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ASYMPTOTIC EXPANSION OF THE INVARIANT MEASURE FOR BALLISTIC RANDOM WALK IN THE LOW DISORDER REGIME[1]

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We consider a random walk in random environment in the low disorder regime on $\mathbb{Z}^d$, that is, the probability that the random walk jumps from a site x to a nearest neighboring site $x+e$ is given by $p(e)+\varepsilon\xi(x,e)$, where $p(e)$ is deterministic, $\{\{\xi(x,e):|e|_1=1\}:x\in\mathbb{Z}^d\}$ are i.i.d. and $\varepsilon>0$ is a parameter, which is eventually chosen small enough. We establish an asymptotic expansion in ε for the invariant measure of the environmental process whenever a ballisticity condition is satisfied. As an application of our expansion, we derive a numerical expression up to first order in ε for the invariant measure of random perturbations of the simple symmetric random walk in dimensions $d=2$.

1. Introduction. We derive an asymptotic expansion for the invariant measure of the environmental process of random walks moving on $\mathbb{Z}^d$ in the low disorder regime within the spirit of previous expansions of Sabot [15] for the velocity. Our result is one of the few instances where explicit quantitative information about the invariant measure of the environmental process is given for random walks in random environments in dimensions $d\geq 2$ with nonvanishing velocity.

For $x\in\mathbb{R}^d$, we denote by $|x|_1$ and $|x|_2$ its l^1 and l^2 norms, respectively. Let $V:=\{e\in\mathbb{Z}^d:|e|_1=1\}$ and $\mathcal{P}:=\{p_e:e\in V\}$ where $p_e\geq 0$ and $\sum_{e\in V}p_e=1$. We define $\Omega:=\mathcal{P}^{\mathbb{Z}^d}$ endowed with its Borel σ-algebra and denote any $\omega=\{\omega(x):x\in\mathbb{Z}^d\}\in\Omega$ where for each $x\in\mathbb{Z}^d$ we let $\omega(x)=\{\omega(x,e):e\in V\}\in\mathcal{P}$, an *environment*. We now define the *random walk in the environment* ω as the Markov chain $\{X_n:n\geq 0\}$ with state space $\mathbb{Z}^d$ defined by the transition probabilities

$$P(X_{n+1}=y+e|X_n=y)=\omega(y,e),$$

for all $e\in V$, $y\in\mathbb{Z}^d$ and $n\geq 0$. For each $x\in\mathbb{Z}^d$, we denote by $P_{x,\omega}$ its law if the random walk starts from x. Throughout, we will assume that the space of environments Ω is endowed with a probability measure $\mathbb{P}$. We will call $P_{x,\omega}$ the *quenched*

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law of the random walk, while $P_x := \int P_{x,\omega}\, d\mathbb{P}$ the *averaged* or *annealed* law of the random walk. We will suppose that $\{\omega(x) : x \in \mathbb{Z}^d\}$ are i.i.d. under $\mathbb{P}$. The law $\mathbb{P}$ is said to be uniformly elliptic if there exists a $\kappa > 0$, which we will call the ellipticity constant, such that for all $x \in \mathbb{Z}^d$ and $e \in V$,

$$\mathbb{P}\big(\omega(x,e) \geq \kappa\big) = 1.$$

Define $\mathcal{P}_0 := \{p \in \mathcal{P} : \min_{e\in V} p(e) > 0\}$. Consider a transition kernel $p_0 = \{p_0(e) : e \in V\} \in \mathcal{P}_0$. For our main result, we will consider laws $\mathbb{P}$, which are perturbations of a simple random walk which jumps according to the transition kernel p_0. To be precise, for each $\varepsilon > 0$ define

$$\Omega_{p_0,\varepsilon} := \big\{\omega \in \Omega : \big|\omega(x,e) - p_0(e)\big| \leq \varepsilon \text{ for all } x \in \mathbb{Z}^d, e \in V\big\}. \tag{1.1}$$

We will consider laws $\mathbb{P}$ which are concentrated on $\Omega_{p_0,\varepsilon}$ for some $\varepsilon > 0$. Let us note that for $\varepsilon < \min_{e\in V} p_0(e)$, each probability measure concentrated on $\Omega_{p_0,\varepsilon}$ is uniformly elliptic with ellipticity constant

$$\kappa = \min_{e\in V} p_0(e) - \varepsilon. \tag{1.2}$$

Throughout this article, κ will be given by (1.2). Recall the definition of the local drift

$$d(x,\omega) := \sum_{e\in V} \omega(x,e)e,$$

for $x \in \mathbb{Z}^d$. For $\omega \in \Omega$, define the canonical shifts $\{\theta_x : x \in \mathbb{Z}^d\}$ as $\theta_x\omega(y) := \omega(x+y)$ for all $y \in \mathbb{Z}^d$. Finally, define the *environmental process* $\{\bar{\omega}_n : n \geq 0\}$ starting from $\bar{\omega}_0 = \omega$ as

$$\bar{\omega}_n := \theta_{X_n}\omega.$$

The transition kernel of this process is defined as the map R from the set of functions $f : \Omega \to \mathbb{R}$ to itself given by

$$Rf(\omega) := \sum_{e\in V} \omega(0,e) f(\theta_e\omega). \tag{1.3}$$

To state the main result of this article, let us define for each $\omega \in \Omega$, $x \in \mathbb{Z}^d$ and $e \in V$,

$$\xi(x,e) := \frac{1}{\varepsilon}\big(\omega(x,e) - p_0(e)\big), \tag{1.4}$$

so that

$$\omega(x,e) = p_0(e) + \varepsilon\xi(x,e), \tag{1.5}$$

and

$$\bar{\xi}(x,e) := \xi(x,e) - \mathbb{E}\big[\xi(x,e)\big],$$

where the notation $\mathbb{E}$ denotes taking expectation with respect to the measure $\mathbb{P}$. Define also

$$p_\varepsilon(e) := p_0(e) + \varepsilon \mathbb{E}\big[\xi(0,e)\big]. \tag{1.6}$$

Furthermore, define for $n \geq 0$ and $x, y \in \mathbb{Z}^d$, $p_n(x,y)$ as the probability that a random walk with transition kernel $p \in \mathcal{P}$ jumps from x at time 0 to site y at time n, and the function:

$$J_{p^*}(x) := \lim_{n\to\infty} \sum_{k=0}^{n} \big(p_k(0,-x) - p_k(0,0)\big), \tag{1.7}$$

where here for $p \in \mathcal{P}$, the subscript p^* in the above expression is the transition kernel $p^* \in \mathcal{P}$ defined by

$$p^*(e) := p(-e) \qquad \text{for } e \in V. \tag{1.8}$$

Note that for each p which defines a transient random walk, the above expression can be written as a difference of a Green function evaluated at different points. On the other hand, in the two-dimensional recurrent case, (1.7) is equal to the negative of the potential kernel of a random walk with transition kernel p^*.

In the main result of this article, we establish an asymptotic expansion for the invariant measure of random walks in environments whose law is supported in $\Omega_{p_0,\varepsilon}$ for a given $p_0 \in \mathcal{P}_0$ and ε small enough. To formulate it, we will assume the following condition on the local drift. For later convenience, we will denote the elements of V also as $e_1, \dots, e_d, -e_1, \dots, -e_d$, where for each $1 \leq i \leq d$, e_i is the vector which has all coordinates equal to 0 except for the ith coordinate which is equal to 1. Given $p_0 \in \mathcal{P}_0$, $C > 0$ and $\varepsilon > 0$ we will say that a probability measure $\mathbb{P}$ defined on Ω satisfies the linear local drift condition $(\mathrm{LLD})_\varepsilon$ with constant C if $\mathbb{P}(\Omega_{p_0,\varepsilon}) = 1$ and

$$\mathbb{E}\big[d(0,\omega)\big] \cdot e_1 \geq C\varepsilon. \tag{1.9}$$

As shown in Lemma 2 and pages 3010 and 3011 of [15], whenever the linear local drift condition $(\mathrm{LLD})_\varepsilon$ is satisfied for p_0 fixed and for ε small enough with constant C (not depending on ε) the random walk satisfies Kalikow's condition (see [19] or [15] for its definition), and hence by Theorem 3.1 of Sznitman and Zerner [19], the environmental process has a marginal law at fixed time which converges in distribution to an invariant measure. We will call this invariant measure, the *limiting invariant measure* of the environmental process. Given a measure μ defined on Ω and a subset $B \subset \mathbb{Z}^d$, we will call the marginal law of μ in $\mathcal{P}^B$ *the restriction of μ to B*.

THEOREM 1. *Let $\eta > 0$, $C > 0$ and B be a finite subset of $\mathbb{Z}^d$. Assume that $p_0 \in \mathcal{P}_0$ and that there is an ε_1 such that for all $\varepsilon \leq \varepsilon_1$, $\mathbb{P}$ satisfies the linear local drift condition* $(\mathrm{LLD})_\varepsilon$ [*cf.* (1.9)] *with constant C. Then there is an $\varepsilon_0 > 0$ such* *that whenever $\varepsilon \le \varepsilon_0$, the limiting invariant measure $\mathbb{Q}$ has a restriction $\mathbb{Q}_B$ to B which is absolutely continuous with respect to the restriction $\mathbb{P}_B$ to B of $\mathbb{P}$, with a Radon–Nikodym derivative admitting $\mathbb{P}$-a.s. the expansion*

$$\frac{d\mathbb{Q}_B}{d\mathbb{P}_B} = 1 + \varepsilon \sum_{z\in B}\sum_{e\in V} \bar{\xi}(z,e) J_{p_\varepsilon^*}(e+z) + O(\varepsilon^{2-\eta}), \tag{1.10}$$

where $|O(\varepsilon^{2-\eta})| \le c_1\varepsilon^{2-\eta}$, for some constant $c_1 = c_1(\eta,\kappa,d,B)$ depending only on η,κ,d and B.

Since explicit formulas for the Radon–Nykodym derivative are available in the case $d = 1$ (see, e.g., [7]), Theorem 1 in that case is not particularly interesting. Nevertheless, it is worth noting and it will be shown in Section 2.1 that when $d = 1$, the expansion (1.10) has an error $O(\varepsilon^{2-\eta})$, which is bounded by $c_1\varepsilon^{2-\eta}$ where c_1 does not depend on B, so that actually the expansion is valid for the full Radon–Nikodym derivative $d\mathbb{Q}/d\mathbb{P}$ with $B = \mathbb{Z}$ in (1.10). This is in accordance with the known fact that under the linear local drift condition (LLD) in dimension $d = 1$, $\mathbb{Q}$ is absolutely continuous with respect to $\mathbb{P}$ [7]. A key point for the sum in (1.10) of Theorem 1 to converge on $\mathbb{Z}$ in dimension $d = 1$, is the appearance of $J_{p_\varepsilon^*}$ instead of $J_{p_0^*}$.

On the other hand, in certain situations, we can expand $J_{p_\varepsilon^*}$. This is the content of the following corollary, where we define

$$d_0 := \sum_{e\in V} e p_0(e). \tag{1.11}$$

COROLLARY 2. *Let $d \ge 2$ or $d = 1$ with $d_0 \ne 0$. Let $\eta > 0$, $C > 0$ and B be a finite subset of $\mathbb{Z}^d$. Assume that $p_0 \in \mathcal{P}_0$ and that there is an ε_1 such that for all $\varepsilon \le \varepsilon_1$, $\mathbb{P}$ satisfies the linear local drift condition* (LLD)$_\varepsilon$ [*cf.* (1.9)] *with constant C. Then there is an $\varepsilon_0 > 0$ such that whenever $\varepsilon \le \varepsilon_0$, the limiting invariant measure $\mathbb{Q}$ has a restriction $\mathbb{Q}_B$ to B which is absolutely continuous with respect to the restriction $\mathbb{P}_B$ to B of $\mathbb{P}$, with a Radon–Nikodym derivative admitting $\mathbb{P}$-a.s. the expansion*

$$\frac{d\mathbb{Q}_B}{d\mathbb{P}_B} = 1 + \varepsilon \sum_{z\in B}\sum_{e\in V} \bar{\xi}(z,e) J_{p_0^*}(e+z) + O(\varepsilon^{2-\eta}), \tag{1.12}$$

where $|O(\varepsilon^{2-\eta})| \le c_2\varepsilon^{2-\eta}$, for some constant $c_2 = c_2(\eta,\kappa,d,B)$ depending only on η,κ,d and B.

From the point of view of its explicitness, a startling consequence of Corollary 2 is stated in Corollary 3 of Section 3, where due to the fact that the potential kernel of a simple symmetric random walk in dimension $d = 2$ can be recursively computed, we can obtain a numerical expression up to first order for the limiting invariant measure. Furthermore, the Radon–Nikodym derivative (1.10) plays an important role in local limit theorems (see, e.g., Theorem 1.11 of [2] valid for $d \geq 4$).

On the other hand, by the fact that the marginal law of the environmental process converges to the limiting invariant measure, and the fact that

$$X_n - \sum_{i=0}^{n-1} d(0, \bar{\omega}_i) n \geq 0,$$

is a P_0-martingale, we can recover through Theorem 1 Sabot's expansion for the limiting velocity [15], defined as the P_0-a.s. limit

$$v := \lim_{n\to\infty} \frac{X_n}{n},$$

under the linear local drift condition $(\mathrm{LLD})_\varepsilon$ for ε small enough with constant C (not depending on ε), so that

$$v = \int d(0, \omega)\, d\mathbb{Q} = d_0 + \varepsilon d_1 + \varepsilon^2 d_2^\varepsilon + O(\varepsilon^{3-\eta}), \tag{1.13}$$

where d_0 is defined in (1.11), $d_1 := \sum_{e\in V} e\mathbb{E}[\xi(0, e)]$ and

$$d_2^\varepsilon := \sum_{e\in V} \sum_{e'\in V} e C_{e,e'} J_{p_\varepsilon^*}(e'),$$

where $C_{e,e'} := \mathrm{Cov}(\xi(0, e), \xi(0, e'))$.

The absolute continuity of the invariant measure $\mathbb{Q}$ of Theorem 1 with respect to the law of the environment restricted to finite sets follows from the proof of Theorem 3.1 of Sznitman and Zerner in [19]. In dimensions $d \geq 4$, since Kalikow's condition is satisfied, by a result of Berger, Cohen and Rosenthal [2] (see also a previous result of Bolthausen and Sznitman [4] valid at low disorder), we also know that $\mathbb{Q}$ is absolutely continuous with respect to $\mathbb{P}$, and in dimensions $d \geq 2$, by [14], we know that it is absolutely continuous with respect to $\mathbb{P}$ in every forward half space perpendicular to e_1. In dimensions $d = 2$ and $d = 3$, it is still an open question wether or not $\mathbb{Q}$ is absolutely continuous with respect to $\mathbb{P}$. On the other hand, with the methods presented in this article, we are not able to derive an expansion of $d\mathbb{Q}/d\mathbb{P}$ on the whole lattice in dimensions $d \geq 4$, because the error term we obtain in (1.10) of Theorem 1 tends to infinity as the cardinality of B grows to infinity. An open problem is to settle wether or not under the linear local drift condition (LLD), at least in dimensions $d \geq 4$, the expansion (1.10) is still valid for $B = \mathbb{Z}^d$.

Besides the work of Sabot [15] for the expansion of the velocity, random perturbations of random walks have been also considered in [8] (see also [12] for perturbations of diffusions in random environment), for perturbations leading to the Einstein relation and in [1, 5, 6, 18] for perturbations of the simple symmetric random walk. The Green function expansion methods of [15] are an important tool used in the proof of Theorem 1. Nevertheless, they have to be adapted to the context of this article, and most importantly, in [15] somehow the use of Kalikow's property for random walks in random environment conceals the fact that underlying the velocity expansion there is really an expansion of the invariant measure. In fact, a key step in the proof of Theorem 1, is to obtain an explicit expression of the invariant measure in terms of the Green function of the random walk. In [8], it is shown that the perturbations over a random environment which satisfies Sznitman's ballisiticity condition (T') [17], produce an invariant measure for the environmental process which is up to first order equal to the unperturbed invariant measure. Nevertheless, although an expression for the error is given in terms of regeneration times, it is not explicit. Furthermore, in [18] it is proven that for perturbations of the simple symmetric random walk, under a weaker condition than the linear local drift condition (LLD), where in the right-hand side of (1.9) $C = 1$, ε is replaced by $\varepsilon^{\alpha(d)}$, with $\alpha(3) = 2.5 - \eta$ and $\alpha(d) = 3 - \eta$ for $d \geq 4$ with $\eta > 0$ arbitrarily small, the perturbed random walk is ballistic. Nevertheless, we do not know if it would be possible to extend Theorem 1 and the expansion of [15] under this weaker assumption in dimensions $d \geq 3$. On the other hand, we would like to emphasize that Theorem 1 is one of the first results for the model of random walks in a random environment in the ballistic regime, and hence nonreversible, giving explicit quantitative information about the invariant measure of the environmental process.

Standard analytic perturbative expansions of invariant measures usually require the existence of a spectral gap for the unperturbed generator of the corresponding process [10], which is not the case in the framework of Theorem 1. The proof of Theorem 1 is based on careful expansions of the Green function of the random walk within the spirit of [15]. Nevertheless, a key ingredient that we have to incorporate here is to obtain an expression for the limiting invariant measure in terms of accumulation points of a Cesàro-type average performed at a stopping time with a geometric distribution. This is the content of Proposition 5 in Section 4. Furthermore, it is necessary to obtain careful expansions of Green functions of random walks perturbed at multiple points.

In the next section of this article, we will derive a more explicit version of Corollary 2 for the case of perturbations of the simple symmetric random walk in dimension $d = 2$. Then, in Section 3, we will give a heuristic explanation of the expansions (1.10) and (1.12) of Theorem 1 and Corollary 2. In Section 4, we will derive an expression for the limiting invariant measure in terms of Cesàro averages of the marginal laws of the environmental process up to a stopping time with a geometric distribution. In Section 5, we will show how to perturb at a finite number of sites the Green function of the random walk. The results of Section 5 will be used to expand a typical term of the expression giving the limiting measure in Proposition 8 of Section 6. Using this expansion, Theorem 1 is proved in Section 7. Finally, Corollary 2 will be proved in Section 8.

2. Some explicit computations.

2.1. *Dimension* $d=1$. Even though in the case $d=1$ the Radon–Nikodym derivative $d\mathbb{Q}/d\mathbb{P}$ can be computed explicitly (see, e.g., [7]), for completeness and as a way to asses the insight given by Theorem 1 and Corollary 2 in higher dimensions, we show to what does the expansion (1.10) of Theorem 1 reduce in that case.

Whenever the linear local drift condition $(\mathrm{LLD})_\varepsilon$ is satisfied for ε small enough with a fixed constant C (not depending on ε), the invariant measure $\mathbb{Q}$ of the environmental process has the explicit Radon–Nikodym derivative

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \frac{1-\mathbb{E}[\rho_0]}{1+\mathbb{E}[\rho_0]}(1+\rho_0)\sum_{j=0}^{\infty}\prod_{k=0}^{j-1}\rho_{k+1}, \tag{2.1}$$

where $\rho_j := \frac{\omega(k,-1)}{\omega(k,1)}$ for $j \in \mathbb{Z}$. Furthermore, we have

$$J_{p_\varepsilon^*}(x) = \begin{cases} 0 & \text{for } x<0, \\ \dfrac{1}{d_0+\varepsilon d_1}\rho_\varepsilon^x - \dfrac{1}{d_0+\varepsilon d_1} & \text{for } x \geq 0, \end{cases} \tag{2.2}$$

where

$$\rho_\varepsilon := \frac{1-p_\varepsilon(1)}{p_\varepsilon(1)}.$$

It folllows from (1.10) of Theorem 1 that for each finite $B \subset \mathbb{Z}$, with $B^+ := \{x \in B : x \geq 0\}$,

$$\frac{d\mathbb{Q}_B}{d\mathbb{P}_B} = 1 + \varepsilon\frac{1}{d_0+\varepsilon d_1}\sum_{z\in B^+}\sum_{e\in V}\bar{\xi}(z,e)\rho_\varepsilon^{z+e} + O(\varepsilon^{2-\eta}), \tag{2.3}$$

which can also be obtained expanding (2.1) in $\varepsilon\bar{\xi}$. Of course, (2.3) is valid even for $B=\mathbb{Z}$.

2.2. *Random perturbations of the simple symmetric random walk in* $d=2$. Here, we will derive explicit numerical expressions for some marginal laws of the invariant measure which will be consequences of Theorem 1 and Corollary 2, for the case in which the perturbations are done on a simple symmetric random walk, so that $p_0(e) = \frac{1}{2d}$ for all $e \in V$. To simplify notation, we will drop the subindex from J_{p_0} writing instead just $J := J_{p_0}$ for this choice of p_0.

First, let us note that when $d=2$, the explicit values

$$J(z) = \begin{cases} 0 & \text{for } z=(0,0), \\ -1 & \text{for } z=(0,\pm1), (\pm1,0), \\ -\dfrac{4}{\pi} & \text{for } z=(1,\pm1), (\pm1,1), \\ \dfrac{8}{\pi}-4 & \text{for } z=(0,\pm2), (\pm2,0), \end{cases}$$

can be recursively derived (see McCrea and Whipple [13] or Spitzer [16]). We hence obtain the following corollary from Corollary 2. Here, we define $z_0 := (0,0)$ and $z_1 := (0,1)$.

COROLLARY 3. *Let p_0 be the jump probabilities of a simple symmetric random walk, $\eta > 0$, $C > 0$ and $d = 2$. Assume that there is an ε_1 such that for all $\varepsilon \le \varepsilon_1$, $\mathbb{P}$ satisfies the linear local drift condition* $(\mathrm{LLD})_\varepsilon$ [*cf.* (1.9)] *with constant C. Then there is an $\varepsilon_0 > 0$ such that whenever $\varepsilon \le \varepsilon_0$, the Radon–Nikodym derivative of the restriction $\mathbb{Q}_{z_0,z_1}$ to $\{z_0, z_1\}$ of the limiting invariant measure $\mathbb{Q}$ with respect to the restriction $\mathbb{P}_{z_0,z_1}$ of $\mathbb{P}$ to $\{z_0, z_1\}$, admits $\mathbb{P}$-a.s. the following expansion*:

$$\frac{d\mathbb{Q}_{z_0,z_1}}{d\mathbb{P}_{z_0,z_1}} = 1 - \frac{4}{\pi}(\bar{\xi}(z_1, e_1) + \bar{\xi}(z_1, -e_1))\varepsilon + \left(\frac{8}{\pi} - 4\right)\bar{\xi}(z_1, e_2)\varepsilon + O(\varepsilon^{2-\eta}).$$

In particular, we have that $\mathbb{P}$-a.s.

$$\frac{d\mathbb{Q}_{z_0}}{d\mathbb{P}_{z_0}} = 1 + O(\varepsilon^{2-\eta}),$$

$$\frac{d\mathbb{Q}_{z_1}}{d\mathbb{P}_{z_1}} = 1 - \left(\frac{4}{\pi}(\bar{\xi}(z_1, e_1) + \bar{\xi}(z_1, -e_1)) + \left(\frac{8}{\pi} - 4\right)\bar{\xi}(z_1, e_2)\right)\varepsilon + O(\varepsilon^{2-\eta}).$$

Here, $|O(\varepsilon^{2-\eta})| \le c_2'\varepsilon^{2-\eta}$, for some constant $c_2' = c_2'(\eta)$ depending only on η.

Similar estimates can be obtained for the marginal law of the limiting invariant measure $\mathbb{Q}$ restricted to other finite subsets of $\mathbb{Z}^2$ using the recursive method presented in [13] (see also [16]) to compute J.

3. Formal derivation of the invariant measure perturbative expansion. Here, we will show how to formally derive the expansion (1.10) of Theorem 1. Given any $p \in \mathcal{P}$, defining a nonvanishing drift $\sum_{e \in V} ep(e) \neq 0$, we define for each $x, y \in \mathbb{Z}^d$ the Green function $g^p(x, y)$ as the expectation of the number of visits to site y of the random walk starting from site x.

Consider a perturbation of an environment p_0 according to (1.5). Let us write the transition kernel of the environmental process [cf. (1.3)] as

$$R = R_0 + \varepsilon A, \tag{3.1}$$

where R_0 is the transition kernel of the deterministic environment p_ε [cf. (1.6)], so that for $f : \Omega \to \mathbb{R}$ we have

$$R_0 f(\omega) := \sum_{e \in V} p_\varepsilon(e) f(\theta_e \omega),$$

and

$$A f(\omega) := \sum_{e \in V} \bar{\xi}(0, e) f(\theta_e \omega).$$

The invariant measure $\mathbb{Q}$ satisfies the equality

$$\int (R - I) f \, d\mathbb{Q} = 0, \tag{3.2}$$

for every continuous function $f : \Omega \to \mathbb{R}$, where I is the identity operator. Now assume that the Radon–Nikodym derivative $h := \frac{d\mathbb{Q}}{d\mathbb{P}}$ exists and that it has an analytic expansion:

$$h = \sum_{i=0}^{\infty} \varepsilon^i h_i.$$

Note that since for $\varepsilon = 0$ the measure $\mathbb{P}$ is the invariant measure for the environmental process with transition kernel R_0, we should have that $h_0 = 1$. Substituting this expansion and R [cf. (3.1)] into (3.2), and matching powers, we conclude that for each $i \geq 0$,

$$h_{i+1} = -\left(R_0^* - I\right)^{-1} A^* h_i, \tag{3.3}$$

where for any linear operator L, L^* denotes its adjoint with respect to the measure $\mathbb{P}$. Now, note that at least for functions f for which the summation $\sum_{z \in \mathbb{Z}^d} g^{p_\varepsilon}(z, 0) f(\theta_z \omega)$ is finite, we have

$$\begin{aligned} \left(R_0^* - I\right)^{-1} f(\omega) &= -\sum_{z \in \mathbb{Z}^d} g^{p_\varepsilon^*}(0, z) f(\theta_z \omega) \\ &= -\sum_{z \in \mathbb{Z}^d} g^{p_\varepsilon}(z, 0) f(\theta_z \omega), \end{aligned}$$

where we recall that $p_\varepsilon^*(e) := p_\varepsilon(-e)$ for $e \in V$ [cf. (1.8)]. Furthermore,

$$A^* f(\omega) := \sum_{e \in V} \bar{\xi}(-e, e) f(\theta_e \omega).$$

From the recursion (3.3), and the fact that $\sum_{e \in V} \bar{\xi}(z, e) = 0$, it follows that

$$\begin{aligned} h_1(\omega) &= \sum_{z \in \mathbb{Z}^d, e \in V} g^{p_\varepsilon^*}(0, z) \bar{\xi}(z - e, e) \\ &= \sum_{z \in \mathbb{Z}^d, e \in V} \bar{\xi}(z, e) g^{p_\varepsilon^*}(0, z + e) \\ &= \sum_{z \in \mathbb{Z}^d, e \in V} \bar{\xi}(z, e) J_{p_\varepsilon^*}(z + e), \end{aligned}$$

which is the factor of the first-order term in the expansion (1.10).

4. Invariant measure as a geometric Cesàro limit. In analogy with the fact that limit points of Cesàro averages of the environmental process give rise to invariant measures (see, e.g., [7]), here we will show that when such an average is done according to a geometric stopping time, its limit points are still invariant measures.

For each $\delta \in (0, 1)$, let us consider the Green function of the random walk before a stopping time τ_δ with geometric distribution of parameter $1 - \delta$, independent of the random walk and of the environment, defined for a given environment $\omega \in \Omega$ and sites $x, y \in \mathbb{Z}^d$ as

$$g_\delta^\omega(x, y) := E'_{x,\omega}\left[\sum_{n=0}^{\tau_\delta - 1} 1_y(X_n)\right],$$

where the expectation $E'_{x,\omega}$ is taken both over the random walk and over the random variable τ_δ. Define now the probability measure μ_δ on Ω as the unique probability measure such that for every continuous function $f : \Omega \to \mathbb{R}$,

$$\int f\, d\mu_\delta = \frac{\sum_{x\in\mathbb{Z}^d} \mathbb{E}[g_\delta^\omega(0, x) f(\theta_x\omega)]}{\sum_{x\in\mathbb{Z}^d} \mathbb{E}[g_\delta^\omega(0, x)]}. \tag{4.1}$$

For the following proposition, we do not require the environment of the random walk to be elliptic, nor any other assumption on the environment.

PROPOSITION 4. *Consider a random walk in random environment. Then each accumulation point of the set of measures* $\{\mu_\delta : \delta > 0\}$ [*cf.* (4.1)] *as* δ *tends to* 1 *from below, is an invariant measure of the environmental process.*

PROOF. Note that for each $\delta > 0$, as in the proof of Proposition 2 of Sabot [15], one can prove that for every continuous function $f : \Omega \to \mathbb{R}$ the following identity is satisfied:

$$\frac{E'_0[\sum_{k=0}^{\tau_\delta-1} f(\theta_{X_k}\omega)]}{E[\tau_\delta]} = \frac{\sum_{y\in\mathbb{Z}^d} \mathbb{E}[g_\delta^\omega(0, y) f(\theta_y\omega)]}{\sum_{y\in\mathbb{Z}^d} \mathbb{E}[g_\delta^\omega(0, y)]}, \tag{4.2}$$

where the notation E'_0 denotes taking the expectation with respect to the annealed law of the random walk and with respect to τ_δ, while E means the expectation with respect to τ_δ. Let μ be an accumulation point of $\{\mu_\delta : \delta > 0\}$ as $\delta \to 1$. Then there exists a sequence $\{\delta_k : k \geq 1\}$ such that $\lim_{k\to\infty} \delta_k = 1$ and such that $\lim_{k\to\infty} \mu_k := \lim_{k\to\infty} \mu_{\delta_k} = \mu$ weakly. Using the Markov property of the quenched random walk, one can deduce that, for all natural m,

$$E_{0,\omega}\big[Rf(\theta_{X_m}\omega)\big] = R^{m+1} f(\omega),$$

where R is the transition kernel defined in (1.3). Hence, by (4.2) and the definition (1.3) we see that

$$\begin{aligned}
\int Rf\,d\mu_k &= \frac{E_0'[\sum_{m=0}^{\tau_{\delta_k}-1} Rf(\theta_{X_m}\omega)]}{E[\tau_{\delta_k}]} \\
&= \frac{\mathbb{E}E[\sum_{m=0}^{\tau_{\delta_k}-1} R^{m+1}f(\omega)]}{E[\tau_{\delta_k}]} \\
&= \frac{\mathbb{E}E[\sum_{m=0}^{\tau_{\delta_k}-1} R^{m}f(\omega)]}{E[\tau_{\delta_k}]} + \frac{1}{E[\tau_{\delta_k}]}\cdot \mathbb{E}E\bigl[R^{\tau_{\delta_k}}f\bigr] - \frac{1}{E[\tau_{\delta_k}]}\cdot \mathbb{E}[f] \\
&= \int f\,d\mu_k + \frac{1}{E[\tau_{\delta_k}]}\cdot \mathbb{E}E\bigl[R^{\tau_{\delta_k}}f\bigr] - \frac{1}{E[\tau_{\delta_k}]}\cdot \mathbb{E}[f].
\end{aligned}$$

Taking the limit when $k \to \infty$ and using the fact that the last two terms tend to zero as $k \to \infty$ by the boundedness of f and the fact that $\lim_{k\to\infty} E[\tau_{\delta_k}] = \infty$, we conclude that

$$\int Rf\,d\mu = \int f\,d\mu. \qquad \square$$

To state the next proposition, we recall the definition of the polynomial ballisticity condition, introduced in [3]. We say that the *polynomial ballisticity condition* $(P)_M$ in direction l is satisfied (see [3]) if for all $L \ge c_0$, where

$$c_0 = 2^{3(d-1)} \wedge \exp\left\{2\left(\ln 90 + \sum_{j=1}^{\infty} \frac{\ln j}{2^j}\right)\right\},$$

it is true that

$$P_0(X_{T_{B_L}} \cdot l < L) \le \frac{1}{L^M},$$

where

$$B_L := \left\{x \in \mathbb{Z}^d : -\frac{L}{2} \le x \cdot l \le L, |\pi_l x|_\infty \le 25L^3\right\},$$

and $\pi_l x$ is the orthogonal projection of x on the subspace perpendicular to l.

PROPOSITION 5. *Consider a random walk in a uniformly elliptic random environment satisfying the polynomial condition* $(P)_M$ *for* $M \ge 15d + 5$. *Then* $\mu := \lim_{\delta\to 1^-} \mu_\delta$ *exists and is an invariant measure for the environmental process. Furthermore, the law of the environmental process at time* n, *converges in distribution to* μ *as* $n \to \infty$.

PROOF. Let us first note that by Theorem 1 of [3], the polynomial condition $(P)_M$ with $M \geq 15d + 5$ implies condition (T') introduced by Sznitman in [17]. On the other hand, Theorem 3.1 of [19], which is formulated under the assumption that Kalikow's condition is satisfied, is still valid if Kalikow's condition is replaced by condition (T'). Therefore, since $\lim_{\delta \to 1} \tau_\delta = \infty$ in probability, we know by Theorem 3.1 of [19] that there exists an invariant measure μ of the environmental process such that in probability

$$\lim_{\delta \to 1^-} \frac{1}{\tau_\delta} E_0\left[\sum_{m=0}^{\tau_\delta - 1} f(\theta_{X_m}\omega)\right] = \int f\, d\mu.$$

Since $\tau_\delta / E[\tau_\delta]$ converges in distribution to an exponential random variable S of parameter 1, it follows that in distribution

$$\lim_{\delta \to 1^-} \frac{1}{E[\tau_\delta]} E_0\left[\sum_{m=0}^{\tau_\delta - 1} f(\theta_{X_m}\omega)\right] = S \int f\, d\mu.$$

Hence,

$$\lim_{\delta \to 1^-} \frac{1}{E[\tau_\delta]} E_0'\left[\sum_{m=0}^{\tau_\delta - 1} f(\theta_{X_m}\omega)\right] = \int f\, d\mu,$$

which proves the claim. □

5. Green function expansion. To prove Theorem 1, we will extend the method presented by Sabot in [15], starting with perturbative estimates for the Green function of the random walk. To do this, we need first the following lemma, which we will use several times. We recall the definition of the ellipticity constant given in (1.2).

LEMMA 6. *For each $\delta \in (0, 1)$, $e \in V$, $y, z \in \mathbb{Z}^d$, with $y \neq z$ and $\omega \in \Omega$, we have that*

$$g_\delta^\omega(y, z) \geq \delta\kappa g_\delta^\omega(y, z + e). \tag{5.1}$$

PROOF. It is enough to note that for all $y, z \in \mathbb{Z}^d$ one has that

$$g_\delta^\omega(y, z) = \delta_{y,z} + \delta \sum_{e \in V} g_\delta^\omega(y, z + e)\omega(z + e, -e),$$

and then use the fact that the environment is uniformly elliptic with ellipticity constant κ. □

The main result of this section is the following lemma which extends Lemma 1 of [15] for perturbations at one site of the Green function, to perturbation at multiple sites.

LEMMA 7. *Consider an environment $\omega \in \Omega$. For $B \subset \mathbb{Z}^d$ consider an environment ω^B which is a perturbation of ω in each of the points in B. In particular, we have for each $e \in V$ that*

$$\omega^B(x,e) := \begin{cases} \omega(x,e) & \text{if } x \notin B, \\ \omega(x,e) + \Delta_x\omega(e) & \text{if } x \in B, \end{cases}$$

for some $\{\Delta_x\omega(e) : e \in V\} \in (-1,1)^V$ for each $x \in B$. Assume that there is a constant $\kappa > 0$ such that $\omega(x,e) \geq \kappa$ and $\omega_B(x,e) \geq \kappa$ for all $x \in \mathbb{Z}^d$ and $e \in U$. Then, for each $\delta \in (0,1)$ and $y, y' \in \mathbb{Z}^d$,

$$\left|g_\delta^{\omega^B}(y,y') - g_\delta^\omega(y,y')\right| \leq nc_3 g_\delta^{\omega^B}(y,y') \tag{5.2}$$

and

$$\begin{aligned} &\Bigl| g_\delta^{\omega^B}(y,y') - g_\delta^\omega(y,y') \\ &\quad - \sum_{x\in B} g_\delta^\omega(y,x) \sum_{e\in V} \Delta_x\omega(e)\bigl[\delta g_\delta^\omega(x+e,y') - g_\delta^\omega(x,x)\bigr]\Bigr| \\ &\quad \leq n^2 c_3^2(1+nc_3) g_\delta^{\omega^B}(y,y'), \end{aligned} \tag{5.3}$$

where

$$c_3 := \frac{2d \sup_{e\in V, x\in B} |\Delta_x\omega(e)|}{\kappa^2} \tag{5.4}$$

and n is the cardinality of B.

PROOF. Using a standard resolvent expansion, (see (9) and (10) of [15] with $n = 0$), we can obtain the following first-order expansion of $g_\delta^{\omega^B}$:

$$\begin{aligned} &g_\delta^{\omega^B}(y,y') - g_\delta^\omega(y,y') \\ &\quad = \delta \sum_{x\in B} g_\delta^\omega(y,x) \sum_{e\in V} \Delta_x\omega(e) g_\delta^{\omega^B}(x+e,y'). \end{aligned}$$

Now, following the proof of Lemma 1 of Sabot [15], it is easy to deduce (5.2). Meanwhile, in order to prove (5.3), we will use (5.2) and the following inequality, which is valid for any environment ω:

$$\left|\delta g_\delta^\omega(z+e,y') - g_\delta^\omega(z,y')\right| \leq \frac{1}{\kappa^2} \frac{g_\delta^\omega(z,y')}{g_\delta^\omega(z,z)}, \qquad \forall z, y' \in \mathbb{Z}^d, e \in V \tag{5.5}$$

(for more details, see Lemma 1 of [15]). Now, expanding again the Green function through the resolvent (see (9) of [15] with $n = 1$), we obtain the second-order expansion

$$
\begin{aligned}
&g_\delta^{\omega^B}(y,y') - g_\delta^\omega(y,y') \\
&\quad - \sum_{x\in B} g_\delta^\omega(y,x) \sum_{e\in V} \Delta_x\omega(e)\big[\delta g_\delta^\omega(x+e,y') - g_\delta^\omega(x,x)\big] \\
&\quad = \sum_{x,z\in B} \sum_{e,e'\in V} g_\delta^\omega(y,x)\Delta_x\omega(e)\big(\delta g_\delta^\omega(x+e,z) - g_\delta^\omega(x,z)\big) \\
&\qquad \times \Delta_z\omega(e')\big(\delta g_\delta^{\omega^B}(z+e',y') - g_\delta^{\omega^B}(z,y')\big),
\end{aligned} \tag{5.6}
$$

where we have used the fact $\sum_{e\in V}\Delta_x\omega(e) = 0$ twice. Hence, with the help of (5.5) and (5.6), we can see that

$$
\begin{aligned}
&\bigg| g_\delta^{\omega^B}(y,y') - g_\delta^\omega(y,y') \\
&\quad - \sum_{x\in B} g_\delta^\omega(y,x) \sum_{e\in V} \Delta_x\omega(e)\big(\delta g_\delta^\omega(x+e,y') - g_\delta^\omega(x,x)\big)\bigg| \\
&\quad \le A \sum_{z\in B}\bigg(\sum_{x\in B} \frac{g_\delta^\omega(y,x) g_\delta^\omega(x,z)}{g_\delta^\omega(x,x)}\bigg) \times \frac{g_\delta^{\omega^B}(z,y')}{g_\delta^{\omega^B}(z,z)} \\
&\quad \le An\bigg(\sum_{z\in B} \frac{g_\delta^\omega(y,z) g_\delta^{\omega^B}(z,y')}{g_\delta^{\omega^B}(z,z)}\bigg),
\end{aligned} \tag{5.7}
$$

where we define

$$
A := \frac{1}{\kappa^4}\Big(2d \sup_{e\in V, z\in B} \big|\Delta_z\omega(e)\big|\Big)^2,
$$

and where in the last step, we have used the fact that for each environment ω and each $y, z \in \mathbb{Z}^d$,

$$
\sum_{x\in B} \frac{g_\delta^\omega(y,x) g_\delta^\omega(x,z)}{g_\delta^\omega(x,x)} \le n g_\delta^\omega(y,z). \tag{5.8}
$$

Now, with the help of (5.2), for each $z \in B$ one can deduce that

$$
\frac{g_\delta^\omega(y,z)}{g_\delta^{\omega^B}(y,z)} \le 1 + nc_3, \tag{5.9}
$$

where c_3 is defined in (5.4). Thus, with the help of (5.8), we can substitute (5.9) into (5.7) to conclude that

$$
\begin{aligned}
&\bigg| g_\delta^{\omega^B}(y,y') - g_\delta^\omega(y,y') \\
&\quad - \sum_{x\in B} g_\delta^\omega(y,x) \sum_{e\in V} \Delta_x\omega(e)\big[\delta g_\delta^\omega(x+e,y') - g_\delta^\omega(x,x)\big]\bigg|
\end{aligned}
$$

$$\leq An(1+nc_3)\sum_{z\in B}\frac{g_\delta^{\omega^B}(y,z)g_\delta^{\omega^B}(z,y')}{g_\delta^{\omega^B}(z,z)}$$
$$\leq n^2c_3^2(1+nc_3)g_\delta^{\omega^B}(y,y'). \qquad \square$$

6. Local function expansions. In this section, we will derive in Proposition 8 which follows, an asymptotic expansion in the perturbation parameter ε for certain expectations of a given local function f, involving the Green function of the random walk. In fact, these expectations are with respect to the so called Kalikow environment [9]. Throughout, we fix a transition kernel $p_0 \in \mathcal{P}_0$.

PROPOSITION 8. *Let $C > 0$. Assume that there is an $\varepsilon_1 > 0$ such that for all $\varepsilon \leq \varepsilon_1$, $\mathbb{P}$ satisfies the linear local drift condition* (LLD)$_\varepsilon$ [*cf.* (1.9)] *with constant C. Let A be a finite fixed subset of $\mathbb{Z}^d$. Consider a continuous function f defined on $\Omega_{p_0,\varepsilon}$, which depends only on sites located at A. Let $\eta > 0$. Then there exists an $\varepsilon_0 > 0$, and a constant $c_4 = c_4(\eta, d, \kappa, A)$, such that for all $0 < \varepsilon \leq \varepsilon_0$, whenever δ is close enough to* 1, *there is a function h_δ such that for each $y \in \mathbb{Z}^d$ the following identity is satisfied*:

$$\begin{aligned}
&\frac{\mathbb{E}[g_\delta^\omega(0,y)f(\theta_y\omega)]}{\mathbb{E}[g_\delta^\omega(0,y)]}\\
&\qquad = \mathbb{E}[f] + \varepsilon\frac{1}{\mathbb{E}[g_\delta^\omega(0,y)]}\\
&\qquad\quad \times \sum_{z\in A}\sum_{e\in V}\operatorname{Cov}[\xi(z,e),f]J_{p_\varepsilon^*}(z+e)\mathbb{E}[g_\delta^\omega(0,z+y)]\\
&\qquad\quad + \mathbb{E}[h_\delta f]\times O(\varepsilon^{2-\eta}),
\end{aligned} \tag{6.1}$$

where h_δ satisfies

$$\bigl|\mathbb{E}[|h_\delta|f]\bigr| \leq \mathbb{E}[|f|], \tag{6.2}$$

$|O(\varepsilon^{2-\eta})| \leq c_4\varepsilon^{2-\eta}$ *and $J_{p_\varepsilon^*}(x)$ is defined in* (1.7).

Let us now prove Proposition 8. For each subset $B \subset \mathbb{Z}^d$, we will define the following perturbation of a given environment $\omega \in \Omega$,

$$\omega^B(x,e) := \begin{cases} \omega(x,e) & \text{if } x \notin B,\\ p_\varepsilon(e) & \text{if } x \in B.\end{cases}$$

Note that trivially we can get

$$\frac{\mathbb{E}[g_\delta^\omega(0,y)f(\theta_y\omega)]}{\mathbb{E}[g_\delta^\omega(0,y)]} = \frac{\mathbb{E}[g_\delta^\omega(0,y)\bar{f}(\theta_y\omega)]}{\mathbb{E}[g_\delta^\omega(0,y)]} + \mathbb{E}[f], \tag{6.3}$$

where $\bar{f} = f - \mathbb{E}[f]$. Next, using the independence between $g_\delta^{\omega^{A+y}}$ and $f \circ \theta_y$ and the fact that $\bar{f}$ is a centered random variable, we can see that

$$\frac{\mathbb{E}[g_\delta^\omega(0,y)\bar{f}(\theta_y\omega)]}{\mathbb{E}[g_\delta^\omega(0,y)]} = \frac{\mathbb{E}[(g_\delta^\omega(0,y) - g_\delta^{\omega^{A+y}}(0,y))\bar{f}(\theta_y\omega)]}{\mathbb{E}[g_\delta^\omega(0,y)]},$$

where $A + y = \{a + y : a \in A\}$. Thus, using inequality (5.3) of Lemma 7, we can deduce that

$$\begin{aligned}
&\frac{\mathbb{E}[(g_\delta^\omega(0,y) - g_\delta^{\omega^{A+y}}(0,y))\bar{f}(\theta_y\omega)]}{\mathbb{E}[g_\delta^\omega(0,y)]} \\
&\qquad = \biggl(\varepsilon\mathbb{E}\biggl[\sum_{z\in A+y}\sum_{e\in V} g_\delta^{\omega^{A+y}}(0,z)\bar{\xi}(z,e)\bigl(\delta g_\delta^{\omega^{A+y}}(z+e,y) \\
&\qquad\qquad - g_\delta^{\omega^{A+y}}(z,z)\bigr)\bar{f}(\theta_y\omega)\biggr]\biggr)\Big/\bigl(\mathbb{E}[g_\delta^\omega(0,y)]\bigr) \\
&\qquad\qquad + \frac{\mathbb{E}[g_\delta^\omega(0,y)\bar{f}(\theta_y\omega) \times O_1(\varepsilon^2)]}{\mathbb{E}[g_\delta^\omega(0,y)]},
\end{aligned} \tag{6.4}$$

where $O_1(\varepsilon)$ satisfies the inequality

$$|O_1(\varepsilon^2)| \le \frac{16d^2n^2}{\kappa^4}(1 + c_5 n)\varepsilon^2, \tag{6.5}$$

n is the cardinality of A and here c_5 is defined by [see (1.1), (1.4) and (5.4)]

$$c_5 = c_5(d,\kappa,A) := \frac{4d\varepsilon}{\kappa^2}.$$

Using now the independence between $\bar{\xi}(z,e)$ for $z \in A + y$ and the Green function $g_\delta^{\omega^{A+y}}$, we can see by (6.4) that

$$\begin{aligned}
&\frac{\mathbb{E}[(g_\delta^\omega(0,y) - g_\delta^{\omega^{A+y}}(0,y))\bar{f}(\theta_y\omega)]}{\mathbb{E}[g_\delta^\omega(0,y)]} \\
&\qquad = \varepsilon\sum_{\substack{z\in A\\ e\in V}} \mathrm{Cov}\bigl[\xi(z,e), f(\omega)\bigr] \\
&\qquad\qquad \times \frac{\mathbb{E}[g_\delta^{\omega^{A+y}}(0,z+y)(\delta g_\delta^{\omega^{A+y}}(z+y+e,y) - g_\delta^{\omega^{A+y}}(z+y,z+y))]}{\mathbb{E}[g_\delta^\omega(0,y)]} \\
&\qquad\qquad + \frac{\mathbb{E}[g_\delta^\omega(0,y)\bar{f}(\theta_y\omega) \times O_1(\varepsilon^2)]}{\mathbb{E}[g_\delta^\omega(0,y)]}.
\end{aligned} \tag{6.6}$$

In addition, with the help of (5.2) of Lemma 7, due to the development of (6.6) we can conclude that

$$
\begin{aligned}
&\frac{\mathbb{E}[(g_\delta^\omega(0,y)-g_\delta^{\omega^{A+y}}(0,y))\bar{f}(\theta_y\omega)]}{\mathbb{E}[g_\delta^\omega(0,y)]}\\
&\quad=\varepsilon\sum_{\substack{z\in A\\ e\in V}}\operatorname{Cov}[\xi(z,e),f(\omega)]\\
&\qquad\times\frac{\mathbb{E}[g_\delta^{\omega^{A+y}}(0,z+y)(\delta g_\delta^{\omega^{A+y}}(z+y+e,y)-g_\delta^{\omega^{A+y}}(z+y,z+y))]}{\mathbb{E}[g_\delta^{\omega^{A+y}}(0,y)]}\\
&\qquad+\sum_{\substack{z\in A\\ e\in V}}\operatorname{Cov}[\xi(z,e),f(\omega)]\\
&\qquad\times\frac{\mathbb{E}[g_\delta^{\omega^{A+y}}(0,z+y)(\delta g_\delta^{\omega^{A+y}}(z+y+e,y)-g_\delta^{\omega^{A+y}}(z+y,z+y))]}{\mathbb{E}[g_\delta^{\omega^{A+y}}(0,y)]}\\
&\qquad\times\frac{\mathbb{E}[g_\delta^\omega(0,y)O_2(\varepsilon^2)]}{\mathbb{E}[g_\delta^\omega(0,y)]}+\frac{\mathbb{E}[g_\delta^\omega(0,y)\bar{f}(\theta_y\omega)O_1(\varepsilon^2)]}{\mathbb{E}[g_\delta^\omega(0,y)]},
\end{aligned}
\tag{6.7}
$$

where

$$
|O_2(\varepsilon^2)|\le\frac{4dn}{\kappa^2}\varepsilon^2. \tag{6.8}
$$

Now, to express the second term of (6.7) in terms of $J_{p_\varepsilon^*}$ [cf. (1.7)], we will require a lemma which is a variation of Lemma 3 of [15]. For $v\in\mathbb{Z}^d$, define

$$
\phi^\varepsilon(v):=\prod_{i=1}^d\left(\sqrt{\frac{p_\varepsilon(-e_i)}{p_\varepsilon(e_i)}}\right)^{v_i}, \tag{6.9}
$$

where v_i are the coordinates of v. Also, for each $z\in A$, $e\in V$ and $y\in\mathbb{Z}^d$, define

$$
\begin{aligned}
&J_e^\delta(y,z)\\
&\quad:=\frac{\mathbb{E}[g_\delta^{\omega^{A+y}}(0,z+y)(\delta g_\delta^{\omega^{A+y}}(z+y+e,y)-g_\delta^{\omega^{A+y}}(z+y,z+y))]}{\mathbb{E}[g_\delta^{\omega^{A+y}}(0,z+y)]}.
\end{aligned}
\tag{6.10}
$$

LEMMA 9. *Let $C>0$, $\eta>0$ and $p_0\in\mathcal{P}_0$. Assume that there is an ε_1 such that for all $\varepsilon\le\varepsilon_1$, $\mathbb{P}$ satisfies* (LLD)$_\varepsilon$ *with constant C. Then there exists an $\varepsilon_0>0$ and a constant $c_6=c_6(\eta)>0$ such that for each $\varepsilon\le\varepsilon_0$ we have that for all $z\in A$, $e\in V$ and $y\in\mathbb{Z}^d$ one has that*

$$
\varlimsup_{\delta\to1}|J_e^\delta(y,z)-J_{p_\varepsilon^*}(z+e)|\le c_6\phi^\varepsilon(z+e)\varepsilon^{1-\eta}. \tag{6.11}
$$

PROOF. We will just give an outline of the proof, stressing the steps where modifications have to be made with respect to the proof of Lemma 3 of [15]. For each $z \in A$, $y \in \mathbb{Z}^d$, we define

$$\tilde{\mathbb{P}} := \frac{g_\delta^{\omega^A}(0, y+z)}{\mathbb{E}[g_\delta^{\omega^A}(0, y+z)]}\mathbb{P}.$$

Now, using a generalized version of a result of Kalikow [9], stated in Proposition 1 of [15], we can see that

$$J_e^\delta(y,z) = \delta g_\delta^{\tilde{\omega}}(z+y+e, y) - g_\delta^{\tilde{\omega}}(z+y, z+y),$$

where $g_\delta^{\tilde{\omega}}$ denotes the Green function of Kalikow random walk, defined by its transition probabilities $\tilde{\omega}(x,e)$ given by

$$\tilde{\omega}(x,e) := \frac{\tilde{\mathbb{E}}[g_\delta^{\omega^A}(y+z, x)\omega^A(x,e)]}{\tilde{\mathbb{E}}[g_\delta^{\omega^A}(y+z, x)]},$$

for each $x \in \mathbb{Z}^d$ and $e \in V$. Here, $\tilde{\mathbb{E}}$ is the expectation with respect to $\tilde{\mathbb{P}}$. It is easy to verify that

$$\tilde{\omega}(x,e) = \begin{cases} \mathbb{E}\big[\omega(x,e)\big] + \varepsilon \dfrac{\tilde{\mathbb{E}}[g_\delta^{\omega^A}(y+z,x)\bar{\xi}(x,e)]}{\tilde{\mathbb{E}}[g_\delta^{\omega^A}(y+z,x)]} & \text{if } x \notin A, \\ \mathbb{E}\big[\omega(x,e)\big] & \text{if } x \in A. \end{cases} \tag{6.12}$$

Using twice (5.9), we can deduce that

$$\begin{aligned} \frac{\tilde{\mathbb{E}}[g_\delta^{\omega^A}(y+z,x)\bar{\xi}(x,e)]}{\tilde{\mathbb{E}}[g_\delta^{\omega^A}(y+z,x)]} &= \frac{\tilde{\mathbb{E}}[g_\delta^{\omega^{A\cup\{x\}}}(y+z,x)\bar{\xi}(x,e)]}{\tilde{\mathbb{E}}[g_\delta^{\omega^{A\cup\{x\}}}(y+z,x)]} + O(\varepsilon) \\ &= \frac{\mathbb{E}[g_\delta^{\omega^A}(0,y+z)g_\delta^{\omega^{A\cup\{x\}}}(y+z,x)\bar{\xi}(x,e)]}{\mathbb{E}[g_\delta^{\omega^A}(0,y+z)g_\delta^{\omega^{A\cup\{x\}}}(y+z,x)]} + O(\varepsilon) \\ &= O(\varepsilon), \end{aligned} \tag{6.13}$$

where in the last step, we used the independence between $g_\delta^{A\cup\{x\}}$ and $\bar{\xi}(x,e)$, and the fact that $|O(\varepsilon)| \le c_7\varepsilon$, where c_7 is a constant, which depends on κ, d and cardinality of A [see (5.4)]. Now, from (6.12) and (6.13), we can deduce that for each $x \in \mathbb{Z}^d$ and $e \in V$ the following identity is satisfied:

$$\tilde{\omega}(x,e) = \mathbb{E}\big[\omega(x,e)\big] + \varepsilon^2 \Delta_x \omega(e),$$

where $\Delta_x\omega(x)$ is uniformly bounded in $x, e, y, z, \delta, \varepsilon$. The following steps of the proof are then identical to steps 2 and 3 of Lemma 3 of [15]. $\square$

We can now continue with the proof of Proposition 8. Note that using the definition (6.10), we can rewrite (6.7) as

$$
\begin{aligned}
&\frac{\mathbb{E}[(g_\delta^\omega(0,y)-g_\delta^{\omega^{A+y}}(0,y))\bar{f}(\theta_y\omega)]}{\mathbb{E}[g_\delta^\omega(0,y)]}\\
&\qquad = \varepsilon \sum_{\substack{z\in A\\ e\in V}} \operatorname{Cov}[\xi(z,e), f(\omega)] J_e^\delta(y,z) \times \frac{\mathbb{E}[g_\delta^{\omega^{A+y}}(0,z+y)]}{\mathbb{E}[g_\delta^{\omega^{A+y}}(0,y)]}\\
&\qquad\quad + \sum_{\substack{z\in A\\ e\in V}} \operatorname{Cov}[\xi(z,e), f(\omega)] J_e^\delta(y,z) \times \frac{\mathbb{E}[g_\delta^{\omega^{A+y}}(0,z+y)]}{\mathbb{E}[g_\delta^{\omega^{A+y}}(0,y)]}\\
&\qquad\quad \times \frac{\mathbb{E}[g_\delta^\omega(0,y)O_2(\varepsilon^2)]}{\mathbb{E}[g_\delta^\omega(0,y)]}\\
&\qquad\quad + \frac{\mathbb{E}[g_\delta^\omega(0,y)\bar{f}(\theta_y\omega)O_1(\varepsilon^2)]}{\mathbb{E}[g_\delta^\omega(0,y)]}.
\end{aligned}
\tag{6.14}
$$

On the other hand, with the help of (6.11) of Lemma 9, it follows that for each $\eta > 0$ we can choose δ_0 such that for $\delta \in (\delta_0, 1)$

$$
\left|J_e^\delta(y,z) - J_{p_\varepsilon^*}(z+e)\right| \le 2c_6\phi^\varepsilon(z+e)\varepsilon^{1-\eta}. \tag{6.15}
$$

Hence, for $\delta \in (\delta_0, 1)$ we conclude from (6.14) that

$$
\begin{aligned}
&\frac{\mathbb{E}[(g_\delta^\omega(0,y)-g_\delta^{\omega^{A+y}}(0,y))\bar{f}(\theta_y\omega)]}{\mathbb{E}[g_\delta^\omega(0,y)]}\\
&\qquad = \varepsilon \sum_{z\in A}\sum_{e\in V} \operatorname{Cov}[\xi(z,e), f(\omega)] J_{p_\varepsilon^*}(z+e) \times \frac{\mathbb{E}[g_\delta^{\omega^{A+y}}(0,z+y)]}{\mathbb{E}[g_\delta^{\omega^{A+y}}(0,y)]}\\
&\qquad\quad + O_3(\varepsilon^{2-\eta}) \sum_{z\in A}\sum_{e\in V} \operatorname{Cov}[\xi(z,e), f(\omega)]\\
&\qquad\quad + \sum_{z\in A}\sum_{e\in V} \operatorname{Cov}[\xi(z,e), f(\omega)] J_{p_\varepsilon^*}(z+e) \frac{\mathbb{E}[g_\delta^\omega(0,y)O_4(\varepsilon^2)]}{\mathbb{E}[g_\delta^\omega(0,y)]}\\
&\qquad\quad + \sum_{z\in A}\sum_{e\in V} \operatorname{Cov}[\xi(z,e), f(\omega)] \frac{\mathbb{E}[g_\delta^\omega(0,y)O_5(\varepsilon^{3-\eta})]}{\mathbb{E}[g_\delta^\omega(0,y)]}\\
&\qquad\quad + \frac{\mathbb{E}[g_\delta^\omega(0,y)\bar{f}(\theta_y\omega)O_1(\varepsilon^2)]}{\mathbb{E}[g_\delta^\omega(0,y)]},
\end{aligned}
\tag{6.16}
$$

where we have used the fact that the expression $\frac{\mathbb{E}[g_\delta^{\omega^{A+y}}(0,z+y)]}{\mathbb{E}[g_\delta^{\omega^{A+y}}(0,y)]}$ can be bounded by $\frac{1}{(\delta\kappa)^{\rho(A)}}$ choosing a nonrandom nearest neighbor self-avoiding path from y to $z+y$

and using (5.1) of Lemma 6 at most $\rho(A)$ times, and where

$$|O_3(\varepsilon^{2-\eta})| \le \frac{2c_6c_8\varepsilon^{2-\eta}}{(\delta\kappa)^{\rho(A)}}, \qquad |O_4(\varepsilon^2)| \le \frac{4dn}{\kappa^2(\delta\kappa)^{\rho(A)}}\varepsilon^2, \tag{6.17}$$

$$|O_5(\varepsilon^{3-\eta})| \le 2\frac{4dn}{\kappa^2(\delta\kappa)^{\rho(A)}}c_6c_8\varepsilon^{3-\eta} \tag{6.18}$$

and

$$c_8 := \sup_{z\in A, e\in V}|\phi^\varepsilon(z+e)|.$$

In addition, if we use again (5.2) of Lemma 7, we can say that

$$\frac{\mathbb{E}[g_\delta^{\omega^{A+y}}(0,z+y)]}{\mathbb{E}[g_\delta^\omega(0,z+y)]} = 1 + \frac{\mathbb{E}[g_\delta^\omega(0,z+y)O_6(\varepsilon)]}{\mathbb{E}[g_\delta^\omega(0,z+y)]}, \tag{6.19}$$

and

$$\frac{\mathbb{E}[g_\delta^\omega(0,y)]}{\mathbb{E}[g_\delta^{\omega^{A+y}}(0,y)]} = \frac{\mathbb{E}[g_\delta^\omega(0,y)]}{\mathbb{E}[g_\delta^\omega(0,y)(1+O_7(\varepsilon))]}, \tag{6.20}$$

provided that for each $\tau > 0$ one has that

$$|O_i(\tau)| \le nc_3\tau, \qquad \forall i = 6, 7. \tag{6.21}$$

Using (6.19) and (6.20) for the first term of the right-hand side of (6.16) and using once more (6.15), we see that for $\delta \ge \delta_0$ one has that

$$\begin{aligned}
&\frac{\mathbb{E}[(g_\delta^\omega(0,y) - g_\delta^{\omega^{A+y}}(0,y))\bar{f}(\theta_y\omega)]}{\mathbb{E}[g_\delta^\omega(0,y)]} \\
&\quad = \varepsilon\frac{1}{\mathbb{E}[g_\delta^\omega(0,y)]}\sum_{\substack{z\in A\\ e\in V}}\mathrm{Cov}[\xi(z,e), f(\omega)]J_{p_\varepsilon^*}(z+e)\mathbb{E}[g_\delta^\omega(0,z+y)] \\
&\qquad + \sum_{\substack{z\in A\\ e\in V}}\mathrm{Cov}[\xi(z,e), f(\omega)]J_{p_\varepsilon^*}(z+e)\frac{\mathbb{E}[O_8(\varepsilon^2)g_\delta^\omega(0,z+y)]}{\mathbb{E}[g_\delta^\omega(0,z+y)]} \\
&\qquad - \sum_{\substack{z\in A\\ e\in V}}\mathrm{Cov}[\xi(z,e), f(\omega)]J_{p_\varepsilon^*}(z+e)\frac{\mathbb{E}[O_9(\varepsilon^2)g_\delta^\omega(0,y)]}{\mathbb{E}[g_\delta^\omega(0,y)(1+O_7(\varepsilon))]} \\
&\qquad - \sum_{\substack{z\in A\\ e\in V}}\mathrm{Cov}[\xi(z,e), f(\omega)]J_{p_\varepsilon^*}(z+e)\frac{\mathbb{E}[O_8(\varepsilon^2)g_\delta^\omega(0,z+y)]}{\mathbb{E}[g_\delta^\omega(0,z+y)]} \\
&\qquad \times \frac{\mathbb{E}[O_9(\varepsilon)g_\delta^\omega(0,y)]}{\mathbb{E}[g_\delta^\omega(0,y)(1+O_7(\varepsilon))]}
\end{aligned} \tag{6.22}$$

$$+ O_3(\varepsilon^{2-\eta}) \sum_{z\in A} \sum_{e\in V} \mathrm{Cov}[\xi(z,e), f(\omega)]$$

$$+ \sum_{\substack{z\in A\\ e\in V}} \mathrm{Cov}[\xi(z,e), f(\omega)] J_{p_\varepsilon^*}(z+e) \frac{\mathbb{E}[g_\delta^\omega(0,y) O_4(\varepsilon^2)]}{\mathbb{E}[g_\delta^\omega(0,y)]}$$

$$+ \sum_{\substack{z\in A\\ e\in V}} \mathrm{Cov}[\xi(z,e), f(\omega)] \frac{\mathbb{E}[g_\delta^\omega(0,y) O_5(\varepsilon^{3-\eta})]}{\mathbb{E}[g_\delta^\omega(0,y)]}$$

$$+ \frac{\mathbb{E}[g_\delta^\omega(0,y) \bar{f}(\theta_y\omega) O_1(\varepsilon^2)]}{\mathbb{E}[g_\delta^\omega(0,y)]},$$

where, by (6.21), we know that for each $\tau > 0$

$$|O_i(\tau)| \le \frac{nc_3}{(\delta\kappa)^{\rho(A)}}\tau \qquad \forall i = 8, 9. \tag{6.23}$$

Defining

$$h_1 := \sum_{\substack{z\in A\\ e\in V}} \bar{\xi}(z,e) J_{p_\varepsilon^*}(z+e) \times \frac{\mathbb{E}[O_8(\varepsilon^2) g_\delta^\omega(0,z+y)]}{E[g_\delta^\omega(0,z+y)]},$$

$$h_2 := -\sum_{\substack{z\in A\\ e\in V}} \bar{\xi}(z,e) J_{p_\varepsilon^*}(z+e) \times \frac{\mathbb{E}[O_9(\varepsilon^2) g_\delta^\omega(0,y)]}{E[g_\delta^\omega(0,y)(1+O_7(\varepsilon))]},$$

$$h_3 := -\sum_{\substack{z\in A\\ e\in V}} \bar{\xi}(z,e) J_{p_\varepsilon^*}(z+e) \times \frac{\mathbb{E}[O_8(\varepsilon^2) g_\delta^\omega(0,z+y)]}{E[g_\delta^\omega(0,z+y)]} \times \frac{\mathbb{E}[O_9(\varepsilon) g_\delta^\omega(0,y)]}{E[g_\delta^\omega(0,y)(1+O_7(\varepsilon))]},$$

$$h_4 := \sum_{\substack{z\in A\\ e\in V}} \bar{\xi}(z,e) O_3(\varepsilon^{2-\eta}),$$

$$h_5 := \sum_{\substack{z\in A\\ e\in V}} \bar{\xi}(z,e) J_{p_\varepsilon^*}(z+e) \times \frac{\mathbb{E}[O_4(\varepsilon^2) g_\delta^\omega(0,y)]}{E[g_\delta^\omega(0,y)]},$$

$$h_6 := \sum_{\substack{z\in A\\ e\in V}} \bar{\xi}(z,e) \times \frac{\mathbb{E}[O_5(\varepsilon^{3-\eta}) g_\delta^\omega(0,y)]}{E[g_\delta^\omega(0,y)]}$$

$$h_7 := \frac{g_\delta^\omega(-y,0)\tilde{O}_1(\varepsilon^2)}{\mathbb{E}[g_\delta^\omega(0,y)]} \qquad \text{with } \tilde{O}_1(\varepsilon^2)(\omega) := O_1(\varepsilon^2)(\theta_{(-y)}\omega),$$

$$\text{and} \quad h_8 := -\frac{\mathbb{E}[g_\delta^\omega(0, y) O_1(\varepsilon^2)]}{\mathbb{E}[g_\delta^\omega(0, y)]},$$

we can rewrite (6.22) in the following way:

$$\begin{aligned}
&\frac{\mathbb{E}[(g_\delta^\omega(0, y) - g_\delta^{\omega^{A+y}}(0, y))\bar{f}(\theta_y\omega)]}{\mathbb{E}[g_\delta^\omega(0, y)]} \\
&\quad = \varepsilon \frac{1}{\mathbb{E}[g_\delta^\omega(0, y)]} \sum_{\substack{z\in A\\ e\in V}} \operatorname{Cov}[\xi(z, e), f(\omega)] J_{p_\varepsilon^*}(z+e) \mathbb{E}[g_\delta^\omega(0, z+y)] \qquad (6.24)\\
&\qquad + E[h_\delta f],
\end{aligned}$$

where $h_\delta := \sum_{i=1}^8 h_i$. Now, in order to show that (6.2) is satisfied, it is necessary to justify that for any $z \in A$ and $e \in V$, $J_{p_\varepsilon^*}(z+e)$ is bounded. If we use (6.5), (6.8), (6.17), (6.18), (6.21), (6.23) and the fact that for each $e \in V$ and $z \in A$, $\bar{\xi}(z, e)$ is bounded by 2, it is easy to deduce that there exists $\varepsilon_0 > 0$ and an constant $c_9 = c_9(\eta)$ such that for all $0 < \varepsilon \le \varepsilon_0$, whenever δ is close enough to 1, the following inequality is satisfied for all $1 \le i \le 8$:

$$\bigl|\mathbb{E}\bigl[|h_i| f\bigr]\bigr| \le \bigl|O\bigl(\varepsilon^{2-\eta}\bigr)\bigr| \mathbb{E}\bigl[|f|\bigr] \qquad \text{with } \bigl|O\bigl(\varepsilon^{2-\eta}\bigr)\bigr| \le c_9 \varepsilon^{2-\eta}. \qquad (6.25)$$

In the case of h_7, if we apply independence and use (5.4), (5.9) and (6.5), one can deduce that for $0 < \varepsilon \le \varepsilon_0$ exists a constant $c_{10} > 0$ such that

$$\begin{aligned}
\bigl|\mathbb{E}\bigl[|h_7| f\bigr]\bigr| &\le \frac{1}{\mathbb{E}[g_\delta^\omega(0, y)]} \\
&\qquad \times \mathbb{E}\biggl[g_\delta^{\omega^{A+y}}(0, y) \cdot \bigl|f(\theta_y\omega)\bigr| \cdot \frac{g_\delta^\omega(0, y)}{g_\delta^{\omega^{A+y}}(0, y)} \bigl|O_1\bigl(\varepsilon^2\bigr)\bigr|\biggr] \qquad (6.26)\\
&\le \bigl|O\bigl(\varepsilon^2\bigr)\bigr| \mathbb{E}\bigl[|f|\bigr],
\end{aligned}$$

where $|O(\varepsilon^2)| \le c_{10}\varepsilon^2$. Finally, with the help of (6.25) and (6.26), Proposition 8 is easily proven.

7. Proof of Theorem 1.

In this section, we will prove Theorem 1. Let A be a finite set. Note that, with the help of (4.1) and Proposition 8, there exists an $\varepsilon_0 > 0$ and a constant c_4 such that for δ close enough to 1, for all $y \in \mathbb{Z}^d$ and $0 < \varepsilon \le \varepsilon_0$ we have that for every continuous function f of $\{\omega(x) : x \in A\}$ that

$$\begin{aligned}
&\int f \, d\mu_\delta \\
&\quad = \biggl(\sum_{y\in\mathbb{Z}^d} \Bigl(\mathbb{E}[g_\delta^\omega(0, y)]\mathbb{E}[f] \\
&\qquad + \varepsilon \sum_{\substack{z\in A\\ e\in V}} \operatorname{Cov}[\xi(z, e), f(\omega)] J_{p_\varepsilon^*}(z+e) \mathbb{E}[g_\delta^\omega(0, z+y)]
\end{aligned}$$

$$+\mathbb{E}[g_\delta^\omega(0,y)]\mathbb{E}[h_\delta f]\times O(\varepsilon^{2-\eta})\Big)\Big)\Big/\Big(\sum_{y\in\mathbb{Z}^d}\mathbb{E}[g_\delta^\omega(0,y)]\Big)$$

$$=\mathbb{E}[f]+\varepsilon\sum_{\substack{z\in A\\ e\in V}}\mathrm{Cov}[\xi(z,e),f(\omega)]J_{p_\varepsilon^*}(z+e)+\mathbb{E}[h_\delta f]\times O(\varepsilon^{2-\eta}),$$

where h_δ satisfies

$$|\mathbb{E}[|h_\delta|f]|\le\mathbb{E}[|f|], \tag{7.1}$$

and $|O(\varepsilon^{2-\eta})|\le c_4\varepsilon^{2-\eta}$. Taking now the limit when $\delta\to 1^-$, by Proposition 5 we conclude that

$$\int f\,d\mathbb{Q}=\mathbb{E}[f]+\varepsilon\sum_{z\in A,e\in V}\mathrm{Cov}[\xi(z,e),f(\omega)]J_{p_\varepsilon^*}(z+e)+\int f\,d\mathbb{V},$$

where by (7.1), $\mathbb{V}$ is a signed measure satisfying

$$\left|\int f\,d|\mathbb{V}|\right|\le|O(\varepsilon^{2-\eta})|\mathbb{E}[|f|],$$

where $|\mathbb{V}|$ is the total variation of $\mathbb{V}$. Hence, the restriction of $\mathbb{Q}$ to A is absolutely continuous with respect to $\mathbb{P}_A$, from where we can conclude that there is a function h, which is defined by

$$\int f\,d\mathbb{Q}=\mathbb{E}[f]+\varepsilon\sum_{z\in A,e\in V}\mathrm{Cov}[\xi(z,e),f(\omega)]J_{p_\varepsilon^*}(z+e)+\mathbb{E}[fh]\times O(\varepsilon^{2-\eta})$$

such that $\mathbb{E}[|h|]<\infty$, and such that for every continuous function f one has that

$$|\mathbb{E}[f|h|]|\le\mathbb{E}[|f|]. \tag{7.2}$$

Now, noting that any function f in L_1 can be approximated by continuous functions, it is easy to check that in fact (7.2) is satisfied for every $f\in L_1$. Therefore, h is bounded, from where we conclude the proof of Theorem 1.

8. Proof of Corollary 2. Here, we prove Corollary 2. It is enough to show that $J_{p_\varepsilon^*}$ [cf. (1.7)] is well approximated by $J_{p_0^*}$. By standard Fourier inversion formulas (see, e.g., display (1) of p. 148 of Spitzer [16] stated for dimension $d=2$ but actually valid in any dimension, or also Proposition 4.2.3 of Lawler and Limic [11]) we can conclude that for each $z\in\mathbb{Z}^d$ and $e\in V$,

$$J_{p_\varepsilon^*}(z+e)$$

$$=\frac{1}{(2\pi)^d}\left(\prod_{j=1}^d\left(\frac{p^\varepsilon(-e_j)}{p^\varepsilon(e_j)}\right)^{\frac{z_j+e_j}{2}}-1\right) \tag{8.1}$$

$$\times \int_{[0,2\pi]^d} \frac{\cos(\sum_{j=1}^d (z_j + e_j)x_j)}{1 - 2\sum_{j=1}^d \sqrt{p^\varepsilon(e_j)p^\varepsilon(-e_j)}\cos(x_j)} \prod dx_j$$

$$+ \frac{1}{(2\pi)^d} \int_{[0,2\pi]^d} \frac{\cos(\sum_{j=1}^d (z_j + e_j)x_j) - 1}{1 - 2\sum_{j=1}^d \sqrt{p^\varepsilon(e_j)p^\varepsilon(-e_j)}\cos(x_j)} \prod dx_j,$$

where for each $1 \le j \le d$, z_j is the jth coordinate of z. When $d_0 = 0$ [cf. (1.11)], we can conclude from (8.1) that

$$J_{p_\varepsilon^*}(z+e) = \begin{cases} J_{p_0^*}(z+e) + O(\varepsilon \log \varepsilon) & \text{if } d = 2, \\ J_{p_0^*}(z+e) + O(\varepsilon) & \text{if } d \ge 3. \end{cases}$$

For the case $d_0 \neq 0$, a simpler estimation gives us that for any dimension $d \ge 1$:

$$J_{p_\varepsilon^*}(z+e) = J_{p_0^*}(z+e) + O(\varepsilon).$$

Note that the case $d = 1$ with $d_0 \neq 0$ can also be deduced from (2.2).

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