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INVARIANCE TIMES[1]

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On a probability space $(\Omega, \mathcal{A}, \mathbb{Q})$, we consider two filtrations $\mathbb{F} \subset \mathbb{G}$ and a $\mathbb{G}$ stopping time θ such that the $\mathbb{G}$ predictable processes coincide with $\mathbb{F}$ predictable processes on $(0, \theta]$. In this setup, it is well known that, for any $\mathbb{F}$ semimartingale X, the process $X^{\theta-}$ (X stopped "right before θ") is a $\mathbb{G}$ semimartingale. Given a positive constant T, we call θ an invariance time if there exists a probability measure $\mathbb{P}$ equivalent to $\mathbb{Q}$ on $\mathcal{F}_T$ such that, for any $(\mathbb{F}, \mathbb{P})$ local martingale X, $X^{\theta-}$ is a $(\mathbb{G}, \mathbb{Q})$ local martingale. We characterize invariance times in terms of the $(\mathbb{F}, \mathbb{Q})$ Azéma supermartingale of θ and we derive a mild and tractable invariance time sufficiency condition. We discuss invariance times in mathematical finance and BSDE applications.

1. Introduction. On a probability space $(\Omega, \mathcal{A}, \mathbb{Q})$, we consider two filtrations $\mathbb{F} \subset \mathbb{G}$ of sub-σ-algebras of $\mathcal{A}$ and a $\mathbb{G}$ stopping time θ. We suppose the condition (B) that the $\mathbb{G}$ predictable processes coincide with the $\mathbb{F}$ predictable processes on $(0, \theta]$. In this setup, it is well known that, for any $(\mathbb{F}, \mathbb{Q})$ local martingale X, the process $X^{\theta-}$ (X stopped "right before θ") is a $(\mathbb{G}, \mathbb{Q})$ special semimartingale, whose drift part can be deduced from the Jeulin and Yor (1978) formula. In this paper, given a positive constant T, we study the condition (A) that there exists a probability measure $\mathbb{P}$ equivalent to $\mathbb{Q}$ on $\mathcal{F}_T$ such that, for any $(\mathbb{F}, \mathbb{P})$ local martingale X, the process $X^{\theta-}$ is a $(\mathbb{G}, \mathbb{Q})$ local martingale. If such a probability measure $\mathbb{P}$ exists, we call θ an invariance time and $\mathbb{P}$ an invariance measure.

One way of dealing with the above martingale invariance problem (we will propose two approaches to it in the paper) is to investigate the conditions under which the enlargement of filtration Jeulin–Yor formula can be compensated by the Girsanov formula of an equivalent change of probability measure. The Jeulin–Yor formula, initially proved by projection computations, has long been supposed to be affiliated with the Girsanov formula. Yoeurp (1985) gives a formal proof that the Jeulin–Yor formula can be obtained by the so-called generalized Girsanov formula (between nonabsolutely continuous probability measures). Actually, according to Song (1987, 2013), not only the Jeulin–Yor formula, but most of the formulas in

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enlargement of filtration, can be retrieved by the generalized Girsanov formula. However, the methodologies of the above-mentioned papers are not applicable to the study of the condition (A), notably because they operate in an enlarged probability space and with a nonabsolutely continuous probability measure. Hence, a new approach is required for our problem.

Besides their theoretical interest, invariance times are closely linked to questions raised from recent mathematical finance developments. Since the occurrence of the global financial crisis, the theoretical research on random times has seen a vigorous revival in relation with the field of default risk modeling, viewed under different angles: no arbitrage, pricing and hedging of credit derivatives, counterparty risk, insider trading, etc. See Nikeghbali and Yor (2005), Jeanblanc and Song (2011, 2015), Aksamit et al. (2013, 2014), Kardaras (2014), Song (2016b), Li and Rutkowski (2014), Fontana, Jeanblanc and Song (2014) and Acciaio, Fontana and Kardaras (2016), among others. Invariance times yield a class of models of default times for which intensity-based defaultable asset pricing formulas can be obtained beyond the classical but restrictive "immersion" setup [see Crépey and Song (2015, 2016)].

Our main results are Theorem 3.1 and Theorems 3.2–3.3, which give necessary and sufficient conditions, stated in terms of the Azéma supermartingale of θ, for $\mathbb{P}$ to be an invariance measure and θ to be an invariance time, respectively. These theorems are established in Sections 3.1 and 3.2 by reduction, a methodology introduced in Song (2016b) for transferring properties in $\mathbb{G}$ into properties in $\mathbb{F}$. However, the ensuing proofs do not directly explain how a Girsanov drift can compensate a Jeulin–Yor drift. Given the importance of that matter for the purpose of this work, we provide in Appendix C an alternative proof of Theorem 3.1 based on that compensation.

The second main contribution of this work is Theorem 3.5. When applied to a defaultable asset, a basic no-arbitrage pricing formula explicitly involves the default time θ, whereas it is only the intensity of θ that can be retrieved, by calibration, from market data. To tackle this issue, Duffie, Schroder and Skiadas (1996) have established a defaultable asset pricing formula stated in terms of the intensity process (assumed to exist) of θ. From a financial interpretation point of view, their intensity-based formula also shows that credit risk can be valued as a shift in interest rates. However, the tractability of this formula is subject to a technical no-jump condition at time θ. In a progressive enlargement of filtration setup satisfying a restrictive immersion assumption, this no-jump condition is satisfied. Alternatively, Collin-Dufresne, Goldstein and Hugonnier (2004) have proposed a reinterpretation of the Duffie, Schroder and Skiadas (1996) formula exempt from no-jump condition, under the so-called survival measure. As explained in Section 4.2, Theorem 3.5 provides a connection between the two approaches by showing that, under mild conditions, the restriction to $\mathcal{F}_T$ of the survival measure yields an invariance measure. Theorem 3.5 also provides a tractable invariance time sufficiency condition.

Several additional results are useful for the establishment of Theorem 3.5. In particular, according to Theorems 3.2–3.3, the positivity of a tentative $\mathbb{Q}$ to $\mathbb{P}$ measure change density on $[0, T]$, as well as the true martingale property on $[0, T]$ of the martingale part $\mathcal{Q}$ in the predictable multiplicative decomposition of the Azéma supermartingale S of θ, are key conditions for θ to be an invariance time. In Theorem 3.4, we characterize the above positivity by the predictability of the time of first zero of S on the time interval $[0, T]$. Theorem 3.6 provides a necessary and sufficient condition for the true martingale property of $\mathcal{Q}$.

Complementary results are provided for a better understanding of invariance times. The starting point of this paper is the condition (B), which is discussed in Section 2.1. As recalled in Section 2.2, the condition (A) appears naturally in the study of counterparty risk. Theorem 3.7 characterizes in terms of S the local martingales under the invariance measure $\mathbb{P}$. This characterization plays a key role in Crépey and Song (2015). Section 4 studies invariance times in several situations, comparing them with the so-called pseudo-stopping times, the stopping times in Collin-Dufresne, Goldstein and Hugonnier (2004) and showing how invariance times are involved in a variety of applications.

1.1. *Basic notation and terminology.* The real line, half-line and the nonnegative integers are respectively denoted by $\mathbb{R}$, $\mathbb{R}_+$ and $\mathbb{N}$; $\mathcal{B}(\mathbb{R})$ and $\mathcal{B}(\mathbb{R}_+)$ are the Borel σ fields on $\mathbb{R}$ and $\mathbb{R}_+$; $\boldsymbol{\lambda}$ is the Lebesgue measure on $\mathbb{R}_+$. Unless otherwise stated, a function (or process) is real-valued; order relationships between random variables (resp., processes) are meant almost surely (resp., in the indistinguishable sense); a time interval is random. We do not explicitly mention the domain of definition of a function (or process) when it is implied by the measurability, for example, we write "a $\mathcal{B}(\mathbb{R})$ measurable function h (or $h(x)$)" rather than "a $\mathcal{B}(\mathbb{R})$ measurable function h defined on $\mathbb{R}$." For a function $h(\omega, x)$ defined on a product space $\Omega \times E$, we usually write $h(x)$ without ω (or h_t in the case of a stochastic process).

We employ the tools and terminology of the general theory of processes and of filtrations as given in the books by Dellacherie and Meyer (1975) and He, Wang and Yan (1992). Footnotes are used for referring to comparatively standard results. For any semimartingale X and any predictable process L integrable with respect to X, the corresponding stochastic integral is denoted by $\int_0^{\cdot} L_t dX_t = \int_{(0,\cdot]} L_t dX_t = L \centerdot X$, with the usual precedence convention $KL \centerdot X = (KL) \centerdot X$ if K is another predictable process such that KL is integrable with respect to X. The stochastic exponential of a semimartingale X is denoted by $\mathcal{E}(X)$ (in particular, $\mathcal{E}(X)_0 = 1$). We denote by $\bar{X} = \frac{1}{X_-} \centerdot X$ (whenever it exists) the so-called stochastic logarithm of a positive semimartingale X, such that $X = X_0 \mathcal{E}(\bar{X})$. Given semimartingales

X and X', the bracket process $[X, X']$ and its predictable counterpart $\langle X, X'\rangle$ are defined as in He, Wang and Yan (1992), Definition 8.2. In particular, we use the convention $[X, X']_0 = 0$.

For any càdlàg process X, for any random time τ (nonnegative random variable), $\Delta_\tau X$ represents the jump of X at τ. Following Dellacherie and Meyer (1975) and He, Wang and Yan (1992), we use the convention that $X_{0-} = X_0$ (hence $\Delta_0 X = 0$) and we write X^τ and $X^{\tau-}$ for the process X stopped at τ and before τ, respectively, that is,

$$X^\tau = X\mathbb{1}_{[0,\tau)} + X_\tau \mathbb{1}_{[\tau,+\infty)}, \qquad X^{\tau-} = X\mathbb{1}_{[0,\tau)} + X_{\tau-}\mathbb{1}_{[\tau,+\infty)}. \tag{1.1}$$

We call compensator of a stopping time τ the compensator of the process $\mathbb{1}_{[\tau,\infty)}$.[2] We say that a stopping time τ is totally inaccessible (resp., has an intensity) if it is positive and if its compensator is continuous (resp., absolutely continuous) on $[0, \tau]$. Given a filtration $\mathbb{G} = (\mathcal{G}_t)_{t\in\mathbb{R}_+}$ and a $\mathbb{G}$ stopping time τ, for A in $\mathcal{G}_\tau$, we denote by τ_A the $\mathbb{G}$ stopping time[3] $\mathbb{1}_A\tau + \mathbb{1}_{A^c}\infty$.

We work with semimartingales on a predictable set of interval type I as defined in He, Wang and Yan (1992), Section VIII.3. In particular, X is a local martingale on I (respectively $Y = L \centerdot X$ on I) means that

$$X^{\tau_n} \text{ local martingale} \qquad \big(\text{respectively } Y^{\tau_n} = L \centerdot (X^{\tau_n})\big) \tag{1.2}$$

holds for at least one, or equivalently any, nondecreasing sequence of stopping times such that $\bigcup[0, \tau_n] = I$. The existence of such a sequence is ensured by He, Wang and Yan (1992), Theorem 8.18(3). From a computational point of view, stochastic calculus on predictable intervals reduces to standard stochastic calculus on $\mathbb{R}_+$ for each of the processes stopped at τ_n. But a process L can be integrable with respect to X on I without being locally integrable on $\mathbb{R}_+$ if the stochastic integrals Y^{τ_n} explode as $n \to \infty$.

Given a filtration $\mathbb{G}$ and a probability measure $\mathbb{Q}$, we denote by $\mathcal{S}_I(\mathbb{G}, \mathbb{Q})$ and $\mathcal{M}_I(\mathbb{G}, \mathbb{Q})$ the respective sets of $(\mathbb{G}, \mathbb{Q})$ semimartingales and local martingales on a predictable interval I (or $\mathbb{R}_+$, when no interval I is mentioned in the notation). The $\mathbb{G}$ predictable and optional σ fields are denoted by $\mathcal{P}(\mathbb{G})$ and $\mathcal{O}(\mathbb{G})$.

Throughout the paper, Ω is a space equipped with a σ field $\mathcal{A}$, $\mathbb{Q}$ is a probability measure on $\mathcal{A}$, $\mathbb{G} = (\mathcal{G}_t)_{t\in\mathbb{R}_+}$ is a filtration of sub σ fields of $\mathcal{A}$, θ is a $\mathbb{G}$ stopping time and $\mathbb{F} = (\mathcal{F}_t)_{t\in\mathbb{R}_+}$ is a subfiltration of $\mathbb{G}$. Both filtrations $\mathbb{F}$ and $\mathbb{G}$ are supposed to satisfy the usual conditions. Regarding the filtration $\mathbb{F}$, we have to deal with two probability measures $\mathbb{Q}$ and $\mathbb{P}$. Accordingly, letters of the family "q" and "p" are respectively used for $(\mathbb{F}, \mathbb{Q})$ and $(\mathbb{F}, \mathbb{P})$ local martingales. By default $\mathbb{E}$ refers to the $\mathbb{Q}$ expectation, whereas the $\mathbb{P}$ expectation is denoted by $\mathbb{E}^{\mathbb{P}}$.

[2]See He, Wang and Yan (1992), Definition 5.21, for the notion of compensator of a locally integrable nondecreasing process.

[3]Cf. Theorem 3.9 in He, Wang and Yan (1992).

2. Preliminaries.

2.1. *The condition* (B). We consider the following.

Condition (B). For any $\mathbb{G}$ predictable process L, there exists an $\mathbb{F}$ predictable process L', called the $\mathbb{F}$ predictable reduction[4] of L, such that $\mathbb{1}_{(0,\theta]}L = \mathbb{1}_{(0,\theta]}L'$.

REMARK 2.1. The equality $\mathbb{1}_{(0,\theta]}L = \mathbb{1}_{(0,\theta]}L'$ is in the sense of indistinguishability. But, as $\mathbb{F}$ satisfies the usual conditions, we can find a version of L' such that the equality holds everywhere.[5]

The condition (B) corresponds to a relaxation of the classical progressive enlargement of filtration setup, where the bigger filtration $\mathbb{G}$ is given as the progressive enlargement by θ of a reference filtration $\mathbb{F}$. As compared with this classical case, the possibility to use a bigger filtration $\mathbb{G}$ in the condition (B) is material, for instance, in the dynamic Marshall–Olkin copula model of Crépey and Song (2016) (see Section 4.4 below).

As an immediate consequence of the condition (B), we have[6]

$$\{0 < \theta < \infty\} \cap \mathcal{G}_{\theta-} = \{0 < \theta < \infty\} \cap \mathcal{F}_{\theta-},$$

where we recall[7] that

$$\mathcal{G}_{\theta-} = \mathcal{G}_0 \vee \sigma\big\{B \cap \{t < \theta\}, B \in \mathcal{G}_t, t \in \mathbb{R}_+\big\},$$
$$\mathcal{F}_{\theta-} = \mathcal{F}_0 \vee \sigma\big\{A \cap \{t < \theta\}, A \in \mathcal{F}_t, t \in \mathbb{R}_+\big\}.$$

But we can say more. We introduce the right-continuous[8] filtration $\overline{\mathbb{F}} = (\overline{\mathcal{F}}_t)_{t\in\mathbb{R}_+}$, where

$$\overline{\mathcal{F}}_t = \big\{B \in \mathcal{A} : \exists A \in \mathcal{F}_t, B \cap \{t < \theta\} = A \cap \{t < \theta\}\big\} \tag{2.1}$$

[see Dellacherie, Maisonneuve and Meyer (1992), Chapitre XX, n°75].

LEMMA 2.1. *$\mathbb{F}$, $\mathbb{G}$ and θ satisfy the condition* (B) *if and only if $\mathbb{G}$ is a subfiltration of $\overline{\mathbb{F}}$.*

[4] Also known as pre-default process of L in the credit risk literature [see, e.g., Bielecki, Jeanblanc and Rutkowski (2009)].

[5] Cf. He, Wang and Yan [(1992), Theorem 4.26] and Lemma 2.2(2).

[6] Cf. He, Wang and Yan (1992), Corollary 3.23(2).

[7] Cf. He, Wang and Yan (1992), Definition 3.3, equation (3.3).

[8] And complete under our assumption that $\mathbb{F}$ satisfies the usual conditions.

PROOF. Suppose the condition (B). For any $t \in \mathbb{R}_+$ and $B \in \mathcal{G}_t$, $\mathbb{1}_B \mathbb{1}_{(t,\infty)}$ is a $\mathbb{G}$ predictable process, having a bounded $\mathbb{F}$ predictable reduction K such that $\mathbb{1}_{(0,\theta]} \mathbb{1}_B \mathbb{1}_{(t,\infty)} = \mathbb{1}_{(0,\theta]} K \mathbb{1}_{(t,\infty)}$. Then $\mathbb{1}_B \mathbb{1}_{\{t<s\le\theta\}} = K_s \mathbb{1}_{\{t<s\le\theta\}}$, hence

$$\liminf_{s\downarrow t} \mathbb{1}_B \mathbb{1}_{\{t<s\le\theta\}} = \liminf_{s\downarrow t} K_s \mathbb{1}_{\{t<s\le\theta\}}.$$

But

$$\liminf_{s\downarrow t} \mathbb{1}_B \mathbb{1}_{\{t<s\le\theta\}} = \mathbb{1}_B \mathbb{1}_{\{t<\theta\}}$$

and

$$\liminf_{s\downarrow t} K_s \mathbb{1}_{\{t<s\le\theta\}} = \Big(\liminf_{s\downarrow t} K_s\Big) \mathbb{1}_{\{t<\theta\}},$$

which proves $B \in \overline{\mathcal{F}}_t$. Conversely [cf. Lemma 1 in Jeulin and Yor (1978)], suppose that $\mathbb{G}$ is a subfiltration of $\overline{\mathbb{F}}$. For any $t > 0$, for any $B \in \mathcal{G}_t$, let $A \in \mathcal{F}_t$ satisfy $B \cap \{t < \theta\} = A \cap \{t < \theta\}$, so that

$$\mathbb{1}_{(0,\theta]} \mathbb{1}_B \mathbb{1}_{(t,\infty)} = \mathbb{1}_{(0,\theta]} \mathbb{1}_A \mathbb{1}_{(t,\infty)}.$$

Note that $\mathbb{1}_A \mathbb{1}_{(t,\infty)}$ is an $\mathbb{F}$ predictable process. For any $B \in \mathcal{G}_0$, $\mathbb{1}_{(0,\theta]} \mathbb{1}_B \mathbb{1}_{\{0\}} = 0$, which is an $\mathbb{F}$ predictable process. Since the processes $\mathbb{1}_B \mathbb{1}_{(t,\infty)}$ ($t > 0$, $B \in \mathcal{G}_t$) and $\mathbb{1}_B \mathbb{1}_{\{0\}}$ ($B \in \mathcal{G}_0$) generate the $\mathbb{G}$ predictable σ-algebra,[9] this proves the condition (B). □

The condition (B) *is assumed everywhere* in the sequel of the paper. Let ${}^{o}\cdot$ and ${}^{p}\cdot$ denote the $(\mathbb{F}, \mathbb{Q})$ optional and predictable projections and let $\langle\cdot,\cdot\rangle$ and $[\cdot,\cdot]$ denote the $(\mathbb{F}, \mathbb{Q})$ optional and predictable brackets. We introduce, denoted by straight capital letters, a number of processes related to θ. We write $\mathtt{J} = \mathbb{1}_{[0,\theta)}$, hence $\mathtt{J}_- = \mathbb{1}_{\{0<\theta\}} \mathbb{1}_{[0,\theta]}$. The fundamental tool to work with the condition (B) is the Azéma supermartingale $\mathtt{S} = {}^{o}\mathtt{J}$ of θ, that is, $\mathtt{S}_t = \mathbb{Q}(\theta > t | \mathcal{F}_t)$, $t \in \mathbb{R}_+$, with canonical Doob–Meyer decomposition $\mathtt{S} = \mathtt{S}_0 + \mathtt{Q} - \mathtt{D}$, where $\mathtt{Q}$ is an $(\mathbb{F}, \mathbb{Q})$ martingale starting from 0 while $\mathtt{D}$ is the $(\mathbb{F}, \mathbb{Q})$ dual predictable projection of $\mathbb{1}_{\{0<\theta\}} \mathbb{1}_{[\theta,\infty)}$. The most classical properties of $\mathtt{S}$ useful for this work are recalled in Section A.

The proofs of the progressive of enlargement results in Jeulin and Yor (1978) or Chapitre XX in Dellacherie, Maisonneuve and Meyer (1992) only require that $\mathbb{G}$ is a subfiltration of $\overline{\mathbb{F}}$. Hence, in view of Lemma 2.1, all these results hold under the condition (B). The next lemma gathers the main ones that we need in the sequel.

LEMMA 2.2. (1) *For any $\mathbb{G}$ stopping time τ, there exists an $\mathbb{F}$ stopping time τ', which we call the $\mathbb{F}$ reduction of τ, such that $\{\tau < \theta\} = \{\tau' < \theta\} \subseteq \{\tau = \tau'\}$.*

[9] Cf. Theorem 3.21 in He, Wang and Yan (1992).

(2) *Let* $(E,\mathcal{E})$ *be a measurable space. Any* $\mathcal{P}(\mathbb{G})\otimes\mathcal{E}$ *(resp.,* $\mathcal{O}(\mathbb{G})\otimes\mathcal{E}$*) measurable function* $g_t(\omega,x)$ *admits a* $\mathcal{P}(\mathbb{F})\otimes\mathcal{E}$ *(resp.,* $\mathcal{O}(\mathbb{F})\otimes\mathcal{E}$*) reduction, that is, a* $\mathcal{P}(\mathbb{F})\otimes\mathcal{E}$ *(resp.,* $\mathcal{O}(\mathbb{F})\otimes\mathcal{E}$*) measurable function* $g'_t(\omega,x)$ *such that* $\mathbb{1}_{(0,\theta]}g=\mathbb{1}_{(0,\theta]}g'$ *(respectively* $\mathbb{1}_{[0,\theta)}g=\mathbb{1}_{[0,\theta)}g'$*) holds almost everywhere. Moreover, these relations can be made to hold everywhere by the choice of suitable versions.*

(3) *We have*

$$Q\in\mathcal{M}(\mathbb{F},\mathbb{Q}) \quad\Longrightarrow\quad Q^{\theta-}-\frac{\mathtt{J}_-}{\mathtt{S}_-}\centerdot\langle\mathtt{S},Q\rangle\in\mathcal{M}(\mathbb{G},\mathbb{Q}). \tag{2.2}$$

(4) *For any* $K\in\mathcal{S}_{\{\mathtt{S}_->0\}}(\mathbb{F},\mathbb{Q})$ *such that* $\mathtt{S}_-\centerdot K+[\mathtt{S},K]\in\mathcal{M}_{\{\mathtt{S}_->0\}}(\mathbb{F},\mathbb{Q})$*, we have* $K^{\theta-}\in\mathcal{M}(\mathbb{G},\mathbb{Q})$.

Conversely, for any $M\in\mathcal{M}(\mathbb{G},\mathbb{Q})$ *with* $\Delta_\theta M=0$ *on* $\{\theta<\infty\}$*, any* $\mathbb{F}$ *optional reduction* M' *of* M *is in* $\mathcal{S}_{\{\mathtt{S}_->0\}}(\mathbb{F},\mathbb{Q})$, $\mathbb{1}_{\{\mathtt{S}_->0\}}M'_-$ *is an* $\mathbb{F}$ *predictable reduction of* M_- *and* $\mathtt{S}_-\centerdot M'+[\mathtt{S},M']\in\mathcal{M}_{\{\mathtt{S}_->0\}}(\mathbb{F},\mathbb{Q})$.

(5) *The Azéma supermartingale* $\mathtt{S}$ *admits the predictable multiplicative decomposition* $\mathtt{S}=\mathtt{S}_0\mathcal{Q}\mathcal{D}$*, for the finite variation predictable factor* $\mathcal{D}=\mathcal{E}(-\mathbb{1}_{\{\mathtt{S}_->0\}}\frac{1}{\mathtt{S}_-}\centerdot\mathtt{D})$ *and the local martingale factor* $\mathcal{Q}$ *defined by the pointwise limit*

$$\mathcal{Q}=\lim_{n\to\infty}\mathcal{E}\left(\frac{1}{{}^p\mathtt{S}}\centerdot\mathtt{Q}\right)^{\zeta_n}, \tag{2.3}$$

where $(\zeta_n)_{n\in\mathbb{N}}$ *is the sequence that appears in* (A.8)*. In particular, on the random set* $\{{}^p\mathtt{S}>0\}$*, we have* $\mathcal{Q}=\mathcal{E}(\frac{1}{{}^p\mathtt{S}}\centerdot\mathtt{Q})$, $\mathcal{D}=\mathcal{E}(-\frac{1}{\mathtt{S}_-}\centerdot\mathtt{D})$ *and*

$$\mathcal{E}\left(\frac{1}{{}^p\mathtt{S}}\centerdot\mathtt{D}\right)\mathcal{E}\left(-\frac{1}{\mathtt{S}_-}\centerdot\mathtt{D}\right)=1. \tag{2.4}$$

If $\mathtt{S}_T$ *is positive, then* $\mathcal{Q}=\mathcal{E}(\frac{1}{{}^p\mathtt{S}}\centerdot\mathtt{Q})>0$ *holds on* $[0,T]$.

PROOF. (1) is proven in Chapitre XX, n°75 (a) of Dellacherie, Maisonneuve and Meyer (1992); (3) is proven in Chapitre XX, n°77 (b) of Dellacherie, Maisonneuve and Meyer (1992); (4) is proven in Song (2016b), Lemmas 6.5 and 6.8.

(5) is proven in Song (2016b), Lemmas 3.9 and 3.10. Specifically, (2.4) is Song (2016b), Lemma 3.9. In (2.3), the limits exist and the local martingale property of $\mathcal{E}(\frac{1}{{}^p\mathtt{S}}\centerdot\mathtt{Q})^{\zeta_n}$ passes to the limit $\mathcal{Q}$ by virtue of Song (2016b), Lemma 3.10. Note that $\frac{1}{\mathtt{S}_-}\centerdot\mathtt{D}$ (resp., $\frac{1}{{}^p\mathtt{S}}\centerdot\mathtt{Q}$) is well defined on $\{\mathtt{S}_->0\}$, by (A.7) [resp., on $\{{}^p\mathtt{S}>0\}$, by (A.8)].

Regarding (2), we first consider the question for processes, that is, without the measurable space $(E,\mathcal{E})$. Let χ be a bounded $\mathcal{G}_\infty$ measurable random variable and Y be the (càdlàg) $\mathbb{G}$ martingale with terminal variable χ. By a classical result[10]

[10] Known as the key lemma in the credit risk literature [see, e.g., Bielecki, Jeanblanc and Rutkowski (2009), Lemma 3.1.2].

[see, e.g., Dellacherie, Maisonneuve and Meyer (1992), Chapitre XX n°75 (75.2)], for every $t \in \mathbb{R}_+$, we have the almost sure identity

$$\mathtt{J}_t Y_t = \mathtt{J}_t \mathbb{E}[\chi | \mathcal{G}_t] = \frac{\mathtt{J}_t}{\mathsf{S}_t} \mathbb{E}[\mathtt{J}_t \chi | \mathcal{F}_t] = \mathtt{J}_t X_t,$$

where

$$X_t = \frac{{}^o(\mathtt{J}\chi)_t}{\mathsf{S}_t} \mathbb{1}_{\{\mathsf{S}_t > 0\}}.$$

By Dellacherie and Meyer (1975), Chapitre VI n°47, the process ${}^o(\mathtt{J}\chi)$ is càdlàg, so that, actually, $\mathtt{J}X = \mathtt{J}Y$ holds in the indistinguishable sense. This proves the existence of an $\mathbb{F}$ optional reduction for the $\mathbb{G}$ martingale Y.

Let $\mathfrak{C}$ denote the class of all the bounded $\mathcal{B}(\mathbb{R}_+) \otimes \mathcal{G}_\infty$ measurable functions L such that ${}^{\mathbb{G}\cdot o}L$ admits an $\mathbb{F}$ optional reduction, where ${}^{\mathbb{G}\cdot o}\cdot$ denotes the $(\mathbb{G}, \mathbb{Q})$ optional projection. We can verify that $\mathfrak{C}$ is a functional monotone class in the sense of the monotone class Theorem 1.4 in He, Wang and Yan (1992). It results from above that $\mathfrak{C}$ contains all the random variables $\mathbb{1}_{(a,b]}\chi$, where $a, b \in \mathbb{R}_+$ and χ is a bounded $\mathcal{G}_\infty$ measurable random variable. Therefore, by the monotone class theorem, $\mathfrak{C}$ contains all the bounded $\mathcal{B}(\mathbb{R}_+) \otimes \mathcal{G}_\infty$-measurable random variables. In particular, every bounded $\mathbb{G}$ optional process admits an $\mathbb{F}$ optional reduction. By taking limits, the result is extended to general $\mathbb{G}$ optional processes.

Since $\mathbb{F}$ satisfies the usual conditions, any evanescent measurable process is $\mathbb{F}$ predictable [Theorem 4.26 in He, Wang and Yan (1992)]. For any $\mathbb{G}$ optional process L with $\mathbb{F}$ optional reduction K, there exists a negligible set O such that $\mathtt{J}_- K = \mathtt{J}_- L$ holds everywhere outside O. Therefore, defining $\tilde{K} = K - (K - L)\mathbb{1}_O$, the process $\tilde{K}$ is $\mathbb{F}$ optional and satisfies $\mathtt{J}_- \tilde{K} = \mathtt{J}_- L$ everywhere.

We have thus proved the optional version of the part (2) of the lemma in the case of processes. A standard reasoning by monotone class theorem proves the result in the presence of the measurable space $(E, \mathcal{E})$.

The predictable version of the part (2) of the lemma can be proved similarly. In fact, the predictable version for processes [i.e., without the measurable space $(E, \mathcal{E})$] is precisely the condition (B) (cf. Remark 2.1). □

Recall the respective $\mathbb{F}$ predictable and optional Girsanov formulas[11] involving the density process q of the measure change from the probability measure $\mathbb{Q}$ to some $\mathbb{Q}$ absolutely continuous probability measure $\mathbb{P}$:

$$Q \text{ bounded in } \mathcal{M}(\mathbb{F}, \mathbb{Q}) \quad \Longrightarrow \quad Q - \frac{1}{q_-} \centerdot \langle q, Q \rangle \in \mathcal{M}(\mathbb{F}, \mathbb{P}), \tag{2.5}$$

respectively,

$$\begin{aligned} &P \in \mathcal{M}(\mathbb{F}, \mathbb{P}) \quad \Longleftrightarrow \\ &\qquad P \in \mathcal{S}_{\{q_- > 0\}}(\mathbb{F}, \mathbb{Q}) \quad \text{and} \quad q_- \centerdot P + [q, P] \in \mathcal{M}_{\{q_- > 0\}}(\mathbb{F}, \mathbb{Q}). \end{aligned} \tag{2.6}$$

[11] Cf. He, Wang and Yan (1992), Theorem 12.18.

Observe that the Jeulin–Yor formula (2.2) and Lemma 2.2(4) are formal analogs, in the field of progressive enlargement of filtration, of these respective Girsanov measure change formulas, the Azéma supermartingale S playing the role of the measure change density q. Starting from the so-called generalized Girsanov formulas (between possibly nonabsolutely continuous probability measures) and representing the Azéma supermartingale S as a "generalized density," these formal analogies can be turned into proofs of the corresponding enlargement of filtration formulas [see Yoeurp (1985) and Song (1987, 2013)].

Note that the classical formulation of the Jeulin–Yor formula is stated in terms of Q^{θ}, instead of $Q^{\theta-}$ in (2.2), as

$$Q \in \mathcal{M}(\mathbb{F}, \mathbb{Q}) \quad \Longrightarrow \quad Q^{\theta} - \frac{\mathsf{J}_-}{\mathsf{S}_-} \boldsymbol{\cdot} \big(\langle \mathsf{S}, Q\rangle + B\big) \in \mathcal{M}(\mathbb{G}, \mathbb{Q}), \tag{2.7}$$

where B is the $(\mathbb{F}, \mathbb{Q})$ dual predictable projection of the process $\Delta_{\theta} Q \mathbb{1}_{\{\theta \le \cdot\}}$ [cf. Jeulin and Yor (1978), Theorem 1 and Lemma 4(b)]. However, as visible in the proof of Theorem 1 in Jeulin and Yor (1978), the bracket $\langle \mathsf{S}, Q\rangle$ in the Jeulin–Yor formula (2.2) is intrinsically linked with $Q^{\theta-}$, rather than with Q^{θ}.

The next result shows that $\mathbb{F}$ predictable and optional reductions of $\mathbb{G}$ predictable and optional processes are uniquely defined on the random intervals $\{\mathsf{S}_- > 0\}$ [which contains θ on $\{0 < \theta < \infty\}$, cf. (A.11)] and $\{\mathsf{S} > 0\}$, respectively.

LEMMA 2.3. *Two $\mathbb{F}$ predictable (respectively optional) processes K and $\tilde{K}$ undistinguishable on $[0, \theta]$ (respectively $[0, \theta)$) are undistinguishable on $\{\mathsf{S}_- > 0\}$ (resp., $\{\mathsf{S} > 0\}$).*

PROOF. Otherwise, the predictable section theorem[12] (considering the predictable case in the lemma) would imply the existence of an $\mathbb{F}$ predictable stopping time σ such that $\mathbb{E}[\mathbb{1}_{K_\sigma \neq \tilde{K}_\sigma} \mathsf{S}_{\sigma-} \mathbb{1}_{\{\sigma < \infty\}}] > 0$, in contradiction with

$$\begin{aligned} \mathbb{E}[\mathbb{1}_{K_\sigma \neq \tilde{K}_\sigma} \mathsf{S}_{\sigma-} \mathbb{1}_{\{\sigma < \infty\}}] &= \mathbb{E}\big[\mathbb{1}_{K \neq \tilde{K}} \mathsf{S}_- \boldsymbol{\cdot} (\mathbb{1}_{\{\sigma > 0\}} \mathbb{1}_{[\sigma, +\infty)})\big] \\ &= \mathbb{E}\big[\mathbb{1}_{K \neq \tilde{K}} \mathsf{J}_- \boldsymbol{\cdot} (\mathbb{1}_{\{\sigma > 0\}} \mathbb{1}_{[\sigma, +\infty)})\big] \\ &= \mathbb{E}[\mathbb{1}_{K_\sigma \neq \tilde{K}_\sigma} \mathsf{J}_{\sigma-} \mathbb{1}_{\{\sigma < \infty\}}] = 0. \end{aligned}$$

The optional version of the lemma can be proven similarly. □

For the (first) proof of Theorem 3.1 below, we need a refined comparison of the random intervals $\{{}^{p}\mathsf{S} > 0\}$ and $\{\mathsf{S}_- > 0\}$. Let

$$\varsigma = \inf\{s > 0; \mathsf{S}_s = 0\}, \qquad \eta = \inf\big\{s > 0; {}^{p}\mathsf{S}_s = 0, \mathsf{S}_{s-} > 0\big\}. \tag{2.8}$$

[12] Cf. He, Wang and Yan (1992), Theorem 4.8.

LEMMA 2.4. *We have*

$$\begin{aligned}\eta &= \inf\{s>0; \mathtt{S}_{s-}=\Delta_s\mathtt{D}>0\}\\ &= \inf\Big\{s\in\{\mathtt{S}_->0\}; \mathcal{E}\Big(-\frac{1}{\mathtt{S}_-}\centerdot\mathtt{D}\Big)_s=0\Big\}.\end{aligned} \tag{2.9}$$

Moreover, we have $\eta\ge\varsigma$, $\eta=\varsigma$ *on* $\{\eta<\infty\}$ *and*

$$\{\mathtt{S}_->0\}\setminus\{{}^p\mathtt{S}>0\}=[\eta]. \tag{2.10}$$

In particular, η *is* $\mathbb{F}$ *predictable.*

PROOF. The first equality in (2.9) results from (A.2). The stochastic exponential $\mathcal{E}(-\frac{1}{\mathtt{S}_-}\centerdot\mathtt{D})$ vanishes at t in $\{\mathtt{S}_->0\}$ if and only if

$$\Delta_t\Big(-\frac{1}{\mathtt{S}_-}\centerdot\mathtt{D}\Big)=\frac{-1}{\mathtt{S}_{t-}}\Delta_t\mathtt{D}=-1\quad\text{that is, }\mathtt{S}_{t-}=\Delta_t\mathtt{D}.$$

Hence,

$$\begin{aligned}\inf\Big\{s\in\{\mathtt{S}_->0\}; \mathcal{E}\Big(-\frac{1}{\mathtt{S}_-}\centerdot\mathtt{D}\Big)_s=0\Big\} &= \inf\big\{s\in\{\mathtt{S}_->0\}; \mathtt{S}_{s-}=\Delta_s\mathtt{D}\big\}\\ &= \inf\big\{s\in\{\mathtt{S}_->0\}; \mathtt{S}_{s-}=\Delta_s\mathtt{D}>0\big\}\\ &= \inf\{s>0; \mathtt{S}_{s-}=\Delta_s\mathtt{D}>0\}=\eta,\end{aligned}$$

which proves the second equality in (2.9). The remainder of the lemma is the consequence of Lemma 3.2 (cf. also Lemmas 3.3 and 3.6) in Song (2016b). □

LEMMA 2.5. *We have* $\mathbb{1}_{\{{}^p\mathtt{S}=0\}}\centerdot\mathtt{Q}=0$.

PROOF. By (A.2), we have $\mathtt{S}_-\ge{}^p\mathtt{S}$. Hence,

$$\begin{aligned}\mathbb{1}_{\{{}^p\mathtt{S}=0\}}\centerdot\mathtt{Q} &= \mathbb{1}_{(\varsigma,\infty)}\centerdot\mathtt{Q}+\mathbb{1}_{(0,\varsigma]}\mathbb{1}_{\{\mathtt{S}_-=0\}}\centerdot\mathtt{Q}+\mathbb{1}_{\{{}^p\mathtt{S}=0,\mathtt{S}_->0\}}\centerdot\mathtt{Q}\\ &= \mathbb{1}_{(\varsigma,\infty)}\centerdot\mathtt{Q}+\mathbb{1}_{\{\mathtt{S}_{\varsigma-}=0\}}\Delta_\varsigma\mathtt{Q}\mathbb{1}_{[\varsigma,\infty)}+\Delta_\eta\mathtt{Q}\mathbb{1}_{[\varsigma,\infty)},\end{aligned}$$

by Lemma 2.4. The first term is null because $\mathtt{Q}$ is constant on (ς,∞) [cf. (A.10)]. The second term is null because of Song (2016b), Lemmas 3.4 and 3.7. The third term is null because [cf. (A.2) and Lemma 2.4] $\mathbb{1}_{\{\eta<\infty\}}\Delta_\eta\mathtt{Q}=\mathbb{1}_{\{\eta<\infty\}}\mathbb{1}_{\{\mathtt{S}_{\varsigma-}>0,{}^p\mathtt{S}_\varsigma=0\}}(\mathtt{S}_\varsigma-{}^p\mathtt{S}_\varsigma)=0$. □

2.2. *Toward the condition* (A): *Counterparty risk BSDEs motivation*. This section, for motivation mainly, can be skipped at no harm from the point of the theoretical developments of Section 3. More on applications will be delivered in Section 4. We consider:

– An exposure at default, or "recovery" of a bank upon the default of its counterparty, of the form $\mathbb{1}_{\{\theta<T\}}G_\theta$, where $T>0$ is some maturity, θ represents the default time of the counterparty and G is a $\mathbb{G}$ predictable process [for simplicity of presentation here, see however the comment following (2.12)],
– A $\mathcal{P}(\mathbb{G})\otimes\mathcal{B}(\mathbb{R})$ funding cost coefficient $g_t(\omega,x)$ *of the bank.*

Assuming θ endowed with an intensity γ, the *counterparty risk backward stochastic differential equation (BSDE)*, which prices the exposure at default G_θ at θ (if $<T$) and the funding costs g until $\theta\wedge T$, can be formulated as the following BSDE for some process Z in $\mathcal{S}(\mathbb{G},\mathbb{Q})$:

$$\begin{cases} Z_{T-}\mathbb{1}_{\{T\le\theta\}}=0,\\ Z^{\theta\wedge T-}+\int_0^{\cdot\wedge\theta\wedge T}\big(g_s(Z_{s-})+(G_s-Z_{s-})\gamma_s\big)\,ds\in\mathcal{M}(\mathbb{G},\mathbb{Q}). \end{cases} \tag{2.11}$$

For the sake of conciseness, we present the counterparty risk BSDE under this slightly unusual appearance, which is the equation (3.8) in Crépey and Song (2015), Theorem 3.1, because this formulation is the most convenient for the discussion that follows. Moreover, we only state the problem in its most basic form here. To be in line with applications, the assumptions in Crépey and Song (2015) cover more general BSDEs with

$$G=G_t(x,u),\qquad g=g_t(x,u), \tag{2.12}$$

where the additional argument u corresponds to integrands in a stochastic integral representation of the martingale part of Z. In particular, dependencies of G as of (2.12), where the dependency in t is not necessarily of predictable type, make the corresponding form of (2.11) a nonstandard BSDE.

In essence, the Duffie, Schroder and Skiadas (1996) approach to (2.11) would consist in forgetting about θ there (or "sending θ to infinity"), which results in a simpler equation "without θ," where θ is only indirectly represented through its intensity γ. One then tentatively sets $Z=\tilde{Z}^{\theta-}$, where $\tilde{Z}$ is a solution $\tilde{Z}$ to the simpler equation without θ. However, this only yields a solution Z to (2.11) if $\tilde{Z}$ does not jump at θ. In the basic reduced-form setup discussed in Section 2.2.1 below, this no-jump condition is satisfied. But, apart from this restrictive situation, the no-jump condition is unverifiable and it does not hold in general.

One way out of this is to introduce a "reduced BSDE," given a smaller filtration $\mathbb{F}$ such that the condition (B) is satisfied. For any càdlàg process X on $\mathbb{R}_+$ (or any predictable set of interval type), we write

$$\overline{X}=X+\big(g'_\cdot(X_-)+\big(G'-X_-\big)\gamma'\big)\centerdot\boldsymbol{\lambda}. \tag{2.13}$$

Observe that, assuming the BSDE (2.11) has a solution Z and letting $U = Z'$ denote an $\mathbb{F}$ optional reduction of Z, the martingale term in the BSDE (2.11) satisfies

$$
\begin{aligned}
&Z_t^{\theta\wedge T-} + \int_0^{t\wedge\theta\wedge T} \big(g_s(Z_{s-}) + (G_s - Z_{s-})\big)\gamma_s\,ds \\
&\quad = U_t^{\theta\wedge T-} + \int_0^{t\wedge\theta\wedge T} \big(g'_s(U_{s-}) + (G'_s - U_{s-})\gamma'_s\big)\,ds \\
&\quad = \overline{U}_t^{\theta\wedge T-} = \big(\overline{U}^{T-}\big)_t^{\theta-}.
\end{aligned}
\tag{2.14}
$$

This suggests to solve the BSDE (2.11) with Lemma 2.2(4). Namely, we consider the following BSDE for some process U in $\mathcal{S}_{\{\mathtt{S}_->0\}}(\mathbb{F},\mathbb{Q})$:

$$
U_{T-}\mathtt{S}_{T-} = 0, \qquad \mathtt{S}_- \boldsymbol{\cdot} \overline{U}^{T-} + \big[\mathtt{S}, \overline{U}^{T-}\big] \in \mathcal{M}_{\{\mathtt{S}_->0\}}(\mathbb{F},\mathbb{Q}).
\tag{2.15}
$$

Based on Lemma 2.2(4), the following result is proved in Crépey and Song (2015).

PROPOSITION 2.1. *The BSDEs* (2.11) *and* (2.15) *are equivalent*, *in the following sense*:

- *If Z is a solution to the BSDE* (2.11), *then $U = Z'$ is a solution to the BSDE* (2.15);
- *Conversely*, *if U is a solution to the BSDE* (2.15), *then $Z = U^{\theta-}$ is a solution to the BSDE* (2.11).

In (2.15), θ is only indirectly represented, through γ' in $\overline{U}$ [cf. (2.13)]. In this sense, passing from (2.11) to (2.15) removes θ from the equation. However, beyond the simple case where $\mathtt{S}$ is continuous and nonincreasing so that $[\mathtt{S},\cdot] = 0$, this comes at the expense of an additional bracket in the martingale condition of (2.15).

To untie the Gordian knot that we are facing here, let us suppose for a moment the condition (A) stated in the beginning of Section 3 below, for some invariance measure $\mathbb{P}$. Under this condition, in view of (2.14), any solution U in $\mathcal{S}_{\{\mathtt{S}_->0\}}(\mathbb{F},\mathbb{P})$ to

$$
U_{T-}\mathtt{S}_{T-} = 0, \qquad \overline{U}^{T-} \in \mathcal{M}_{\{\mathtt{S}_->0\}}(\mathbb{F},\mathbb{P})
\tag{2.16}
$$

yields a solution $Z = U^{\theta-}$ to (2.11) [since $\mathtt{S}_{\theta-} > 0$, cf. (A.11)].

As compared with (2.11), this approach allows getting rid of θ in the equation and it no longer comes at the expense of a more complicated martingale condition. In fact, the martingale condition in (2.16) is essentially the same as the one in (2.11), modulo reduction. From this point of view, the condition (A) and the related invariance measure $\mathbb{P}$ appear as "deus ex machina" for dealing with (2.11). See Section 4.3 for a concrete application.

However, such an approach postulating the condition (A) raises two important issues:

1. How strong is the condition (A)?
2. Do we have equivalence, under the condition (A), between (2.16) and (2.11), and not only (2.16) implies (2.11)? (for instance, for the application mentioned in Section 4.3 below, one really needs the equivalence).

These two questions were our initial motivation for the introduction and study of the condition (A). Regarding the first one, a complete characterization of the condition (A) and a mild sufficiency condition for it are established as Theorems 3.2–3.3 and Theorem 3.5, respectively. The second question is given a positive answer in Crépey and Song (2015), Theorems 3.1 and 4.3, based on Theorem 3.7 below regarding local martingales under an invariance measure $\mathbb{P}$.

2.2.1. *Basic reduced-form setup.* The immersion property, first introduced under the name of $(\mathcal{H})$ hypothesis in Brémaud and Yor (1978), page 284, means that all $\mathbb{F}$ local martingales are $\mathbb{G}$ local martingales. In the historically much considered case of a Brownian filtration $\mathbb{F}$ [see, e.g., Mansuy and Yor (2006)], the immersion property implies that $\mathtt{S}$ is a finite variation and predictable process. In this case, "$\overline{U}^{T-} \in \mathcal{M}_{\{\mathtt{S}_->0\}}(\mathbb{F},\mathbb{Q})$" suffices to ensure the more involved martingale condition in (2.15), by Yoeurp's lemma.[13] Accordingly, people tend to identify immersion as the "easy" case where an enlargement of filtration approach without measure change is successful for dealing with pricing equations such as (2.11) [see the comments before Section 3 in Duffie, Schroder and Skiadas (1996) or in Collin-Dufresne, Goldstein and Hugonnier (2004), page 1379, and see the comments following (3.22) and (H.3) or the remarks following Proposition 6.1 in Bielecki and Rutkowski (2001)]. However, on the one hand, immersion is unnecessarily strong for that purpose, because whatever happens after θ is irrelevant here. On the other hand, even in the Brownian case, immersion is not enough to grant the converse implication from (2.11) to (2.15). In fact, the key property granting the equivalence between (2.11) and (2.15), including in models with jumps, is not immersion, but rather the property that $\mathtt{S}$ is continuous and nonincreasing (see the comments following Proposition 2.1), which corresponds to the case of a pseudo-stopping time avoiding $\mathbb{F}$ stopping times (cf. Section 4.1). In the sequel, we call "basic reduced-form setup," by contrast with the "extended reduced-form setup" provided by the invariance times of this paper, the case where $\mathtt{S}$ has no martingale component (i.e., $\mathtt{Q}=0$), θ has an intensity (hence $\mathtt{S}=\mathtt{S}_0+\mathtt{D}$ is continuous, by Lemma A.1) and $\mathbb{G}$ is $\mathbb{F}$ progressively enlarged by θ. The simplest situation of this kind is the Cox process framework [see Bielecki, Jeanblanc and Rutkowski (2009), Chapter 3], in which case immersion holds, but this does not necessarily need to be the case even in a basic reduced-form setup.

[13] See He, Wang and Yan (1992), Exercise 9.4(1).

3. Invariance measures and invariance times. For a given triplet $(\mathbb{F}, \mathbb{G}, \theta)$ satisfying the condition (B), for a given positive constant T, we introduce the following.

Condition (A). There exists a probability measure $\mathbb{P}$ equivalent to $\mathbb{Q}$ on $\mathcal{F}_T$ such that, for any $(\mathbb{F}, \mathbb{P})$ local martingale P, $P^{\theta-}$ is a $(\mathbb{G}, \mathbb{Q})$ local martingale on $[0, T]$, that is,

$$P \in \mathcal{M}(\mathbb{F}, \mathbb{P}) \quad \Longrightarrow \quad P^{\theta-} \in \mathcal{M}_{[0,T]}(\mathbb{G}, \mathbb{Q}).$$

If this condition is satisfied, we call the random time θ an invariance time and the related probability measure $\mathbb{P}$ an invariance measure.

If θ is $\mathbb{G}$ predictable and $\mathbb{F} = \mathbb{G}$, then $\mathbb{P} = \mathbb{Q}$ is an invariance measure. But we are mostly interested in the case where θ has a nontrivial totally inaccessible part and θ is not an $\mathbb{F}$ stopping time. The possibility to change the measure in the condition (A) is material, for instance, in the dynamic Gaussian copula model of Crépey and Song (2016) (see Section 4.4 below).

To the best of our knowledge, the condition (A) has not been considered before in the probabilistic literature. In relation with it, one may think of the density hypothesis of Jacod (1987), initially formulated in a setup of initial enlargement and reconsidered in a progressive enlargement setup in Jeanblanc and Le Cam (2009), under which there exists a measure change to a probability that makes the reference filtration $\mathbb{F}$ and the random time θ independent. However, under an invariance measure $\mathbb{P}$, $\mathbb{F}$ and θ do not need to be independent. The spirit of invariance times is not to extend the case of independence by a measure change.

Stopping before θ in the condition (A), rather than at θ in the case of pseudo-stopping times (cf. Section 4.1), appears naturally in the motivating application of Section 2.2. On top of that, there are (at least) two reasons for stopping before θ rather than at θ in the condition (A). First, in view of the optional version of Lemmas 2.2(2) [under the condition (B) which is assumed throughout the paper], $P^{\theta-}$ is determined by the information of $\mathbb{F}$, which is not the case of P^{θ}. Second, as explained after equation (2.7), the bracket $\langle \mathsf{S}, Q\rangle$ in the Jeulin–Yor formula (2.2) is intrinsically linked with $Q^{\theta-}$, rather than with Q^{θ}.

This section is a theoretical study of the condition (A). Sections 3.1 and 3.2 establish the invariance measure and invariance time characterizations of Theorems 3.1 and 3.2–3.3. Section 3.3 studies the positivity of the stochastic exponential $\mathcal{E}(\mathbb{1}_{\{p\mathsf{S}>0\}} \frac{1}{p\mathsf{S}} \centerdot \mathsf{Q})$ that appears as the tentative density process of the measure change in Theorem 3.2. Section 3.4 is about the true martingale property of the multiplicative martingale part $\mathcal{Q}$ of S, which, under the condition (A), will be seen in Theorems 3.2–3.3 to coincide with $\mathcal{E}(\mathbb{1}_{\{p\mathsf{S}>0\}} \frac{1}{p\mathsf{S}} \centerdot \mathsf{Q})$ on $[0, T]$. Section 3.5 yields a characterization of local martingales under an invariance measure.

3.1. *Azéma supermartingale characterization of invariance measures.* The condition (A) is an existence condition for invariance measures. Before characterizing this existence, we consider in this section the conditions for a given probability measure $\mathbb{P}$, equivalent to $\mathbb{Q}$ on $\mathcal{F}_T$, to be an invariance measure.

Given a probability measure $\mathbb{P}$ equivalent to $\mathbb{Q}$ on $\mathcal{F}_T$, we denote by q the $(\mathbb{F}, \mathbb{Q})$ martingale of the density functions $\frac{d\mathbb{P}}{d\mathbb{Q}}|_{\mathcal{F}_{t\wedge T}}$, $t \in \mathbb{R}_+$. We also introduce $p = \frac{1}{q}$ and the stochastic logarithms $\bar{p}$ and $\bar{q}$ such that

$$p = p_0 \mathcal{E}(\bar{p}), \qquad q = q_0 \mathcal{E}(\bar{q}), \qquad \bar{p}_0 = \bar{q}_0 = 0. \tag{3.1}$$

In particular, p and $\bar{p}$ (resp., q and $\bar{q}$) are $(\mathbb{F}, \mathbb{P})$ [resp., $(\mathbb{F}, \mathbb{Q})$] local martingales on $[0, T]$.

LEMMA 3.1. *We consider a probability measure $\mathbb{P}$ equivalent to $\mathbb{Q}$ on $\mathcal{F}_T$ with the notation introduced in* (3.1). *The two conditions that follow are equivalent*:

$$q = q_0 \mathcal{E}\left(\frac{1}{{}^p\mathsf{S}} \centerdot \mathsf{Q}\right) \qquad \text{on } \{{}^p\mathsf{S} > 0\} \cap [0, T], \tag{3.2}$$

$${}^p\mathsf{S} \centerdot \bar{q} = \mathsf{Q} \qquad \text{on } [0, T]. \tag{3.3}$$

If they hold, then $\mathbb{1}_{\{{}^p\mathsf{S}>0\}} \frac{1}{{}^p\mathsf{S}}$ is $(\mathbb{F}, \mathbb{Q})$ integrable with respect to Q on $[0, T]$.

PROOF. Recalling (1.2) regarding the notion of a stochastic integral on the predictable interval $I = \{{}^p\mathsf{S} > 0\} \cap [0, T]$, we can interpret (3.2) by its versions stopped at each of the ζ_n, where $(\zeta_n)_{n\in\mathbb{N}}$ is the sequence that appears in (A.8). Therefore, given the relationship between the stochastic exponential and the stochastic logarithm, (3.2) is equivalent to

$$\mathbb{1}_{(0,\zeta_n\wedge T]} \centerdot \bar{q} = \mathbb{1}_{(0,\zeta_n\wedge T]} \frac{1}{{}^p\mathsf{S}} \centerdot \mathsf{Q}, \qquad \forall n \in \mathbb{N}, \tag{3.4}$$

which in turn is equivalent to

$${}^p\mathsf{S}\mathbb{1}_{(0,\zeta_n\wedge T]} \centerdot \bar{q} = \mathbb{1}_{(0,\zeta_n\wedge T]} \centerdot \mathsf{Q}, \qquad \forall n \in \mathbb{N}.$$

In virtue of the dominated convergence theorem for stochastic integrals,[14] this is equivalent to

$${}^p\mathsf{S} \centerdot \bar{q} = {}^p\mathsf{S}\mathbb{1}_{\{{}^p\mathsf{S}>0\}} \centerdot \bar{q} = \mathbb{1}_{\{{}^p\mathsf{S}>0\}} \centerdot \mathsf{Q} \quad \text{on } [0, T],$$

which is equivalent to (3.3) because of Lemma 2.5. The first part of the lemma is proved. The second part holds because $\mathbb{1}_{\{{}^p\mathsf{S}>0\}} \frac{1}{{}^p\mathsf{S}}$ is $(\mathbb{F}, \mathbb{Q})$ integrable with respect to ${}^p\mathsf{S} \centerdot \bar{q}$ on $[0, T]$. □

A similar reasoning can be used for proving the following result.

[14] Cf. He, Wang and Yan (1992), Theorem 9.30.

LEMMA 3.2. *The process* $\mathbb{1}_{\{^p\mathsf{S}>0\}}\frac{1}{^p\mathsf{S}}$ *is* $(\mathbb{F},\mathbb{Q})$ *integrable with respect to* Q *on* $[0,T]$ *if and only if the multiplicative martingale part* $\mathcal{Q}$ *of* S *as of* (2.3) *is a Doléans–Dade exponential on* $[0,T]$, *in which case the identity* $\mathcal{Q}=\mathcal{E}(\mathbb{1}_{\{^p\mathsf{S}>0\}}\frac{1}{^p\mathsf{S}}\centerdot\mathsf{Q})$ *holds on* $[0,T]$.

PROOF. By (2.3), we have

$$\mathbb{1}_{[0,\zeta_n]}\frac{1}{^p\mathsf{S}}\centerdot\mathsf{Q}=\mathbb{1}_{[0,\zeta_n]}\mathbb{1}_{\{\mathcal{Q}_->0\}}\frac{1}{\mathcal{Q}_-}\centerdot\mathcal{Q},\qquad\forall n\in\mathbb{N}.$$

Hence, by He, Wang and Yan (1992), Theorem 9.2, if $\mathbb{1}_{\{\mathcal{Q}_->0\}}\frac{1}{\mathcal{Q}_-}\centerdot\mathcal{Q}$ exists on $[0,T]$, then the process $\mathbb{1}_{\{^p\mathsf{S}>0\}}\frac{1}{^p\mathsf{S}}$ is $(\mathbb{F},\mathbb{Q})$ integrable with respect to Q on $[0,T]$. In addition, by the dominated convergence theorem for stochastic integrals, we have on $[0,T]$

$$\mathbb{1}_{\{^p\mathsf{S}>0\}}\frac{1}{^p\mathsf{S}}\centerdot\mathsf{Q}=\mathbb{1}_{\{^p\mathsf{S}>0\}}\mathbb{1}_{\{\mathcal{Q}_->0\}}\frac{1}{\mathcal{Q}_-}\centerdot\mathcal{Q}=\mathbb{1}_{\{\mathcal{Q}_->0\}}\frac{1}{\mathcal{Q}_-}\centerdot\mathcal{Q},$$

as $\mathrm{supp}(d[\mathcal{Q},\mathcal{Q}])\subset\{^p\mathsf{S}>0\}$, by (2.3) and Lemma 2.5.

Conversely, suppose that $\mathbb{1}_{\{^p\mathsf{S}>0\}}\frac{1}{^p\mathsf{S}}$ is $(\mathbb{F},\mathbb{Q})$ integrable with respect to Q on $[0,T]$. Then, on $[0,T]$, we have by (2.3) the pointwise limits

$$\mathcal{Q}=\lim_{n\to\infty}\mathcal{E}\left(\frac{1}{^p\mathsf{S}}\centerdot\mathsf{Q}\right)^{\zeta_n}=\lim_{n\to\infty}\mathcal{E}\left(\mathbb{1}_{\{^p\mathsf{S}>0\}}\frac{1}{^p\mathsf{S}}\centerdot\mathsf{Q}\right)^{\zeta_n}=\mathcal{E}\left(\mathbb{1}_{\{^p\mathsf{S}>0\}}\frac{1}{^p\mathsf{S}}\centerdot\mathsf{Q}\right),$$

by stochastic dominated convergence. □

Theorem 3.1 below relates the conditions introduced in Lemma 3.1 to the invariance measure property. The proof that follows is based on a reduction of all the computations from $\mathbb{G}$ to $\mathbb{F}$. The basic idea is to make use of Lemma 2.2(4) for establishing an equivalence between the invariance measure property and an SDE (3.6) for the process p. Being based on this reduction methodology, this proof does not directly explain how a Girsanov drift can compensate a Jeulin–Yor drift. Given the importance of that matter for the purpose of this paper, we provide in Section C an alternative proof of Theorem 3.1 based on this compensation.

THEOREM 3.1. *A probability measure* $\mathbb{P}$ *equivalent to* $\mathbb{Q}$ *on* $\mathcal{F}_T$ *is an invariance measure if and only if* (3.2) [*i.e.*, (3.3)] *holds.*

PROOF. The invariance measure property for $\mathbb{P}$ is equivalent to

$$\left(P^{\theta-}\right)^T=\left(P^T\right)^{\theta-}\in\mathcal{M}(\mathbb{G},\mathbb{Q}),\qquad\forall P\in\mathcal{M}_{[0,T]}(\mathbb{F},\mathbb{P}).$$

By Lemma 2.2(4), this holds if and only if

$$\mathsf{S}_-\centerdot P^T+\left[\mathsf{S},P^T\right]\in\mathcal{M}_{\{\mathsf{S}_->0\}}(\mathbb{F},\mathbb{Q}),\qquad\forall P\in\mathcal{M}_{[0,T]}(\mathbb{F},\mathbb{P}).$$

We proceed to transform this property into an SDE (3.6) for p. The integration by parts formula and the Doob–Meyer decomposition of $\mathtt{S}$ yield

$$P^T\mathtt{S} = P^T_- \centerdot \mathtt{S} + \mathtt{S}_- \centerdot P^T + [P^T, \mathtt{S}] = P^T_- \centerdot \mathtt{Q} - P^T_- \centerdot \mathtt{D} + \mathtt{S}_- \centerdot P^T + [P^T, \mathtt{S}].$$

Hence, the preceding property is equivalent to

$$P^T\mathtt{S} + P^T_- \centerdot \mathtt{D} \in \mathcal{M}_{\{\mathtt{S}_->0\}}(\mathbb{F}, \mathbb{Q}), \qquad \forall P \in \mathcal{M}_{[0,T]}(\mathbb{F}, \mathbb{P}). \tag{3.5}$$

Note that $\mathcal{M}_{[0,T]}(\mathbb{F}, \mathbb{P}) = \{Qp; Q \in \mathcal{M}_{[0,T]}(\mathbb{F}, \mathbb{Q})\}$.[15] For any $Q \in \mathcal{M}_{[0,T]}(\mathbb{F}, \mathbb{Q})$, an application of the integration by parts formula to $Q^T(p^T_- \centerdot \mathtt{D})$ yields

$$Q^T(p^T\mathtt{S} + p^T_- \centerdot \mathtt{D}) = (Qp)^T\mathtt{S} + (Qp)^T_- \centerdot \mathtt{D} + (p^T_- \centerdot \mathtt{D}) \centerdot Q^T + [Q^T, p^T_- \centerdot \mathtt{D}].$$

On the right-hand side, by Yoeurp's lemma,[16] the bracket $[Q^T, p^T_- \centerdot \mathtt{D}]$ is in $\mathcal{M}(\mathbb{F}, \mathbb{Q})$, as is also $(p^T_- \centerdot \mathtt{D}) \centerdot Q^T$. Hence, (3.5) is equivalent to

$$Q^T(p^T\mathtt{S} + p^T_- \centerdot \mathtt{D}) \in \mathcal{M}_{\{\mathtt{S}_->0\}}(\mathbb{F}, \mathbb{Q}), \qquad \forall Q \in \mathcal{M}_{[0,T]}(\mathbb{F}, \mathbb{Q}).$$

By Lemma B.1, this in turn is equivalent to

$$p^T\mathtt{S} + p^T_- \centerdot \mathtt{D} = p_0\mathtt{S}_0 \qquad \text{on } \{\mathtt{S}_- > 0\} \cap [0, T]. \tag{3.6}$$

Noting that $p\mathtt{S} + p_- \centerdot \mathtt{D} = p\mathtt{S} + (p\mathtt{S})_- \frac{1}{\mathtt{S}_-} \centerdot \mathtt{D}$, we recognize in (3.6) the linear SDE for the stochastic exponential of $(-\frac{1}{\mathtt{S}_-} \centerdot \mathtt{D})$ with the initial condition $p_0\mathtt{S}_0$ on $\{\mathtt{S}_- > 0\} \cap [0, T]$, that is, (3.6) is equivalent to

$$p\mathtt{S} = p_0\mathtt{S}_0\mathcal{E}\left(-\frac{1}{\mathtt{S}_-} \centerdot \mathtt{D}\right) \qquad \text{on } \{\mathtt{S}_- > 0\} \cap [0, T]. \tag{3.7}$$

Recall $\{\mathtt{S}_- > 0\} \setminus \{{}^p\mathtt{S} > 0\} = [\eta]$ [cf. (2.10) and (2.8)]. Actually, the identity (3.7) is equivalent to the analogous identity on the smaller set $\{{}^p\mathtt{S} > 0\} \cap [0, T]$. To understand why, note that, if η is finite, then $\mathtt{S}_\eta = 0$, whereas (2.9) yields $\mathcal{E}(-\frac{1}{\mathtt{S}_-} \centerdot \mathtt{D})_\eta = 0$, so that one has the trivial equality

$$(p\mathtt{S})_\eta = p_0\mathtt{S}_0\mathcal{E}\left(-\frac{1}{\mathtt{S}_-} \centerdot \mathtt{D}\right)_\eta = 0.$$

Hence, the identity (3.7) is equivalent to

$$p\mathtt{S} = p_0\mathtt{S}_0\mathcal{E}\left(-\frac{1}{\mathtt{S}_-} \centerdot \mathtt{D}\right) \qquad \text{on } \{{}^p\mathtt{S} > 0\} \cap [0, T], \tag{3.8}$$

that is, in view of Lemma 2.2 5), to

$$p\mathtt{S}_0\mathcal{E}\left(-\frac{1}{\mathtt{S}_-} \centerdot \mathtt{D}\right)\mathcal{E}\left(\frac{1}{{}^p\mathtt{S}} \centerdot \mathtt{Q}\right) = p_0\mathtt{S}_0\mathcal{E}\left(-\frac{1}{\mathtt{S}_-} \centerdot \mathtt{D}\right) \qquad \text{on } \{{}^p\mathtt{S} > 0\} \cap [0, T],$$

which is equivalent to (3.2), because $\mathtt{S}_0\mathcal{E}(-\frac{1}{\mathtt{S}_-} \centerdot \mathtt{D})$ is positive on $\{{}^p\mathtt{S} > 0\}$, by (2.10). □

[15] Cf. He, Wang and Yan (1992), Theorem 12.12.
[16] See He, Wang and Yan (1992), Exercise 9.4(1).

3.2. *Azéma supermartingale characterization of invariance times.* In this section, we provide two (closely related) Azéma supermartingale characterizations of the condition (A). Given Theorem 3.1, which designates $\mathcal{E}(\mathbb{1}_{\{p_{\mathsf{S}}>0\}}\frac{1}{p_{\mathsf{S}}}\centerdot \mathsf{Q})$ as the tentative $\mathbb{F}$ density process of $\frac{d\mathbb{P}}{d\mathbb{Q}}$ for some invariance measure $\mathbb{P}$, all needs be done for verifying the condition (A) is to check the positivity and true martingale property of $\mathcal{E}(\mathbb{1}_{\{p_{\mathsf{S}}>0\}}\frac{1}{p_{\mathsf{S}}}\centerdot \mathsf{Q})$.

THEOREM 3.2. *The condition* (A) *holds if and only if*

$$\begin{cases} \mathbb{1}_{\{p_{\mathsf{S}}>0\}}\dfrac{1}{p_{\mathsf{S}}} \text{ is } (\mathbb{F},\mathbb{Q}) \text{ integrable with respect to } \mathsf{Q} \text{ on } [0,T] \text{ and} \\ \mathcal{E}\left(\mathbb{1}_{\{p_{\mathsf{S}}>0\}}\dfrac{1}{p_{\mathsf{S}}}\centerdot \mathsf{Q}\right) \text{ is a positive } (\mathbb{F},\mathbb{Q}) \text{ true martingale on } [0,T]. \end{cases} \tag{3.9}$$

If this is satisfied, then an invariance measure $\mathbb{P}$ is defined by the $\mathbb{Q}$ density $\mathcal{E}(\mathbb{1}_{\{p_{\mathsf{S}}>0\}}\frac{1}{p_{\mathsf{S}}}\centerdot \mathsf{Q})_T$ on $\mathcal{F}_T$ and any invariance measure $\mathbb{P}$ is such that

$$q_T = q_0\mathcal{E}\left(\mathbb{1}_{\{p_{\mathsf{S}}>0\}}\frac{1}{p_{\mathsf{S}}}\centerdot \mathsf{Q}\right)_T \mathcal{E}(\mathbb{1}_{\{p_{\mathsf{S}}=0\}}\centerdot \bar{q})_T. \tag{3.10}$$

PROOF. Suppose the existence of an invariance measure $\mathbb{P}$. By Theorem 3.1, the $\mathbb{F}$ density process q of $\frac{d\mathbb{P}}{d\mathbb{Q}}$ satisfies (3.2). By Lemma 3.1, the $\mathbb{F}$ predictable process $\mathbb{1}_{\{p_{\mathsf{S}}>0\}}\frac{1}{p_{\mathsf{S}}}$ is $(\mathbb{F},\mathbb{Q})$ integrable with respect to Q on $[0,T]$.

In order to show that the process $\mathcal{Q} = \mathcal{E}(\mathbb{1}_{\{p_{\mathsf{S}}>0\}}\frac{1}{p_{\mathsf{S}}}\centerdot \mathsf{Q})$ (cf. Lemma 3.2) is a positive $(\mathbb{F},\mathbb{Q})$ true martingale on $[0,T]$, it is enough to represent it as the $(\mathbb{F},\mathbb{Q})$ conditional expectation process of a positive and $\mathbb{Q}$ integrable $\mathcal{F}_T$ measurable random variable. By (3.2) and (A.8), for any $n\in\mathbb{N}$ and $t\in[0,T]$, we have

$$\begin{aligned} q_0\mathcal{E}\left(\mathbb{1}_{\{p_{\mathsf{S}}>0\}}\frac{1}{p_{\mathsf{S}}}\centerdot \mathsf{Q}\right)_t^{\zeta_n\wedge T} &= q_0\mathcal{E}\left(\frac{1}{p_{\mathsf{S}}}\centerdot \mathsf{Q}\right)_t^{\zeta_n\wedge T} \\ &= q_t^{\zeta_n\wedge T} = \mathbb{E}[q_{\zeta_n\wedge T}|\mathcal{F}_t] \\ &= \mathbb{E}\big[\mathbb{E}[q_T|\mathcal{F}_{\zeta_n\wedge T}]|\mathcal{F}_t\big]. \end{aligned} \tag{3.11}$$

On the one hand, noting that $\mathcal{Q} = 1 + \mathcal{Q}_-\mathbb{1}_{\{p_{\mathsf{S}}>0\}}\frac{1}{p_{\mathsf{S}}}\centerdot \mathsf{Q}$, the dominated convergence theorem for stochastic integrals yields

$$\begin{aligned} \lim_{n\to\infty}\mathcal{Q}^{\zeta_n\wedge T} &= 1 + \lim_{n\to\infty}\mathbb{1}_{(0,\zeta_n\wedge T]}\centerdot \mathcal{Q} \\ &= 1 + \lim_{n\to\infty}\mathbb{1}_{(0,\zeta_n\wedge T]}\mathcal{Q}_-\mathbb{1}_{\{p_{\mathsf{S}}>0\}}\frac{1}{p_{\mathsf{S}}}\centerdot \mathsf{Q} \\ &= 1 + \mathcal{Q}_-\mathbb{1}_{\{p_{\mathsf{S}}>0\}}\frac{1}{p_{\mathsf{S}}}\centerdot \mathsf{Q} = \mathcal{Q}. \end{aligned}$$

On the other hand,

$$\lim_{n\to\infty}\mathbb{E}[q_T|\mathcal{F}_{\zeta_n\wedge T}] = \mathbb{E}\left[q_T \,\middle|\, \bigvee_{n\in\mathbb{N}}(\mathcal{F}_{\zeta_n\wedge T})\right] \qquad \text{holds in } L^1.$$

Hence, for $t \in [0, T]$, we obtain by passing to the limit in (3.11)

$$\mathcal{E}\left(\mathbb{1}_{\{^p\mathtt{S}>0\}} \frac{1}{^p\mathtt{S}} \cdot \mathtt{Q}\right)_t = \mathbb{E}\left[\mathbb{E}\left[\frac{q_T}{q_0} \,\Big|\, \bigvee_{n\in\mathbb{N}} \mathcal{F}_{\zeta_n \wedge T}\right] \Big|\, \mathcal{F}_t\right], \tag{3.12}$$

where $\mathbb{E}[\frac{q_T}{q_0} | \bigvee_{n\in\mathbb{N}} \mathcal{F}_{\zeta_n\wedge T}]$ is a positive and $\mathbb{Q}$ integrable $\mathcal{F}_T$ measurable random variable. As a consequence, the process $\mathcal{E}(\mathbb{1}_{\{^p\mathtt{S}>0\}} \frac{1}{^p\mathtt{S}} \cdot \mathtt{Q})$ is a positive $(\mathbb{F}, \mathbb{Q})$ uniformly integrable martingale on $[0, T]$, which proves (3.9).

Conversely, supposing (3.9), we can define a probability measure $\mathbb{P}$ by an $\mathbb{F}$ density process of $\frac{d\mathbb{P}}{d\mathbb{Q}}$ given as $q = \mathcal{E}(\mathbb{1}_{\{^p\mathtt{S}>0\}} \frac{1}{^p\mathtt{S}} \cdot \mathtt{Q})$ on $[0, T]$, hence $\bar{q} = \mathbb{1}_{\{^p\mathtt{S}>0\}} \frac{1}{^p\mathtt{S}} \cdot \mathtt{Q}$ on $[0, T]$. Given Lemma 2.5, this establishes (3.3). By Theorem 3.1, $\mathbb{P}$ is therefore an invariance measure, which proves the condition (A).

Assuming the condition (A), (3.10) is the consequence of (3.3) (which implies $\mathbb{1}_{\{^p\mathtt{S}>0\}}\bar{q} = \mathbb{1}_{\{^p\mathtt{S}>0\}} \frac{1}{^p\mathtt{S}} \cdot \mathtt{Q}$) and of the formula $\mathcal{E}(\bar{q}) = \mathcal{E}(\mathbb{1}_{\{^p\mathtt{S}>0\}}\bar{q})\mathcal{E}(\mathbb{1}_{\{^p\mathtt{S}=0\}}\bar{q})$ [cf. Jacod (1979), Proposition 6.4]. $\square$

In view of Lemma 3.2, Theorem 3.2 can be restated in the following form.

THEOREM 3.3. *The condition* (A) *holds if and only if the multiplicative martingale part $\mathcal{Q}$ of* $\mathtt{S}$ *as of* (2.3) *is a positive* $(\mathbb{F}, \mathbb{Q})$ *true martingale on* $[0, T]$. *In this case, we have* $\mathcal{Q} = \mathcal{E}(\mathbb{1}_{\{^p\mathtt{S}>0\}} \frac{1}{^p\mathtt{S}} \cdot \mathtt{Q})$ *on* $[0, T]$.

In what follows, we examine in the light of Theorems 3.1 and 3.2–3.3 the extreme cases where $\mathbb{F} = \mathbb{G}$ or $\mathbb{P} = \mathbb{Q}$ in the condition (A).

PROPOSITION 3.1. (1) $\mathbb{Q}$ *itself is an invariance measure for all* $T > 0$ *if and only if* $\mathtt{Q} = 0$.

(2) *If* $\mathbb{F} = \mathbb{G}$ *and* θ *is totally inaccessible with* $\mathbb{Q}(\theta \le T)$ *positive, then the condition* (A) *cannot hold on* $[0, T]$.

PROOF. (1) In the case where $\mathbb{P} = \mathbb{Q}$, we have $q = q_0$ on $[0, T]$, that is, $\bar{q} = 0$ on $[0, T]$. Hence, in view of Theorem 3.1 and of (3.3), $\mathbb{P}$ is an invariance measure for all $T > 0$ if and only if $\mathtt{Q} = 0$ on $[0, T]$.

(2) In the case where $\mathbb{F} = \mathbb{G}$ and θ is totally inaccessible, we have by (A.2) and Lemma A.1:

$$\mathtt{S} = \mathtt{J}, \qquad \mathtt{D} \text{ is continuous}, \qquad {}^p\mathtt{S} = \mathtt{J}_- \quad \text{and} \quad \mathtt{Q} = \mathtt{J} + \mathtt{D} - 1.$$

Hence, using the stochastic exponential formula,[17]

$$\mathcal{E}\left(\mathbb{1}_{\{^p\mathtt{S}>0\}} \frac{1}{^p\mathtt{S}} \cdot \mathtt{Q}\right)_t = \mathcal{E}(\mathtt{Q})_t = e^{\mathtt{Q}_t} \prod_{s\le t} (1 + \Delta_s \mathtt{Q}) e^{-\Delta_s \mathtt{Q}} = e^{\mathtt{J}_t + \mathtt{D}_t - 1} \mathtt{J}_t = e^{\mathtt{D}_t} \mathtt{J}_t,$$

[17]See, for example, He, Wang and Yan (1992), Theorem 9.39.

which vanishes at θ on $\{\theta \le T\}$. Therefore, in view of Theorem 3.2, the condition (A) cannot hold on $[0, T]$ unless $\mathbb{Q}(\theta \le T) = 0$. $\square$

EXAMPLE 3.1. Let $\mathbb{G}$ be the augmentation of the natural filtration of the jump process at an exponential time θ relative to some probability measure $\mathbb{Q}$.

For $\mathbb{F} = \mathbb{G}$ [so that the condition (B) holds trivially], Proposition 3.1(2) shows that the condition (A) does not hold. This can also be recovered directly from the definitions. In fact, for any probability measure $\mathbb{P}$ equivalent to $\mathbb{Q}$ on $\mathcal{F}_T$, each $(\mathbb{F}, \mathbb{P})$ local martingale P has to be an $(\mathbb{F}, \mathbb{P})$ stochastic integral with respect to the compensated jump process of θ [cf. He, Wang and Yan (1992), Theorem 13.20]. Thus, the process $P^{\theta-}$ is absolutely continuous, hence it is not a $(\mathbb{G}, \mathbb{Q})$ local martingale on $[0, T]$ unless it is constant on $[0, T]$. Therefore, $\mathbb{P}$ is not an invariance measure for $\mathbb{F} = \mathbb{G}$.

For $\mathbb{F}$ trivial, any $\mathbb{G}$ predictable process coincides with a Borel function before θ, so that the condition (B) is satisfied. The constants are the only $(\mathbb{F} = \{\varnothing, \Omega\}, \mathbb{Q})$ local martingales, so that $\mathbb{Q}$ itself is an invariance measure and the condition (A) is satisfied. Consistent with this conclusion in regard of Theorem 3.2, S is deterministic (equal to the survival function of θ), $\mathsf{Q} = 0$ and $q \equiv 1$, hence (3.2) is satisfied.

In conclusion, an exponential time θ in its own filtration $\mathbb{G}$ is an invariance time for $\mathbb{F}$ trivial but not for $\mathbb{F} = \mathbb{G}$.

3.3. *Positivity of* $\mathcal{E}(\mathbb{1}_{\{^p\mathsf{S}>0\}}\frac{1}{^p\mathsf{S}} \centerdot \mathsf{Q})$ *on* $[0, T]$. The positivity condition of $\mathcal{E}(\mathbb{1}_{\{^p\mathsf{S}>0\}}\frac{1}{^p\mathsf{S}} \centerdot \mathsf{Q})$ on $[0, T]$ is a key element in the characterization of Theorem 3.2. There exist general results on the positivity of a stochastic exponential.[18] But, for our proof of Theorem 3.5 below, we need a different characterization. In this section, we show that the positivity of $\mathcal{E}(\mathbb{1}_{\{^p\mathsf{S}>0\}}\frac{1}{^p\mathsf{S}} \centerdot \mathsf{Q})$ (assuming that $\mathbb{1}_{\{^p\mathsf{S}>0\}}\frac{1}{^p\mathsf{S}}$ is $(\mathbb{F}, \mathbb{Q})$ integrable with respect to Q on $[0, T]$) can be characterized in terms of the time ς of first zero of S [cf. (A.5)]. Specifically, the positivity of $\mathcal{E}(\mathbb{1}_{\{^p\mathsf{S}>0\}}\frac{1}{^p\mathsf{S}} \centerdot \mathsf{Q})$ reduces to the predictability of the $\mathbb{F}$ stopping time $\varsigma_{\{\varsigma \le T\}}$.

LEMMA 3.3. *Let σ be an $\mathbb{F}$ predictable stopping time. Then, on the set $\{\sigma < \infty\}$, we have $^p\mathsf{S}_\sigma = 0$ if and only if $\sigma \ge \varsigma$.*

PROOF. By definition of the predictable projection and by nonnegativity of S, on the set $\{\sigma < \infty\}$, it holds

$$^p\mathsf{S}_\sigma = 0 \iff \mathbb{E}[\mathsf{S}_\sigma | \mathcal{F}_{\sigma-}] = 0 \iff \mathsf{S}_\sigma = 0 \iff \sigma \ge \varsigma. \qquad \square$$

LEMMA 3.4. *The $\mathbb{F}$ stopping times $\varsigma_{\{\varsigma<\infty, \mathsf{S}_{\varsigma-}=0\}}$ and $\varsigma_{\{\varsigma<\infty, {}^p\mathsf{S}_\varsigma=0\}}$ are predictable.*

[18] Cf. He, Wang and Yan (1992), Lemma 9.40.

PROOF. The assertion regarding $\varsigma_{\{\varsigma<\infty,\mathrm{S}_{\varsigma-}=0\}}$ comes from the proof of Theorem 9.41 in He, Wang and Yan (1992). For $\varsigma_{\{\varsigma<\infty,{}^p\mathrm{S}_\varsigma=0\}}$, it suffices to note that

$$\varsigma_{\{\varsigma<\infty,{}^p\mathrm{S}_\varsigma=0\}}=\varsigma_{\{\varsigma<\infty,\mathrm{S}_{\varsigma-}=0\}}\wedge\varsigma_{\{\varsigma<\infty,\mathrm{S}_{\varsigma-}>0,{}^p\mathrm{S}_\varsigma=0\}}=\varsigma_{\{\varsigma<\infty,\mathrm{S}_{\varsigma-}=0\}}\wedge\eta,$$

where η is the $\mathbb{F}$ predictable time introduced in Lemma 2.4. □

THEOREM 3.4. *Assuming that* $\mathbb{1}_{\{{}^p\mathrm{S}>0\}}\frac{1}{{}^p\mathrm{S}}$ *is* $(\mathbb{F},\mathbb{Q})$ *integrable with respect to* Q *on* $[0,T]$, *the following conditions are equivalent to each other*:

(i) $\mathcal{E}(\mathbb{1}_{\{{}^p\mathrm{S}>0\}}\frac{1}{{}^p\mathrm{S}}\centerdot\mathrm{Q})>0$ *on* $[0,T]$,
(ii) ${}^p\mathrm{S}_\varsigma=0$ *on* $\{\varsigma\le T\}$,
(iii) $\varsigma_{\{\varsigma\le T\}}$ *is a predictable stopping time.*

PROOF. Note that

$$\mathbb{1}_{\{{}^p\mathrm{S}_t>0\}}\Delta_t\left(\frac{1}{{}^p\mathrm{S}}\centerdot\mathrm{Q}\right)=\mathbb{1}_{\{{}^p\mathrm{S}_t>0\}}\frac{1}{{}^p\mathrm{S}_t}\Delta_t\mathrm{Q}=\mathbb{1}_{\{{}^p\mathrm{S}_t>0\}}\left(\frac{\mathrm{S}_t}{{}^p\mathrm{S}_t}-1\right),\qquad\forall t\in[0,T],$$

where (A.2) was used in the last equality. Hence, $\mathcal{E}(\mathbb{1}_{\{{}^p\mathrm{S}>0\}}\frac{1}{{}^p\mathrm{S}}\centerdot\mathrm{Q})$ is positive on $[0,T]$ if and only if

$$\mathbb{1}_{\{{}^p\mathrm{S}_t>0\}}\left(\frac{\mathrm{S}_t}{{}^p\mathrm{S}_t}-1\right)>-1,\qquad\forall t\in[0,T].\tag{3.13}$$

Recalling (A.10) and (A.9), the only way (3.13) can break down is if $\varsigma\le T$ and ${}^p\mathrm{S}_\varsigma>0$, which shows the equivalence between (i) and (ii).

To prove the equivalence between (ii) and (iii), let $\xi=\varsigma_{\{\varsigma<\infty,{}^p\mathrm{S}_\varsigma=0\}}$, which is predictable by Lemma 3.4. If

$${}^p\mathrm{S}_\varsigma=0\qquad\text{on }\{\varsigma\le T\},\tag{3.14}$$

then

$$\varsigma_{\{\varsigma\le T\}}=(\varsigma_{\{\varsigma<\infty,{}^p\mathrm{S}_\varsigma=0\}})_{\{\varsigma\le T\}}=\xi_{\{\varsigma\le T\}}$$

and

$$\{\varsigma\le T\}=\{\varsigma_{\{\varsigma<\infty,{}^p\mathrm{S}_\varsigma=0\}}\le T\}=\{\xi\le T\}$$

is $\mathcal{F}_{\xi-}$ measurable. Hence, by He, Wang and Yan (1992), Theorem 3.29(7), $\varsigma_{\{\varsigma\le T\}}=\xi_{\{\varsigma\le T\}}$ is predictable. Conversely, if $\varsigma_{\{\varsigma\le T\}}$ is predictable, as $\varsigma_{\{\varsigma\le T\}}\ge\varsigma$, the condition (3.14) holds by Lemma 3.3. □

In particular, if S_T is positive almost surely, then $\varsigma_{\{\varsigma\le T\}}=\infty$, which is a predictable stopping time. Therefore, $\mathcal{E}(\frac{1}{{}^p\mathrm{S}}\centerdot\mathrm{Q})$ is positive on $[0,T]$, consistently with the last statement in Lemma 2.2(5).

EXAMPLE 3.2. Consider $\mathbb{F}=\mathbb{G}$ and θ given as a totally inaccessible stopping time such that $\mathbb{Q}(\theta\le T)>0$. Then $\varsigma=\theta$, so that the stopping time $\varsigma_{\{\varsigma\le T\}}$ is not predictable. Accordingly, as seen in the proof of Proposition 3.1, $\mathcal{E}(\mathbb{1}_{\{{}^p\mathrm{S}>0\}}\frac{1}{{}^p\mathrm{S}}\centerdot\mathrm{Q})$ vanishes at θ on $\{\theta\le T\}$.

3.4. *True martingale property of $\mathcal{Q}$.* The second key ingredient in the characterization of Theorem 3.2 is the $(\mathbb{F},\mathbb{Q})$ true martingale property of $\mathcal{E}(\mathbb{1}_{\{^p\mathtt{S}>0\}}\frac{1}{^p\mathtt{S}}\centerdot\mathtt{Q})$ on $[0,T]$ (on top of its positivity). Regarding the true martingale property of a Doléans–Dade exponential, one immediately thinks of Novikov–Kazamaki-type conditions (see Larsson and Ruf (2014) for a survey). However, under the condition (B), the true martingale property of $\mathcal{E}(\mathbb{1}_{\{^p\mathtt{S}>0\}}\frac{1}{^p\mathtt{S}}\centerdot\mathtt{Q})$ or, more precisely, of the multiplicative martingale part $\mathcal{Q}$ of $\mathtt{S}$ as of (2.3) (cf. Theorems 3.2–3.3), can be better studied by means of Azéma supermartingale computations. Note that, in the case of any nonnegative $(\mathbb{F},\mathbb{Q})$ local martingale (hence supermartingale) Q, the $(\mathbb{F},\mathbb{Q})$ true martingale property of Q on $[0,T]$ is equivalent to $\mathbb{E}Q_T=\mathbb{E}Q_0$.

The following example shows that there exists times θ for which $\mathcal{Q}$ is not a true martingale.

EXAMPLE 3.3. Let X be the inverse of a three-dimensional $(\mathbb{F},\mathbb{Q})$ Bessel process[19] starting from: 1. Define $X^*_t=\sup_{0\le s\le t}X_s$, for $t\ge 0$, and let $\theta=\sup\{t\ge 0: X_t=X^*_t\}$. According to Nikeghbali and Yor (2006), $\mathtt{S}=\frac{X}{X^*}$ is the Azéma supermartingale of θ. We have $\mathcal{Q}=X$, which is not an $(\mathbb{F},\mathbb{Q})$ true martingale on any nonempty interval $[0,T]$.

Our ensuing study of the true martingale property of $\mathcal{Q}$ on $[0,T]$ is based on the following properties. Note that $\mathcal{E}(\frac{1}{^p\mathtt{S}}\centerdot\mathtt{D})$ is well defined on the set $\{^p\mathtt{S}>0\}$ and, since $\theta\in\{\mathtt{S}_->0\}$ on $\{0<\theta<\infty\}$ [cf. (A.11)], we have $[0,\theta)\subset\{^p\mathtt{S}>0\}$.

LEMMA 3.5. (1) *The* $(\mathbb{F},\mathbb{Q})$ *optional projection of the process* $\mathcal{E}(\frac{1}{^p\mathtt{S}}\centerdot\mathtt{D})\mathbb{1}_{[0,\theta)}$ *is equal to the process* $\mathcal{E}(\frac{1}{^p\mathtt{S}}\centerdot\mathtt{D})\mathtt{S}\mathbb{1}_{\{^p\mathtt{S}>0\}}=\mathtt{S}_0\mathcal{E}(\frac{1}{^p\mathtt{S}}\centerdot\mathtt{Q})\mathbb{1}_{\{^p\mathtt{S}>0\}}=\mathtt{S}_0\mathcal{Q}\mathbb{1}_{\{^p\mathtt{S}>0\}}$.

(2) *The process* $\mathcal{E}(\frac{1}{^p\mathtt{S}}\centerdot\mathtt{D})\mathbb{1}_{[0,\theta)}$ *is a* $(\mathbb{G},\mathbb{Q})$ *local martingale on* $\{^p\mathtt{S}>0\}$.

(3) *If the family of the random variables* $\mathcal{E}(\frac{1}{^p\mathtt{S}}\centerdot\mathtt{D})_\sigma\mathbb{1}_{\{\sigma<\theta\}}$, *for any* $\mathbb{F}$ *stopping time* σ *such that* $[0,\sigma]\subset\{^p\mathtt{S}>0\}\cap[0,T]$, *is* $\mathbb{Q}$ *uniformly integrable, then* $\mathcal{Q}$ *is an* $(\mathbb{F},\mathbb{Q})$ *true martingale on* $[0,T]$.

PROOF. (1) Let σ be any $\mathbb{F}$ stopping time and χ be any bounded $\mathcal{F}_\sigma$ measurable random variable. We have (using $[0,\theta)\subset\{^p\mathtt{S}>0\}$)

$$
\begin{aligned}
\mathbb{E}\Big[\chi\mathcal{E}\Big(\frac{1}{^p\mathtt{S}}\centerdot\mathtt{D}\Big)_\sigma\mathbb{1}_{\{\sigma<\theta\}}\Big]&=\mathbb{E}\Big[\chi\mathcal{E}\Big(\frac{1}{^p\mathtt{S}}\centerdot\mathtt{D}\Big)_\sigma\mathbb{1}_{\{\sigma<\theta\}}\mathbb{1}_{\{\sigma\in\{^p\mathtt{S}>0\}\}}\Big]\\
&=\mathbb{E}\Big[\chi\mathcal{E}\Big(\frac{1}{^p\mathtt{S}}\centerdot\mathtt{D}\Big)_\sigma\mathtt{S}_\sigma\mathbb{1}_{\{\sigma\in\{^p\mathtt{S}>0\}\}}\mathbb{1}_{\{\sigma<\infty\}}\Big].
\end{aligned}
\tag{3.15}
$$

Hence, the $(\mathbb{F},\mathbb{Q})$ optional projection of the process $\mathcal{E}(\frac{1}{^p\mathtt{S}}\centerdot\mathtt{D})\mathbb{1}_{[0,\theta)}$ is equal to the process $\mathcal{E}(\frac{1}{^p\mathtt{S}}\centerdot\mathtt{D})\mathtt{S}\mathbb{1}_{\{^p\mathtt{S}>0\}}$. Moreover, by Lemma 2.2(5), on $\{^p\mathtt{S}>0\}$, we have

$$
\mathcal{E}\Big(\frac{1}{^p\mathtt{S}}\centerdot\mathtt{D}\Big)\mathtt{S}=\mathcal{E}\Big(\frac{1}{^p\mathtt{S}}\centerdot\mathtt{D}\Big)\mathtt{S}_0\mathcal{E}\Big(-\frac{1}{\mathtt{S}_-}\centerdot\mathtt{D}\Big)\mathcal{E}\Big(\frac{1}{^p\mathtt{S}}\centerdot\mathtt{Q}\Big)=\mathtt{S}_0\mathcal{E}\Big(\frac{1}{^p\mathtt{S}}\centerdot\mathtt{Q}\Big)=\mathtt{S}_0\mathcal{Q}.
$$

[19]Cf., Revuz and Yor (1999), Chapter V, Exercice (2.13).

(2) Note that $\mathrm{S}_0\mathcal{E}(\frac{1}{p_\mathrm{S}}\centerdot \mathrm{Q})$ is an $(\mathbb{F},\mathbb{Q})$ local martingale on $\{^p\mathrm{S}>0\}$. Let σ be a finite $\mathbb{F}$ stopping time such that $[0,\sigma]\subset\{^p\mathrm{S}>0\}$ and $\mathrm{S}_0\mathcal{E}(\frac{1}{p_\mathrm{S}}\centerdot \mathrm{Q})^\sigma$ is an $(\mathbb{F},\mathbb{Q})$ uniformly integrable martingale. Consider any $\mathbb{G}$ stopping time τ. Recalling that τ' is an $\mathbb{F}$ stopping time such that $\tau\wedge\theta=\tau'\wedge\theta$ [cf. Lemma 2.2(1)], we have

$$\begin{aligned}\mathbb{E}\Big[\mathcal{E}\Big(\frac{1}{p_\mathrm{S}}\centerdot \mathrm{D}\Big)_{\tau\wedge\sigma}\mathbb{1}_{\{\tau\wedge\sigma<\theta\}}\Big]&=\mathbb{E}\Big[\mathcal{E}\Big(\frac{1}{p_\mathrm{S}}\centerdot \mathrm{D}\Big)_{\tau'\wedge\sigma}\mathbb{1}_{\{\tau'\wedge\sigma<\theta\}}\Big]\\&=\mathbb{E}\Big[\mathcal{E}\Big(\frac{1}{p_\mathrm{S}}\centerdot \mathrm{D}\Big)_{\tau'\wedge\sigma}\mathrm{S}_{\tau'\wedge\sigma}\Big]\\&=\mathbb{E}\Big[\mathrm{S}_0\mathcal{E}\Big(\frac{1}{p_\mathrm{S}}\centerdot \mathrm{Q}\Big)_{\tau'\wedge\sigma}\Big]=\mathbb{E}[\mathrm{S}_0],\end{aligned}\tag{3.16}$$

where part (1) was used in the second line. As a consequence, according to He, Wang and Yan (1992), Theorem 4.40, $(\mathcal{E}(\frac{1}{p_\mathrm{S}}\centerdot \mathrm{D})\mathbb{1}_{[0,\theta)})^\sigma$ is a $(\mathbb{G},\mathbb{Q})$ uniformly integrable martingale, which proves the second assertion of the lemma.

(3) By virtue of de la Vallée–Poussin theorem, there exists a nonnegative increasing convex function ϕ such that

$$\lim_{t\to\infty}\frac{\phi(t)}{t}=\infty\quad\text{and}\quad\sup_\sigma\mathbb{E}\Big[\phi\Big(\mathcal{E}\Big(\frac{1}{p_\mathrm{S}}\centerdot \mathrm{D}\Big)_\sigma\mathbb{1}_{\{\sigma<\theta\}}\Big)\Big]<\infty,$$

where σ runs over the family of $\mathbb{F}$ stopping times such that $[0,\sigma]\subset\{^p\mathrm{S}>0\}\cap[0,T]$. Applying the part (1) of the lemma and Jensen's inequality, we obtain

$$\sup_\sigma\mathbb{E}\big[\phi(\mathrm{S}_0\mathcal{Q}_\sigma\mathbb{1}_{\{^p\mathrm{S}_\sigma>0\}})\big]<\infty.$$

Hence, another application of the de la Vallée–Poussin theorem yields that $\mathrm{S}_0\mathcal{Q}$ is a martingale of class (D) on the set $\{^p\mathrm{S}>0\}\cap[0,T]$. In view of (2.3), we have

$$\mathcal{Q}=\lim_{n\to\infty}\mathcal{E}\Big(\mathbb{1}_{\{^p\mathrm{S}>0\}}\frac{1}{p_\mathrm{S}}\centerdot \mathrm{Q}\Big)^{\zeta_n}.$$

As $\zeta_n\in\{^p\mathrm{S}>0\}$ for every $n\in\mathbb{N}$ [cf. (A.8)], then, on the one hand, the family of random variables $\{\mathrm{S}_0\mathcal{E}(\mathbb{1}_{\{^p\mathrm{S}>0\}}\frac{1}{p_\mathrm{S}}\centerdot \mathrm{Q})_{\zeta_n\wedge T},n\in\mathbb{N}\}$ is uniformly integrable and, on the other hand, $\mathrm{S}_0\mathcal{E}(\mathbb{1}_{\{^p\mathrm{S}>0\}}\frac{1}{p_\mathrm{S}}\centerdot \mathrm{Q})^{\zeta_n\wedge T}$ is a uniformly integrable martingale, for every $n\in\mathbb{N}$. These two properties imply the two following equalities:

$$\mathbb{E}[\mathrm{S}_0\mathcal{Q}_T]=\lim_{n\to\infty}\mathbb{E}\Big[\mathrm{S}_0\mathcal{E}\Big(\mathbb{1}_{\{^p\mathrm{S}>0\}}\frac{1}{p_\mathrm{S}}\centerdot \mathrm{Q}\Big)_{\zeta_n\wedge T}\Big]=\mathbb{E}[\mathrm{S}_0],$$

which proves that the nonnegative $(\mathbb{F},\mathbb{Q})$ local martingale $\mathrm{S}_0\mathcal{Q}$ is an $(\mathbb{F},\mathbb{Q})$ true martingale on $[0,T]$. We now write, for any $0\le t\le T$,

$$\mathrm{S}_0\mathbb{E}[\mathcal{Q}_T|\mathcal{F}_t]=\mathbb{E}[\mathrm{S}_0\mathcal{Q}_T|\mathcal{F}_t]=\mathrm{S}_0\mathcal{Q}_t.$$

As $\mathcal{Q}\equiv 1$ on $\{\mathrm{S}_0=0\}$, we have in turn

$$\begin{aligned}\mathbb{E}[\mathcal{Q}_T|\mathcal{F}_t]&=\mathbb{E}[\mathcal{Q}_T|\mathcal{F}_t]\mathbb{1}_{\{\mathrm{S}_0>0\}}+\mathbb{E}[\mathcal{Q}_T|\mathcal{F}_t]\mathbb{1}_{\{\mathrm{S}_0=0\}}\\&=\mathcal{Q}_t\mathbb{1}_{\{\mathrm{S}_0>0\}}+\mathbb{1}_{\{\mathrm{S}_0=0\}}=\mathcal{Q}_t,\end{aligned}$$

which finishes the demonstration. $\square$

We now state a sufficient condition for the true martingale property of $\mathcal{Q}$.

THEOREM 3.5. *If* $\mathbb{E}[\mathcal{E}(\mathbb{1}_{\{^p\mathtt{S}>0\}}\frac{1}{^p\mathtt{S}}\centerdot\mathtt{D})_{\theta\wedge T}]<\infty$, *then* $\mathcal{Q}$ *is an* $(\mathbb{F},\mathbb{Q})$ *true martingale on* $[0,T]$. *Assuming* $\mathtt{S}_T$ *positive, then* θ *is an invariance time. Assuming further* θ *positive, then the nonnegative* $(\mathbb{G},\mathbb{Q})$ *local martingale* $\mathcal{E}(\mathbb{1}_{\{^p\mathtt{S}>0\}}\frac{1}{^p\mathtt{S}}\centerdot\mathtt{D})\mathbb{1}_{[0,\theta)}$ [*by Lemma* 3.5(2)] *is a* $(\mathbb{G},\mathbb{Q})$ *true martingale on* $[0,T]$, *and an invariance measure* $\mathbb{P}$ *is provided by the restriction to* $\mathcal{F}_T$ *of the probability measure* $\mathbb{S}$ *with* $\mathcal{E}(\mathbb{1}_{\{^p\mathtt{S}>0\}}\frac{1}{^p\mathtt{S}}\centerdot\mathtt{D})\mathbb{1}_{[0,\theta)}$ *as* $\mathbb{G}$ *density process of* $\frac{d\mathbb{S}}{d\mathbb{Q}}$.

PROOF. If $\mathbb{E}[\mathcal{E}(\mathbb{1}_{\{^p\mathtt{S}>0\}}\frac{1}{^p\mathtt{S}}\centerdot\mathtt{D})_{\theta\wedge T}]<\infty$, then the nonnegative $(\mathbb{F},\mathbb{Q})$ local martingale $\mathcal{E}(\frac{1}{^p\mathtt{S}}\centerdot\mathtt{D})\mathbb{1}_{[0,\theta)}$ is a $(\mathbb{G},\mathbb{Q})$ martingale of class (D) on $\{^p\mathtt{S}>0\}\cap[0,T]$. The $(\mathbb{F},\mathbb{Q})$ true martingale property of $\mathcal{Q}$ on $[0,T]$ is then the consequence of Lemma 3.5(3). If, in addition, the positivity of $\mathtt{S}_T$ is assumed, then (A.10) implies that $T<\varsigma$ and $[0,T]\subset\{^p\mathtt{S}>0\}$, so that $\mathcal{Q}=\mathcal{E}(\frac{1}{^p\mathtt{S}}\centerdot\mathtt{Q})>0$ holds on $[0,T]$, by Lemma 2.2(5). Hence, Lemma 3.2 and Theorems 3.2–3.3 imply that θ is an invariance time. Assuming further θ positive, that is, $\mathtt{S}_0=1$, Lemma 3.5(1) implies that the $(\mathbb{F},\mathbb{Q})$ optional projection of the $\mathbb{G}$ density process of $\frac{d\mathbb{S}}{d\mathbb{Q}}$ is $\mathcal{Q}$. But, in view of Theorems 3.2–3.3, $\mathcal{Q}$ is the $\mathbb{F}$ density process of $\frac{d\mathbb{P}}{d\mathbb{Q}}$ for some invariance measure $\mathbb{P}$, which is therefore the restriction to $\mathcal{F}_T$ of the probability measure $\mathbb{S}$. $\square$

On top of the above sufficient condition for the true martingale property of $\mathcal{Q}$ on $[0,T]$, we now look for a necessary and sufficient condition. According to He, Wang and Yan (1992), Theorem 8.18, and Jacod (1979), Lemme 1.37, the first two conditions can always be made to hold in what follows. Hence, only the third one is really material for the true martingale property.

THEOREM 3.6. *Suppose* $\mathtt{S}_T$ *positive. Then the* $(\mathbb{F},\mathbb{Q})$ *true martingale property of* $\mathcal{Q}$ *on* $[0,T]$ *holds if and only if there exists a nondecreasing sequence of* $\mathbb{F}$ *stopping times* $(\sigma_n)_{n\in\mathbb{N}}$ *such that*:

- $\bigcup_n[0,\sigma_n]=[0,T]$,
- $\mathcal{E}(\frac{1}{^p\mathtt{S}}\centerdot\mathtt{D})_{\sigma_n}$ *is bounded, for any* $n\in\mathbb{N}$,
- *The family of random variables* $(\mathtt{D}_T-\mathtt{D}_{\sigma_n})\mathcal{E}(\frac{1}{^p\mathtt{S}}\centerdot\mathtt{D})_{\sigma_n}$, $n\in\mathbb{N}$, *is* $\mathbb{Q}$ *uniformly integrable.*

In this case, the condition (A) *is satisfied.*

PROOF. By the argument used at the end of the proof of Lemma 3.5, the $(\mathbb{F},\mathbb{Q})$ true martingale property of $\mathcal{Q}$ on $[0,T]$ is equivalent to $\mathbb{E}[\mathtt{S}_0\mathcal{Q}_T]=\mathbb{E}[\mathtt{S}_0]$.

Let there be given a nondecreasing sequence of $\mathbb{F}$ stopping times $(\sigma_n)_{n\in\mathbb{N}}$ satisfying the first two conditions in the above.

On the one hand, by Lemma 2.2(5) (having assumed $\mathtt{S}_T > 0$), we have

$$\mathbb{E}[\mathtt{S}_0 \mathcal{Q}_T] = \mathbb{E}\left[\mathtt{S}_0 \mathcal{E}\left(\frac{1}{{}^p\mathtt{S}} \centerdot \mathtt{Q}\right)_T\right] = \mathbb{E}\left[\mathcal{E}\left(\frac{1}{{}^p\mathtt{S}} \centerdot \mathtt{D}\right)_T \mathtt{S}_T\right]$$
$$= \lim_{n\to\infty} \mathbb{E}\left[\mathcal{E}\left(\frac{1}{{}^p\mathtt{S}} \centerdot \mathtt{D}\right)_{\sigma_n} \mathtt{S}_T\right],$$

by the monotone convergence theorem.

On the other hand, we have $[0, \sigma_n] \subset [0, T] \subset \{\mathtt{S} > 0\} \subset \{{}^p\mathtt{S} > 0\}$ (as $\mathtt{S}_T > 0$), so that, for any $n \in \mathbb{N}$, $\mathtt{S}_0\mathcal{E}(\frac{1}{{}^p\mathtt{S}} \centerdot \mathtt{Q})^{\sigma_n}$ is a well-defined $(\mathbb{F}, \mathbb{Q})$ local martingale. Actually, $\mathtt{S}_0\mathcal{E}(\frac{1}{{}^p\mathtt{S}} \centerdot \mathtt{Q})^{\sigma_n}$ is a bounded martingale because, by Lemma 2.2(5), it is equal to $\mathcal{E}(\frac{1}{{}^p\mathtt{S}} \centerdot \mathtt{D})^{\sigma_n} \mathtt{S}^{\sigma_n}$, which is bounded by assumption. As a consequence,

$$\mathbb{E}[\mathtt{S}_0] = \mathbb{E}\left[\mathtt{S}_0 \mathcal{E}\left(\frac{1}{{}^p\mathtt{S}} \centerdot \mathtt{Q}\right)_{\sigma_n}\right] = \mathbb{E}\left[\mathcal{E}\left(\frac{1}{{}^p\mathtt{S}} \centerdot \mathtt{D}\right)_{\sigma_n} \mathtt{S}_{\sigma_n}\right]. \tag{3.17}$$

Hence, $\mathcal{Q}$ is an $(\mathbb{F}, \mathbb{Q})$ true martingale on $[0, T]$ if and only if

$$\lim_{n\to\infty}\left(\mathbb{E}\left[\mathcal{E}\left(\frac{1}{{}^p\mathtt{S}} \centerdot \mathtt{D}\right)_{\sigma_n} \mathtt{S}_{\sigma_n}\right] - \mathbb{E}\left[\mathcal{E}\left(\frac{1}{{}^p\mathtt{S}} \centerdot \mathtt{D}\right)_{\sigma_n} \mathtt{S}_T\right]\right) = 0.$$

But, by definition of $\mathtt{S}$, we have

$$\mathbb{E}\left[\mathcal{E}\left(\frac{1}{{}^p\mathtt{S}} \centerdot \mathtt{D}\right)_{\sigma_n} \mathtt{S}_{\sigma_n}\right] - \mathbb{E}\left[\mathcal{E}\left(\frac{1}{{}^p\mathtt{S}} \centerdot \mathtt{D}\right)_{\sigma_n} \mathtt{S}_T\right] = \mathbb{E}\left[\mathcal{E}\left(\frac{1}{{}^p\mathtt{S}} \centerdot \mathtt{D}\right)_{\sigma_n} \mathbb{1}_{\{\sigma_n < \theta \le T\}}\right]$$
$$= \mathbb{E}\left[(\mathtt{D}_T - \mathtt{D}_{\sigma_n})\mathcal{E}\left(\frac{1}{{}^p\mathtt{S}} \centerdot \mathtt{D}\right)_{\sigma_n}\right],$$

by definition of $\mathtt{D}$ as $(\mathbb{F}, \mathbb{Q})$ dual predictable projection of $\mathbb{1}_{\{0<\theta\}}\mathbb{1}_{[\theta,\infty)}$.[20]

Hence, the true martingale property of $\mathcal{Q}$ on $[0, T]$ is equivalent to the L^1 convergence to zero of the sequence of random variables $(\mathtt{D}_T - \mathtt{D}_{\sigma_n})\mathcal{E}(\frac{1}{{}^p\mathtt{S}} \centerdot \mathtt{D})_{\sigma_n}$. Now, as $\mathtt{S}_T > 0$, we have $\mathcal{E}(\frac{1}{{}^p\mathtt{S}} \centerdot \mathtt{D})_T < \infty$. The random variables $(\mathtt{D}_T - \mathtt{D}_{\sigma_n})\mathcal{E}(\frac{1}{{}^p\mathtt{S}} \centerdot \mathtt{D})_{\sigma_n}$ converge in probability to zero. Therefore, their L^1 convergence to zero is equivalent to their uniform integrability,[21] which concludes the proof of the equivalence stated in the theorem.

Moreover, having assumed $\mathtt{S}_T$ positive, Lemma 2.2(5) implies that $\mathcal{Q} = \mathcal{E}(\frac{1}{{}^p\mathtt{S}} \centerdot \mathtt{Q}) > 0$ holds on $[0, T]$. Hence, the condition (A) reduces to the $(\mathbb{F}, \mathbb{Q})$ true martingale property of $\mathcal{Q}$ on $[0, T]$, by Theorems 3.2–3.3. □

As shown in Lemma A.1, whenever θ has a $(\mathbb{G}, \mathbb{Q})$ intensity γ, we have $\mathtt{D} = \gamma'\mathtt{S}_{-}.\boldsymbol{\lambda}$ and $\mathcal{E}(\frac{1}{{}^p\mathtt{S}} \centerdot \mathtt{D}) = e^{\gamma'\centerdot\boldsymbol{\lambda}}$ on $[0, T]$ (assuming $\mathtt{S}_T > 0$). Hence, in this case, Theorem 3.6 reduces to a condition on γ.

[20] Cf. He, Wang and Yan (1992), Theorem 5.26(2).

[21] Cf. He, Wang and Yan (1992), Theorem 1.11.

REMARK 3.1. In the case $\mathtt{S}_T > 0$, Theorem 3.5 can be deduced from Theorem 3.6. In fact, given a sequence $(\sigma_n)_{n\in\mathbb{N}}$ (which exists) satisfying the first two conditions in Theorem 3.6, we have the inequalities

$$(\mathtt{D}_T - \mathtt{D}_{\sigma_n})\mathcal{E}\Big(\frac{1}{{}^p\mathtt{S}} \centerdot \mathtt{D}\Big)_{\sigma_n} \le \int_{\sigma_n}^T \mathcal{E}\Big(\frac{1}{{}^p\mathtt{S}} \centerdot \mathtt{D}\Big)_s \, d\mathtt{D}_s \le \int_0^T \mathcal{E}\Big(\frac{1}{{}^p\mathtt{S}} \centerdot \mathtt{D}\Big)_s \, d\mathtt{D}_s. \tag{3.18}$$

Since $\mathtt{D}$ is the $(\mathbb{F}, \mathbb{Q})$ dual predictable projection of $\mathbb{1}_{\{\theta>0\}}\mathbb{1}_{[\theta,\infty)}$, we have

$$\mathbb{E}\Big[\mathcal{E}\Big(\frac{1}{{}^p\mathtt{S}} \centerdot \mathtt{D}\Big)_{\theta\wedge T}\Big] = \mathbb{E}[\mathbb{1}_{\{\theta=0\}}] + \mathbb{E}\Big[\int_0^T \mathcal{E}\Big(\frac{1}{{}^p\mathtt{S}} \centerdot \mathtt{D}\Big)_s \, d\mathtt{D}_s\Big] + \mathbb{E}\Big[\mathcal{E}\Big(\frac{1}{{}^p\mathtt{S}} \centerdot \mathtt{D}\Big)_T \mathbb{1}_{\{T\le\theta\}}\Big].$$

Hence, the assumption of Theorem 3.5 implies that $\mathbb{E}[\int_0^T \mathcal{E}(\frac{1}{{}^p\mathtt{S}} \centerdot \mathtt{D})_s \, d\mathtt{D}_s]$ is finite. As a consequence in view of (3.18), the sequence of random variables $(\mathtt{D}_T - \mathtt{D}_{\sigma_n})\mathcal{E}(\frac{1}{{}^p\mathtt{S}} \centerdot \mathtt{D})_{\sigma_n}$ tends to zero in L^1, by the dominated convergence theorem. One can then apply Theorem 3.6 to deduce that $\mathcal{Q}$ is an $(\mathbb{F}, \mathbb{Q})$ true martingale on $[0, T]$, as concluded in Theorem 3.5.

3.5. *Local martingales under an invariance measure.* In this section, we characterize the local martingales under an invariance measure $\mathbb{P}$.

On top of the theoretical interest, the following result is the key for the demonstration of the equivalence in Crépey and Song (2015) between the counterparty risk "full" $(\mathbb{G}, \mathbb{Q})$ BSDE (2.11) and the "reduced" $(\mathbb{F}, \mathbb{P})$ BSDE (2.16) (cf. Section 2.2).

THEOREM 3.7. *Assume the condition* (A), *with an invariance measure* $\mathbb{P}$:

(1) *A process* P *is in* $\mathcal{M}_{\{{}^p\mathtt{S}>0\}\cap[0,T]}(\mathbb{F}, \mathbb{P})$ *if and only if* ${}^p\mathtt{S} \centerdot P + [\mathtt{Q}, P]$ *is in* $\mathcal{M}_{\{{}^p\mathtt{S}>0\}\cap[0,T]}(\mathbb{F}, \mathbb{Q})$.

(2) *If* $\mathtt{D}$ *is continuous*, *then* ${}^p\mathtt{S} = \mathtt{S}_-$ *and the previous condition becomes*

$$\mathtt{S}_- \centerdot P + [\mathtt{S}, P] \in \mathcal{M}_{\{\mathtt{S}_->0\}\cap[0,T]}(\mathbb{F}, \mathbb{Q}). \tag{3.19}$$

In addition, *we have*

$$\{\mathtt{S} > 0\} = \{{}^p\mathtt{S} > 0\} = \{\mathtt{S}_- > 0\} = [0, \varsigma), \tag{3.20}$$

where ς *is the time of first zero of* $\mathtt{S}$ [*cf.* (A.8)].

PROOF. (1) On $\{{}^p\mathtt{S} > 0\} \cap [0, T]$, we have

$$\begin{aligned} qP &= P_- \centerdot q + q_- \centerdot P + [q, P] \\ &= P_- \centerdot q + q_- \centerdot P + q_- \frac{1}{{}^p\mathtt{S}} \centerdot [\mathtt{Q}, P], \end{aligned}$$

where the second equality comes from the formula (3.10) in Theorem 3.2. Hence, P is in $\mathcal{M}_{\{^p\mathtt{S}>0\}\cap[0,T]}(\mathbb{F},\mathbb{P})$, that is, qP is in $\mathcal{M}_{\{^p\mathtt{S}>0\}\cap[0,T]}(\mathbb{F},\mathbb{Q})$,[22] if and only if $q_- \centerdot P + q_- \frac{1}{^p\mathtt{S}} \centerdot [\mathtt{Q}, P]$ is in $\mathcal{M}_{\{^p\mathtt{S}>0\}\cap[0,T]}(\mathbb{F},\mathbb{Q})$.

Consider the sequence $(\zeta_n)_{n\in\mathbb{N}}$ introduced in (A.8). Let $(\sigma_n)_{n\in\mathbb{N}}$ be a nondecreasing sequence of $\mathbb{F}$ stopping times tending to infinity and reducing q_- and its inverse to bounded processes on $[0,T]$. By definition of a local martingale on a predictable interval [cf. (1.2)], the above condition on P can be stated as

$$q_- \centerdot P^{\zeta_n\wedge\sigma_n} + q_- \frac{1}{^p\mathtt{S}} \centerdot [\mathtt{Q}, P]^{\zeta_n\wedge\sigma_n} \in \mathcal{M}_{[0,T]}(\mathbb{F},\mathbb{Q}), \qquad \forall n \in \mathbb{N}.$$

As $^p\mathtt{S}$ and q_- are bounded away from 0 on $[0, \zeta_n \wedge \sigma_n]$, this is equivalent to

$$^p\mathtt{S} \centerdot P^{\zeta_n\wedge\sigma_n} + [\mathtt{Q}, P]^{\zeta_n\wedge\sigma_n} \in \mathcal{M}_{[0,T]}(\mathbb{F},\mathbb{Q}), \qquad \forall n \in \mathbb{N},$$

that is,

$$^p\mathtt{S} \centerdot P + [\mathtt{Q}, P] \in \mathcal{M}_{\{^p\mathtt{S}>0\}\cap[0,T]}(\mathbb{F},\mathbb{Q}).$$

(2) In the case where $\mathtt{D}$ is continuous, we have $[\mathtt{D},\cdot]=0$ and $^p\mathtt{S} = \mathtt{S}_-$ [cf. (A.3)], so that the condition in part (1) is rewritten as (3.19). In order to establish (3.20), in view of (A.9), all we need is showing $\{\mathtt{S}_- > 0\} \subseteq [0,\varsigma)$. Toward this aim, we apply the local martingale characterization (3.19) to $P = \mathbb{1}_{\{0<\varsigma_n=\varsigma<C\}}\mathbb{1}_{[\varsigma,\infty)}$, where $(\varsigma_n)_{n\in\mathbb{N}}$ is the sequence defined in (A.5) and C is a positive constant. Namely, we compute

$$\mathtt{S}_- \centerdot P + [\mathtt{S}, P] = \mathtt{S}_{\varsigma-}\mathbb{1}_{\{0<\varsigma_n=\varsigma<C\}}\mathbb{1}_{[\varsigma,\infty)} + \Delta_\varsigma \mathtt{S}\mathbb{1}_{\{0<\varsigma_n=\varsigma<C\}}\mathbb{1}_{[\varsigma,\infty)} = 0$$

(since $\mathtt{S}_\varsigma = 0$). Hence, P satisfies (3.19), so that it is a bounded $(\mathbb{F},\mathbb{P})$ martingale on $\{\mathtt{S}_- > 0\}\cap[0,T]$. Noting that $P_0 = 0$ and

$$P_{\varsigma_n\wedge T} = \mathbb{1}_{\{0<\varsigma_n=\varsigma<C\}}\mathbb{1}_{\{\varsigma_n\wedge T\geq\varsigma\}} = \mathbb{1}_{\{0<\varsigma_n=\varsigma<C,\varsigma\leq T\}},$$

we conclude that

$$0 = \mathbb{E}^{\mathbb{P}}[P_{\varsigma_n\wedge T}] = \mathbb{P}[0<\varsigma_n=\varsigma<C, \varsigma\leq T],$$

for every positive constant C. As a consequence, we have $\mathbb{P}[0<\varsigma_n=\varsigma\leq T]=0$. Thus, $\varsigma_n<\varsigma$ holds whenever $\varsigma\leq T$, under $\mathbb{P}$ as under $\mathbb{Q}$. Hence, in view of (A.7):

– On $\{\mathtt{S}_0 > 0, \varsigma\leq T\}$, we have

$$\{\mathtt{S}_- > 0\} = \bigcup_n [0,\varsigma_n] \subseteq [0,\varsigma),$$

– On $\{\mathtt{S}_0 = 0\}$, we have

$$\{\mathtt{S}_- > 0\} = \varnothing = [0,\varsigma).$$

In both cases, we have $\{\mathtt{S}_- > 0\} \subseteq [0,\varsigma)$, which finishes the demonstration. □

[22] cf. He, Wang and Yan (1992), Theorem 12.12.

4. Invariance times in different situations. In this section, we review a variety of situations involving invariance times.

4.1. *Comparison with pseudo-stopping times.* In this section, we study the connection between invariance times and pseudo-stopping times as of Nikeghbali and Yor (2005). In addition, we assess the materiality of stopping before θ as opposed to at θ in the condition (A).

Consider a $(0, +\infty)$-valued random time θ. By Nikeghbali and Yor (2005), it is an $(\mathbb{F}, \mathbb{Q})$ pseudo-stopping time if and only if Q^{θ} is a local martingale for any $(\mathbb{F}, \mathbb{Q})$ local martingale Q. Clearly, if a pseudo-stopping time θ avoids the $\mathbb{F}$ stopping times, then it is an invariance time satisfying the condition (A) for any positive constant T, with invariance measure $\mathbb{P} = \mathbb{Q}$. It is also shown in Nikeghbali and Yor (2005), Theorem 1(3), that θ is a pseudo-stopping time if and only if $\mathtt{A}_{\infty} \equiv 1$, that is, if and only if $\mathtt{S} = 1 - \mathtt{A}$, where $\mathtt{A}$ denotes the $\mathbb{F}$ dual optional projection of $\mathbb{1}_{[\theta,\infty)}$. By contrast, Proposition 3.1(2) shows that $\mathbb{Q}$ itself is an invariance measure for any positive constant T if and only if $\mathtt{S} = 1 - \mathtt{D}$ (noting that $\mathtt{S}_0 = 1$ here, as $\theta > 0$). Both conditions coincide if and only if $\mathtt{A} = \mathtt{D}$. We recall that in the case where $\mathtt{S}_0 = 1$ and $\mathtt{D}$ is continuous [i.e., in our setup, whenever θ is a $(\mathbb{G}, \mathbb{Q})$ totally inaccessible stopping time], then $\mathtt{A} = \mathtt{D}$ if and only if θ avoids the $\mathbb{F}$ stopping times. Hence, in the case where $\mathtt{S}_0 = 1$ and $\mathtt{D}$ is continuous, there are two "orthogonal" cases:

– If θ has the avoidance property, then θ is a pseudo-stopping time if and only if $\mathbb{Q}$ itself is an invariance measure;
– If θ is a pseudo-stopping time without the avoidance property, then $\mathbb{Q}$ itself cannot be an invariance measure. This is due to the fact that a pseudo-stopping time is defined in terms of stopping at θ, whereas invariance is defined in terms of stopping before θ.

Moreover, by comparison with pseudo-stopping times that are defined with respect to the fixed probability measure $\mathbb{Q}$, the additional flexibility of invariance times lies in the possibility to consider the martingale property under a changed measure $\mathbb{P}$. In fact, the pseudo-stopping time condition is very restrictive. By contrast, Theorem 3.5 shows that invariance times are the rule rather the exception. Actually, the spirit of invariance times is not to define one more fancy class of random times, but rather to show that a reduction of filtration methodology can be useful very broadly, much beyond the basic reduced-form setup of Section 2.2.1.

The following examples provides more insight on the relationship between pseudo-stopping times and invariance times.

EXAMPLE 4.1. Fix a filtration $\mathbb{F}$ satisfying the usual conditions. For $i = 1, 2$, let $\sigma_i > 0$ be a finite $\mathbb{F}$ stopping time with bounded and continuous compensator $\mathbf{v}_i$. Assuming $\sigma_2 > T$, define $\theta = \mathbb{1}_A \sigma_1 + \mathbb{1}_{A^c} \sigma_2$, for some random event A independent of $\mathcal{F}_{\infty}$ such that $\alpha = \mathbb{Q}(A) \in (0, 1)$. Let $\mathbb{G}$ be the progressive enlargement of $\mathbb{F}$ with θ. Hence, θ intersects the $\mathbb{F}$ stopping times σ_i. By independence of A, on $[0, T]$, we have

$$\begin{aligned} \mathtt{S} &= \mathbb{1}_{[0,\sigma_1)}\alpha + \mathbb{1}_{[0,\sigma_2)}(1-\alpha), \\ \mathtt{D} &= \alpha \mathbf{v}_1 + (1-\alpha)\mathbf{v}_2, \\ \mathtt{Q} &= \alpha(\mathbb{1}_{[0,\sigma_1)} + \mathbf{v}_1) + (1-\alpha)(\mathbb{1}_{[0,\sigma_2)} + \mathbf{v}_2). \end{aligned}$$

Hence, since $\sigma_2 > T$, we have $\mathtt{S}$ and in turn ${}^p\mathtt{S} \geq 1-\alpha > 0$ on $[0, T]$, so that

$$\mathcal{E}\left(\frac{1}{{}^p\mathtt{S}} \boldsymbol{\cdot} \mathtt{D}\right) = e^{\frac{1}{{}^p\mathtt{S}}\boldsymbol{\cdot}\mathtt{D}} \leq e^{\frac{1}{1-\alpha}\mathtt{D}}$$

is bounded on $[0, T]$. Therefore, the conditions of Theorem 3.5 are fulfilled and θ is an invariance time. In addition, for every bounded $\mathbb{F}$ optional process K, by independence of A, we have

$$\mathbb{E}[K_\theta] = \mathbb{E}[\mathbb{1}_A K_{\sigma_1} + \mathbb{1}_{A^c} K_{\sigma_2}] = \mathbb{E}\big[\alpha K_{\sigma_1} + (1-\alpha) K_{\sigma_2}\big],$$

hence

$$\mathtt{A} = (\mathbb{1}_{[\theta,\infty)})^o = \alpha \mathbb{1}_{[\sigma_1,\infty)} + (1-\alpha)\mathbb{1}_{[\sigma_2,\infty)}.$$

As the σ_i are finite, it follows that $\mathtt{A}_\infty \equiv 1$ and θ is a pseudo-stopping time.

EXAMPLE 4.2. To obtain an invariance time θ intersecting $\mathbb{F}$ stopping times without being a pseudo-stopping time, we set

$$\theta = \mathbb{1}_{A_1}\sigma_1 + \mathbb{1}_{A_2}\sigma_2 + \mathbb{1}_{A_3}\sigma_3,$$

where σ_1 and σ_2 are as in Example 4.1 and where the finite random time $\sigma_3 > 0$ is not an $\mathbb{F}$ pseudo-stopping time, for a partition A_i, $i = 1, 2, 3$, independent of $\mathcal{F}_\infty$ and of σ_3. Assuming $\alpha_i = \mathbb{Q}(A_i) > 0$, we have

$$\mathtt{A} = (\mathbb{1}_{[\theta,\infty)})^o = \alpha_1 \mathbb{1}_{[\sigma_1,\infty)} + \alpha_2 \mathbb{1}_{[\sigma_2,\infty)} + \alpha_3 (\mathbb{1}_{[\sigma_3,\infty)})^o,$$

where $(\mathbb{1}_{[\sigma_3,\infty)})^o_\infty \neq 1$, hence $\mathtt{A}_\infty \neq 1$, with positive $\mathbb{Q}$ probability. Hence, θ is not a pseudo-stopping time. But the Azéma supermartingale of θ is given by

$$\mathtt{S} = \mathbb{1}_{[0,\sigma_1)}\alpha_1 + \mathbb{1}_{[0,\sigma_2)}\alpha_2 + {}^o(\mathbb{1}_{[0,\sigma_3)})\alpha_3 \geq \alpha_2 \qquad \text{on } [0, T],$$

so that θ is an invariance time, by the same arguments as in Example 4.1.

4.2. *Connection with the survival measure.* In the financial context of defaultable asset pricing, for deriving an intensity-based pricing formula exempt from the no-jump condition in Duffie, Schroder and Skiadas (1996), an alternative to the reduction of filtration approach of this paper is to work purely in the filtration $\mathbb{G}$, but to switch from the risk-neutral measure $\mathbb{Q}$ to some nonequivalent pricing measure.

To establish the connection between the two frameworks, let us postulate a $\mathbb{G}$ stopping time θ in an enlargement setup satisfying the condition (B) (as everywhere in this paper), with a $(\mathbb{G},\mathbb{Q})$ intensity γ and $\mathtt{S}_T > 0$, so that $\mathcal{E}(\frac{1}{p\mathtt{S}} \centerdot \mathtt{D}) = \mathcal{E}(\frac{1}{\mathtt{S}_-} \centerdot \mathtt{D}) = e^{\gamma' \centerdot \boldsymbol{\lambda}}$ holds on $[0,T]$, by Lemma A.1. Assuming further that $e^{\gamma \centerdot \boldsymbol{\lambda}_{\theta\wedge T}}$ is $\mathbb{Q}$ integrable [this is the basic assumption in Collin-Dufresne, Goldstein and Hugonnier (2004)], then all the conditions of Theorem 3.5 are satisfied. Hence, by application of this theorem, we conclude that θ is an invariance time, that the process $e^{\gamma \centerdot \boldsymbol{\lambda}} \mathbb{1}_{[0,\theta)}$ is a $(\mathbb{G},\mathbb{Q})$ true martingale on $[0,T]$ and that an invariance measure $\mathbb{P}$ is provided by the restriction to $\mathcal{F}_T$ of the "survival measure" $\mathbb{S}$ with $e^{\gamma \centerdot \boldsymbol{\lambda}} \mathbb{1}_{[0,\theta)}$ as $\mathbb{G}$ density process of $\frac{d\mathbb{S}}{d\mathbb{Q}}$ [the "survival measure" idea and terminology were first introduced and used for different purposes in Schönbucher (1999, 2004)].

Put differently, if the default time θ in Collin-Dufresne, Goldstein and Hugonnier (2004) also satisfies the condition (B), then the condition (A) is satisfied and the restriction to $\mathcal{F}_T$ of the survival measure $\mathbb{S}$ is an invariance measure. This establishes the connection between the notions of invariance and survival measures. In particular, this shows that, even though $\mathbb{S}$ and $\mathbb{Q}$ are not equivalent (not even in a basic reduced-form setup as of Section 2.2.1), their restrictions to $\mathcal{F}_T$ are equivalent.

Assuming that $e^{\gamma \centerdot \boldsymbol{\lambda}_{\theta\wedge T}}$ is $\mathbb{Q}$ integrable, Collin-Dufresne, Goldstein and Hugonnier (2004) obtain an intensity-based $(\mathbb{G},\mathbb{S})$ pricing formula exempt from the no-jump condition in Duffie, Schroder and Skiadas (1996). A no-arbitrage interpretation of the survival measure $\mathbb{S}$ is given in Fisher, Pulido and Ruf (2015), who use it to model financial assets that may potentially lose value relative to each other at time θ.

4.3. *BSDEs of counterparty risk.* Invariance times make the filtration reduction technique particularly efficient for dealing with BSDEs with random terminal time (this was actually our initial motivation for the introduction of invariance times, cf. Section 2.2). For instance, in the two-part paper by Bichuch, Capponi and Sturm (2015), where $\mathbb{F}$ is a Brownian filtration and $\mathbb{G}$ is $\mathbb{F}$ progressively enlarged by the default times of a bank and its counterparty, the analysis can be drastically simplified by converting, by an application of Crépey and Song (2015), Theorem 4.3, their main jump BSDE (17) [or likewise (18)] in Part I, formulated with respect to the enlarged filtration $\mathbb{G}$ on the time interval $[0,\theta\wedge T]$, into a continuous BSDE with respect to the reference filtration $\mathbb{F}$ on the time interval $[0,T]$. The two BSDEs are equivalent but the continuous one is of course much simpler to study. The related PDE also becomes continuous. Without jumps, the proofs of all the technical results in their work, that is, the BSDE well-posedness and comparison Theorems A.2 and A.3 in part I and the PDE unique viscosity solution Theorem 3.2 in part II (or rather their continuous analogues), become a matter of referring to standard continuous BSDE and PDE results.

The reformulation of the "full" $(\mathbb{G}, \mathbb{Q})$ BSDE (2.11) as the "reduced" $(\mathbb{F}, \mathbb{P})$ BSDE (2.16) can also represent a significant improvement with numerical solutions in view. In the above example, it implies that standard numerical schemes without jumps can be used for solving the problem [cf. Bichuch, Capponi and Sturm (2015), Remark 3.4 in Part II].

4.4. *Dynamic copulas.* A singular point in the field of counterparty risk is when the underlying contracts are credit derivatives, due to the extreme wrong-way risk effects that occur, in the form of frailty and contagion, between the credit risk of the counterparty and the credit risk of the underlying market (credit in this case) exposure. In models of "instantaneous contagion" where the counterparty and the reference credit entities may default at the same time, wrong-way risk may take the extreme form of an instantaneous gap risk, which is not only a surge in the value of credit protection, but also credit protection cash flows that fail to be paid to the bank by the defaulted counterparty at time θ.

To embed such credit frailty and contagion effects in their models, market practitioners use (static) copulas for the default times of the reference entities. In the case of counterparty risk on credit derivatives, one needs to model the default time of the counterparty jointly with these, that is, one needs a model for $(\theta_0, \theta_1, \dots, \theta_n)$, where $\theta_0 = \theta$ corresponds to the default time of the counterparty of a bank involved in credit derivatives on reference entities with the respective default times θ_i, $i = 1, \dots, n$. Moreover, as the counterparty risk and funding valuation adjustments are in fact options on these, one needs to make the model dynamic in some way. One possibility is to use "dynamic copula models" resulting from the introduction of a suitable filtration on top of a static copula model. Related references in the literature include the seminal working paper by Schönbucher and Schubert (2001) as well as, among others, Brigo, Capponi and Pallavicini (2014), Section 3, Bo and Capponi (2015), Lee and Capriotti (2015) or Crépey and Song (2016) [cf. also Kusuoka and Nakashima (2012) and Bielecki, Jeanblanc and Rutkowski (2009), Section 7.3].

Now, in order to understand the nature of the credit dependence in a dynamic copula model, one would like to address the following question: Given a stopping time θ relative to the full model filtration $\mathbb{G}$, when and how can one separate the information that comes from θ from a "market filtration" $\mathbb{F}$, such that, for consistency and tractability, some kind of local martingale invariance property holds between $\mathbb{F}$ and $\mathbb{G}$? Invariance times are precisely designed in this spirit. Specifically, it is shown in Crépey and Song (2016) that:

– In the context of the dynamic Gaussian copula model, the condition (A) is achieved by $\theta = \theta_0$, for a suitable $\mathbb{P} \neq \mathbb{Q}$ and for $\mathbb{G}$ given as the progressive enlargement of $\mathbb{F}$ by θ. This model can be used as a "wrong-way risk model" of counterparty risk embedded in credit derivatives, where the default intensities of the surviving reference names and therefore the value of credit protection spike, as an effect of $\mathbb{P} \neq \mathbb{Q}$, at the default time of the counterparty.

– In the context of the dynamic Marshall–Olkin copula (or "common-shock" model, where the credit dependence stems from the possibility of simultaneous defaults), the condition (A) is achieved by $\theta = \theta_0$, for $\mathbb{P} = \mathbb{Q}$ but $\mathbb{G}$ greater than the progressive enlargement of $\mathbb{F}$ by θ, reflecting various possible joint default scenarios that may prompt the default of the counterparty. This model can be used as a "gap risk model" of counterparty risk embedded in credit derivatives, where promised cash-flows fail to be paid at the counterparty default time θ in joint default scenarios.

[Cf. Figure 7 in Crépey and Song (2016), preprint version.]

4.4.1. *An open problem.* A subfiltration $\mathbb{F}$ satisfying the condition (B) is considered as given everywhere in this paper. As an open problem related to invariance times, there is the question of finding a subfiltration $\mathbb{F}$ of a given full model filtration $\mathbb{G}$ such that the condition (B) [before considering (A)] is satisfied. Suitable subfiltrations $\mathbb{F}$ can be worked out on a case-by-case basis in the above dynamic copula models. But, beyond a necessary condition which is derived in Song (2016b), Section 9, we have no constructive methodology to offer in this regard.

Conclusion. From the enlargement of filtration literature of the seventies, two facts were known. First, the study of progressive enlargement of a filtration with a random time θ is "easy" before θ. Second, the Jeulin–Yor formulas for the semimartingale decomposition, in a progressively enlarged filtration, of martingales in a smaller filtration $\mathbb{F}$, are similar to the Girsanov measure change formulas, but the Jeulin–Yor and Girsanov formulas are not equivalent to each other, mainly due to integrability reasons at time θ.

In this paper, we characterize the random times θ such that the local martingale property is preserved for a process stopped before θ under filtration enlargement and equivalent change of measure. The corresponding random times, called invariance times, make the filtration reduction technique particularly efficient for dealing with BSDEs with random terminal time.

More broadly, this paper is a contribution to two-parallel progressive enlargement of filtration literatures, depending on whether a predictable or optional point of view is considered:

– Ends of sets: predictable in Azéma (1972) versus optional in Jeulin and Yor (1978).
– Multiplicative decompositions of the Azéma supermartingale S of θ: predictable in Jacod (1979) and Song (2016b) versus optional in Kardaras (2015).
– Representations of S under the form $\frac{X}{X^*}$, for some positive local martingale X and $X^* = \sup_{0 \le s \le \cdot} X_s$, following Nikeghbali and Yor (2006) (cf. Example 3.3): for honest times avoiding $\mathbb{F}$ (optional) stopping times in Kardaras (2014) versus $\mathbb{F}$ predictable stopping times in Acciaio and Penner (2016) [and for general honest times in Song (2016a)].

– Deflators: predictable in Song (2016b) versus optional in Aksamit et al. (2013) or Acciaio, Fontana and Kardaras (2016).
– Local martingale invariance properties: for processes stopped before θ, based on the dual predictable projection $\mathtt{D}$ of θ, regarding the invariance times of this paper, versus processes stopped at θ, based on the optional projection $\mathtt{A}$ of θ (and without measure change), in the case of pseudo-stopping times in Nikeghbali and Yor (2005).

From an application point of view, the relevance of stopping at θ [as for the study of no arbitrage for an insider observing θ in Aksamit et al. (2013) or Acciaio, Fontana and Kardaras (2016)] or before θ (as for pricing counterparty risk) depends on the problem at hand. At the technical level, the best approach to deal with the process $X^{\theta-}$ (as in the present paper) is to make use of the predictable multiplicative decomposition of the Azéma supermartingale and more generally of the reduction of filtration methodology. By contrast, the consideration of X^{θ} leads naturally to optional computations.

As far as the pricing of defaultable securities is concerned, the $(\mathbb{F},\mathbb{P})$ (or "extended reduced-form") approach based on this paper sheds light on the relation between three streams of mathematical finance literature, namely the $(\mathbb{G},\mathbb{Q})$ seminal approach by Duffie, Schroder and Skiadas (1996), the nowadays standard $(\mathbb{F},\mathbb{Q})$ approach in a basic reduced-form setup and the $(\mathbb{G},\mathbb{S})$ survival measure approach of Collin-Dufresne, Goldstein and Hugonnier (2004).

APPENDIX A: ABOUT THE AZÉMA SUPERMARTINGALE

This section recapitulates the most classical properties of the Azéma supermartingale $\mathtt{S} = {}^{o}\mathtt{J}$ of a random time θ, where $\mathtt{J} = \mathbb{1}_{[0,\theta)}$, with canonical Doob–Meyer decomposition $\mathtt{S} = \mathtt{S}_0 + \mathtt{Q} - \mathtt{D}$ (cf. Section 2.1). For more information, see Jeulin (1980) and Jeulin and Yor (1978).

We have

$$^{p}(\mathtt{J}_-) = \mathtt{S}_- \qquad \text{on } (0,\infty) \tag{A.1}$$

[see Jeulin (1980), page 63] and

$$^{p}\mathtt{S} = \mathtt{Q}_- - \mathtt{D} = \mathtt{S} - \Delta\mathtt{Q} = \mathtt{S}_- - \Delta\mathtt{D} \le \mathtt{S}_-. \tag{A.2}$$

In particular, if $\mathtt{D}$ is continuous, then

$$^{p}\mathtt{S} = \mathtt{S}_-. \tag{A.3}$$

The $(\mathbb{G},\mathbb{Q})$ compensator $\mathbf{v}$ of θ satisfies

$$\mathbf{v} = \int_0^{\cdot\wedge\theta} \frac{1}{\mathtt{S}_{s-}}\, d\mathtt{D}_s \tag{A.4}$$

[see Jeulin (1980), Remark 4.5].

LEMMA A.1. *The* $\mathbb{G}$ *stopping time* θ *is* $(\mathbb{G},\mathbb{Q})$ *totally inaccessible if and only if* $\mathtt{S}_0 = 1$ *and* $\mathtt{D}$ *is continuous* (*so that* $\mathcal{E}(\pm\frac{1}{\mathtt{S}_-} \centerdot \mathtt{D}) = e^{\pm\frac{1}{\mathtt{S}_-}\centerdot\mathtt{D}}$ *holds on* $\{\mathtt{S}_- > 0\}$). *In the special case where* θ *has a* $(\mathbb{G},\mathbb{Q})$ *intensity* γ, *then* $\mathbb{1}_{(0,\theta]}\frac{1}{\mathtt{S}_-} \centerdot \mathtt{D} = \gamma \centerdot \boldsymbol{\lambda}$ *and* $\mathtt{D} = \gamma'\mathtt{S}_- \centerdot \boldsymbol{\lambda}$ *hold on* $\mathbb{R}_+$ (*where* γ' *denotes an* $\mathbb{F}$ *predictable reduction of* γ), *hence* $\frac{1}{\mathtt{S}_-} \centerdot \mathtt{D} = \gamma' \centerdot \boldsymbol{\lambda}$ *holds on* $\{\mathtt{S}_- > 0\}$.

PROOF. By definition in this paper (cf. Section 1.1), θ is $(\mathbb{G},\mathbb{Q})$ totally inaccessible if and only if $\mathtt{S}_0 = 1$ and the $(\mathbb{G},\mathbb{Q})$ compensator $\mathbf{v}$ of θ is continuous on $[0,\theta]$. In view of (A.4), this happens if and only if $\mathtt{S}_0 = 1$ and $\Delta\mathtt{D} = 0$ on $[0,\theta)$ or, equivalently by the predictable version of Lemma 2.3, if $\Delta\mathtt{D} = 0$ on $\{\mathtt{S}_- > 0\}$, that is, if $\mathtt{D}$ is continuous on $\mathbb{R}_+$ [cf. Song (2016b), Lemma 3.7]. If θ has an intensity γ, then (A.4) and Lemma A.1 show that $\mathbb{1}_{(0,\theta]}\frac{1}{\mathtt{S}_-} \centerdot \mathtt{D} = \gamma \centerdot \boldsymbol{\lambda}$ and $\mathtt{D} = \gamma'\mathtt{S}_- \centerdot \boldsymbol{\lambda}$. $\square$

Define

$$\varsigma = \inf\{s > 0; \mathtt{S}_s = 0\}, \qquad \varsigma_n = \inf\left\{s > 0; \mathtt{S}_s \le \frac{1}{n}\right\} \qquad (n > 0). \tag{A.5}$$

Then

$$\varsigma = \inf\{s > 0; \mathtt{S}_{s-} = 0\} \tag{A.6}$$

(since $\mathtt{S}$ is a nonnegative supermartingale)[23] and

$$\varsigma = \sup_n \varsigma_n, \qquad \{\mathtt{S}_- > 0\} \cup [0] = \bigcup_n [0, \varsigma_n]. \tag{A.7}$$

Likewise, $\{{}^p\mathtt{S} > 0\}$ is a predictable interval and, according to Jacod (1979), (6.24) and (6.28), there exists a nondecreasing sequence $(\zeta_n)_{n\in\mathbb{N}}$ of $\mathbb{F}$ stopping times such that

$$\{{}^p\mathtt{S} > 0\} \cup [0] = \bigcup_n [0, \zeta_n] \quad \text{and} \quad \frac{1}{{}^p\mathtt{S}} \text{ is bounded on } (0, \zeta_n] \text{ for every } n. \tag{A.8}$$

The following relations hold:

$$[0, \varsigma) = \{\mathtt{S} > 0\} \subset \{{}^p\mathtt{S} > 0\} \subset \{\mathtt{S}_- > 0\}, \tag{A.9}$$

whereas, on (ς, ∞),

$$\mathtt{S} = \mathtt{S}_- = {}^p\mathtt{S} = 0, \qquad \mathtt{D} \text{ and } \mathtt{Q} \text{ are constant.} \tag{A.10}$$

We have

$$\mathtt{S}_{\theta-} > 0 \qquad \text{on } \{0 < \theta < \infty\} \tag{A.11}$$

[cf. Yor (1978), p. 63 and see also Song (2016b), Lemma 3.7].

[23] Cf. n°17 Chapitre VI in Dellacherie and Meyer (1975).

APPENDIX B: EQUIVALENT CHANGES OF PROBABILITY MEASURES

The following folklore result of local martingales plays an important role in the paper.

LEMMA B.1. *Let U be a stopping time. Let X be a local martingale on* $[0, U]$ *such that, for any locally bounded local martingale Y on* $[0, U]$, *X is orthogonal to Y, that is,* $[X, Y]$ *is a local martingale on* $[0, U]$. *Then $X = X_0$ on* $[0, U]$.

PROOF. Suppose without loss of generality that X is uniformly integrable. For any stopping time $\sigma > 0$, let ϵ denote the sign of $\Delta_\sigma X$. First assuming $\sigma > 0$ totally inaccessible, let $\mathbf{u}$ be the compensator of $\epsilon \mathbb{1}_{[\sigma,\infty)}$, so that $(\epsilon \mathbb{1}_{[\sigma,\infty)} - \mathbf{u})$ is a locally bounded local martingale. By the assumed orthogonality, as $\mathbf{u}$ is continuous [cf. He, Wang and Yan (1992), Corollary 5.28], the process $|\Delta_\sigma X| \mathbb{1}_{[\sigma,\infty)} = [X, \epsilon \mathbb{1}_{[\sigma,\infty)} - \mathbf{u}]$ must be a local martingale on $[0, U]$. It is therefore null. Likewise, for any predictable stopping time $\sigma > 0$ and bounded random variable χ such that $\mathbb{E}[\chi | \mathcal{F}_{\sigma-}] = 0$ [denoting the filtration by $\mathbb{F} = (\mathcal{F}_t)_{t \in \mathbb{R}_+}$], $\chi \mathbb{1}_{[\sigma,\infty)}$ is a bounded martingale and the orthogonality assumption implies that $\Delta_\sigma X \chi \mathbb{1}_{[\sigma,\infty)}$ is a martingale on $[0, U]$. Considering $\chi = \epsilon \mathbb{1}_{\{\sigma<\infty\}} - \mathbb{E}[\epsilon \mathbb{1}_{\{\sigma<\infty\}} | \mathcal{F}_{\sigma-}]$, we have

$$\mathbb{E}\big[|\Delta_\sigma X| \mathbb{1}_{\{\sigma<\infty\}}\big] = \mathbb{E}[\Delta_\sigma X \chi \mathbb{1}_{\{\sigma<\infty\}}] = 0,$$

because $\mathbb{E}[\epsilon \mathbb{1}_{\{\sigma<\infty\}} | \mathcal{F}_{\sigma-}]$ is $\mathcal{F}_{\sigma-}$ measurable and $\mathbb{E}[\Delta_\sigma X | \mathcal{F}_{\sigma-}] = 0$, by the predictability of σ. In conclusion, the local martingale X is continuous at any predictable or totally inaccessible stopping time. Hence, X is a continuous local martingale on $[0, U]$. As a consequence, by the orthogonality assumption applied to X itself, $[X, X]$ is a continuous local martingale on $[0, U]$. Therefore, $[X, X] = [X, X]_0 = 0$ on $[0, U]$ and the lemma is proved. □

In the rest of this section, we derive measure change results of independent interest (unrelated to enlargement of filtration), used for our second proof of Theorem 3.1 in Section C. Given a probability measure $\mathbb{P}$ equivalent to the probability measure $\mathbb{Q}$ on $\mathcal{F}_T$, we use the notation $p, \bar{p}, q, \bar{q}$ introduced in (3.1). The $(\mathbb{F}, \mathbb{P})$ predictable bracket is denoted by $\langle \cdot, \cdot \rangle^{\mathbb{P}}$, whereas the $(\mathbb{F}, \mathbb{Q})$ predictable bracket is denoted as before by $\langle \cdot, \cdot \rangle$ (of course, when one argument is continuous, the two brackets coincide and we typically write $\langle \cdot, \cdot \rangle$). The continuous and purely discontinuous parts (starting from 0) of a local martingale are denoted by $\cdot^c$ and $\cdot^d$.

There are two forms of the Girsanov theorem.[24] The optional bracket Girsanov formula states that, for any P in $\mathcal{M}(\mathbb{F}, \mathbb{P})$,

$$P - q \centerdot [p, P] \tag{B.1}$$

[24]Cf. (2.6) and (2.5) and see He, Wang and Yan (1992), Theorem 12.18.

is in $\mathcal{M}_{[0,T]}(\mathbb{F},\mathbb{Q})$. The predictable bracket Girsanov formula states that, if P in $\mathcal{M}(\mathbb{F},\mathbb{P})$ is such that $[p,P]$ is $(\mathbb{F},\mathbb{P})$ locally integrable,[25] then

$$\mathfrak{Q}(P) := P - q_- \centerdot \langle p, P\rangle^{\mathbb{P}} = P - \langle \bar{p}, P\rangle^{\mathbb{P}} \tag{B.2}$$

is in $\mathcal{M}_{[0,T]}(\mathbb{F},\mathbb{Q})$. Lemmas B.2 and B.3 study the class of $(\mathbb{F},\mathbb{Q})$ local martingales issued from predictable bracket Girsanov transforms.

LEMMA B.2. *Let Q be an $(\mathbb{F},\mathbb{Q})$ uniformly integrable martingale such that $Q_0 = 0$ and, for any bounded $(\mathbb{F},\mathbb{P})$ martingale P starting from 0, $\mathfrak{Q}(P)Q$ is an $(\mathbb{F},\mathbb{Q})$ local martingale on $[0,T]$. Then $Q = 0$ on $[0,T]$.*

PROOF. For any bounded $(\mathbb{F},\mathbb{P})$ martingale P starting from 0 and any $\mathbb{F}$ stopping time $\sigma \le T$ reducing the involved processes to integrability, we have

$$\begin{aligned} 0 &= \mathbb{E}[\mathfrak{Q}(P)_\sigma Q_\sigma] = \mathbb{E}^{\mathbb{P}}\big[\big(P - q_- \centerdot \langle p, P\rangle^{\mathbb{P}}\big)_\sigma Q_\sigma p_\sigma\big] \\ &= \mathbb{E}^{\mathbb{P}}\big[[Qp, P]_\sigma - Q_- \centerdot \langle p, P\rangle^{\mathbb{P}}_\sigma\big] = \mathbb{E}^{\mathbb{P}}\big[[Qp - Q_- \centerdot p, P]_\sigma\big], \end{aligned} \tag{B.3}$$

where the $(\mathbb{F},\mathbb{P})$ martingale properties of P^σ and $(Qp)^\sigma$ were used to pass to the second line, in combination with the following predictable projection formula:[26]

$$\mathbb{E}^{\mathbb{P}}\big[q_- \centerdot \langle p, P\rangle^{\mathbb{P}}_\sigma Q_\sigma p_\sigma\big] = \mathbb{E}^{\mathbb{P}}\big[{}^{p\cdot\mathbb{P}}(Q_\sigma p_\sigma) q_- \centerdot \langle p, P\rangle^{\mathbb{P}}_\sigma\big] = \mathbb{E}^{\mathbb{P}}\big[Q_- \centerdot \langle p, P\rangle^{\mathbb{P}}_\sigma\big],$$

where $\cdot^{p\cdot\mathbb{P}}$ denotes the $(\mathbb{F},\mathbb{P})$ predictable dual projection.

Hence, according to He, Wang and Yan (1992), Theorem 4.40, $[Qp - Q_- \centerdot p, P]$ is in $\mathcal{M}_{[0,T]}(\mathbb{F},\mathbb{P})$. Since this must hold for every bounded P in $\mathcal{M}_{[0,T]}(\mathbb{F},\mathbb{P})$, Lemma B.1 implies

$$0 = Q_0 p_0 = Qp - Q_- \centerdot p = Qp - (Qp)_- \centerdot (q_- \centerdot p) \qquad \text{on } [0,T]. \tag{B.4}$$

Consequently, $Qp = Q_0 p_0 \mathcal{E}(q_- \centerdot p) = 0$ (having assumed $Q_0 = 0$). □

LEMMA B.3. *For any $(\mathbb{F},\mathbb{P})$ local martingale P starting from 0 such that $[p,P]$ is $(\mathbb{F},\mathbb{P})$ locally integrable, $\mathfrak{Q}(P^c)$ and $\mathfrak{Q}(P^d)$ are a continuous $(\mathbb{F},\mathbb{Q})$ martingale and a purely discontinuous $(\mathbb{F},\mathbb{Q})$ martingale on $[0,T]$, respectively. Hence, $\mathfrak{Q}(P^c) = (\mathfrak{Q}(P))^c$ and $\mathfrak{Q}(P^d) = (\mathfrak{Q}(P))^d$ hold on $[0,T]$.*

PROOF. The predictable bracket Girsanov formula (B.2) shows that $\mathfrak{Q}(P^c)$ is a continuous local martingale on $[0,T]$. Regarding $\mathfrak{Q}(P^d)$, it is enough to prove that, for any continuous $(\mathbb{F},\mathbb{Q})$ martingale Q starting from 0, $\mathfrak{Q}(P^d)Q$ is in $\mathcal{M}_{[0,T]}(\mathbb{F},\mathbb{Q})$. [27]

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[25]Cf. He, Wang and Yan (1992), Theorem 12.13.
[26]Cf. He, Wang and Yan (1992), Remark 5.3 and Theorem 5.26.
[27]Cf. He, Wang and Yan (1992), Theorem 7.34.

In fact, for any $\mathbb{F}$ stopping time $\sigma \le T$ reducing the concerned processes to integrability, as in the proof of Lemma B.2, we can write

$$\mathbb{E}[\mathfrak{Q}(P^d)_\sigma Q_\sigma] = \mathbb{E}^{\mathbb{P}}[[Qp - Q_- \centerdot p, P^d]_\sigma],$$

where, by the integration by parts formula on $[0, T]$,

$$[Qp - Q_- \centerdot p, P^d] = [p_- \centerdot Q + [Q, p], P^d] = 0,$$

because $p_- \centerdot Q + [Q, p]$ is continuous, like Q. By He, Wang and Yan (1992), Theorem 4.40, this proves that $\mathfrak{Q}(P^d)Q$ is in $\mathcal{M}_{[0,T]}(\mathbb{F}, \mathbb{Q})$. □

LEMMA B.4. *We have the following relationships between the measure change density processes q, p and their stochastic logarithms $\bar{q}$, $\bar{p}$*:

$$\bar{q}^c = -\mathfrak{Q}(\bar{p}^c), \qquad \Delta\bar{q}^d = -q\Delta p \qquad \text{on } [0, T]. \tag{B.5}$$

PROOF. Note that, for any bounded $\mathbb{F}$ predictable process K and $\mathbb{F}$ predictable stopping time $\sigma \le T$,

$$\mathbb{E}[K_\sigma q_\sigma \Delta_\sigma p] = \mathbb{E}^{\mathbb{P}}[K_\sigma \Delta_\sigma p] = 0.$$

As a consequence, by Theorem 7.42 in He, Wang and Yan (1992), there exists an $(\mathbb{F}, \mathbb{Q})$ purely discontinuous local martingale $\mathbf{q}$ on $[0, T]$ such that

$$\Delta_s \mathbf{q} = q_s \Delta_s p. \tag{B.6}$$

By Lemma 3.4 in Karatzas and Kardaras (2007), we have the following relationship between $\bar{q}$ and $\bar{p}$:

$$\bar{q} = -\bar{p} + \langle \bar{p}^c, \bar{p}^c \rangle + \sum_{s \le \cdot} \frac{(\Delta_s \bar{p})^2}{1 + \Delta_s \bar{p}}. \tag{B.7}$$

Moreover, on $[0, T]$, we have

$$\begin{aligned} \Delta_t \mathbf{q} = \frac{\Delta_t p}{p_t} = \frac{p_{t-}\Delta_t \bar{p}}{p_{t-} + \Delta_t p} = \frac{\Delta_t \bar{p}}{1 + \Delta_t \bar{p}} \\ \text{hence } [\mathbf{q}, \bar{p}^d] = \sum_{s \le \cdot} \frac{(\Delta_s \bar{p})^2}{1 + \Delta_s \bar{p}}. \end{aligned} \tag{B.8}$$

Using (B.7) and (B.8), we obtain

$$\begin{aligned} \bar{q} &= -\bar{p} + \langle \bar{p}^c, \bar{p}^c \rangle + [\mathbf{q}, \bar{p}^d] \\ &= -\bar{p}^c + \langle \bar{p}^c, \bar{p}^c \rangle - \bar{p}^d + [\mathbf{q}, \bar{p}^d] \\ &= -\mathfrak{Q}(\bar{p}^c) - \bar{p}^d + [\mathbf{q}, \bar{p}^d] \end{aligned}$$

on $[0,T]$ [cf. (B.2)]. As a consequence together with Lemma B.3, we have $\bar{q}^c = -\mathfrak{Q}(\bar{p}^c)$. In addition, by (B.7),

$$\Delta_t \bar{q}^d = \Delta_t \bar{q} = -\Delta_t \bar{p} + \frac{(\Delta_t \bar{p})^2}{1+\Delta_t \bar{p}} = -\frac{\Delta_t \bar{p}}{1+\Delta_t \bar{p}} = -\Delta_t \mathbf{q},$$

by the first part in (B.8). Since $\bar{q}^d$ and $(-\mathbf{q})$ are both $(\mathbb{F},\mathbb{Q})$ purely discontinuous local martingales, they coincide on $[0,T]$, by Corollary 7.23 in He, Wang and Yan (1992). In view of (B.6), this proves that $\Delta\bar{q}^d = -q\Delta p$. $\square$

APPENDIX C: CHARACTERIZATION OF INVARIANCE MEASURES VIA THE GIRSANOV AND JEULIN–YOR FORMULAS

In this section, we combine the measure change lemmas B.2 through B.4 (still using the notation introduced in Section B) with enlargement of filtration computations for obtaining our second proof of Theorem 3.1. The goal is to relate the invariance measure property to (3.3). Passing from $(\mathbb{F},\mathbb{P})$ to $(\mathbb{F},\mathbb{Q})$ creates a Girsanov's drift while passing from $(\mathbb{F},\mathbb{Q})$ to $(\mathbb{G},\mathbb{Q})$, in combination with stopping before θ, creates a Jeulin–Yor's drift. Therefore, to have the invariance probability property, the two drifts must cancel out each other. The projection methodology, a classical approach systematically used in Jeulin (1980), can then be used then to transform this cancellation condition in the filtration $\mathbb{G}$ into a cancellation condition in the filtration $\mathbb{F}$, which, with the help of Lemmas B.2 through B.4, will be seen to be equivalent to (3.3).

Recall $\mathrm{J} = \mathbb{1}_{[0,\theta)}$. For any $P \in \mathcal{M}(\mathbb{F},\mathbb{P})$, the Girsanov formula (B.1) says that $P - q \centerdot [p,P] \in \mathcal{M}_{[0,T]}(\mathbb{F},\mathbb{Q})$. The Jeulin–Yor formula (2.2) applied to $Q = P - q \centerdot [p,P]$ then yields that

$$\begin{aligned}(P - q \centerdot [p,P])^{\theta-} &- \mathrm{J}_- \frac{1}{\mathrm{S}_-} \centerdot \langle \mathrm{Q}, P - q \centerdot [p,P]\rangle \\ &= P^{\theta-} - \Big(\mathrm{J} q \centerdot [p,P] + \mathrm{J}_- \frac{1}{\mathrm{S}_-} \centerdot \langle \mathrm{Q}, P - q \centerdot [p,P]\rangle\Big) \in \mathcal{M}_{[0,T]}(\mathbb{G},\mathbb{Q}).\end{aligned}$$

Therefore, the invariance measure property that the first term $P^{\theta-}$ in the right-hand side is in $\mathcal{M}_{[0,T]}(\mathbb{G},\mathbb{Q})$ is equivalent to the condition that the second term (in parentheses) is in $\mathcal{M}_{[0,T]}(\mathbb{G},\mathbb{Q})$, for any $P \in \mathcal{M}(\mathbb{F},\mathbb{P})$. In other words, we conclude from the above that $\mathbb{P}$ is an invariance measure if and only if, for any $P \in \mathcal{M}(\mathbb{F},\mathbb{P})$,

$$\mathrm{J} q \centerdot [p,P] + \mathrm{J}_- \frac{1}{\mathrm{S}_-} \centerdot \langle \mathrm{Q}, P - q \centerdot [p,P]\rangle \in \mathcal{M}_{[0,T]}(\mathbb{G},\mathbb{Q}). \tag{C.1}$$

In the next steps, we derive, by projection, an $\mathbb{F}$ counterpart of the condition (C.1). For doing so, we need an expression of the condition (C.1) in the form of expectations. In fact, (C.1) is equivalent to

$$\text{(C.2)}\quad \begin{aligned}&\exists \text{ a nondecreasing sequence of } \mathbb{G} \text{ stopping times } (\tau_n)_{n\in\mathbb{N}} \text{ tending to}\\ &\text{infinity such that, } \forall \text{ bounded } \mathbb{G} \text{ predictable process } L \text{ and } n \in \mathbb{N},\\ &\mathbb{E}\Big[L\mathtt{J}q \centerdot [p,P]_{\tau_n\wedge T} + L\mathtt{J}_-\frac{1}{\mathtt{S}_-} \centerdot \langle \mathtt{Q}, P - q \centerdot [p,P]\rangle_{\tau_n\wedge T}\Big] = 0.\end{aligned}$$

By Lemma 2.2(1), the nondecreasing sequence of $\mathbb{G}$ stopping times $(\tau_n)_{n\in\mathbb{N}}$ is associated by reduction with a nondecreasing sequence of $\mathbb{F}$ stopping times $(\sigma_n)_{n\in\mathbb{N}}$ such that, for every $n \in \mathbb{N}$, $\theta \wedge \sigma_n = \theta \wedge \tau_n$. We have necessarily $\bigcup_{n\in\mathbb{N}}[0,\sigma_n] \supseteq [0,\theta]$. By Jeulin (1980), Lemma (4,3), $\{{}^p(\mathbb{1}_{(\theta,\infty)}) = 1\}$ is the largest $\mathbb{F}$ predictable set contained in (θ,∞). As a consequence, $\{{}^p(\mathbb{1}_{(\theta,\infty)}) = 1\} \supseteq (\mathbb{R}_+ \times \Omega) \setminus (\bigcup_{n\in\mathbb{N}}[0,\sigma_n])$ or, equivalently

$$\text{(C.3)}\quad \{0\} \cup \{\mathtt{S}_- > 0\} = \{0\} \cup \{1 - \mathtt{S}_- < 1\} = \{{}^p(\mathbb{1}_{(\theta,\infty)}) < 1\} \subset \bigcup_{n\in\mathbb{N}}[0,\sigma_n].$$

Conversely, if (C.3) holds for a nondecreasing sequence of $\mathbb{F}$ stopping times $(\sigma_n)_{n\in\mathbb{N}}$, then, in view of (A.11), the nondecreasing sequence of $\mathbb{G}$ stopping times $\tau_n = (\sigma_n)_{\{\sigma_n<\theta\}}$ tends to infinity. In addition, by the condition (B), the bounded $\mathbb{G}$ predictable processes L are associated with the bounded $\mathbb{F}$ predictable processes K through the reduction identity $K\mathbb{1}_{(0,\theta]} = L\mathbb{1}_{(0,\theta]}$.

Through these correspondences and the projection identities ${}^o(\mathtt{J}) = \mathtt{S}$ and ${}^p(\mathtt{J}_-) = \mathtt{S}_-$ on $(0,\infty)$, (C.2) can be rewritten in terms of $\mathbb{F}$ stopping times and adapted processes, as

$$\text{(C.4)}\quad \begin{aligned}&\exists \text{ a nondecreasing sequence of } \mathbb{F} \text{ stopping times } (\sigma_n)_{n\in\mathbb{N}} \text{ satisfying}\\ &\text{(C.3) such that, } \forall \text{ bounded } \mathbb{F} \text{ predictable process } K \text{ and } n \in \mathbb{N},\\ &\mathbb{E}\big[K\mathtt{S}q \centerdot [p,P]_{\sigma_n\wedge T} + K \centerdot \langle \mathtt{Q}, P - q \centerdot [p,P]\rangle_{\sigma_n\wedge T}\big] = 0.\end{aligned}$$

We are now transferred into the filtration $\mathbb{F}$. To establish the desired connection with (3.3), we interpret (C.4) as a local martingale condition in $\mathbb{F}$. In fact, (C.4) is equivalent to the following $\mathbb{F}$ counterpart of (C.1):

$$\text{(C.5)}\quad \mathtt{S}q \centerdot [p,P] + \langle \mathtt{Q}, P - q \centerdot [p,P]\rangle \in \mathcal{M}_{\{\mathtt{S}_->0\}\cap[0,T]}(\mathbb{F},\mathbb{Q}),$$

which is equivalent to

$$\text{(C.6)}\quad \mathtt{S}q \centerdot [p,P] + \big[\mathtt{Q}, P - q \centerdot [p,P]\big] \in \mathcal{M}_{\{\mathtt{S}_->0\}\cap[0,T]}(\mathbb{F},\mathbb{Q}).$$

Operating separately on the continuous parts P^c and the purely discontinuous parts P^d of the $(\mathbb{F},\mathbb{P})$ local martingales P, we conclude that $\mathbb{P}$ is an invariance measure if and only if, for any $P \in \mathcal{M}(\mathbb{F},\mathbb{P})$,

$$\text{(C.7)}\quad \begin{cases}\mathtt{S}q \centerdot \langle p^c, P^c\rangle + \langle \mathtt{Q}^c, P^c - q \centerdot \langle p, P^c\rangle\rangle \in \mathcal{M}_{\{\mathtt{S}_->0\}\cap[0,T]}(\mathbb{F},\mathbb{Q}),\\ \mathtt{S}q \centerdot [p^d, P^d] + [\mathtt{Q}^d, P^d - q \centerdot [p, P^d]] \in \mathcal{M}_{\{\mathtt{S}_->0\}\cap[0,T]}(\mathbb{F},\mathbb{Q}).\end{cases}$$

We are now in a position to deduce the equivalent condition (3.3). Using the identities ${}^p\mathtt{S} = \mathtt{S} - \Delta\mathtt{Q}$ and $\bar{q}^c = -\mathfrak{Q}(\bar{p}^c)$ [cf. (A.2) and Lemma B.4], by continuity, we have

$$
\begin{aligned}
&\mathtt{S}q \centerdot \langle p^c, P^c\rangle + \langle \mathtt{Q}^c, P^c - q \centerdot \langle p, P^c\rangle\rangle \\
&\quad = {}^p\mathtt{S} \centerdot \langle \bar{p}^c, P^c\rangle + \langle \mathtt{Q}^c, P^c - q \centerdot \langle p, P^c\rangle\rangle \\
&\quad = {}^p\mathtt{S} \centerdot \langle \mathfrak{Q}(\bar{p}^c), \mathfrak{Q}(P^c)\rangle + \langle \mathtt{Q}^c, \mathfrak{Q}(P^c)\rangle = \langle -{}^p\mathtt{S} \centerdot \bar{q}^c + \mathtt{Q}^c, \mathfrak{Q}(P^c)\rangle.
\end{aligned}
$$

Hence, the first line in (C.7) means that

$$\langle -{}^p\mathtt{S} \centerdot \bar{q}^c + \mathtt{Q}^c, \mathfrak{Q}(P^c)\rangle \in \mathcal{M}_{\{\mathtt{S}_- > 0\}\cap[0,T]}(\mathbb{F}, \mathbb{Q}).$$

This holding for all the $(\mathbb{F}, \mathbb{P})$ local martingales P, including the bounded ones for which

$$\langle -{}^p\mathtt{S} \centerdot \bar{q}^c + \mathtt{Q}^c, \mathfrak{Q}(P^c)\rangle = \langle -{}^p\mathtt{S} \centerdot \bar{q}^c + \mathtt{Q}^c, \mathfrak{Q}(P)\rangle$$

(cf. Lemma B.3), is equivalent by Lemma B.2 to

$$-{}^p\mathtt{S} \centerdot \bar{q}^c + \mathtt{Q}^c = 0 \qquad \text{on } \{\mathtt{S}_- > 0\} \cap [0, T]. \tag{C.8}$$

Likewise, we have

$$
\begin{aligned}
&\mathtt{S}q \centerdot [p^d, P^d] + [\mathtt{Q}^d, P^d - q \centerdot [p, P^d]] \\
&\quad = \sum (\mathtt{S}q \Delta p \Delta P + \Delta \mathtt{Q}(\Delta P - q \Delta p \Delta P)) \\
&\quad = \sum ((\mathtt{S} - \Delta\mathtt{Q}) q \Delta p \Delta P + \Delta\mathtt{Q}\Delta P) \\
&\quad = -{}^p\mathtt{S} \centerdot [\bar{q}^d, P] + [\mathtt{Q}^d, P] = [-{}^p\mathtt{S} \centerdot \bar{q}^d + \mathtt{Q}^d, P],
\end{aligned}
$$

where the identities ${}^p\mathtt{S} = \mathtt{S} - \Delta\mathtt{Q}$ and $\Delta\bar{q}^d = -q\Delta p$ [cf. (A.2) and Lemma B.4] are used in the next-to-last equality. Hence, the second line in (C.7) means that

$$[-{}^p\mathtt{S} \centerdot \bar{q}^d + \mathtt{Q}^d, P] \in \mathcal{M}_{\{\mathtt{S}_- > 0\}\cap[0,T]}(\mathbb{F}, \mathbb{Q}).$$

For P bounded [so that the predictable bracket Girsanov formula (B.2) is applicable], in view of Yoeurp's lemma, this means that

$$[-{}^p\mathtt{S} \centerdot \bar{q}^d + \mathtt{Q}^d, \mathfrak{Q}(P)] \in \mathcal{M}_{\{\mathtt{S}_- > 0\}\cap[0,T]}(\mathbb{F}, \mathbb{Q}).$$

Using Lemma B.2, we conclude that the second line in (C.7) holds for all the $(\mathbb{F}, \mathbb{P})$ local martingales P (including the bounded ones) if and only if

$$-{}^p\mathtt{S} \centerdot \bar{q}^d + \mathtt{Q}^d = 0 \qquad \text{on } \{\mathtt{S}_- > 0\} \cap [0, T]. \tag{C.9}$$

By Lemma 2.5, $\mathtt{Q}$ is constant on $\{\mathtt{S}_- = 0\} \subset \{{}^p\mathtt{S} = 0\}$. Hence, (C.8) and (C.9) are respectively the continuous and purely discontinuous parts of (3.3), so that Theorem 3.1 is proved.

C.1. A by-product. Inspecting the above proof of Theorem 3.1, we see that the implication from the invariance measure property to (3.3), which is based on Lemma B.2, only makes use of the bounded $(\mathbb{F}, \mathbb{P})$ martingales. We can therefore introduce a seemingly weaker condition.

Condition (A′). There exists a probability measure $\mathbb{P}$ equivalent to $\mathbb{Q}$ on $\mathcal{F}_T$ such that, for any bounded $(\mathbb{F}, \mathbb{P})$ martingale P, $P^{\theta-}$ is a $(\mathbb{G}, \mathbb{Q})$ local martingale on $[0, T]$.

COROLLARY C.1. *The condition* (A) *is equivalent to the condition* (A′).

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