realized that there is a typo in the first inequality of page 498:
$$E\left\{(\Pi(W_{1n}\cap G_1|Y_1,\dots,Y_n))^d\right\} \le \frac{C'}{(n\Delta_n^2)^{2+2d}}e^{-\frac{dL\cdot n\Delta_n^2}{4}}.$$
The right hand side should be $\frac{C'}{(\Delta_n^2)^{2+2d}}e^{-\frac{dL\cdot n\Delta_n^2}{4}}$, i.e. without the "$n$" in the denominator term. With this correction, the arguments for proving Theorem 1 as well as Theorem 2 part (a) still hold good. However, we note that the argument for Theorem 2 part (b) (when $\Delta_n = M_n n^{-1/2}$) does not go through for any general $M_n \to \infty$, but holds when $M_n^2 > C'' \log(n)$ for a sufficiently large C'' (to be precise $C'' > \frac{8(1+d)}{dL}$).

*Main article DOI: 10.1214/13-BA817.
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 DOI: 10.1214/17-BA1067

For easy reference, we have created a version of the paper that incorporates these corrections. This can be accessed at the link https://goo.gl/KLz9gV or by contacting the first author.[1]

2 Correction to Lemma 4

Here, we restate and present the corrected proof of Lemma 4. Using notations in the paper, let $T_i = \log \frac{f_{(i,\alpha,\beta,1)}(Y_i)}{f_{(i,\alpha_0,\beta_0,1)}(Y_i)}$, $Z_i = Y_i - \alpha_0 - \beta_0 X_i$ and $b_i = (\alpha + \beta X_i) - (\alpha_0 + \beta_0 X_i)$. We recall Lemma 1a of the paper:

$$T_i = \begin{cases} -b_i(1-\tau), & \text{if } Y_i \le \min(\alpha+\beta X_i, \alpha_0+\beta_0 X_i) \\ (Y_i - \alpha_0 - \beta_0 X_i) - b_i(1-\tau), & \text{if } \alpha_0+\beta_0 X_i < Y_i \le \alpha+\beta X_i \\ b_i\tau - (Y_i - \alpha_0 - \beta_0 X_i), & \text{if } \alpha+\beta X_i < Y_i \le \alpha_0+\beta_0 X_i \\ b_i\tau, & \text{if } Y_i \ge \max(\alpha+\beta X_i, \alpha_0+\beta_0 X_i). \end{cases}$$

We will assume $(\alpha,\beta) \in G \cap W_{1n}$, where G is compact and $W_{1n} = \{(\alpha,\beta) : \alpha - \alpha_0 \ge \Delta_n, \beta \ge \beta_0\}$. We recall that Δ_n is a constant while proving just consistency and $\Delta_n \to 0$ while considering rates. Our Lemma 4 can be restated as follows:

Lemma 4. *Let $G \subseteq \Theta$ be compact and assumption* **2** *hold. Let $\epsilon_0 > 0$ be as in assumption 3(i) and $C > 0$ be as in assumption 3(ii). Then $\exists\ 0 < d < 1$ such that for $K = \frac{C\tau(1-\tau)}{2} > 0$ and $\forall(\alpha,\beta) \in G \cap W_{1n}$,*

$$E\left[e^{dT_i}\right] \le e^{-dK\Delta_n^2 I_{X_i > \epsilon_0}}.$$

Proof. We will assume $b_i \ge 0$ as the argument is similar when $b_i < 0$. We note by Lemma 1a that when $\alpha_0 + \beta_0 X_i < Y_i \le \alpha + \beta X_i$,

$$T_i = (Y_i - \alpha_0 - \beta_0 X_i) - b_i(1-\tau) = Y_i - q_i$$
$$\text{where } q_i = (\alpha_0 + \beta_0 X_i)\tau + (\alpha + \beta X_i)(1-\tau).$$
$$\text{So, } (Y_i - q_i) \le \begin{cases} 0, & \text{if } \alpha_0 + \beta_0 X_i < Y_i \le q_i \\ (\alpha + \beta X_i - q_i) = b_i\tau, & \text{if } q_i < Y_i < \alpha + \beta X_i. \end{cases}$$

This observation along with Lemma 1a, implies

$$T_i \le -b_i(1-\tau) \times I_{Y_i \le \alpha_0 + \beta_0 X_i} + 0 \times I_{\alpha_0+\beta_0 X_i < Y_i \le q_i} + b_i\tau \times I_{Y_i > q_i}. \tag{2.1}$$

Denoting $\tau_i^* = P(Y_i \le q_i)$ and recalling that $\tau = P(Y_i \le \alpha_0 + \beta_0 X_i)$,

$$E\left[e^{dT_i}\right] \le \tau e^{-db_i(1-\tau)} + (\tau_i^* - \tau) + e^{db_i\tau}(1 - \tau_i^*). \tag{2.2}$$

Let $g_i(t) = e^{-tb_i(1-\tau)}\tau + (\tau_i^* - \tau) + e^{tb_i\tau}(1-\tau_i^*)$. By Taylor's formula,

$$g_i(t) = 1 + g_i'(0)t + g_i''(\xi)t^2/2, \quad \text{for some } 0 < \xi < t. \tag{2.3}$$

[1] Also accessible from the first author's webpage https://www.iima.ac.in/web/faculty/faculty-profiles/karthik-sriram under the link to "Publications".

In equation (2.3), we first note that $g_i'(0) = -b_i\tau(\tau_i^* - \tau)$. Suppose, C, Δ_0 be as in Assumption 3(ii), i.e. $P(0 < Y_i - \alpha_0 - \beta_0 X_i < \Delta) > C\Delta\ \forall\Delta \le \Delta_0$. Defining $b_i^* = \min(b_i, \Delta_0)$ and noting that $q_i - \alpha_0 - \beta_0 X_i = b_i(1-\tau)$, we have

$$\begin{aligned}\tau_i^* - \tau &= P(\alpha_0 + \beta_0 X_i < Y_i \le q_i) = P(0 < Z_i \le b_i(1-\tau))\\ &\ge P(0 < Z_i \le b_i^*(1-\tau)) > C\ b_i^*\ (1-\tau).\end{aligned}$$

$$\text{Hence } g_i'(0) \le -C\tau(1-\tau){b_i^*}^2. \tag{2.4}$$

Further, we note $g_i''(t) = b_i^2 \times (\tau(1-\tau)^2 e^{-tb_i(1-\tau)} + \tau^2(1-\tau_i^*)e^{tb_i\tau})$. Since G is compact and hence b_i is uniformly bounded, say $b_i \le M_1\ \forall i$, the term within the parenthesis in the above expression can be bounded by some constant $K_1 > 0$. Further, note by definition that $b_i^* = b_i I_{b_i \le \Delta_0} + \Delta_0 I_{b_i > \Delta_0}$. Therefore, if we choose $K_2 > 1$, such that $K_2\Delta_0 > M_1$, then we would have $K_2 b_i^* = (K_2 b_i I_{b_i \le \Delta_0} + K_2\Delta_0 I_{b_i > \Delta_0}) \ge b_i$. In other words, $\exists\ K_2$ such that $b_i \le K_2 b_i^*$ or $b_i^2 \le K_2^2 {b^*}_i^2$. Therefore, by taking $2K_3 = K_1 \cdot K_2^2$, we get

$$g_i''(t) \le 2K_3 \cdot {b_i^*}^2, \quad \forall 0 \le t \le 1. \tag{2.5}$$

Equations (2.3), (2.4) and (2.5) together give

$$g_i(t) \le 1 - {b_i^*}^2 \cdot t \cdot (C\tau(1-\tau) - K_3 t)\,.$$

Let $t_0 < \min(\frac{1}{2}, \frac{1}{2}\frac{C\tau(1-\tau)}{K_3})$ and $K = \frac{C\tau(1-\tau)}{2}$ then $\forall t < t_0$ we have,

$$g_i(t) \le 1 - tK{b_i^*}^2 \le e^{-tK{b_i^*}^2}. \tag{2.6}$$

We have $b_i^* \ge 0$, $\forall i$. Further, when $X_i > \epsilon_0$ and $(\alpha, \beta) \in W_{1n}$, we have $b_i \ge \Delta_n$. So, if we assume without loss of generality that $\Delta_0 > \Delta_n$, then $b_i^* \ge \Delta_n I_{(X_i > \epsilon_0)}$, $\forall i$. It follows therefore that for $(\alpha, \beta) \in W_{1n} \cap G$,

$$\forall d < t_0,\ E\left[e^{dT_i}\right] \le e^{-dK{b_i^*}^2} \le e^{-dK\Delta_n^2 I_{(X_i > \epsilon_0)}}. \qquad \square$$

Acknowledgments

We thank Michael Guggisberg, a PhD candidate at University of California, Irvine for bringing one of the erros to our attention.