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Barrier estimates for a critical Galton–Watson process and the cover time of the binary tree

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Abstract. For the critical Galton–Watson process with geometric offspring distributions we provide sharp barrier estimates for barriers which are (small) perturbations of linear barriers. These are useful in analyzing the cover time of finite graphs in the critical regime by random walk, and the Brownian cover times of compact two-dimensional manifolds. As an application of the barrier estimates, we prove that if C_L denotes the cover time of the binary tree of depth L by simple walk, then $\sqrt{C_L/2^{L+1}} - \sqrt{2\log 2}L + \log L/\sqrt{2\log 2}$ is tight. The latter improves results of Aldous (*J. Math. Anal. Appl.* **157** (1991) 271–283), Bramson and Zeitouni (*Ann. Probab.* **37** (2009) 615–653) and Ding and Zeitouni (*Stochastic Process. Appl.* **122** (2012) 2117–2133). In a subsequent article we use these barrier estimates to prove tightness of the Brownian cover time for compact two-dimensional manifolds.

Résumé. Pour le processus critique de Galton–Watson avec loi de reproduction géométrique de la progéniture, nous fournissons des estimations fines de barrière pour des obstacles qui sont des (petites) perturbations de barrières linéaires. Les estimations sont utiles pour analyser le temps de recouvrement, par une marche aleatoire, de graphes finis dans le régime critique, et les temps de recouvrement brownien de variétés bidemensionelles compactes. Comme application des estimations de barrière, nous prouvons que si C_L dénote le temps de recouvrement de l'arbre binaire de profondeur L par une marche aleatoire simple, la suite $\sqrt{C_L/2^{L+1}} - \sqrt{2\log 2}L + \log L/\sqrt{2\log 2}$ est tendue. Ce dernier resultat amélioré les résultats d'Aldous (*J. Math. Anal. Appl.* **157** (1991) 271–283), Bramson et Zeitouni (*Ann. Probab.* **37** (2009) 615–653) et Ding et Zeitouni (*Stochastic Process. Appl.* **122** (2012) 2117–2133). Dans un article compagnon, nous utilisons ces estimations de barrière pour prouver la tension du temps de recouvrement brownien pour des variétés riemanniennes compactes en deux dimensions.

1. Introduction and statement of main results

Let P_n be the law of the critical Galton–Watson process $(T_l)_{l \geq 0}$ with inital population $T_0 = n$ and geometric offspring distribution. For any a, b, L let

$$f_{a,b}(l; L) = a + (b-a)\frac{l}{L},$$

be the line interpolating a and b over the interval $[0, L]$. Abbreviate $l_L = l \wedge (L - l)$. Let $\mathbb{N} = \{1, 2, \ldots\}$ and $\mathbb{Z}^+ = \{0, 1, 2, \ldots\}$. For $y, \delta \geq 0$, let $H_{y,\delta} = [y, y + \delta]$ and set $H_y = H_{y,1}$. The main result of this article is the following barrier estimates for the process T_l. It is motivated by applications to the study of the cover time of the binary tree by a random walk and of two-dimensional compact manifolds by a Brownian sausage, see the discussion following Remark 1.2 below.

Theorem 1.1.

(a) *For all fixed* $\delta, C > 0$, $\eta > 1$ *and* $\varepsilon \in (0, \frac{1}{2})$ *we have, uniformly in* $\sqrt{2} \leq x, y \leq \eta L$ *such that* $x^2/2 \in \mathbb{N}$, *any* $0 \leq a \leq x, 0 \leq b \leq y$, *that*

$$P_{x^2/2}\big(f_{a,b}(l; L) - Cl_L^{\frac{1}{2}-\varepsilon} \leq \sqrt{2T_l}, l = 1, \ldots, L - 1, \sqrt{2T_L} \in H_{y,\delta}\big)$$
$$\leq c\frac{(1 + x - a)(1 + y - b)}{L}\sqrt{\frac{x}{yL}}e^{-\frac{(x-y)^2}{2L}}, \tag{1.1}$$

(see Figure 1). If in addition $a \geq L/\eta$, *similar bounds hold for* $y \leq \sqrt{2}$, *with* $\sqrt{\frac{x}{yL}}$ *replaced by* 1.

(b) *For any* $\tilde{C} \geq 2C + 2\delta + \eta + \sqrt{2}$, *if, in addition to the conditions in part* (a), *we also have* $(1 + x - a)(1 + y - b) \leq \eta L$, $\max(ab, |a - b|) \geq L/\eta$ *and* $[y, y + \delta] \cap \sqrt{2}\mathbb{Z}^+ \neq \varnothing$ *then*

$$P_{x^2/2}\big(f_{a,b}(l; L) + Cl_L^{\frac{1}{2}-\varepsilon} \leq \sqrt{2T_l} \leq f_{x,y}(l; L) + \tilde{C}l_L^{\frac{1}{2}+\varepsilon}, l = 1, \ldots, L - 1, \sqrt{2T_L} \in H_{y,\delta}\big)$$
$$\geq c\frac{(1 + x - a)(1 + y - b)}{L} \times \left(\sqrt{\frac{x}{yL}} \wedge 1\right)e^{-\frac{(x-y)^2}{2L}}, \tag{1.2}$$

(see Figure 2), and the estimate is uniform in such x, y, a, b *and all* L.
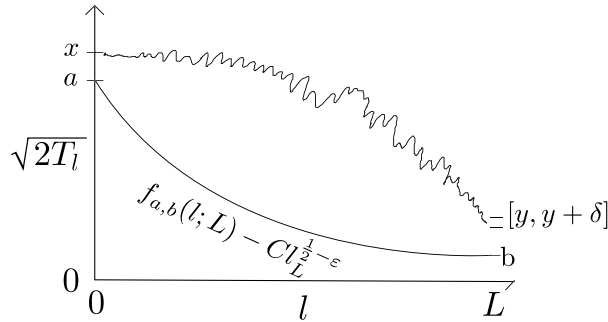

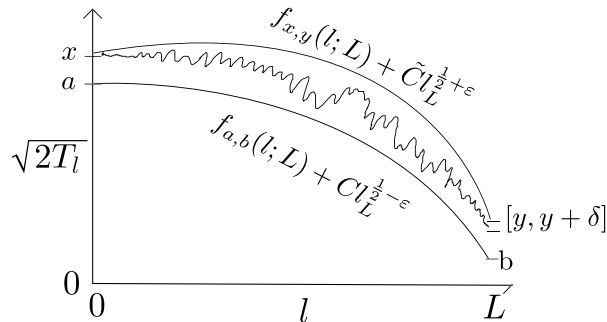Fig. 1. Illustration of barrier event in Theorem 1.1(a).

Fig. 2. Illustration of barrier event in Theorem 1.1(b).

Remark 1.2.

(a) In this paper constants, whose value may change from occurence to occurence, that depend at most on C, δ, η and ε, are denoted by c. The notation $a \asymp b$ means that $a \leq c \cdot b$ and $b \leq c \cdot a$.

(b) When $\delta < \sqrt{2}$ and $y = 0$ the terminal condition $\sqrt{2T_L} \in H_{y,\delta}$ is equivalent to $T_L = 0$. Less precise barrier estimates for the process T_l conditioned on $T_L = 0$ appear in [6, Proposition 7.1].

We will put Theorem 1.1 in the context of estimates for Bessel processes, see Proposition 1.4 below. Before doing so, we emphasize that the main application of Theorem 1.1 is in a paper by the authors [7] which proves tightness of the (centered) square-root of the Brownian cover time of two-dimensional compact manifolds. In the current paper, we illustrate the use of Theorem 1.1 by presenting a quick proof of a similar result for the cover time of the binary tree. Let $\mathcal{T}_L$ be the tree of depth L, with a root of degree one attached to the top. (Formally, begin with a binary tree rooted at vertex o, and attach to it a vertex ρ, the root of $\mathcal{T}_L$, connected by an edge to o.) The tree $\mathcal{T}_L$ has 2^{L+1} vertices and $2^L + 1$ leaves (including ρ). Let $\mathbb{P}$ be the law of discrete time simple random walk $(X_n)_{n \geq 0}$ on $\mathcal{T}_L$ starting at the root. Let $\tau_y, y \in \mathcal{T}_L$, be the hitting time of the vertex y by the random walk. The cover time

$$C_L = \max_{y \in \mathcal{T}_L} \tau_y = \max_{y \in \mathcal{T}_L, y \text{ is a leaf}} \tau_y$$

is the first time the random walk has visited every vertex of $\mathcal{T}_L$. We prove the following estimate.

Theorem 1.3. *There exist constants c such that for all $x > 0$,*

$$\limsup_{L \to \infty} \mathbb{P}\left(\sqrt{\frac{C_L}{2^{L+1}}} \geq \sqrt{2 \log 2} L - \frac{1}{\sqrt{2 \log 2}} \log L + x\right) \leq cx e^{-x\sqrt{2 \log 2}}, \tag{1.3}$$

$$\liminf_{L \to \infty} \mathbb{P}\left(\sqrt{\frac{C_L}{2^{L+1}}} \geq \sqrt{2 \log 2} L - \frac{1}{\sqrt{2 \log 2}} \log L + x\right) \geq cx e^{-x\sqrt{2 \log 2}}, \tag{1.4}$$

and

$$\limsup_{L \to \infty} \mathbb{P}\left(\sqrt{\frac{C_L}{2^{L+1}}} \leq \sqrt{2 \log 2} L - \frac{1}{\sqrt{2 \log 2}} \log L - x\right) \leq e^{-cx}. \tag{1.5}$$

In particular, Theorem 1.3 shows that

$$\sqrt{\frac{C_L}{2^{L+1}}} = \sqrt{2 \log 2} L - \frac{1}{\sqrt{2 \log 2}} \log L + O(1), \tag{1.6}$$

that is, tightness of a centered, scaled version of the square root of the cover time. Equivalently one could state tightness directly in terms of a centered and scaled version of the cover time itself, as

$$\frac{C_L}{2^{L+1}L} = 2\big(\log 2 \cdot L - \log L + O(1)\big),$$

and the corresponding tail bounds can also be written in a similar way. The statement (1.6) improves on the estimate from [14], which has $O((\log \log L)^8)$ in place of $O(1)$. (Earlier results of Aldous [4] give the leading order $\sqrt{2 \log 2} L(1 + o(1))$.) Theorem 1.3 also provides a new proof that after appropriate centering, $\sqrt{C_L/2^{L+1}}$ is tight, a result proven in [10] using a recursion (in a way that avoids computing the centering term).

In light of recent works [6,14], it appears that cover times of homogeneous trees and of two dimensional graphs or manifolds are related to the extrema of certain critical hierarchical random fields, a universality class which contains Gaussian logarithmically correlated fields and branching random walks.[3] The logarithmic correction term in (1.6),

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[3]More generally, isomorphism theorems have been used to show that for any finite connected graph for which hitting times are asymptotically shorter than cover times, the cover time divided by the number of edges is of the same order as the square of the maximum of the Gaussian Free Field on the same graph [13]; for arbitrary trees or for bounded degree graphs, they were shown to match to leading order [12], and for general graphs in [21].

whose form was proven in [14], is, up to constant multiple, the universal correction term[4] for fields in this universality class. That the minimum (or maximum) is tight after centering by the leading term together with this logarithmic correction term is another conjectured universal feature of these fields, as is the decay xe^{-cx}, for some c, of the right tail, which we verify in (1.3) and (1.4). (By contrast, the left tail for which we have only the rough bound (1.5), is not expected to be universal, and in general is not even of exponential form.) Our approach builds on previous works on branching random walk [2,3,5,8,9] and cover times [6,11,14]. More precisely, we use a second moment method with a truncation involving the process of discrete edge local times staying above certain barriers. The barrier estimates that are the main results of this paper are a crucial technical input.

The main step of the proof of the cover time result is the analysis of the *minimum* of the discrete local times among the leaves at the time a certain local time is reached at the root. For the *maximum* of continuous local times on the leaves more precise results have been obtained [1].

The intuition behind Theorem 1.1 is that the process $l \to \sqrt{2T_l}$ behaves like a Bessel-0 process, i.e. a process Y_t which satisfies the SDE

$$dY_t = dW_t - \frac{1}{2Y_t}\, dt, \tag{1.7}$$

for a Brownian motion W_t (the drift is only significant if Y_t is close to zero, otherwise the process Y_t behaves like a Brownian motion). The Galton–Watson process T_l can be thought of as a discrete version of a squared Bessel-0 process. Indeed, if $f_{l,j}$, $j \geq 1$, are the number of offspring of each individual in generation l, letting $g_{l,j} = f_{l,j} - 1$ we can also write $\sqrt{2T_{l+1}}$ as

$$\sqrt{2T_{l+1}} = \sqrt{2T_l}\sqrt{1 + \frac{1}{T_l}\sum_{j=1}^{T_l} g_{l,j}}. \tag{1.8}$$

When T_l is large one can Taylor expand the square root to obtain that the increment $\sqrt{2T_{l+1}} - \sqrt{2T_l}$ equals

$$\frac{1}{\sqrt{2T_l}}\sum_{j=1}^{T_l} g_{l,j} - \frac{1}{2\sqrt{2T_l}} \tag{1.9}$$

plus terms that can be shown to be negligible. The distribution of the normalized sum in (1.9) will be close to Gaussian by the central limit theorem (the $g_{l,j}$ are independent, have mean zero, and are independent of T_l). Thus if we let $Z_l = \sqrt{2T_l}$ we can informally write

$$Z_{l+1} - Z_l \approx N_l - \frac{1}{2Z_l},$$

for approximately Gaussian independent N_l, making the heuristic link to (1.7) apparent. A proof of results similar to Theorem 1.1, that could be extended to other Galton–Watson processes, could be provided[5] by following these ideas. Instead, in this paper, we use exact equalities in law to provide a shorter proof.

Theorem 1.1 should be understood in the light of the following precise large deviation estimate for $\sqrt{2T_L}$.

Proposition 1.4. *For all fixed $\delta > 0$ and $\eta > 1$ we have, uniformly in $\sqrt{2} \leq x$, $y \leq \eta L$ such that $x^2/2$ is an integer, that*

$$P_{x^2/2}(\sqrt{2T_L} \in H_{y,\delta}) \leq c\sqrt{\frac{x}{yL}}e^{-\frac{(x-y)^2}{2L}}. \tag{1.10}$$

If, in addition, $[y, y + \delta] \cap \sqrt{2}\mathbb{Z}^+ \neq \varnothing$ and $L/\eta \leq xy$, then the corresponding lower bound

$$P_{x^2/2}(\sqrt{2T_L} \in H_{y,\delta}) \geq c\sqrt{\frac{x}{yL}} e^{-\frac{(x-y)^2}{2L}}, \tag{1.11}$$

also holds. Also, for any $0 < x \leq \eta L$,

$$P_{x^2/2}(\sqrt{2T_L} = 0) \asymp e^{-\frac{x^2}{2L}}. \tag{1.12}$$

The bounds (1.10)–(1.12) are the same as the ones satisfied by Y_L if the process Y_t, under P_x^Y, is a Bessel process of dimension zero, since

$$P_x^Y(Y_L \in \cdot) = \delta_0 e^{-\frac{x^2}{2L}} + 1_{(0,\infty)} \frac{x}{L} e^{-\frac{x^2+y^2}{2L}} I_1\left(\frac{xy}{L}\right) dy, \tag{1.13}$$

where I_1 is the first Bessel function, see (2.8) and (2.9). Furthermore, e.g., Theorem 1.1(a) can be written as

$$P_{x^2/2}\left(f_{a,b}(l; L) - Cl_L^{\frac{1}{2}-\varepsilon} \leq \sqrt{2T_l}, l = 1, \ldots, L - 1 \mid \sqrt{2T_L} \in H_{y,\delta}\right)$$
$$\leq c\frac{(1 + x - a)(1 + y - b)}{L}. \tag{1.14}$$

The probability that a Brownian bridge in the time interval $[0, L]$ stays above a linear barrier (or a small perturbation thereof) during $[1, L - 1]$ as in the event in (1.14), when it starts at distance $x - a$ from the line and ends at distance $y - b$ from it is of the order of the right-hand side of (1.14) (see (2.2) and (2.3) below). Our Theorem 1.1 can thus be thought of as a Galton–Watson process version of barrier results for the Brownian bridge. Heuristically, it arises from approximating $\sqrt{2T_l}$ conditioned on its end point by a Brownian bridge.

In the rest of the paper, we prove Theorems 1.1 and 1.3. We begin by proving, in Section 2, barrier estimates for 0-dimensional Bessel processes. In Section 3, we show that traversal counts and local times of a random walk on $\mathbb{Z}^+$ give a Markovian structure closely related to the sampling of a Bessel-0 process. In Section 4, we use the latter structure to transfer barrier estimates for Bessel-0 processes to the setting of Theorem 1.1. Finally, in Section 5 we use a first/second moment method, together with the barrier estimates of Theorem 1.1, to obtain Theorem 1.3.

2. Barrier estimates for the 0-dimensional Bessel process

We will derive the barrier estimate in Theorem 1.1 from similar results for a Brownian motion. Let P_x^W be the law of a Brownian motion W_t, $t \geq 0$, starting at x, and let $P_x^W(\cdot | W_L = y)$ be the law of a Brownian bridge starting at x and ending at y at time L. The probability that a Brownian bridge stays above a linear barrier is explicit: If $a \leq x$ and $b \leq y$ then by the reflection principle, see e.g. [8, Lemma 2.2],

$$P_x^W\left(W_l \geq f_{a,b}(l; L), l \in [0, L] | W_L = y\right) = 1 - \exp\left(-2\frac{(x - a)(y - b)}{L}\right). \tag{2.1}$$

If also $(x - a)(y - b) = O(L)$ this implies

$$P_x^W\left(W_l \geq f_{a,b}(l; L), l \in [0, L] | W_L = y\right) \asymp \frac{(x - a)(y - b)}{L}.$$

and by conditioning on W_1 and W_{L-1} one easily derives that

$$P_x^W\left(W_l \geq f_{a,b}(l; L), l \in [1, L - 1] | W_L = y\right) \asymp \frac{(1 + x - a)(1 + y - b)}{L}. \tag{2.2}$$

The next lemma shows that the probability has the same order of magnitude if a "bump" is added or subtracted from the straight line.

Lemma 2.1. *For all fixed $\varepsilon \in (0, \frac{1}{2})$, $\delta > 0$, $\eta > 1$, $\tilde{C} \geq C + \delta > 0$, one has uniformly in $a \leq x$, $b \leq y$, $(x-a)(y-b) \leq \eta L$, $|x-y| \leq \eta L$, and L large enough,*

$$P_x^W\big(f_{a,b}(l; L) + Cl_L^{\frac{1}{2}-\varepsilon} \leq W_l \leq f_{x,y}(l; L) + \tilde{C}l_L^{\frac{1}{2}+\varepsilon}, l \in [1, L-1], W_L \in H_{y,\delta}\big)$$

$$\asymp P_x^W\big(f_{a,b}(l; L) - Cl_L^{\frac{1}{2}-\varepsilon} \leq W_l, l \in [1, L-1], W_L \in H_{y,\delta}\big)$$

$$\asymp \frac{(1 + x - a)(1 + y - b)}{L} \frac{1}{\sqrt{L}} e^{-\frac{(x-y)^2}{2L}}. \tag{2.3}$$

Remark 2.2. The condition $(x-a)(y-b) \leq \eta L$ is not necessary for the upper bounds. One can simply drop the barrier. If we eliminate the condition $|x-y| \leq \eta L$ then the last line in (2.3) would become

$$\frac{(1 + x - a)(1 + y - b)}{L} \frac{1}{\sqrt{L}} \sup_{z \in H_{y,\delta}} e^{-\frac{(x-z)^2}{2L}}. \tag{2.4}$$

for the upper bound, and inf instead of sup for the lower bound. The condition $|x-y| \leq \eta L$ is only used to guarantee that the supremum in (2.4) has the same order as $e^{-(x-y)^2/2L}$.

Proof of Lemma 2.1. It follows from Lemma 2.6, Lemma 2.7 and Proposition 6.1 of [8] that for any $C > 0$ there is an r large enough so that for all $L > 2r$

$$P_x^W\big(f_{a,b}(l; L) + Cl_L^{\frac{1}{2}-\varepsilon} \leq W_l \leq f_{x,y'}(l; L) + Cl_L^{\frac{1}{2}+\varepsilon}, l \in [r, L-r] | W_L = y'\big)$$

$$\geq c P_x^W\big(f_{a,b}(l; L) - Cl_L^{\frac{1}{2}-\varepsilon} \leq W_l, l \in [r, L-r] | W_L = y'\big).$$

Let

$$A = \big\{f_{a,b}(l; L) + Cl_L^{\frac{1}{2}-\varepsilon} \leq W_l \leq f_{x,y'}(l; L) + Cl_L^{\frac{1}{2}+\varepsilon}, l \in [1, r] \cup [L-r, L-1]\big\}.$$

For any fixed $r > 0$ and and for all $l \geq 1$ we have

$$P_x^W\big(A | f_{a,b}(l; L) + Cl_L^{\frac{1}{2}-\varepsilon} \leq W_l \leq f_{x,y'}(l; L) + Cl_L^{\frac{1}{2}+\varepsilon}, l \in [r, L-r], W_L = y'\big)$$

$$\geq c > 0,$$

for a constant c depending only on r and C. Therefore also

$$P_x^W\big(f_{a,b}(l; L) + Cl_L^{\frac{1}{2}-\varepsilon} \leq W_l \leq f_{x,y'}(l; L) + \tilde{C}l_L^{\frac{1}{2}+\varepsilon}, l \in [1, L-1] | W_L = y'\big)$$

$$\geq c P_x^W\big(f_{a,b}(l; L) - Cl_L^{\frac{1}{2}-\varepsilon} \leq W_l, l \in [1, L-1] | W_L = y'\big).$$

Since the inequality in the opposite direction, with $c = 1$, is trivial, we see by sandwiching that we now have shown

$$P_x^W\big(f_{a,b}(l; L) + Cl_L^{\frac{1}{2}-\varepsilon} \leq W_l \leq f_{x,y'}(l; L) + Cl_L^{\frac{1}{2}+\varepsilon}, l \in [1, L-1] | W_L = y'\big)$$

$$\asymp P_x^W\big(f_{a,b}(l; L) \leq W_l, l \in [1, L-1] | W_L = y'\big)$$

$$\asymp P_x^W\big(f_{a,b}(l; L) - Cl_L^{\frac{1}{2}-\varepsilon} \leq W_l, l \in [1, L-1] | W_L = y'\big).$$

The claim (2.3) then follows by first using (2.2), then multiplying by the Gaussian density, and then integrating over $y' \in H_{y,\delta}$, using the fact that $f_{x,y'}(l; L) \leq f_{x,y+\delta}(l; L) \leq f_{x,y}(l; L) + \delta$. $\qquad\square$

The next lemma shows that we can replace the barrier $f_{a,b}(l; L) - Cl_L^{\frac{1}{2}-\varepsilon}$ "checked" for all $l \in [1, L-1]$ with one "checked" only at integer times $l = 1, \ldots, L - 1$.

Lemma 2.3. *For all fixed $C > 0$, $\varepsilon \in (0, \frac{1}{2})$, $\delta > 0$ and $\eta > 1$, one has uniformly in $a \le x$, $b \le y$, $|x - y| \le \eta L$ and L large enough,*

$$P_x^W \big(f_{a,b}(l; L) - Cl_L^{\frac{1}{2}-\varepsilon} \le W_l, l = 1, \ldots, L - 1, W_L \in H_{y,\delta} \big)$$
$$\le c \frac{(1 + x - a)(1 + y - b)}{L} \frac{1}{\sqrt{L}} e^{-\frac{(x-y)^2}{2L}}. \tag{2.5}$$

Proof. Let

$$A_L = \big\{ f_{a,b}(l; L) - Cl_L^{\frac{1}{2}-\varepsilon} \le W_l, l = 1, \ldots, L - 1 \big\},$$

and for a $\theta > 1$ to be fixed later

$$h_l(u, v) = P_u^W \big(f_{a,b}(l + t; L) - \theta C(l + t)_L^{\frac{1}{2}-\varepsilon} \le W_t, t \in [0, 1] | W_1 = v \big)$$
$$\ge P_u^W \big(f_{a',b'}(t; 1) \le W_t, t \in [0, 1] | W_1 = v \big), \tag{2.6}$$

where $a' = a'(a, b) = a + (b - a)l/L - \theta Cl_L^{1/2-\varepsilon}$, $b' = b'(a, b) = a + (b - a)(l + 1)/L - \theta C(l + 1)_L^{1/2-\varepsilon}$ and the inequality in (2.6) is due to the convexity of the function $x \mapsto -x^{1/2-\varepsilon}$. Note that $h_l(u, v)$ is monotone increasing in u, v. Hence using the Markov property we have

$$P_x^W \big(f_{a,b}(l, L) - \theta Cl_L^{\frac{1}{2}-\varepsilon} \le W_l, l \in [1, L - 1], W_L \in H_{y,\delta} \big)$$
$$= E_x^W \left(\prod_{l=1}^{L-2} h_l(W_l, W_{l+1}); W_L \in H_{y,\delta} \right)$$
$$\ge E_x^W \left(\prod_{l=1}^{L-2} h_l(W_l, W_{l+1}); A_L; W_L \in H_{y,\delta} \right)$$
$$\ge \prod_{l=1}^{L-2} h_l \big(f_{a,b}(l) - Cl_L^{\frac{1}{2}-\varepsilon}, f_{a,b}(l + 1) - C(l + 1)_L^{\frac{1}{2}-\varepsilon} \big) P_x^W (A_L; W_L \in H_{y,\delta}). \tag{2.7}$$

But using (2.6) and (2.1) with the L there equal to 1, we have

$$\sum_{l=1}^{L-2} \big(1 - h_l \big(f_{a,b}(l) - Cl_L^{\frac{1}{2}-\varepsilon}, f_{a,b}(l + 1) - C(l + 1)_L^{\frac{1}{2}-\varepsilon} \big) \big)$$
$$\le \sum_{l=1}^{L-2} \exp \big(-c(\theta - 1)^2 C^2 l_L^{\frac{1}{2}-\varepsilon} \big) \le \frac{1}{2},$$

if we let θ be large enough depending on C, but independently of L. Thus (2.5) follows from (2.3), Remark 2.2 and (2.7). $\qquad\square$

Remark 2.4. As in Remark 2.2, the condition $|x - y| \le \eta L$ can be dropped, at the cost of replacing the right hand side of (2.5) by (2.4).

We now derive the corresponding estimates for the 0-dimensional Bessel process by applying a change of measure. To formally introduce the Bessel process recall that the squared Bessel process Y_t^2 is a Markov process, BESQ^0, with semigroup, for $x > 0$,

$$V_t(x, y) = e^{-x/2t}\delta_0(dy) + 1_{(0,\infty)}(y)\frac{1}{2t}\left(\frac{x}{y}\right)^{1/2}I_1(\sqrt{xy}/t)e^{-(x+y)/2t}\,dy, \tag{2.8}$$

see [19, Chapter IX, Corollary (1.4)], where

$$I_1(z) = \sum_{k=0}^{\infty}\frac{(z/2)^{2k+1}}{k!(k+1)!}, \tag{2.9}$$

see [17, 8.447.2], is the first Bessel function. Let P_x^Y denote the law of the process Y_t starting at $Y_0 = x$. From (2.8) the equality (1.13) easily follows.

Lemma 2.5. *For all fixed $\delta > 0$, $\varepsilon \in (0, \frac{1}{2})$ and $\eta > 1$, there exists $c = c(\delta, \eta, \varepsilon, C)$ so that for $1 \leq x, y, 0 \leq a \leq x$, $0 \leq b \leq y$, and $|x - y| \leq \eta L$,*

$$P_x^Y\left(f_{a,b}(l; L) - Cl_L^{\frac{1}{2}-\varepsilon} \leq Y_l, l = 1, \ldots, L-1, Y_L \in H_{y,\delta}\right)$$
$$\leq c\frac{(1+x-a)(1+y-b)}{L}\sqrt{\frac{x}{yL}}e^{-\frac{(x-y)^2}{2L}}. \tag{2.10}$$

If in addition we assume that $(x - a)(y - b) \leq \eta L$, and $\max(ab, |a - b|) \geq L/\eta$, then for any $\tilde{C} \geq C + \delta > 0$ there exists $c' = c'(\delta, \eta, \varepsilon, C, \tilde{C})$ so that for

$$P_x^Y\left(f_{a,b}(l; L) + Cl_L^{\frac{1}{2}-\varepsilon} \leq Y_l \leq f_{x,y}(l; L) + \tilde{C}l_L^{\frac{1}{2}+\varepsilon}, l = 1, \ldots, L-1, Y_L \in H_{y,\delta}\right)$$
$$\geq c'\frac{(1+x-a)(1+y-b)}{L}\sqrt{\frac{x}{yL}}e^{-\frac{(x-y)^2}{2L}}. \tag{2.11}$$

Remark 2.6.

(a) As in Remark 2.2, the condition $|x - y| \leq \eta L$ can be dropped in (2.10), at the cost of replacing the right hand side by

$$\frac{(1+x-a)(1+y-b)}{L}\sqrt{\frac{x}{yL}}\sup_{z\in H_{y,\delta}}e^{-\frac{(x-z)^2}{2L}}.$$

(b) This is where we need the condition $\max(ab, |a - b|) \geq L/\eta$ from Theorem 1.1(b). It guarantees that

$$\int_1^{L-1}\frac{1}{f_{a,b}^2(s; L)}\,ds = \frac{L-2}{ab + \frac{(a-b)^2}{L}(1 - \frac{1}{L})} \leq c(\eta) \tag{2.12}$$

independent of $L \geq 3$. The bound (2.12) is used below to bound the Radon–Nikodym derivative of a 0-Bessel process with respect to Brownian motion.

Proof of Lemma 2.5. Recall that Y_t solves the SDE $dY_t = dW_t - \frac{1}{2Y_t}\,dt$ for a Brownian motion W_t, until τ_0, the time it hits 0 (not to be confused with the hitting time τ_x of a vertex x of the binary tree, a notation used elsewhere in this paper). By Girsanov's theorem (applied until Y_t hits ϵ) and monotone convergence it follows that for any $F \in \mathcal{F}_t$,

$$E_x^Y(1_{\{\tau_0 > t\}}F) = E_x^W\left(1_{\{\tau_0 > t\}}F\sqrt{\frac{x}{W_t}}\exp\left(-\frac{3}{8}\int_0^t\frac{1}{W_s^2}\,ds\right)\right). \tag{2.13}$$

We also note that 0 is an absorbing boundary for the 0-dimensional Bessel process, so that $Y_L > 0$ implies that $\tau_0 > L$. Therefore, since $y \geq 1$,

$$
\begin{aligned}
\mathcal{I}_1 &:= P_x^Y\big(f_{a,b}(l;L) - Cl_L^{\frac{1}{2}-\varepsilon} \leq Y_l, l = 1, \ldots, L-1, Y_L \in H_{y,\delta}\big) \\
&= P_x^Y\big(f_{a,b}(l;L) - Cl_L^{\frac{1}{2}-\varepsilon} \leq Y_l, l = 1, \ldots, L-1, Y_L \in H_{y,\delta}, \tau_0 > L\big) \\
&\leq \sqrt{\frac{x}{y}} P_x^W\big(f_{a,b}(l;L) - Cl_L^{\frac{1}{2}-\varepsilon} \leq W_l, l = 1, \ldots, L-1, W_L \in H_{y,\delta}\big),
\end{aligned}
$$

by (2.13). Then using (2.5),

$$
\mathcal{I}_1 \leq c \sqrt{\frac{x}{y}} \frac{(1+x-a)(1+y-b)}{L} \frac{1}{\sqrt{L}} e^{-\frac{(x-y)^2}{2L}}. \tag{2.14}
$$

This completes the proof of (2.10).

We turn to the proof of the lower bound (2.11). Letting

$$
\widetilde{A} = \big\{ f_{a,b}(l;L) + Cl_L^{\frac{1}{2}-\varepsilon} \leq W_l \leq f_{x,y}(l;L) + \tilde{C}l_L^{\frac{1}{2}+\varepsilon}, l \in [1, L-1] \big\},
$$

we have, by (2.13),

$$
\begin{aligned}
P_x^Y(\widetilde{A}, Y_L \in H_{y,\delta}) &\geq \sqrt{\frac{x}{y+\delta}} E_x^W\left(\exp\left(-\frac{3}{8} \int_0^L \frac{1}{W_t^2} dt\right); \widetilde{A}; W_L \in H_{y,\delta} \right) \\
&\geq c \sqrt{\frac{x}{y}} E_x^W\left(\exp\left(-\frac{3}{8} \int_0^1 \frac{1}{W_t^2} dt\right) \cdot \exp\left(-\frac{3}{8} \int_{L-1}^L \frac{1}{W_t^2} dt\right); \widetilde{A}; W_L \in H_{y,\delta} \right),
\end{aligned} \tag{2.15}
$$

where the last inequality follows because on the event $W_l \geq f_{a,b}(l;L) + Cl_L^{\frac{1}{2}-\varepsilon}, l \in [1, L-1]$, we have $\int_1^{L-1} \frac{1}{W_t^2} dt \leq c(\eta)$ by (2.12).

Furthermore for any $u, v \geq 1/\eta$ we have

$$
\begin{aligned}
J_{u,v} &:= E_u^W\left(\exp\left(-\frac{1}{2} \int_0^1 \frac{1}{W_t^2} dt\right) \Big| W_1 = v \right) \\
&= E_0^W\left(\exp\left(-\frac{1}{2} \int_0^1 \frac{1}{(W_t + u + t(v-u))^2} dt\right) \Big| W_1 = 0 \right) \geq c > 0,
\end{aligned}
$$

where the constant depends only on η, since the standard Brownian bridge has positive probability of staying between $1/2\eta$ and $-1/2\eta$. We apply this to (2.15), where, after the Markov property we need to lower bound J_{x,W_1} and $J_{W_{L-1},v}$ for $v \in [y, y+\delta]$. Since $x, y \geq 1$ and $\max(ab, |a-b|) \geq L/\eta$ implies that $f_{a,b}(1;L), f_{a,b}(L-1;L) \geq 1/\eta$, we therefore have in fact

$$
\begin{aligned}
&P_x^Y\big(f_{a,b}(l;L) + Cl_L^{\frac{1}{2}-\varepsilon} \leq Y_l \leq f_{x,y}(l;L) + \tilde{C}l_L^{\frac{1}{2}+\varepsilon}, l \in [1, L-1], Y_L \in H_{y,\delta}\big) \\
&\geq c \sqrt{\frac{x}{y}} P_x^W(\widetilde{A},; W_L \in H_{y,\delta}) \geq c \sqrt{\frac{x}{y}} \frac{(1+x-a)(1+y-b)}{L} \frac{1}{\sqrt{L}} e^{-\frac{(x-y)^2}{2L}},
\end{aligned} \tag{2.16}
$$

using (2.3). This yields (2.10). $\square$

3. Local times and traversal counts

Let $X_t, t \geq 0$, be the continuous time random walk on the weighted graph $\{0, 1, 2, \ldots\}$ with unit weights on each edge. That is, the continuous time random walk on $\{0, 1, 2, \ldots\}$ with exponential holding times of mean 1 at 0 and mean

1/2 at all other points, and whose jump chain is simple random walk reflected at the origin. Let L_t^x denote the local time at x and $\tau(s) = \inf\{t > 0 | L_t^0 = s\}$, the inverse local time at 0. Fix $u > 0$ once and for all. We use $\mathbb{P}_u$ to denote the probability for the process $\bar{X}_t = X_{t \wedge \tau(u)}, t \geq 0$. Let $\mathcal{L}_l = L_{\tau(u)}^l$, the total local time of $\bar{X}$ at l, and let $T_l, l \geq 0$, be the discrete traversal count $l \to l + 1$. Under $\mathbb{P}_u$ the $T_l, l \geq 0$, and $\mathcal{L}_l, l \geq 0$, together have a Markovian structure, as stated by the following lemma.

Lemma 3.1 (Markovian structure).

(a) *The sequence $(Z_n)_{n \geq 0}$ defined by*

$$(Z_0, Z_1, Z_2, \ldots) = (\mathcal{L}_0, T_0, \mathcal{L}_1, T_1, \mathcal{L}_2, T_2, \ldots)$$

is a time inhomogeneous Markov chain under $\mathbb{P}_u$. When $n = 2k$, the law of $Z_{n+1} = T_k$ conditioned on $Z_n = \mathcal{L}_k = v$ is Poisson with parameter v, and when $n = 2k + 1$, the law of $Z_{n+1} = \mathcal{L}_{k+1}$ conditioned on $Z_n = T_k = m$ is δ_0 if $m = 0$ and is the gamma distribution with shape parameter m and scale parameter 1 if $m \geq 1$.

(b) *Under $\mathbb{P}_u(\cdot | T_1 = m_1, \ldots, T_L = m_L)$ the $\mathcal{L}_1, \ldots, \mathcal{L}_{L+1}$ are independent, $\mathcal{L}_l$ depends only on T_{l-1} and T_l, and the $\mathbb{P}_u(\cdot | T_{l-1} = m_{l-1}, T_l = m_l)$-law of $\mathcal{L}_l$ is the gamma distribution with shape parameter $m_{l-1} + m_l$ and scale parameter $\frac{1}{2}$.*

(c) *Under $\mathbb{P}_u(\cdot | \mathcal{L}_1 = y_1, \ldots, \mathcal{L}_{L+1} = y_{L+1})$ the $T_1, \ldots, T_L$ are independent, T_l depends only on $\mathcal{L}_l$ and $\mathcal{L}_{l+1}$, and with $u_l, u_{l+1} > 0$,*

$$\mathbb{P}_u(T_l = m | \mathcal{L}_l = u_l, \mathcal{L}_{l+1} = u_{l+1}) = \frac{(u_l u_{l+1})^m / ((m-1)! m!)}{\sqrt{u_l u_{l+1}} I_1(2\sqrt{u_l u_{l+1}})}. \tag{3.1}$$

(d) *$(T_l)_{l \geq 0}$ under $\mathbb{P}_u$ is a Markov chain with the same transition kernel as that of $(T_l)_{l \geq 0}$ under $P_{\cdot}$.*

(e) *$(\mathcal{L}_l)_{l \geq 0}$ under $\mathbb{P}_u$ is $\frac{1}{2}\text{BESQ}^0(2u)$ at integer times. That is, $(\mathcal{L}_l)_{l \geq 0}$ under $\mathbb{P}_u$ is a Markov chain with the same transition kernel as $(\frac{1}{2}Y_l^2)_{l \geq 0}$ under $P_{\cdot}^Y$.*

Remark 3.2. The distribution in (3.1) is a generalization to $\nu = -1$ of the Bessel distribution $\text{Bessel}(\nu, z)$ defined in [18] for $\nu > -1$. This distribution appears in [15]. See also [20].

Proof of Lemma 3.1. Construct a collection of independent standard Poisson processes, one for each directed edge. The random walk is constructed by placing a local time clock at each vertex which only advances when the walker is at that vertex. When the walker is at a given vertex its local time clock advances until one of the two independent Poisson processes associated with edges originating at the vertex register an arrival, and then the walker traverses the corresponding edge. Note that the minimum of two independent exponentials of mean 1 is an exponential of mean 1/2. From this construction it is clear that given u the count T_0 is the number of arrivals up to time u of the point process associated to the edge $0 \to 1$, and therefore is Poisson with parameter u. Given $T_0 = m$, then $\mathcal{L}_1$, the amount of local time spent at vertex 1 until the walker has returned to 0 a total of m times, is the time until the mth arrival of the Poisson process associated to the edge $1 \to 0$. (Note that the transitions $1 \to 2$ have no effect on this description of the local time at 1.) Therefore, this local time is gamma distributed with shape parameter m and scale parameter 1. Iterating this (a) follows and (d) is immediate.

(b) It follows from the fact that Z_n is a Markov chain that under $\mathbb{P}_u(\cdot | T_1 = m_1, \ldots, T_L = m_L)$ the $\mathcal{L}_1, \ldots, \mathcal{L}_{L+1}$ are independent and that the conditional law of $\mathcal{L}_l$ depends only on T_{l-1} and T_l. The characterization of the conditional law follows immediately since each local time $\mathcal{L}_l$ is the sum of independent mean 1/2 exponentials, one for every visit to l, and $m_{l-1} + m_l$ is the number of such visits.

(c) Similarly to above the conditional independence and dependence of the conditional law of T_l only on $\mathcal{L}_l$ and $\mathcal{L}_{l+1}$ follows from the Markovian structure. From the joint law, with $u_l > 0$ and $m_l \geq 1$,

$$\mathbb{P}_u(T_l = m_l, \mathcal{L}_{l+1} = du_{l+1} | \mathcal{L}_l = u_l) = e^{-u_l} \frac{u_l^{m_l}}{m_l!} \times \frac{u_{l+1}^{m_l-1}}{(m_l - 1)!} e^{-u_{l+1}},$$

$$= \frac{1}{u_{l+1}} \times \frac{(u_l u_{l+1})^{m_l}}{m_l!(m_l - 1)!} e^{-(u_l + u_{l+1})},$$

while

$$\mathbb{P}_u(T_l = 0, \mathcal{L}_{l+1} = du_{l+1} | \mathcal{L}_l = u_l) = e^{-u_l} \delta_0(du_{l+1}). \tag{3.2}$$

Hence, using (2.9), we obtain

$$\mathbb{P}_u(\mathcal{L}_{l+1} = du_{l+1} | \mathcal{L}_l = u_l)$$

$$= e^{-u_l} \delta_0(du_{l+1}) + \frac{1}{u_{l+1}} \sum_{m \geq 1} \frac{(u_l u_{l+1})^m}{(m-1)! m!} e^{-(u_l + u_{l+1})}$$

$$= e^{-u_l} \delta_0(du_{l+1}) + \left(\frac{u_l}{u_{l+1}}\right)^{1/2} I_1(2\sqrt{u_l u_{l+1}}) e^{-(u_l + u_{l+1})}, \tag{3.3}$$

and one concludes that for $m \geq 1$, which is equivalent to $u_{l+1} > 0$,

$$\mathbb{P}_u(T_l = m | \mathcal{L}_l = u_l, \mathcal{L}_{l+1} = u_{l+1}) = \frac{(u_l u_{l+1})^m / ((m-1)! m!)}{\sqrt{u_l u_{l+1}} I_1(2\sqrt{u_l u_{l+1}})}.$$

(e) This follows by comparing (3.3) with (2.8). $\qquad\square$

We will use $\mathbb{Q}$ to denote the law of the time inhomogeneous Markov chain $(Z_n)_{n \geq 0}$ under $\mathbb{P}_u$. For $F \in \sigma(Z_n, Z_{n+1}, \ldots)$ let

$$\mathbb{Q}_n^x = \mathbb{Q}(F | Z_n = x). \tag{3.4}$$

Thus $\mathbb{Q} = \mathbb{Q}_0^u$.

For future reference we restate part (e) of Lemma 3.1 as

$$\sqrt{2Z_{2l}}, \quad l = 0, 1, \ldots \text{ under } \mathbb{Q} \text{ is a } \mathrm{Bes}^0(\sqrt{2u})|_{\mathbb{N}}. \tag{3.5}$$

In the following we often replace the Z's by the equivalent $\mathcal{L}$'s and T's.

The following lemma gives some estimates for one step transitions of the Markov chain.

Lemma 3.3 (One step estimates).

(a) *For all $x > 0$ such that $x^2/2$ is an integer and all $l \geq 1$,*

$$\mathbb{Q}_{2l-1}^{x^2/2}\big(|\sqrt{2Z_{2l}} - x| \geq z\big)$$

$$= \mathbb{Q}\big(|\sqrt{2\mathcal{L}_l} - x| \geq z | \sqrt{2T_{l-1}} = x\big) \leq e^{-cz^2}. \tag{3.6}$$

(b) *For all $-\sqrt{2} \leq a < b$,*

$$\inf_{\frac{x^2}{2} \in \{1,2,\ldots\}} \mathbb{Q}\big(\sqrt{2\mathcal{L}_l} - x \in [a, b] | \sqrt{2T_{l-1}} = x\big) \geq c > 0, \tag{3.7}$$

with a constant c depending only on a, b.

(c) *For all $x > 0$, and all $l \geq 1$,*

$$\mathbb{Q}_{2l}^{x^2/2}\big(|\sqrt{2Z_{2l+1}} - x| \geq z\big)$$

$$= \mathbb{Q}\big(|\sqrt{2T_l} - x| \geq z | \sqrt{2\mathcal{L}_l} = x\big) \leq e^{-cz^2}. \tag{3.8}$$

(d) *For all* $-1 \leq a < b$,

$$\inf_{x \geq 1: x+[a,b] \cap \sqrt{2\mathbb{N}} \neq \varnothing} \mathbb{Q}\left(\sqrt{2T_l} - x \in [a,b] | \sqrt{2\mathcal{L}_l} = x\right) \geq c > 0, \tag{3.9}$$

with a constant c depending only on a, b.

Proof of Lemma 3.3. Standard large deviation bounds for the gamma and Poisson distributions give (3.6) and (3.8). Indeed, to see (3.8), recall that by Lemma 3.1(a), the conditional law of T_l, when conditioned on $\mathcal{L}_{l-1} = x^2/2$, is Poisson of parameter $x^2/2$. Since the Legendre transform of the logarithmic moment generating function of the Poisson distribution of parameter λ is

$$I_{P,\lambda}(x) = \lambda - x + x \log(x/\lambda), \tag{3.10}$$

we obtain from Chebyshev's inequality that, for $z \geq 0$,

$$\log \mathbb{Q}(\sqrt{2T_l} - x \geq z | \sqrt{2\mathcal{L}_l} = x) \leq -I_{P,x^2/2}\left(\frac{(z+x)^2}{2}\right)$$

$$= -\left[\frac{x^2}{2} + \frac{(x+z)^2}{2}\left(2\log\left(1 + \frac{z}{x}\right) - 1\right)\right], \tag{3.11}$$

and

$$\log \mathbb{Q}(\sqrt{2T_l} - x \leq -z | \sqrt{2\mathcal{L}_l} = x) \leq -I_{P,x^2/2}\left(\frac{(x-z)^2}{2}\right)$$

$$= -\left[\frac{x^2}{2} + \frac{(x-z)^2}{2}\left(2\log\left(1 - \frac{z}{x}\right) - 1\right)\right]. \tag{3.12}$$

Separating according to whether $z/x > 1$ (only relevant for (3.11)) or $z/x \leq 1$, and performing some algebra yields (3.8). The argument for (3.6) is similar: recall, again from Lemma 3.1(a), that the conditional law of $\mathcal{L}_l$, when conditioned on $T_{l-1} = x^2/2$, is gamma of shape parameter $x^2/2$ and scale parameter 1. Recall that the Legendre transform of the logarithmic moment generating function of a gamma variable of shape parameter m and scale parameter 1 equals

$$I_{\Gamma,\lambda}(x) = x - m + m \log(m/x). \tag{3.13}$$

Replacing the expression in (3.10) with (3.13) and then using Chebyshev's inequality to obtain the analogue of (3.11) and (3.12) (with $m = x^2/2$), the proof then proceeds similarly to the Poisson case.

To see (3.7), note that by the central limit theorem for a sum of independent exponentials,

$$\lim_{x \to \infty, \frac{x^2}{2} \in \{1,2,\dots\}} \mathbb{Q}_1^{x^2/2}\left(\mathcal{L}_1 \in \frac{1}{2}[(x+a)^2, (x+b)^2]\right) = \int_a^b \frac{1}{\sqrt{4\pi}} e^{-\frac{u^2}{4}} \, du > 0.$$

Thus for a large enough $\tilde{c} > 0$ we have for $x^2/2 \geq \tilde{c}$ that

$$\mathbb{Q}_1^{x^2/2}\left(\sqrt{2\mathcal{L}_1} - x \in [a,b]\right) \geq c > 0.$$

Since the density $f(z)$ of a gamma random variable with shape parameter $x^2/2$ and scale parameter 1 satisfies $f(z) \geq c > 0$ for all $0 \leq z \leq \frac{1}{2}(\tilde{c} + b)^2$ if $1 \leq x^2/2 \leq \tilde{c}$, it follows that the same holds for $1 \leq x^2/2 \leq \tilde{c}$. This proves (3.7). Similarly (3.9) follows from the central limit theorem for the Poisson distribution, since our assumptions on $[a,b]$ imply that the interval always contains an element from $\sqrt{2\mathbb{N}}$. $\qquad \square$

The following lemma gives some estimates for conditioning the Markov chain at two times, before and after.

Lemma 3.4 (Local time and transversal bounds).

(a) *For all m_{l-1}, m_l and $0 \le z \le \sqrt{m_{l-1} + m_l}$,*

$$\mathbb{Q}_1^{m_0}(\sqrt{2\mathcal{L}_l} \ge \sqrt{m_{l-1} + m_l} - z | T_{l-1} = m_{l-1}, T_l = m_l) \ge 1 - \exp(-z^2). \tag{3.14}$$

(b) *For all $-\sqrt{2} \le a < b$,*

$$\inf_{\substack{m_{l-1} > 0 \\ m_l \ge 0}} \mathbb{Q}_1^{m_0}(\sqrt{2\mathcal{L}_l} - \sqrt{m_{l-1} + m_l} \in [a, b] | T_{l-1} = m_{l-1}, T_l = m_l) \ge c, \tag{3.15}$$

for all l, and a constant $c > 0$ depending only on a, b.

(c) *For all u_l, u_{l+1} and $z \ge 0$*

$$\mathbb{Q}_1^{m_0}(|\sqrt{2T_l} - (4u_l u_{l+1})^{1/4}| \ge z | \mathcal{L}_l = u_l, \mathcal{L}_{l+1} = u_{l+1}) \le c \exp(-cz^2). \tag{3.16}$$

(d) *For all $a < b$, with*

$$D_{a,b} = \{u_l > 0, u_{l+1} \ge 0 | (4u_l u_{l+1})^{1/4} + [a, b] \cap \sqrt{2\mathbb{N}} \ne \varnothing\},$$

we have

$$\inf_{u_l, u_{l+1} \in D_{a,b}} \mathbb{Q}_1^{m_0}(\sqrt{2T_l} - (4u_l u_{l+1})^{1/4} \in [a, b] | \mathcal{L}_l = u_l, \mathcal{L}_{l+1} = u_{l+1}) \ge c, \tag{3.17}$$

for some $c > 0$ depending only on a, b.

Proof. Similarly to (3.7), the claim (3.15) follow from the central limit theorem for a sum of exponential random variables, since under the conditioning $\mathcal{L}_l$ have the gamma distribution with an integer shape parameter (see Lemma 3.1(b). A large deviation bound for the gamma distribution that can be proved using the exponential Chebyshev inequality gives

$$\mathbb{Q}_1^{m_0}\left(\mathcal{L}_l \le \frac{m_{l-1} + m_l}{2} - a \Big| T_1 = m_1, \dots, T_L = m_L\right) \le \exp\left(-2\frac{a^2}{m_{l-1} + m_l}\right)$$

for $0 \le a \le \frac{m_{l-1} + m_l}{2}$. By taking $a = z\sqrt{\frac{m_{l-1} + m_l}{2}}$ this implies (3.14).

A basic large deviation estimate for the Bessel distribution (see [6, Lemma 7.12] and its proof) gives

$$\mathbb{Q}_1^{m_0}(|T_l - \sqrt{u_l u_{l+1}}| \ge a | \mathcal{L}_l = u_l, \mathcal{L}_{l+1} = u_{l+1})$$

$$\le c \exp\left(-c\frac{a^2}{\sqrt{u_l u_{l+1}}} + c\frac{a}{\sqrt{u_l u_{l+1}}}\right) \quad \text{for all } a \ge 0. \tag{3.18}$$

By taking $a = z\sqrt{u_l u_{l+1}}$ this implies (3.16).

For (3.17), set $z = \sqrt{u_l u_{l+1}}$ and $m = \frac{(\sqrt{2z} + v)^2}{2}$ in (3.1) and use the estimate $I_1(z) \sim \frac{e^z}{\sqrt{2\pi z}}$ for $z \to \infty$, see [17, 8.451.5], and Stirling's formula to see that for $v \in [a, b]$ and $z \ge z_0$ large enough

$$\mathbb{Q}_1^{m_0}(\sqrt{2T_l} = \sqrt{2z} + v | \mathcal{L}_l = u_l, \mathcal{L}_{l+1} = u_{l+1}) \ge c\frac{e^{-3v^2}}{\sqrt{z}}.$$

There are at least $c\sqrt{z}$ integers of the form $m = \frac{(\sqrt{2z} + v)^2}{2}$ for $v \in [a, b]$, so one obtains (3.17) for $z \ge z_0$. For smaller $z \le z_0$ one simply uses the fact that the right-hand side of (3.1) is bounded away from zero for the finite number of $m = \frac{(\sqrt{2z} + v)^2}{2}$ with $v \in [a, b]$ and $u_l, u_{l+1} \in D_{a,b}$. $\qquad\square$

4. Galton–Watson process proofs

We now have the necessary tools to prove the precise large deviation estimate Proposition 1.4 and the barrier estimate Theorem 1.1 for the process T_l. We start with the former.

Proof of Proposition 1.4. Since $T_L = 0$ is the event that none of $x^2/2$ independent excursions from 1 to 0 of a simple random walk on $\{0, 1, 2, \ldots\}$ hit $L + 1$, and this has probability $1 - 1/(L + 1)$, the estimate (1.12) follows using the assumption $x \leq \eta L$.

Turning to (1.10), we use that

$$\mathbb{Q}_1^{x^2/2}(\sqrt{2T_L} \in H_{y,\delta}) \leq \frac{\mathbb{Q}_1^{x^2/2}(\sqrt{2\mathcal{L}_{L+1}} \in H_{y,2\delta})}{\mathbb{Q}_1^{x^2/2}(\sqrt{2\mathcal{L}_{L+1}} \in H_{y,2\delta} | \sqrt{2T_L} \in H_{y,\delta})}.$$

By (3.7) we have that

$$\mathbb{Q}_1^{x^2/2}(\sqrt{2\mathcal{L}_{L+1}} \in H_{y,2\delta} | \sqrt{2T_L} \in H_{y,\delta}) \geq c > 0,$$

for a constant c depending only on δ. Also

$$\mathbb{Q}_1^{x^2/2}(\sqrt{2\mathcal{L}_L} \in H_{y,\delta/2}) \leq \frac{\mathbb{Q}_1^{x^2/2}(\sqrt{2T_L} \in H_{y,\delta})}{\mathbb{Q}_1^{x^2/2}(\sqrt{2T_L} \in H_{y,\delta} | \sqrt{2\mathcal{L}_L} \in H_{y,\delta/2})},$$

and by (3.9)

$$\mathbb{Q}_1^{x^2/2}(\sqrt{2T_L} \in H_{y,\delta} | \sqrt{2\mathcal{L}_L} \in H_{y,\delta/2}) \geq c > 0,$$

for a constant c depending only on δ. Thus we have shown that

$$\begin{aligned}
c\mathbb{Q}_1^{x^2/2}(\sqrt{2\mathcal{L}_L} \in H_{y,\delta/2}) &\leq \mathbb{Q}_1^{x^2/2}(\sqrt{2T_L} \in H_{y,\delta}) \\
&\leq \frac{1}{c}\mathbb{Q}_1^{x^2/2}(\sqrt{2\mathcal{L}_{L+1}} \in H_{y,2\delta}).
\end{aligned} \tag{4.1}$$

From Lemma 3.1(e) and (1.13) one obtains

$$\begin{aligned}
\mathbb{Q}_1^{x^2/2}(\sqrt{2\mathcal{L}_{L+1}} \in H_{y,2\delta}) &= \mathbb{Q}_1^{x^2/2}\big(\mathbb{Q}_2^{\mathcal{L}_1}(\sqrt{2\mathcal{L}_L} \in H_{y,2\delta})\big) \\
&= \mathbb{Q}_1^{x^2/2}\big(P_{\sqrt{2\mathcal{L}_1}}^Y(Y_L \in H_{y,2\delta})\big) \\
&= \mathbb{Q}_1^{x^2/2}\bigg(\int_y^{y+2\delta} \frac{\sqrt{2\mathcal{L}_1}}{L} e^{-\frac{2\mathcal{L}_1+z^2}{2L}} I_1\bigg(\frac{\sqrt{2\mathcal{L}_1}z}{L}\bigg) dz\bigg).
\end{aligned}$$

The function $I_1(z)$, see (2.9), is continuous, and clearly non-zero for $z > 0$, $I_1(z) \sim z/2$ for $z \to 0$ and $I_1(z) \sim \frac{e^z}{\sqrt{2\pi z}}$ for $z \to \infty$, see [17, 8.451.5]. In particular, $I_1(z) \leq C\frac{e^z}{\sqrt{z}}$. Using our assumptions on x, y and (3.6), one sees that

$$\mathbb{Q}_1^{x^2/2}(\sqrt{2\mathcal{L}_{L+1}} \in H_{y,2\delta}) \leq c\sqrt{\frac{x/y}{L}} e^{-\frac{x^2+y^2}{2L}} e^{\frac{xy}{L}}.$$

Similarly, since $I_1(z) \geq C'\frac{e^z}{\sqrt{z}}$ for $z \geq \delta > 0$, it follows from (3.6) that if $L/\eta \leq xy$

$$\mathbb{Q}_1^{x^2/2}(\sqrt{2\mathcal{L}_L} \in H_{y,\delta/2}) \geq c'\sqrt{\frac{x/y}{L}} e^{-\frac{x^2+y^2}{2L}} e^{\frac{xy}{L}}.$$

Together with (4.1) this proves (1.10). $\square$

Proof of Theorem 1.1. We begin with the proof of the upper bound (1.1). Fix $C > 0$ and let

$$A = \left\{ f_{a,b}(l; L) - C l_L^{\frac{1}{2}-\varepsilon} \leq \sqrt{2T_l}, l = 1, \dots, L-1, \sqrt{2T_L} \in H_{y,\delta} \right\}.$$

We will first prove that

$$\mathbb{Q}_1^{x^2/2}(A) \leq c \frac{(1 + x - a)(1 + y - b)}{L^{3/2}} \sqrt{\frac{x}{y}} e^{-\frac{(x-y)^2}{2L}} \quad \text{for all } y \geq \sqrt{2}. \tag{4.2}$$

Consider the event

$$B = \left\{ \sqrt{2\mathcal{L}_1} \geq a' + \frac{x-a}{2}, \sqrt{2\mathcal{L}_l} \geq f_{a'b}(l-1; L-1) - \bar{C}(l-1)_{L-1}^{\frac{1}{2}-\varepsilon}, \right.$$

$$\left. \text{for } l = 2, \dots, L-1; \sqrt{2\mathcal{L}_L} \geq \frac{y+b}{2} \right\}$$

for a fixed $\bar{C} \geq 1$ to be specified later, where

$$a' = f_{a,b}(1; L), \quad \text{so that } f_{a'b}(l-1; L-1) = f_{a,b}(l; L). \tag{4.3}$$

By Lemma 3.1(b),

$$\mathbb{Q}_1^{x^2/2}(B \cap A) = \mathbb{Q}_1^{x^2/2} \left\{ 1_A \mathbb{Q}_1^{x^2/2} \left(\sqrt{2\mathcal{L}_1} \geq a' + \frac{x-a}{2} \Big| T_0, T_1 \right) \right.$$

$$\times \left(\prod_{l=2}^{L-1} \mathbb{Q}_1^{x^2/2} \left(\sqrt{2\mathcal{L}_l} \geq f_{a'b}(l-1; L-1) - \bar{C}(l-1)_{L-1}^{\frac{1}{2}-\varepsilon} | T_{l-1}, T_l \right) \right)$$

$$\times \left. \mathbb{Q}_1^{x^2/2} \left(\sqrt{2\mathcal{L}_L} \geq \frac{y+b}{2} \Big| T_{L-1}, T_L \right) \right\}. \tag{4.4}$$

On the event A we have

$$\sqrt{T_0 + T_1} \geq \frac{\sqrt{T_0} + \sqrt{T_1}}{\sqrt{2}} \geq \frac{x + a' - C}{2} \geq a' + \frac{x-a}{2} - C - C',$$

for a constant $C' \geq 0$ depending only on η, where we used that $|a' - a| \leq \eta$ by our assumptions on x, y. Thus we have by (3.15) with $C + C'$ in place of a and $C + C' + 1$ in place of b that for some $c_1 = c_1(C, \eta) > 0$,

$$1_A \mathbb{Q}_1^{x^2/2} \left(\sqrt{2\mathcal{L}_1} \geq a' + \frac{x-a}{2} \Big| T_0, T_1 \right) \geq 1_A c_1. \tag{4.5}$$

Similarly for some $c_2 > 0$,

$$1_A \mathbb{Q}_1^{x^2/2} \left(\sqrt{2\mathcal{L}_L} \geq \frac{y+b}{2} \Big| T_{L-1}, T_L \right) \geq 1_A c_2. \tag{4.6}$$

Note that on the event A,

$$\sqrt{\frac{T_{l-1} + T_l}{2}} \geq \min(\sqrt{T_{l-1}}, \sqrt{T_l}) \geq \frac{1}{\sqrt{2}} \left\{ f_{a,b}(l; L) - C l_L^{\frac{1}{2}-\varepsilon} - c \right\},$$

for $2 \le l \le L - 1$ for a constant c depending only on η and C. Since $f_{a,b}(l; L) = f_{a',b}(l-1; L-1)$ and $l_L, (l-1)_{L-1} \ge 1$ for $2 \le l \le L - 1$ we have if we pick $\bar{C}$ large enough that on A,

$$\sqrt{\frac{T_{l-1} + T_l}{2}} \ge \frac{1}{\sqrt{2}} \left\{ f_{a'b}(l-1; L-1) - \frac{\bar{C}}{2}(l-1)_{L-1}^{\frac{1}{2}-\varepsilon} \right\},$$

for $2 \le l \le L - 1$. Then by (3.14) we have

$$1_A \prod_{l=2}^{L-1} \mathbb{Q}_1^{x^2/2}\left(\sqrt{2\mathcal{L}_l} \ge f_{a'b}(l-1; L-1) - \bar{C}(l-1)_{L-1}^{\frac{1}{2}-\varepsilon} | T_{l-1}, T_l\right)$$

$$\ge 1_A \prod_{l=2}^{L-1} \left(1 - \exp\left(-\frac{\bar{C}^2}{4}(l-1)_{L-1}^{1-\varepsilon}\right)\right) \ge \frac{1_A}{2}, \tag{4.7}$$

where the last inequality follows by making $\bar{C}$ large enough (the choice can be made independent of L). Combining (4.4), (4.5), (4.6) and (4.7) we obtain $c\mathbb{Q}_1^{x^2/2}(A) \le \mathbb{Q}_1^{x^2/2}(A \cap B)$ with $c = \frac{1}{2}c_1 c_2 > 0$, so that

$$c\mathbb{Q}_1^{x^2/2}(A) \le \mathbb{Q}_1^{x^2/2}(A \cap B) \le \mathbb{Q}_1^{x^2/2}(B, \sqrt{2T_L} \in H_{y,\delta}).$$

Recalling that $T_k = Z_{2k+1}$ and $\mathcal{L}_k = Z_{2k}$, the right-hand side can be written as

$$J_1 := \mathbb{Q}_1^{x^2/2}\left(\sqrt{2Z_2} \ge a' + \frac{x-a}{2}, \sqrt{2Z_{2l}} \ge f_{a'b}(l-1; L-1) - \bar{C}(l-1)_{L-1}^{\frac{1}{2}-\varepsilon},\right.$$

$$\left. \text{for } l = 2, \dots, L-1; \sqrt{2Z_{2L}} \ge \frac{y+b}{2}, \sqrt{2Z_{2L+1}} \in H_{y,\delta}\right). \tag{4.8}$$

Using the Markov property

$$J_1 = \mathbb{Q}_1^{x^2/2}\left(\sqrt{2Z_2} \ge a' + \frac{x-a}{2};\right.$$

$$\mathbb{Q}_2^{Z_2}\left(\sqrt{2Z_{2l}} \ge f_{a'b}(l-1; L-1) - \bar{C}(l-1)_{L-1}^{\frac{1}{2}-\varepsilon}, l = 2, \dots, L-1;\right.$$

$$\left.\left.\sqrt{2Z_{2L}} \ge \frac{y+b}{2}; \mathbb{Q}_{2L}^{Z_{2L}}\left(\sqrt{2Z_{2L+1}} \in H_{y,\delta}\right)\right)\right). \tag{4.9}$$

By (3.6), for all $j \ge a' + \frac{x-a}{2}$,

$$\mathbb{Q}_1^{x^2/2}(\sqrt{2Z_2} \in H_j) \le e^{-c(j-x)^2}, \tag{4.10}$$

and by (3.8), for all $k \ge \frac{y+b}{2}$,

$$\sup_{v \in H_k} \mathbb{Q}_{2L}^{v^2/2}(\sqrt{2Z_{2L+1}} \in H_{y,\delta}) \le e^{-c(k-y)^2}. \tag{4.11}$$

This shows that

$$J_1 \le \sum_{j \ge a' + \frac{x-a}{2}, k \ge \frac{y+b}{2}} \sup_{z \in H_j} \mathbb{Q}_2^{z^2/2}\left(\sqrt{2Z_{2l}} \ge f_{a'b}(l-1; L-1) - \bar{C}(l-1)_{L-1}^{\frac{1}{2}-\varepsilon},\right.$$

$$\left.\text{for } l = 2, \dots, L-1; \sqrt{2Z_{2L}} \in H_k\right)e^{-c(j-x)^2}e^{-c(k-y)^2}. \tag{4.12}$$

By (3.5),

$$K_1 := \mathbb{Q}_2^{z^2/2}\left(\sqrt{2Z_{2l}} \geq f_{a'b}(l-1; L-1) - \bar{C}(l-1)_{L-1}^{\frac{1}{2}-\varepsilon},\right.$$

$$\left. \text{for } l = 2, \ldots, L-1; \sqrt{2Z_{2L}} \in H_k\right)$$

$$= P_z^Y\left(Y_{l-1} \geq f_{a'b}(l; L-1) - \bar{C}l_{L-1}^{\frac{1}{2}-\varepsilon},\right.$$

$$\left. \text{for } l = 1, \ldots, L-2; Y_{L-1} \in H_k\right), \tag{4.13}$$

where Y_t is a 0-dimensional Bessel process. Thus using (2.10) we obtain that if $j, k \geq 1$ and $|j - k| \leq 2\eta L$

$$K_1 \leq C \frac{(1 + z - a')(1 + k - b)}{(L-1)^{3/2}} \sqrt{\frac{z}{k}} e^{-\frac{(z-k)^2}{2(L-1)}}. \tag{4.14}$$

Bounding K_1 by 1 when $|j - k| > 2\eta L$, it then follows that J_1 is bounded by

$$c \sum_{\substack{j \geq a' + \frac{x-a}{2} \\ k \geq \frac{y+b}{2}, |j-k| \leq 2\eta L}} \frac{(1 + j - a')(1 + k - b)}{(L-1)^{3/2}} \sqrt{\frac{j}{k}} e^{-\frac{(j-k)^2}{2(L-1)}} e^{-c(j-x)^2} e^{-c(k-y)^2}$$

$$+ \sum_{\substack{j \geq a' + \frac{x-a}{2} \\ k \geq \frac{y+b}{2}, |j-k| > 2\eta L}} e^{-c(j-x)^2} e^{-c(k-y)^2}. \tag{4.15}$$

The first sum is bounded by

$$c \frac{(1 + x - a)(1 + y - b)}{L^{3/2}} \sqrt{\frac{x}{y}} e^{-\frac{(x-y)^2}{2L}}. \tag{4.16}$$

As for the second sum, our assumption that $x, y \leq \eta L$ together with $|j - k| > 2\eta L$ implies that either $|x - j| \geq \eta L$ or $|y - k| \geq \eta L$, (or both), in which case the second sum in (4.15) is bounded by $ce^{-c'L^2}$. From this (4.2) follows.

The case $y = 0$ requires special care. Note that when $y = b = 0$ then $L/\eta \leq a \leq x \leq \eta L$ so $f_{a,b}(l; L) - Cl_L^{\frac{1}{2}-\varepsilon} > 1$ for all $l \leq L - K$, for a constant K depending on C and η. Also for a large enough constant C', we have for all $1 \leq k \leq K$ that

$$f_{a0}(l; L) - Cl_L^{\frac{1}{2}-\varepsilon} \geq f_{a0}(l; L-k) - C'l_{L-k}^{\frac{1}{2}-\varepsilon} \quad \text{for } l = 1, \ldots, L-k-1.$$

Thus letting

$$A_k = \left\{\sqrt{2T_l} \geq f_{a,b}(l; L-k) - C'l_{L-k}^{\frac{1}{2}-\varepsilon} \text{ for } l = 1, \ldots, L-k-1\right\},$$

we have

$$\mathbb{Q}_1^{x^2/2}(A) \leq \sum_{k=1}^K \mathbb{Q}_1^{x^2/2}(A_k; \sqrt{2T_{L-k}} > 1, T_{L-k+1} \in H_{0,\delta\vee 1}).$$

Using the Markov property we have

$$\mathbb{Q}_1^{x^2/2}(A_k; \sqrt{2T_{L-k}} > 1, T_{L-k+1} \in H_{0,\delta\vee 1}) \leq \sum_{j=1}^\infty \mathbb{Q}_1^{x^2/2}(A_k; \sqrt{2T_{L-k}} \in H_j)e^{-c'j^2},$$

for some constant $c' > 0$, where the last exponential term follows from a standard concentration estimate. Now applying (4.2) with $L - k$ in place of L and using $L/\eta \le a \le x$ gives that this sum is at most

$$\sum_{j=1}^{\infty} c \frac{(1+x-a)(1+j)}{(L-k)^{3/2}} \sqrt{\frac{x}{j}} e^{-\frac{(x-j)^2}{2(L-k)}} e^{-c'j^2} \le c \frac{(1+x-a)}{(L-k)} e^{-\frac{x^2}{2(L-k)}}$$

$$\le c \frac{(1+x-a)}{L} e^{-\frac{x^2}{2L}},$$

for $1 \le k \le K$, so the desired upper bound follows also in the case $y = 0$. In general, if $y \le \sqrt{2}$ then $[y, y + \delta] \subseteq [0, \sqrt{2} + \delta]$ so the upper bound follows from the case $y = 0$. This completes the proof of the upper bound (1.1).

We now prove the lower bound. Recall that $x \ge \sqrt{2}$ and we first assume that also $y \ge \sqrt{2}$. Let

$$\text{Tube}_{C,\widetilde{C}}(l, L) = \left\{ z \mid f_{a,b}(l; L) + Cl_L^{\frac{1}{2}-\varepsilon} \le z \le f_{x,y}(l; L) + \widetilde{C}l_L^{\frac{1}{2}+\varepsilon} \right\}$$

and set

$$D = \left\{ \sqrt{2T_l} \in \text{Tube}_{C,\widetilde{C}}(l, L), l = 1, \ldots, L - 1, \sqrt{2T_L} \in H_{y,\delta} \right\}.$$

We claim that

$$\text{Tube}_{C,\widetilde{C}}(l, L) \cap \sqrt{2}\mathbb{Z}^+ \ne \varnothing, \quad \text{for all } l = 1, \ldots, L - 1, \tag{4.17}$$

so that $D \ne \varnothing$. By the concavity of the square root, it suffices to show that $\widetilde{C}l_L^{\frac{1}{2}+\varepsilon} - Cl_L^{\frac{1}{2}-\varepsilon} \ge \sqrt{2}$ for all $l = 1, \ldots,$ $L - 1$. This is certainly true for $l = 1$ or $l = L - 1$ by our requirements that imply that $\widetilde{C} \ge C + \sqrt{2}$, and the general case follows since $f(z) = \widetilde{C}z^{\frac{1}{2}+\varepsilon} - Cz^{\frac{1}{2}-\varepsilon}$ is monotone increasing in $z \ge 1$.

We now show that for $x, y \ge \sqrt{2}$

$$\mathbb{Q}_1^{x^2/2}(D) \ge c \frac{(x-a)(y-b)}{L} \sqrt{\frac{x/y}{L}} e^{-\frac{(x-y)^2}{2L}}. \tag{4.18}$$

To this end consider the event

$$E = \left\{ \sqrt{2\mathcal{L}_1} \in H_x, \sqrt{2\mathcal{L}_l} \in \text{Tube}_{C',C''}(l, L), l = 2, \ldots, L - 1; \sqrt{2\mathcal{L}_L} \in \left[y + \frac{\delta}{3}, y + \frac{2\delta}{3} \right] \right\},$$

where $C' = C + \delta/3$ and $C'' = C + \delta$, so that $C'' - C' = 2\delta/3$.

We show below that for some $\hat{c} > 0$

$$\mathbb{Q}_1^{x^2/2}(D \cap E) \ge \hat{c} \mathbb{Q}_1^{x^2/2}(E), \tag{4.19}$$

from which it will follow that

$$\mathbb{Q}_1^{x^2/2}(D) \ge \hat{c} \mathbb{Q}_1^{x^2/2}(E). \tag{4.20}$$

As before, using the Markov property

$$\mathbb{Q}_1^{x^2/2}(E) = \mathbb{Q}_1^{x^2/2}\left(\sqrt{2\mathcal{L}_1} \in H_x, \mathbb{Q}_2^{\mathcal{L}_1}\left(\sqrt{2\mathcal{L}_l} \in \text{Tube}_{C',C''}(l, L), \right. \right.$$

$$\left. \left. l = 2, \ldots, L - 1; \sqrt{2\mathcal{L}_L} \in \left[y + \frac{\delta}{3}, y + \frac{2\delta}{3} \right] \right) \right)$$

$$\ge \mathbb{Q}_1^{x^2/2}(\sqrt{2\mathcal{L}_1} \in H_x) \inf_{z \in H_x} \mathbb{Q}_2^{z^2/2}\left(\sqrt{2\mathcal{L}_l} \in \text{Tube}_{C',C''}(l, L), \right.$$

$$l = 2, \ldots, L-1; \sqrt{2\mathcal{L}_L} \in \left[y + \frac{\delta}{3}, y + \frac{2\delta}{3}\right]\right)$$

$$=: \mathbb{Q}_1^{x^2/2}(\sqrt{2\mathcal{L}_1} \in H_x) \inf_{z \in H_x} K_2(z). \tag{4.21}$$

By (3.7) we have that $\mathbb{Q}_1^{x^2/2}(\sqrt{2\mathcal{L}_1} \in H_x) \geq c > 0$ uniformly in all possible x. Once again, by (3.5), if $Y = Y_t$ is a 0-dimensional Bessel process then

$$K_2(z) = P\left(Y_l \in \text{Tube}_{C',C''}(l+1, L),\right.$$

$$\left. l = 1, \ldots, L-2, Y_{L-1} \in \left[y + \frac{\delta}{3}, y + \frac{2\delta}{3}\right]\Big| Y_0 = z\right), \tag{4.22}$$

and using the Markov property and then (2.11) (with y replaced by $y + \delta/3$ and with δ taken as $2\delta/3$), which requires $x, y \geq 1$ and $C'' - C' \geq 2\delta/3$, we obtain

$$K_2 \geq c\frac{(1 + x - a)(1 + y - b)}{L-1}\sqrt{\frac{x/y}{L-1}}e^{-\frac{(x-y)^2}{2(L-1)}}, \tag{4.23}$$

which because of our assumptions on x, y, a, b is greater than or equal to the right-hand side of (4.18). Together with (4.20) this proves (4.18) for $x, y \geq \sqrt{2}$.

Assume now that $y \leq \sqrt{2}$, which by our assumption requires that $x \geq L/\eta$. The tube at $l = L-1$ contains some interval $[z, z + \bar{\delta}]$ with $z \geq \sqrt{2}$ such that $[z, z + \bar{\delta}] \cap \sqrt{2}\mathbb{Z}^+ \neq \varnothing$. We then apply (4.18) for the interval of length $L-1$ with $\sqrt{2\mathcal{L}_{L-1}} \in [z, z + \bar{\delta}]$. Since now $\sqrt{\frac{x/z}{L-1}} \asymp 1$, it is easy to see that our desired lower bound follows from the fact that there is some positive probability to go from $[z, z + \bar{\delta}]$ to some point in $[y, y + \delta] \cap \sqrt{2}\mathbb{Z}^+$ in a single step.

We now turn to the proof of (4.19). By part (c) of Lemma 3.1,

$$\mathbb{Q}_1^{x^2/2}(D \cap E) = \mathbb{Q}_1^{x^2/2}\left\{1_E \mathbb{Q}_1^{x^2/2}\left(\sqrt{2T_1} \in \text{Tube}_{C,\widetilde{C}}(1, L)|\mathcal{L}_1, \mathcal{L}_2\right)\right.$$

$$\times \left(\prod_{l=2}^{L-1}\mathbb{Q}_1^{x^2/2}\left(\sqrt{2T_l} \in \text{Tube}_{C,\widetilde{C}}(l, L)|\mathcal{L}_l, \mathcal{L}_{l+1}\right)\right)$$

$$\left. \times \mathbb{Q}_1^{x^2/2}(\sqrt{2T_L} \in H_y|\mathcal{L}_L)\right\}. \tag{4.24}$$

Recall from (3.1) that the law of $\sqrt{2T_l}$ depends only on the product $\mathcal{L}_l\mathcal{L}_{l+1}$. Note that

$$\left|f_{x,y}(l+1; L) - f_{x,y}(l; L)\right| \leq \eta \tag{4.25}$$

with a similar bound for $f_{a,b}(l; L)$. It follows that on the event E

$$\left\{f_{a,b}(l; L) + C'l_L^{\frac{1}{2}-\varepsilon} - c'\right\} \leq (4\mathcal{L}_l\mathcal{L}_{l+1})^{1/4} \leq \left\{f_{x,y}(l; L) + C''l_L^{\frac{1}{2}+\varepsilon} + c''\right\}, \tag{4.26}$$

for $2 \leq l \leq L-1$ with $c' = \eta + C'$ and $c'' = \eta + C''$. Thus we have that on E

$$\left\{\left|\sqrt{2T_l} - (4\mathcal{L}_l\mathcal{L}_{l+1})^{1/4}\right| \leq \frac{\delta}{10}l_L^{\frac{1}{2}-\varepsilon}\right\} \subseteq \left\{\sqrt{2T_l} \in \text{Tube}_{C,\widetilde{C}}(l, L)\right\}, \tag{4.27}$$

for $k \le l \le L - k$ for some fixed k independent of L. Then by (3.16) we have

$$
1_E \prod_{l=k}^{L-k} \mathbb{Q}_1^{x^2/2} \big(\sqrt{2T_l} \in \text{Tube}_{C,\widetilde{C}}(l, L) | \mathcal{L}_l, \mathcal{L}_{l+1} \big)
$$

$$
\ge 1_E \prod_{l=k}^{L-k} \bigg(1 - \exp \Big(-c \Big(\frac{\delta}{10} \Big)^2 l_L^{1-2\varepsilon} \Big) \bigg) \ge 1_E c_*, \tag{4.28}
$$

for some $c_* > 0$ independent of L.

Consider now $2 \le l \le k$ or $L - k \le l \le L - 1$. By (4.26), on the event E we have

$$
\big\{ \sqrt{2T_l} \in (4\mathcal{L}_l \mathcal{L}_{l+1})^{1/4} + [r, s] \big\} \subseteq \big\{ \sqrt{2T_l} \in \text{Tube}_{C,\widetilde{C}}(l, L) \big\}, \tag{4.29}
$$

with

$$
[r, s] = \big[\eta + C', \eta + C'' + \sqrt{2} \big].
$$

Hence using (3.17) we see that for any such l,

$$
1_E \mathbb{Q}_1^{x^2/2} \big(\sqrt{2T_l} \in \text{Tube}_{C,\widetilde{C}}(l, L) | \mathcal{L}_l, \mathcal{L}_{l+1} \big) \ge 1_E c_l, \tag{4.30}
$$

for some $c_l > 0$ which depends on η, δ and C.

In addition, on the event E we have

$$
\sqrt{a} \sqrt{f_{a,b}(2; L) + C'} \le (4\mathcal{L}_1 \mathcal{L}_2)^{1/4} \le \sqrt{x + 1} \sqrt{f_{x,y}(2; L) + 2C''}.
$$

Thus, as in the last paragraph, we have by (3.17) that

$$
1_E \mathbb{Q}_1^{x^2/2} \big(\sqrt{2T_1} \in \text{Tube}_{C,\widetilde{C}}(1, L) | \mathcal{L}_1, \mathcal{L}_2 \big) \ge 1_E c_1, \tag{4.31}
$$

for some $c_1 > 0$ which depends on η, δ and C.

Finally, on the event E we have

$$
\sqrt{2\mathcal{L}_L} \in \bigg[y + \frac{\delta}{3}, y + \frac{2\delta}{3} \bigg].
$$

For all $y \ge y_0$ sufficiently large, for any $v \in [y + \frac{\delta}{3}, y + \frac{2\delta}{3}]$, we will have that $v + [0, \delta/3] \cap \sqrt{2\mathbb{N}} \ne \varnothing$. Then we have by (3.9) that for some for some $c_L > 0$

$$
1_E \mathbb{Q}_1^{x^2/2} \big(\sqrt{2T_L} \in H_{y,\delta} | \mathcal{L}_L \big) \ge 1_E c_L. \tag{4.32}
$$

On the other hand, we know from Lemma 3.1(a) that T_L conditioned on $\mathcal{L}_L = u$ is Poisson with parameter u, and since by our assumption $H_{y,\delta} \cap \sqrt{2\mathbb{Z}^+} \ne \varnothing$, it follows that for some possibly smaller $c_L > 0$, (4.32) also holds if $y \le y_0$.

This completes the proof of (4.19) with $\hat{c} = c_* \prod_{l=1}^{k} c_l \prod_{j=L-k}^{L} c_j > 0$, and hence of (4.18). $\qquad\square$

5. Tree cover time

In this section we prove Theorem 1.3 about the tightness of the cover time of the regular tree. We begin this section by introducing notation. Recall that $\mathcal{T}_L$ denotes a rooted tree, with root ρ to which one attaches a standard binary tree of depth L. Recall also that $\mathbb{P}$ is the probability of simple random walk on this tree starting from ρ. For a vertex $v \in \mathcal{T}_L$, we write $|v|$ for the tree distance of v from the root ρ. We define level l of the tree ($l = -1, 0, 1, \ldots, L$) as

those vertices at distance $l + 1$ from ρ, and write $\mathbf{L}_l$ for the vertices of $\mathcal{T}_L$ at level l. For $y \in \mathbf{L}_L$, we let $[y]_l$ denote the ancestor of y at level l, i.e. the unique vertex in $\mathbf{L}_l$ on the geodesic connecting y and ρ; thus $[y]_{-1}$ is the root ρ.

Let D_n be the time of the nth return to the root ρ. For each $y \in \mathbf{L}_L$, define

$$T_l^{y,n} = \text{\# of traversals of the edge } [y]_{l-1} \to [y]_l \text{ up to time } D_n.$$

The count $T_l^{y,n}$ is the edge local time of the directed edge $([y]_{l-1}, [y]_l)$ accumlated during the first n excursions from the root. Note that for $y \in \mathbf{L}_L$ we have

$$\tau_y > D_n \quad \text{if and only if} \quad T_L^{y,n} = 0. \tag{5.1}$$

When we observe the random walk on the tree only when it is on the path from ρ to a leaf $y \neq \rho$, we obtain a one dimensional simple random walk on $\{-1, 0, 1, \ldots, L\}$. As such the following is simply a restatement of the classical fact that the directed edge local times of 1D simple random walk are a Markov process, and more precisely a critical Galton–Watson with geometric offspring distribution (this is the discrete equivalent of the second Ray–Knight theorem).

Lemma 5.1. *For each n and each leaf $y \in \mathbf{L}_L$ the process $(T_l^{y,n})_{l=0,\ldots,L}$ has law P_n, i.e. is a Galton–Watson process with offspring distribution given by the geometric law with mean* 1.

Also note that

if y, z are two different leaves in $\mathbf{L}_L$ whose paths from root to leaf

overlap up to level k, then $T_l^{y,n} = T_l^{n,z}$ for $l \leq k$, $\tag{5.2}$

and conditionally on $T_k^{y,n} = T_k^{n,z}$ the processes $(T_l^{y,n})_{l \geq k}$, $(T_l^{n,z})_{l \geq k}$ are independent. Thus, in fact, the $T_l^{y,n}$ are a collection of branching Galton–Watson process, like a branching random walk where the random walk is replaced by the Galton–Watson process. This analogy allows us to use methods from branching random walks to prove Theorem 1.3.

Let $(T_l)_{l \geq 0}$ denote a generic Galton–Watson process with this offspring distribution. A basic bound for the increments of T_l is given by the following

$$P_n\big(|\sqrt{2T_l} - \sqrt{2T_0}| \geq z\big) \leq c e^{-\frac{z^2}{2l}}, \quad z \geq 0 \tag{5.3}$$

which is easily proved by computing the exponential moments of T_l using the representation of T_l as the sum of T_0 random variables, each being the product of a Bernoulli with parameter $\frac{1}{l}$ and a Geometric with parameter $\frac{1}{l}$, see [6, Lemma 4.6] (for $n = O(l^2)$ and $z = O(l)$ it is a special case of the more precise bound (1.10)).

Throughout this section we set

$$\kappa = \kappa_L = \sqrt{2 \log 2} - \frac{\log L}{\sqrt{2 \log 2L}}.$$

The main step in proving the upper bound (1.3) on the right tail is the following proposition which gives an upper bound in terms of the excursion counts – later we relate the excursion counts to the actual clock of the random walk X_n to obtain from this the upper bound of Theorem 1.3.

Proposition 5.2. *There is a constant c such that for all $L \geq 1$ and $x \geq 1$ we have*

$$\mathbb{P}\Big(\min_{y \in \mathbf{L}_L} T_L^{y,n} = 0\Big) \leq cx e^{-\kappa x}, \tag{5.4}$$

where $\sqrt{2n} = \kappa L + x$.

Proof. The probability in (5.4) is at most

$$
\underbrace{\mathbb{P}\Big(\exists y \in \mathbf{L}_L, l \le L - 1 \text{ s.t. } \sqrt{2T_l^{y,n}} < \alpha(l)\Big)}_{\text{(I)}}
$$

$$
+ \underbrace{\mathbb{P}\Big(\exists y \in \mathbf{L}_L \text{ s.t. } \sqrt{2T_l^{y,n}} \ge \alpha(l), l = 1, \ldots, L - 1, T_L^{y,n} = 0\Big)}_{\text{(II)}}, \tag{5.5}
$$

where we have split the event according to whether or not the processes $T_l^{y,n}$ stay above the barrier

$$
\alpha(l) = \kappa(L - l) - l_L^{1/6}
$$

(as typically they should).

By a union bound over the levels of the tree we have,

$$
\text{(I)} \le \sum_{k=1}^{L-2} \mathbb{P}\Big(\exists y \in \mathbf{L}_L \text{ s.t. } \sqrt{2T_l^{y,n}} \ge \alpha(l), l = 1, \ldots, k, \sqrt{2T_{k+1}^{y,n}} < \alpha(k+1)\Big). \tag{5.6}
$$

By Lemma 5.1, (5.2) and a union bound over the 2^k vertices in the kth level of the tree the kth summand is at most

$$
2^k P_n\big(\sqrt{2T_l} \ge \alpha(l), l = 1, \ldots, k, \sqrt{2T_{k+1}} < \alpha(k+1)\big). \tag{5.7}
$$

We condition on the height of the process at time k to obtain an upper bound of

$$
\sum_{j \ge 0} P_n\big(\sqrt{2T_l} \ge \alpha(l), l = 1, \ldots, k - 1, \sqrt{2T_k} \in H_{\alpha(k)+j}\big)
$$

$$
\times \sup_{z \in H_{\alpha(k)+j}} P_{\frac{z^2}{2}}\big(\sqrt{2T_1} \le \alpha(k+1)\big), \tag{5.8}
$$

the probability in the summand in (5.6). By (5.3),

$$
\sup_{z \in H_{\alpha(k)+j}} P_{\frac{z^2}{2}}\big(\sqrt{2T_1} \le \alpha(k+1)\big) \le ce^{-j^2/2}. \tag{5.9}
$$

To bound the first probability in (5.8), we use (1.1) with modified parameters, as follows. First, note that for any $\gamma < 1$,

$$
l_L^\gamma \le k_L^\gamma + l_k^\gamma.
$$

Setting, for $l = 0, \ldots, k$, $\hat{\alpha}(l) = f_{a,b}(l, L) - l_k^{1/6}$ with $a = \kappa L - k_L^{1/6}$ and $b = \kappa(L - k) - k_L^{1/6}$, we thus conclude that $\hat{\alpha}(l) \le \alpha(l)$ with $\alpha(k) = \hat{\alpha}(k)$. Therefore,

$$
P_n\big(\sqrt{2T_l} \ge \alpha(l), l = 1, \ldots, k - 1, \sqrt{2T_k} \in H_{\alpha(k)+j}\big)
$$

$$
\le P_n\big(\sqrt{2T_l} \ge \hat{\alpha}(l), l = 1, \ldots, k - 1, \sqrt{2T_k} \in H_{\hat{\alpha}(k)+j}\big). \tag{5.10}
$$

We now apply (1.1) with L replaced by k, the x of (1.1) replaced by $\sqrt{2n}$, $y = \hat{\alpha}(k) + j$, and these values of a, b, and obtain that the right hand side in (5.10) is bounded by

$$
c \frac{(x + k_L^{1/6})(j + 1)}{k} e^{-(\kappa k + x - j + k_L^{1/6})^2/2k} \le c\big(x + k_L^{1/6}\big)(j + 1)2^{-k}e^{\kappa j - \kappa x - \kappa k_L^{1/6}}.
$$

Combining with (5.9) and summing over j we get that (5.8) is at most $c2^{-k}(x + k_L^{1/6})e^{-\kappa(x+k_L^{1/6})}$. Thus from (5.6)

$$\text{(I)} \leq c \sum_{k=1}^{L-2} 2^k \left(2^{-k}(x + k_L^{1/6})e^{-\kappa x - \kappa k_L^{1/6}}\right) = ce^{-\kappa x} \sum_{k=1}^{L} (x + k_L^{1/6})e^{-\kappa k_L^{1/6}} \leq cxe^{-\kappa x}. \tag{5.11}$$

By Lemma 5.1 and a union bound over all leaves in $\mathbf{L}_L$ and (1.1) we have

$$\text{(II)} \leq 2^L P_n\left(\sqrt{2T_l} \geq \alpha(l), l = 1, \ldots, L, T_L = 0\right) \leq cxe^{-\kappa x},$$

if $x \geq 1$. Together with (5.11) and (5.5), this proves (5.4). $\qquad\square$

To prove a lower bound in terms of excursion counts we will consider the processes

$$T_l^{y,k,m} = \text{\# of traversals from } [y]_{l-1} \text{ to } [y]_l \text{ during}$$
$$\qquad \text{the first } m \text{ excursions from } [y]_k \text{ to } [y]_{k-1}.$$

Like $(T_l^{y,n})_{l\geq 0}$, the process $(T_{l+k}^{y,k,m})_{l\geq 0}$ is a critical Galton–Watson process with geometric offspring, but with initial population m i.e.

$$\text{the law of } \left(T_{l+k}^{y,k,m}\right)_{l\geq 0} \text{ under } \mathbb{P} \text{ is } P_m. \tag{5.12}$$

Also

$$\left(T_{l+k}^{y,k,m}\right)_{l\geq 0} \text{ and } \left(T_{l+k}^{z,k,m}\right)_{l\geq 0} \text{ are independent if the latest common}$$
$$\text{ancestor of leaves } y \neq z \text{ in the tree is in or above level } k. \tag{5.13}$$

Note further that

$$\text{on the event } T_k^{y,n} = m \text{ we have } T_{k+l}^{y,n} = T_{k+l}^{y,k,m} \text{ for } l = 0, \ldots L - k. \tag{5.14}$$

The main step in proving the lower bound on the right tail (1.4) and the upper bound on the left tail (1.5) is the following result for the processes $T_l^{y,k,m}$.

Lemma 5.3. *There is a constant c such that for all $x \geq 0$ and $k \geq 1$ we have*

$$\liminf_{L\to\infty} \mathbb{P}\left(\inf_{y\in\mathbf{L}_L} T_L^{y,k,m} = 0\right) \geq \frac{2^k(1+x)e^{-x\sqrt{2\log 2}}}{2^k(1+x)e^{-\sqrt{2\log 2}x} + c}, \tag{5.15}$$

where

$$\sqrt{2m} = \bar{\kappa}\bar{L} + x, \quad \text{for } \bar{L} = L - k, \text{ and } \bar{\kappa} = \kappa_{\bar{L}} = \sqrt{2\log 2} - \frac{1}{\sqrt{2\log 2}} \frac{\log \bar{L}}{\bar{L}}.$$

Proof. Consider the event

$$I_y = \left\{\bar{\kappa}(\bar{L} - l) + (l_{\bar{L}})^{1/4} \leq \sqrt{2T_{k+l}^{y,k,m}} \text{ for } l = 1, \ldots, \bar{L} - 1, T_{k+\bar{L}}^{y,k,m} = 0\right\},$$

that $(T_{l+k}^{y,k,m})_{l\geq 0}$ stays above the barrier $\bar{\kappa}(\bar{L} - l) + (l_{\bar{L}})^{1/4}$ and ends up at zero, and the counting random variable

$$\mathcal{N}_k = \sum_{y\in\mathbf{L}_L} 1_{I_y}.$$

If $\mathcal{N}_k \geq 1$ then $\min_{y \in \mathbf{L}_L} T_L^{y,k,m} = 0$ so to prove (5.15) it suffices to show that for large enough L

$$\mathbb{P}(\mathcal{N}_k \geq 1) \geq \frac{2^k(1+x)e^{-\bar{\kappa}x}}{2^k(1+x)e^{-\bar{\kappa}x} + c}. \tag{5.16}$$

Using (1.1)–(1.2) one has

$$\mathbb{E}(\mathcal{N}_k) \asymp 2^L 2^{-\bar{L}}(1+x)e^{-\bar{\kappa}x} = 2^k(1+x)e^{-\bar{\kappa}x},$$

and by the Paley–Zygmund inequality

$$\mathbb{P}(\mathcal{N}_k \geq 1) \geq \frac{\mathbb{E}(\mathcal{N}_k)^2}{\mathbb{E}(\mathcal{N}_k^2)}. \tag{5.17}$$

To prove (5.16) we estimate the second moment of $\mathcal{N}_k$. We have

$$\begin{aligned}
\mathbb{E}(\mathcal{N}_k^2) &= \sum_{y,z} \mathbb{P}(I_y \cap I_z) \\
&= \sum_{y,z:\text{ branch early}} \mathbb{P}(I_y)\mathbb{P}(I_z) + \sum_{y,z:\text{ branch late}} \mathbb{P}(I_y \cap I_z),
\end{aligned}$$

where the first sum is over pairs of leaves whose most recent common ancestor lie in level k of the tree or above, so that the events I_y and I_z are independent, and the second sum is over all other pairs of leaves. The first sum is at most $2^{2L}\mathbb{P}(I_y)^2 = \mathbb{E}(\mathcal{N}_k)^2$. Thus (5.17) implies (5.16) once we have shown that for large enough L

$$\sum_{y,z:\text{ branch late}} \mathbb{P}(I_y \cap I_z) \leq c 2^k(1+x)e^{-\bar{\kappa}x}. \tag{5.18}$$

To bound this sum we sum over the possible level of the common ancestor of y, z:

$$\sum_{y,z:\text{ branch late}} \mathbb{P}(I_y \cap I_z) \leq 2^k \sum_{j=0}^{\bar{L}} 2^{2\bar{L}-j} p_j, \tag{5.19}$$

where $p_j = \mathbb{P}(I_y \cap I_z)$ for y, z whose common ancestor is in level $k + j$ of the tree and we have used that there are at most $2^k 2^{2\bar{L}-j}$ such pairs. Now by conditioning on the value of the processes at the point where they branch, we have

$$p_j \leq \sum_{u \geq 0} q_j(u) r_j(u)^2, \tag{5.20}$$

where we sum over positive integers u and where

$$q_j(u) = P_m\left(\bar{\kappa}(\bar{L} - l) \leq \sqrt{2T_l}, l = 1, \ldots, j-1, \sqrt{2T_j} \in H_{m_u}\right),$$

and

$$r_j(u) = \max_{z \in H_{m_u}} P_{\frac{z^2}{2}}\left(\bar{\kappa}(\bar{L} - j - l) \leq \sqrt{2T_l}, l = 1, \ldots, \bar{L} - j - 1, T_{\bar{L}-j} = 0\right),$$

with

$$m_u = \bar{\kappa}(\bar{L} - j) + j_{\bar{L}}^{1/4} + u.$$

By (1.1),

$$q_j(u) \le c\frac{1}{j}(1+x)\big(1+j_{\bar{L}}^{1/4}+u\big)e^{-\frac{(j\bar{\kappa}+x-j_{\bar{L}}^{1/4}-u)^2}{2j}},$$

and

$$r_j(u) \le c\frac{1}{\bar{L}-j}\big(1+j_{\bar{L}}^{1/4}+u\big)e^{-\frac{((\bar{L}-j)\bar{\kappa}+j_{\bar{L}}^{1/4}+u)^2}{2(\bar{L}-j)}}.$$

Since

$$\frac{(j\bar{\kappa}+x-j_{\bar{L}}^{1/4}-u)^2}{2j}+2\frac{((\bar{L}-j)\bar{\kappa}+j_{\bar{L}}^{1/4}+u)^2}{2(\bar{L}-j)}$$

$$\ge \frac{\bar{\kappa}^2}{2}(2\bar{L}-j)+\bar{\kappa}x+cj_{\bar{L}}^{1/4}+cu,$$

and $e^{-\frac{\bar{\kappa}^2}{2}(2\bar{L}-j)} = \bar{L}^{\frac{2\bar{L}-j}{L}}2^{-(2\bar{L}-j)} \le \bar{L}(\bar{L}-j)2^{-2(\bar{L}-j)}$ we thus have from (5.20) that

$$p_j \le c2^{-(2\bar{L}-j)}\bar{L}(\bar{L}-j)\frac{1}{j(\bar{L}-j)^2}(1+x)\big(1+j_{\bar{L}}^{1/4}\big)^3 e^{-\bar{\kappa}x-cj_{\bar{L}}^{1/4}}.$$

Thus the right-hand side of (5.19) is at most

$$2^k\sum_{j=1}^{\bar{L}}\frac{\bar{L}}{j(\bar{L}-j)}(1+x)\big(1+j_{\bar{L}}^{1/4}\big)^3 e^{-\bar{\kappa}x-cj_{\bar{L}}^{1/4}}.$$

Since the sum is bounded by a constant for all $\bar{L}$ we get that

$$\sum_{y,z:\text{ branch late}}\mathbb{P}(I_y\cap I_z)\le c2^k(1+x)e^{-\bar{\kappa}x},$$

which completes the proof of (5.18), and therefore also of (5.16) and (5.22). $\qquad\square$

By simply setting $k=0$ and bounding $xe^{-x\sqrt{2\log 2}}$ by a constant we obtain a lower bound on the right tail in terms of excursions. This will later lead to (1.4).

Corollary 5.4. *For all $x \ge 1$,*

$$\liminf_{L\to\infty}\mathbb{P}\Big(\inf_{y\in\mathbf{L}_L}T_L^{y,n}=0\Big)\ge cxe^{-x\sqrt{2\log 2}}, \tag{5.21}$$

where $\sqrt{2n}=\kappa L+x$.

A lower bound for the cover time in terms of excursions, which will later lead to (1.5) is given by the following proposition.

Proposition 5.5. *There is a constant c such for all $x \ge 0$,*

$$\liminf_{L\to\infty}\mathbb{P}\Big(\min_{y\in\mathbf{L}_L}T_L^{y,n}=0\Big)\ge 1-ce^{-cx}, \tag{5.22}$$

where $\sqrt{2n}=\kappa L-x$.

Proof. We will fix $k = k(x)$ below. From (5.14) it follows that

$$T_k^{y,n} \leq m \text{ and } T_{L-k}^{y,k,m} = 0 \quad \Longrightarrow \quad T_L^{y,n} = 0. \tag{5.23}$$

We consider the processes $(T_{k+l}^{y,k,m})_{l=0,\dots,\bar{L}}$ where $\bar{L} = L - k$. Using (5.3) we have for any $k, a, L \geq 1$ that

$$\mathbb{P}\Big(\max_{y \in \mathbf{L}_L} \sqrt{2T_k^{y,n}} > (\kappa L - x) + \kappa k + a\Big) \leq 2^k e^{-\frac{(\kappa k + a)^2}{2k}} \leq cL^{k/L} e^{-\kappa a},$$

by a union bound over the 2^k vertices in level k of the tree, which is justified by (5.2). Setting $k = cx$ and $a = x - 2\kappa k - \frac{k}{c} \frac{\log L}{L}$ one has $\bar{\kappa} \bar{L} \geq \kappa L - x + \kappa k + a$ and $a \geq cx$ for small enough c and large enough L, so that

$$\limsup_{L \to \infty} \mathbb{P}\Big(\max_{y \in \mathbf{L}_L} \sqrt{2T_k^{y,n}} > \bar{\kappa} \bar{L}\Big) \leq ce^{-cx}. \tag{5.24}$$

Setting $\sqrt{2m} = \bar{\kappa} \bar{L}$ and using Lemma 5.3 with 0 in place of x it follows that

$$\liminf_{L \to \infty} \mathbb{P}\Big(\inf_{y \in \mathbf{L}_L} T_L^{y,k,m} = 0\Big) \geq \frac{2^k}{2^k + c} \geq 1 - c2^{-k}. \tag{5.25}$$

Together with (5.24) this implies

$$\liminf_{L \to \infty} \mathbb{P}\big(\{\forall y \in \mathbf{L}_L \ T_k^{y,k} \leq m\} \cap \{\exists y \in \mathbf{L}_L \text{ s.t. } T_L^{y,k,m} = 0\}\big)$$

$$\geq 1 - c2^{-k} - ce^{-cx},$$

which implies (5.22), because of (5.23) and since $k = cx$. $\qquad\square$

We now relate excursion time to real time to derive Theorem 1.3 from Proposition 5.2 and Proposition 5.5. Note that

$$D_n = S_1 + S_2 + \cdots + S_n,$$

where S_i is the length of the ith excursion from the root. By the strong Markov property the S_i, $i \geq 1$, are i.i.d. Elementary 1D random walk computations show that $\mathbb{E}(S_i) = 2^{L+2} - 2$ and by Khasminskii's lemma (a consequence of Kac's moment formula, see [16, (6)]), we have $\mathbb{E}(S_i^k) \leq k!(\max_{v \in \mathcal{T}_L} \mathbb{E}_v(\tau_\rho))^k$, where τ_ρ is the hitting time of the root and $\mathbb{E}_v$ is the expectation for X_n starting at v. A standard computation thus gives that $\mathbb{E}(S_i^k) \leq c_k 2^{kL}$, so that by the central limit theorem, $\frac{D_n - n\mathbb{E}(S_1)}{\sqrt{n}\sqrt{\mathrm{Var}(S_1)}}$ converges to a normal distribution as $n \to \infty$. Also $\mathbb{E}(S_i^3) \leq c2^{3L} \leq c(\sqrt{\mathrm{Var}(S_i)})^3$ (it is easily seen that $\mathrm{Var}(S_i) \geq c2^{2L}$), so that by the Berry–Essen theorem in addition

$$\sup_{x \in \mathbb{R}} \left|\mathbb{P}\Big(\frac{D_n - n\mathbb{E}(S_1)}{\sqrt{n}\sqrt{\mathrm{Var}(S_1)}} \leq x\Big) - \Phi(x)\right| \leq \frac{c}{\sqrt{n}}, \tag{5.26}$$

uniformly in L. We now prove the estimate Theorem 1.3 for the cover time.

Proof of Theorem 1.3. For the upper bound, let

$$\sqrt{2n} = \sqrt{2\log 2}L - \frac{1}{\sqrt{2\log 2}}\log L + \frac{x}{2}.$$

Because of (5.1) we have

$$\mathbb{P}\bigg(C_L > 2^{L+1}\Big(\sqrt{2\log 2}L - \frac{1}{\sqrt{2\log 2}}\log L + x\Big)^2\bigg)$$

$$\leq \mathbb{P}\bigg(D_n > 2^{L+1}\Big(\sqrt{2\log 2}L - \frac{1}{\sqrt{2\log 2}}\log L + x\Big)^2\bigg) + \mathbb{P}\Big(\min_{y \in \mathbf{L}_L} T_L^{y,n} = 0\Big). \tag{5.27}$$

Since

$$2^{L+1}\left(\sqrt{2\log 2}\,L - \frac{1}{\sqrt{2\log 2}}\log L + x\right)^2$$
$$= 2^{L+2}L\big((\log 2)L - \log L + \sqrt{2\log 2}\,x + o(1)\big),$$

and

$$n = L\left((\log 2)L - \log L + \sqrt{2\log 2}\frac{x}{2} + o(1)\right),$$

we have from (5.26) that

$$\mathbb{P}\left(D_n > 2^{L+1}\left(\sqrt{2\log 2}\,L - \frac{1}{\sqrt{2\log 2}}\log L + \sqrt{2\log 2}\,x\right)^2\right) \le ce^{-cx^2} + \frac{c}{\sqrt{n}},$$

and combining this with Proposition 5.2 we get the claim (1.3) from (5.27). Since the right-hand side tends to zero as $x \to \infty$ this proves the upper bound on the right tail (1.3). The lower bound (1.4) on the right tail follows similarly from (5.21) and (5.26).

For the bound (1.5) on the left tail let

$$\sqrt{2n} = \sqrt{2\log 2}\,L - \frac{1}{\sqrt{2\log 2}}\log L - \frac{x}{2}.$$

We have that

$$\mathbb{P}\left(C_L \le 2^{L+1}\left(\sqrt{2\log 2}\,L - \frac{1}{\sqrt{2\log 2}}\log L - x\right)^2\right)$$
$$\le \mathbb{P}\left(D_n \le 2^{L+1}\left(\sqrt{2\log 2}\,L - \frac{1}{\sqrt{2\log 2}}\log L - x\right)^2\right) + \mathbb{P}\left(\min_{y\in\mathbf{L}_L} T_L^{y,n} > 0\right).$$

Similarly to above (5.26) implies that is at most ce^{-x^2}, and the second term is at most ce^{-cx} by Proposition 5.5, which implies (1.5). $\qquad\square$

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